Loop Quantum Cosmology

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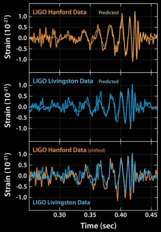
Outline:

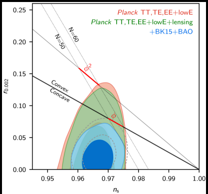
- Two important developments 60 years ago: genericness of singularities in GR and the cosmic microwave background

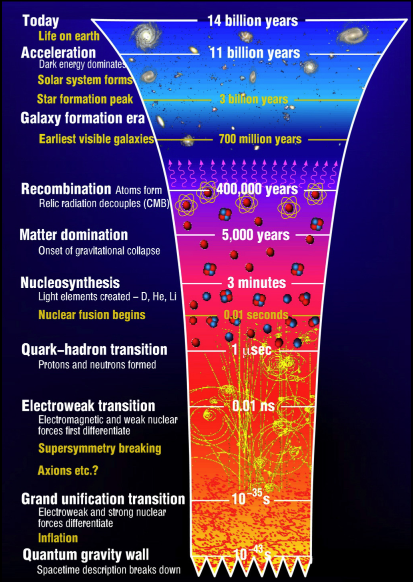
- Part A: an overview of singularity resolution in LQC

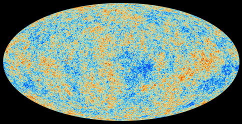
- Part B: Hamiltonian framework for cosmological perturbations and how to explore quantum geometry effects in CMB?

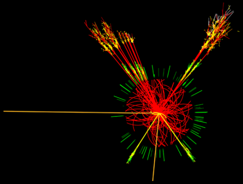
An almost complete history of our Universe











The Expanding Universe

From the spectra emitted by the galaxies, Hubble in 1920's discovered that they are moving farther from each other. Fainter the galaxy, faster it recedes.

Friedmann and Lemaitre found a solution in GR in which the universe was expanding. Primevial Atom as the seed of the Universe.

The expansion of the universe led to many questions. How did the universe begin and how did the galaxies form?

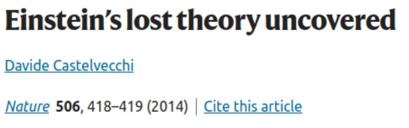
And the problem of initial singularity.

Universe without initial singularity

Eddington believed there is nothing wrong with GR and there is no initial singularity. It was believed to be an artifact of simplifying assumptions of isotropy and homogeneity.

"Philosophically, the notion of a beginning of the present order of Nature is repugnant to me. I should like to find a genuine loophole." (Eddington, 1931)

Eddington's ideas greatly influenced Hoyle who later pioneered the Steady State Theory based on Perfect Cosmological Principle – the universe looks same not only in space but also in time. No initial singularity.



Einstein's lost theory uncovered

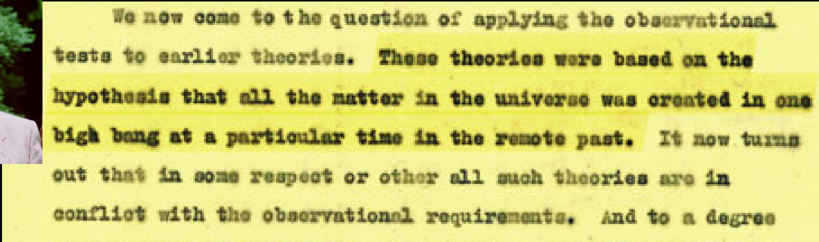
Davide Castelvecchi

Nature **506**, 418–419 (2014) | Cite this article

The "Big Bang"

Gamow developed in detail Lemaitre's preliminary ideas and explored the ultra dense state of matter from which elements would originate. Predicted the universe will have a very cold background radiation (CMB).

Big Bang was coined by (his arch rival) Hoyle in a BBC interview in April 1949.



We now come to the question of applying the observational tests to earlier theories. These theories were based on the hypothesis that all the matter in the universe was created in one big bang at a particular time in the remote past. It now turns out that in some respect or other all such theories are in conflict with the observational requirements. And to a degree

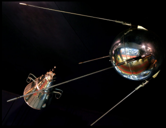
Genericness of Singularities

In 1950's Raychaudhuri proved Eddington wrong by showing existence of singularities in anisotropic and homogeneous spacetimes. Discovered Raychaudhri equation which plays an important role in understanding the attractive nature of gravity and divergence of geodesics.

In 1960's Geroch, Penrose and Hawking proved that singularities are generic in GR. Null energy condition must be violated to avoid singularities; $(\rho + P \geq 0)$

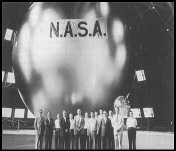
Something unexpected from geopolitics

1945: Arthur C Clarke conceptualized communication satellites.



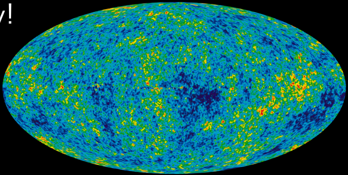
1957: Sputnik 1 launched.

1958: NASA born.



1959: Project Echo (balloon satellites)

1964: Penzias and Wilson found a mysterious noise coming from outside our galaxy!



Fundamental questions:

Is our universe described by the classical continuum spacetime at all the scales?

What is the quantum nature of spacetime?

If the spacetime has an "atomic structure", what is the fate of big bang and black hole singularities?

How does a quantum spacetime affect the physics of very early universe and in the interior of black holes?

Does quantum spacetime leave any signatures in the phenomenology of very early universe in the CMB?

Part A

An Overview of Singularity Resolution in LQC

Useful References for Part A

- A. Ashtekar, M. Bojowald and J. Lewandowski, "Mathematical structure of loop quantum cosmology," Adv. Theor. Math. Phys. **7**, no.2, 233-268 (2003)
- A. Ashtekar, T. Pawlowski and P. Singh, "Quantum Nature of the Big Bang: Improved dynamics," Phys. Rev. D **74**, 084003 (2006)
- A. Ashtekar and P. Singh, "Loop Quantum Cosmology: A Status Report," Class. Quant. Grav. **28**, 213001 (2011)
- I. Agullo and P. Singh, "Loop Quantum Cosmology," [arXiv:1612.01236 [gr-qc]]
- B. F. Li and P. Singh, "Loop Quantum Cosmology: Physics of Singularity Resolution and its Implications," [arXiv:2304.05426 [gr-qc]]

Motivation

Symmetry reduced spacetimes, such as cosmological, anisotropic and black hole interiors provide a tractable, non-trivial and rich setting to implement techniques of LQG. Kindergarten to learn valuable lessons for quantization of spacetimes and to gain insights on the physics at the Planck scale.

Motivation

Symmetry reduced spacetimes, such as cosmological, anisotropic and black hole interiors provide a tractable, non-trivial and rich setting to implement techniques of LQG. Kindergarten to learn valuable lessons for quantization of spacetimes and to gain insights on the physics at the Planck scale.

What can one learn in this quantum gravity playground?

- Rigorous construction of self consistent model quantum spacetimes. (Physical Hilbert space, observables).
- Develop and rigorously test different tools and techniques to extract reliable physics. (Bridges quantum gravity with other areas such as numerical relativity).
- How to rule out different quantizations using internal consistency and physical predictions.
- Understand potential quantum gravity implications for early universe and test them using astronomical observations.
- Provide insights on fundamental issues. (Consistent quantum probabilities, Black hole information loss).

Main Caveat: Quantization of homogeneous spacetimes is "quantum mechanics of spacetime." Where as full quantum gravity is "QFT of spacetime." Assuming homogeneity of spacetime, various hurdles of the full quantum gravity can be bypassed. Hope is that some qualitative aspects captured.

Cosmological model: some preliminaries

Spatially flat homogeneous and isotropic FLRW spacetime

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 \; + \; a^2(t)\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) \qquad (1)$$

where lapse $N = 1$ and $a(t)$ is the scale factor. Consider this universe filled with a perfect fluid

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \qquad (2)$$

where u^μ is the velocity relative to the comoving observer (which follows Hubble flow). Energy density ρ and pressure $P = -\frac{\partial H_m}{\partial a^3}$ satisfy the conservation law obtained from $T^\mu_{\nu;\mu} = 0$

$$\dot{\rho} + 3H(\rho + P) = 0. \qquad (3)$$

For a fixed equation of state $w = P/\rho$, $\rho \propto a^{-3(1+w)}$.

Pressurless dust: $\rho \propto a^{-3}$, $w = 0$

Relativistic matter/radiation: $\rho \propto a^{-4}$, $w = -1/3$

Stiff matter: $\rho \propto a^{-6}$, $w = 1$; Dark energy: $\rho \propto a^{0\pm\epsilon}$, $w \approx -1$
Cosmological constant: $\rho = \mathrm{constt.}$, $w = -1$.

Cosmological model: some preliminaries

From the Einstein's field equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{4}$$

we can obtain the Friedmann equation:

$$H^2 := \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho, \tag{5}$$

where H is the Hubble rate, and the Raichaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{6}$$

Gravity is attractive when $\rho + 3P > 0$. This is strong energy condition (SEC). The Ricci scalar is

$$R = 6\left(H^2 + \frac{\ddot{a}}{a}\right) = 8\pi G(\rho - 3P) \tag{7}$$

Integrating dynamical equations gives $a \propto t^{2/3(1+w)}$ for $w \neq -1$, and $a \propto e^{\sqrt{\Lambda}t}$ for $w = -1$ and $\Lambda = \mathrm{constt.}$.
When $a \to 0$, $\rho \to 0$ in finite time and big bang is reached.

Big bang is not the only kind of cosmological singularity!

Depending on the equation of state there can be various types of singularities which can be classified as strong and weak.

Big Bang/Crunch: ρ, P, R diverge when $a \rightarrow 0$ in finite time. NEC: $(\rho + P) \geq 0$, is always satisfied. Strong singularity.

Big Rip/Type I singularity: NEC violated. In finite time, $a(t) \rightarrow \infty$. Accompanied with a divergence of ρ, P, R. Strong singularity.

Sudden or Type II singularity: At a finite value of the scale factor and energy density, R diverges. Needs exoitic equation of state. Weak singularity.

Type III singularity: Occurs at a finite value of scale factor. Both the energy density and pressure diverge. Strong singularity.

Type IV singularity: Only curvature derivatives blow up. Weak singularity.

Horizon Problem and Inflation

Particle horizon: Maximum comoving distance light can travel in given time. For fixed equation of state $w = P/\rho$:

$$\eta = \int_{t_i}^{t_f} \frac{\mathrm{d}t}{a(t)} = \int_{\ln a_i}^{\ln a_f} \frac{\mathrm{d}\ln a}{aH} = H_0^{-1} \int_{\ln a_i}^{\ln a_f} a^{(1+3w)/2}\,\mathrm{d}\ln a \qquad (8)$$

If strong energy condition (SEC) is satisfied, comoving Hubble radius $(aH)^{-1}$ increases during expansion. For dust, radiation, massless scalar, the Hubble sphere grows.

Horizon Problem: How do we explain almost perfect isotropic nature of CMB in standard big bang model? There are roughly 10000 disconnected patches! Any two points which are more than a degree apart were never in causal contact.

If Hubble sphere decreases during expansion in the early universe, it can explain causal connection between different points in CMB.

Implies violation of SEC. Results in necessity of **inflation**.

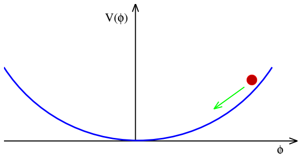
Inflation

A phase of accelerated expansion in the early universe where SEC is violated. Popular paradigm to explain observations by introducing scalar field potentials.

$$\rho = \dot{\phi}^2/2 + U, \quad P = \dot{\phi}^2/2 - U \tag{9}$$

Conservation law results in Klein-Gordon eq:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + U_{,\phi} = 0 \tag{10}$$

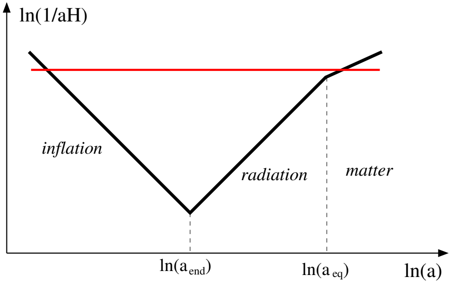


Past incomplete **(Borde, Guth, Vilenkin (03))**

Inflaton slow rolls down: $\dot{\phi}^2 \ll U$ implying $w \approx -1$ and causing an almost exponential expansion measured in number of e-foldings $N := \ln(a_e/a_i)$.

Inflation

Comoving horizon shrinks allowing causal contact between different points in the CMB. Inhomogenities arise from the quantum fluctuations of the inflaton which freeze out on exiting horizon and generate classical density perturbations on re-entry.



In the Fourier space, the power spectrum of these primordial perturbations turns out to be almost scale invariant – that is, almost independent of the wavenumber of the Fourier modes in the observational regime.

Strategy to extract quantum cosmological effects

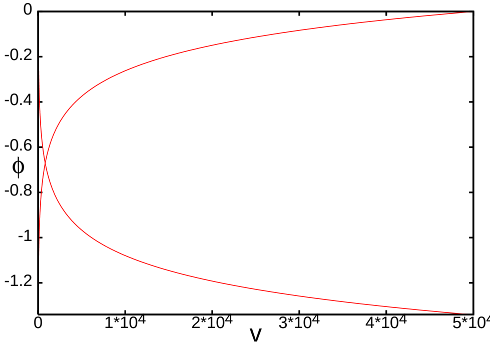
- Quantize the classical system. Find physical Hilbert space: inner product, Dirac observables, physical states.
- Consider physical initial states (such as in the GR epoch) and evolve using quantum Hamiltonian constraint. Almost on all occasions, models not exactly solvable therefore numerical simulations necessary.
- Compute expectation values of observables (and their fluctuations). Compare with the classical trajectory. Obtain departures between GR and quantum model.
- Make precise statements about how singularity resolution occurs. Behavior of energy density, shear scalar etc.
- Extract robust phenomenological predictions.

Due to the underlying symmetries, spatial diffeomorphism constraint is satisfied and the only non-trivial constraint is the Hamiltonian constraint.

Matter Hamiltonian: $\mathcal{H}_\phi = P_\phi^2/2V$

$V = V_o\, a^3$ where V_o is the volume of the fiducial cell introduced to define symplectic structure.

Hamilton's equations yield: $P_\phi = \text{constant}, \;\; \phi \sim \log v, \;\; \rho \propto a^{-6}$



ϕ acts as a "clock." Two solutions: an expanding and a contracting universe (both solutions are singular).

Wheeler-DeWitt quantization

Quantize geometry and matter for a homogeneous universe.
Only finite number of degrees of freedom, system can be treated quantum mechanically.

Earlier attempts based on treating spatial metric and its conjugate as phase space variables (Misner, Wheeler, DeWitt 1970's):

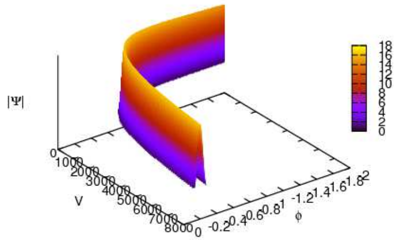
- Basic variables: $v, p_v \propto \dot{v}$ (geometry), ϕ, p_ϕ (matter).
- Operators: $\hat{v}\,\Psi(v,\phi) = v\,\Psi(v,\phi)$, $\hat{p}_v\,\Psi(v,\phi) = -i\hbar\frac{\partial}{\partial v}\Psi(v,\phi)$
- Hamiltonian: $(\hat{v}\,\hat{p}_v)^2\,\Psi(v,\phi) = \hat{\mathcal{H}}_\phi\,\Psi(v,\phi)$
- For a massless scalar, quantum Hamiltonian is:

$$\frac{\partial^2}{\partial\alpha^2}\,\Psi(\alpha,\phi) = \frac{\partial^2}{\partial\phi^2}\,\Psi(\alpha,\phi), \qquad \alpha = \log v$$

- Observables, inner product available.
- To extract departures from General Relativity, consider a semi-classical state at late times (present epoch) and evolve backwards towards big bang.

Is singularity resolved in the backward evolution?

Wheeler-DeWitt states just follow the classical trajectory, all the way to the big bang.



Singularity is not resolved! What went wrong?

- A straight forward union of quantum theory and gravity does not work. No guidance from a full theory of quantum gravity.
- Properties of spacetime same as in the classical theory.

Towards loop quantization

Due to the symmetries of the isotropic and homogeneous spacetime, the connection A^i_a and triad E^a_i can be written as

$$A^i_a \,=\, c\, V_o^{-1/3}\, \mathring{\omega}^i_a, \quad E^a_i \,=\, p\, V_o^{-2/3}\, \sqrt{\mathring{q}}\, \mathring{e}^a_i \;, \tag{11}$$

where c and p denote the isotropic connection and triad, and $\mathring{e}^a_i$ and $\mathring{\omega}^i_a$ are the fiducial triads and co-triads compatible with the fiducial metric $\mathring{q}_{ab}$. The pair (c, p) satisfies

$$\{c, p\} = \frac{8\pi G\gamma}{3}; \qquad \gamma \approx 0.2375(\text{from BH thermo}) \tag{12}$$

Related to the metric variables as

$$|p| = V_o^{2/3} a^2 \tag{13}$$

and (only in GR as)

$$c = \gamma V_o^{1/3} \frac{\dot{a}}{N}. \tag{14}$$

Promote the classical phase variables and the classical Hamiltonian constraint to their quantum operator analogs. Holonomies of the connection A_a^i along edges, and the fluxes of the triads along 2-surfaces. Due to homogeneity the latter is proportioinal to triad.

The holonomy of the symmetry reduced connection A_a^i along a straight edge $\mathring{e}_k^a$ with fiducial length μ is,

$$h_k^{(\mu)} \; = \; \cos\left(\frac{\mu c}{2}\right)\mathbb{I} + 2\sin\left(\frac{\mu c}{2}\right)\tau_k \tag{15}$$

$\mathbb{I}$ is a unit 2×2 matrix and $\tau_k = -i\sigma_k/2$, where σ_k are the Pauli spin matrices.

Captured by functions $N_\mu(c) := e^{i\mu c/2}$. Since μ can take arbitrary values, N_μ are almost periodic functions of the connection c.

There exists a unique kinematical representation of algebra generated by these functions (Ashtekar, Campiglia (12); Engle, Hanusch, Thiemann (16)). Parallels existence of a unique irreducible representation of the holonomy-flux algebra in full LQG (Lewandowski, Okolow, Sahlmann, Thiemann (06); Fleishchack (09))

The gravitational sector of $\mathcal{H}_{\rm kin}$ is $L^2(\mathbb{R}_{\rm Bohr}, {\rm d}\mu_{\rm Bohr})$. The kinematical Hilbert space in LQC is fundamentally different from one in the Wheeler-DeWitt theory. von-Neumann theorem is bypassed.

Action of operators:

$$\hat{N}_\zeta \, \Psi(\mu) = \Psi(\mu + \zeta), \tag{16}$$

where ζ is a constant

$$\hat{p} \, \Psi(\mu) = \frac{8\pi\gamma l_{\rm Pl}^2}{6} \, \mu\Psi(\mu) \ . \tag{17}$$

Change in the orientation of the triads in absence of fermions is a large gauge transformation by a parity operator: $\hat{\Pi}\Psi(\mu) = \Psi(-\mu)$. We choose symmetric states satisfying $\Psi(\mu) = \Psi(-\mu)$.

The field strength $F_{ab}^{\ k}$ in the Hamiltonian constraint is expressed in terms of the holonomies over a square plaquette $\Box_{ij}$ with length $\bar{\mu} V_o^{1/3}$ in the $i - j$ plane spanned by fiducial triads:

$$F_{ab}^{\ k} = -2 \lim_{Ar\Box \to 0} \mathrm{Tr} \left(\frac{h_{\Box_{ij}} - \mathbb{I}}{Ar\Box} \tau^k \right) \mathring{\omega}_a^i \, \mathring{\omega}_b^j \ . \qquad (18)$$

Ar$\Box$ denotes the area of the square plaquette, and $h_{\Box_{ij}} = h_i^{(\bar{\mu})} h_j^{(\bar{\mu})} (h_i^{\bar{\mu}})^{-1} (h_j^{\bar{\mu}})^{-1}$, with $\bar{\mu}$ denoting the edge length of the plaquette.

Note that due to the underlying quantum geometry, the limit Ar$\Box \to 0$ does not exist. Shrink the area of the loop to the minimum non-zero eigenvalue of the area operator in LQG.

Denote this minimum area as Δl_{Pl}^2 where $\Delta = 4\sqrt{3}\pi\gamma$.

Using physical area of the loop equalling $\bar{\mu}^2 |p|$ results in **(Ashtekar, PS, Pawlowski (06))**

$$\bar{\mu}^2 = \frac{\Delta l_{\mathrm{Pl}}^2}{|p|} \qquad (19)$$

Action of $N_{\bar{\mu}}$ on the triad eigenstates is not by a simple translation.

More convenient to work with variables b and v which are defined in terms of c and p as:

$$\mathrm{b} := \frac{c}{|p|^{\frac{1}{2}}}, \quad \mathrm{v} := \mathrm{sgn}(p)\frac{|p|^{\frac{3}{2}}}{2\pi G}, \tag{20}$$

which when promoted to operators have action:

$$\widehat{\exp(i\lambda \mathrm{b})}\,|\nu\rangle \;=\; |\nu - 2\lambda\rangle, \qquad \hat{V}\,|\nu\rangle \;=\; 2\pi\gamma l_{\mathrm{Pl}}^2\,|\nu|\,|\nu\rangle \tag{21}$$

where $\nu = \mathrm{v}/\gamma\hbar$. Quantum Hamiltonian constraint equation:

$$\partial_\phi^2\,\Psi(\nu,\phi) \;=\; 3\pi G\,\nu\,\frac{\sin\lambda \mathrm{b}}{\lambda}\nu\,\frac{\sin\lambda \mathrm{b}}{\lambda}\,\Psi(\nu,\phi) \;=:\; -\Theta\Psi(\nu,\phi) \tag{22}$$

where Θ is a positive definite, second order difference operator:

$$\Theta\Psi(\nu,\phi) \;:=\; -\frac{3\pi G}{4\lambda^2}\,\nu\left((\nu+2\lambda)\Psi(\nu+4\lambda) - 2\nu\Psi(\nu,\phi) + (\nu-2\lambda)\Psi(\nu-4\lambda)\right) \tag{23}$$

(Ashtekar, PS, Pawlowski (06); Ashtekar, Corichi, PS (08))

Quantum difference equation resulting from quantum geometry results in Wheeler-DeWitt differential equation at large volumes.

Quantum constraint similar to the Klein-Gordon theory, ϕ plays the role of time and Θ acts like a spatial Laplacian operator. Physical states can be chosen as solutions of the positive frequency square root of Θ:

$$-i\,\partial_\phi \Psi(\nu,\phi) = \sqrt{\Theta}\,\Psi(\nu,\phi)\ . \tag{24}$$

Inner product:

$$\langle \Psi_1 | \Psi_2 \rangle \;=\; \sum_{\nu}\, \bar{\Psi}_1(\nu,\phi_o)|\nu|^{-1}\Psi(\nu_2,\phi_o)\ . \tag{25}$$

Dirac observables:

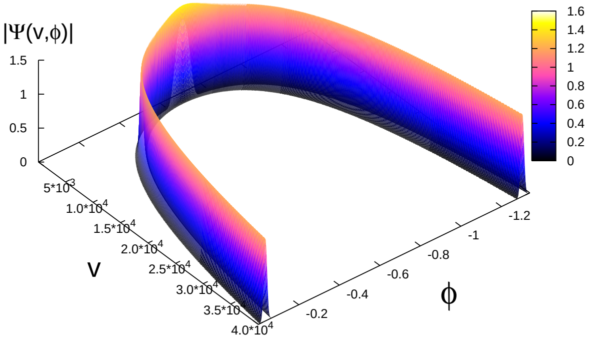
$$\hat{V}|_{\phi_o}\Psi(\nu,\phi) \;=\; 2\pi\gamma l_{\rm Pl}^2\, e^{i\sqrt{\Theta}(\phi-\phi_o)}|\nu|\Psi(\nu,\phi_o) \tag{26}$$

and

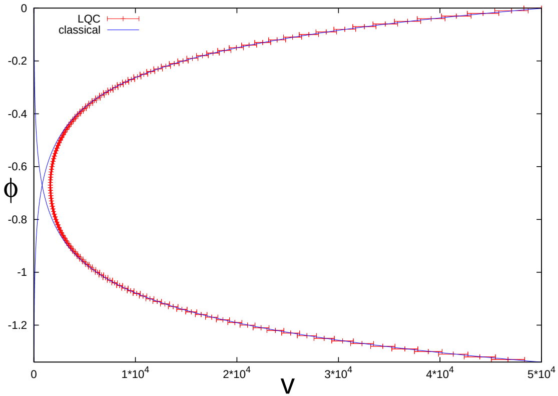
$$\hat{p}_\phi\Psi(\nu,\phi) \;=\; -i\hbar\,\partial_\phi\,\Psi(\nu,\phi) = \hbar\sqrt{\Theta}\Psi(\nu,\phi)\ . \tag{27}$$

Quantum Bounce

Due to quantum geometry effects in loop quantum gravity, big bang is replaced by a quantum bounce! *(Ashtekar, Pawlowski, PS, (06))*



For sharply peaked states universe bounces at a maximum of energy density $\rho = \rho_{\rm max} = 3/8\pi G\Delta^2 \approx 0.41\rho_{\rm Planck}$



Universe follows classical trajectory till curvature is approximately a percent of the Planck curvature. GR an excellent approximation when gravity is weak. Singularity recovered when $\Delta \to 0$.

Probability of bounce to occur is unity! **(Craig, PS (2013))**

Robustness of singularity resolution: examples

- Exactly solvable model (flat, isotropic with a massless scalar) **(Ashtekar, Corichi, PS (2008))**
- In presence of spatial curvature $k = \pm 1$ **(Ashtekar, Pawlowski, PS, Vandersloot (2007); Kaminski, Lewandowski, Szulc (2007); Vandersloot (2007); Szulc (2009))**
- Bianchi models **(Ashtekar, Wilson-Ewing (2009-2010); Martin-Benito, Mena-Marugan, Pawlowski (2009); Diener, Joe, Megevand, PS (2018))**
- Negative cosmological constant **(Bentivegna, Pawlowski (2007))**
- Positive cosmological constant **(Pawlowski, Ashtekar (2012))**
- ϕ^2 inflationary potential **(Ashtekar, Pawlowski, PS (unpublished); Giesel, Li, PS (2021))**
- Extremely wide states not corresponding to a classical universe at late times **(Diener, Gupt, PS (2014))**
- Non-gaussian and highly squeezed states corresponding to highly quantum universes **(Diener, Gupt, PS (2014))**
- Radiation **(Pawlowski, Pierini, Wilson-Ewing (2014))**

Effective dynamics

For suitably chosen coherent states, following geometric formulation of QM, one can obtain an effective description

(Taveras (2008); Taveras, Singh (unpublished))

$$C_H^{(\mathrm{eff})} = -\frac{3\hbar}{4\gamma\lambda^2}\nu \sin^2(\lambda \mathrm{b}) + \frac{1}{4\pi\gamma l_{\mathrm{Pl}}^2}\frac{P_\phi^2}{\nu} \; . \qquad (28)$$

Vanishing of Hamiltonian constraint yields

$$\frac{3}{8\pi G\gamma^2\lambda^2} V \sin^2(\lambda \mathrm{b}) = \frac{P_\phi^2}{2V} \; . \qquad (29)$$

The modified Friedmann and Raychaudhuri equations can be found using Hamilton's equation for V and b

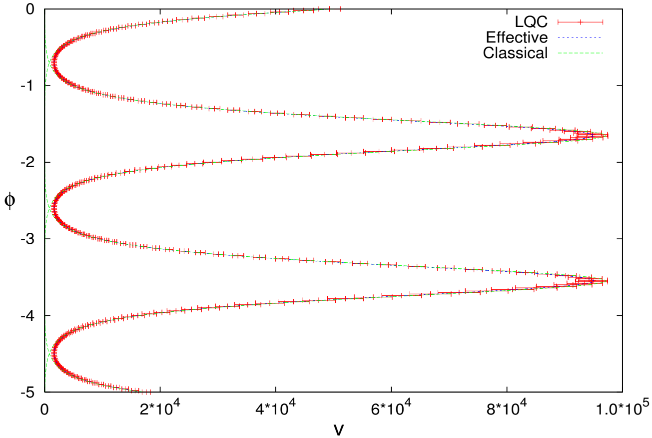
$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\mathrm{max}}}\right) \quad \mathrm{with} \quad \rho_{\mathrm{max}} = \frac{3}{8\pi G\gamma^2\lambda^2} \; . \qquad (30)$$

The quantum gravitational correction thus appears as a ρ^2 modification to the classical Friedmann equation. Bounce occurs when $\rho = \rho_{\mathrm{max}}$. Gravity becomes very repulsive for $\rho > \rho_{\mathrm{max}}/2$.

An accurate test of recovering GR: $k = 1$ FLRW model

(Ashtekar, Pawlowski, PS, Vandersloot; Szulc, Kaminski, Lewandowski (2007))

The closed model has past and future singularities.



LQC predicts recollapse accurately and avoids both big bang and big crunch in various cycles. Effective trajectory completely captures quantum evolution in all cycles.

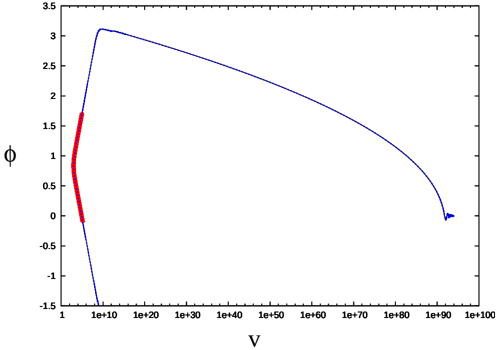
- Is quantum bounce a generic feature of states in the theory?
  Bounce happens for all the states in the physical Hilbert space for the spatially flat, isotropic model (Ashtekar, Corichi, PS (2008)).

- What is the state of the universe on the other side? Is it a quantum foam or a classical spacetime?
  A macroscopic universe, such as ours, bounces from a macroscopic universe similar to ours. Spacetime fluctuations severely constrained on both sides of the bounce (Corichi, PS; Kaminski, Pawlowski; Corichi, Montoya (2008-2010))

- What about quantization ambiguities? Do they affect results?
  Surprisingly, an enormous number of 'possible' quantizations in LQC can be ruled out. Consistency conditions have been proposed, which restrict many mathematically possible choices (Corichi, PS (2008-2009)). Recently new regularizations studied where bounce is asymmetric for massless scalar field (mLQC-I (Li, PS, Wang (2020)))

Does quantum gravity resolve problems of inflation?

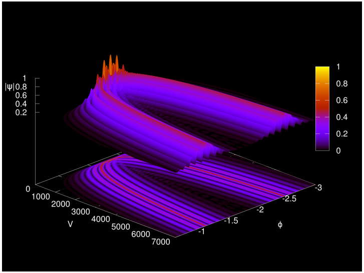
Quantum gravity resolves the past singularity in inflation (Ashtekar, Pawlowski, PS (unpublished)).

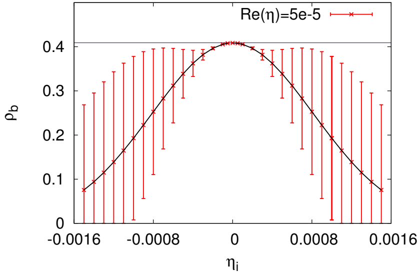


Loop quantum effects also help in setting right initial conditions for inflation (PS, Vandersloot, Vereschagin (2007); Gupt, PS (2013); Ranken, PS (2012)) and give valuable insights on the probability for inflation to occur (Ashtekar, Sloan; Corichi, Karami (2009)). Kinetic dominated bounce only for Starobinsky inflation (Bhardwaj, Copeland, Louko (2018))

Bounce not restricted to any special states. Even occurs for states which are highly non-Gaussian or squeezed.

(Diener, Gupt, Megevand, PS (2014))



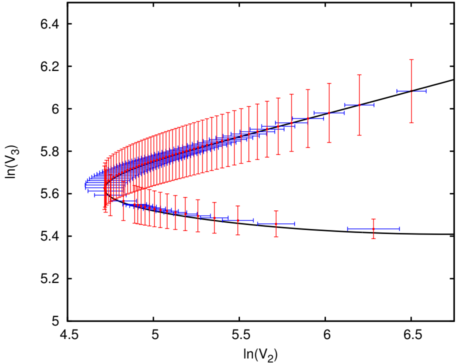


In the isotropic model, quantum fluctuations are found to always lower the curvature scale at which the bounce occurs. Quantum fluctuations in the state enhance the "repulsive nature of gravity" in the quantum regime.

Anisotropic quantum bounce

Computationally challenging and expensive. Limited early results on bounce in Bianchi-I vacuum model **(Martin-Benito, Mena Marugan, Pawlowski (2008))**. Using Cactus framework, physics of quantum bounce in Bianchi-I vacuum spacetime rigorously understood

(Diener, Joe, Megevand, PS (2018))



Anisotropic shear remains bounded throughout the evolution.

Effective description turns out to be a good approximation.

Modifications of standard LQC

(Yang, Ding, Ma (09); Li, PS, Wang (18); Assanioussi, Dapor, Liegener, Pawlowski (18))

Hamiltonian constraint composed of Euclidean and Lorentzian terms:

$$\mathcal{C}_{\rm grav} = \mathcal{C}^{(E)}_{\rm grav} - (1+\gamma^2)\mathcal{C}^{(L)}_{\rm grav}$$

where

$$\mathcal{C}^{(E)}_{\rm grav} = \frac{1}{2}\int \mathrm{d}^3x\, \epsilon_{ijk} F^i_{ab} \frac{E^{aj}E^{bk}}{\sqrt{\det(q)}}$$

and

$$\mathcal{C}^{(L)}_{\rm grav} = \int \mathrm{d}^3x\, K^j_{[a} K^k_{b]} \frac{E^{aj}E^{bk}}{\sqrt{\det(q)}}$$

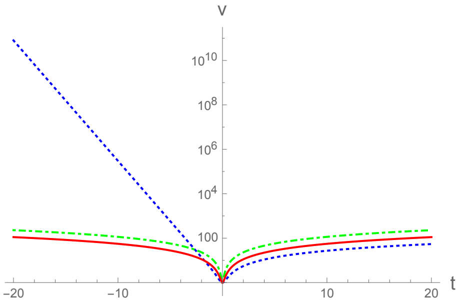
In LQC, quantization of spatially flat models obtained after combining $\mathcal{C}^{(E)}_{\rm grav}$ and $\mathcal{C}^{(L)}_{\rm grav}$. If terms are treated distinct, then form of quantum Hamiltonian constraint significantly different.

Two ambiguities at this level:

- Quantize $\mathcal{C}^{(L)}_{\rm grav}$ as above after using identities on classical phase space and expressing in terms of holonomies. Leads to mLQC-I.
- Use $K^i_a = \gamma^{-1} A^i_a$ in $\mathcal{C}^{(L)}_{\rm grav}$, and then quantize. Results in mLQC-II.

Comparison of mLQC-I and mLQC-II with LQC

Non-trivial modifications to Friedmann dynamics for mLQC-I and mLQC-II in comparison to LQC in Planck regime **(Li, PS, Wang (18))**



In mLQC-I, spacetime curvature remains Planckian before the bounce yet satisfies Einstein field equations but with a quantum gravitational origin matter.

In mLQC-II, spacetime curvature decreases quickly on both sides of the bounce as in LQC. No emergent matter or a rescaled G.

No cyclic models possible in mLQC-I **(Li, PS (22))**

Does LQC resolve all the singularities?

Spacetime curvature invariants can in principle diverge for for various spacetimes in loop quantum gravity **(PS (09,11); Saini, PS (16-17))**

Example: In the spatially flat isotropic model in loop quantum cosmology, spacetime curvature captured by

$$R = 6\left(H^2 + \frac{\ddot{a}}{a}\right) = 8\pi G\rho\left(1 - 3w + 2\frac{\rho}{\rho_{\rm max}}(1+3w)\right), \quad w = p/\rho$$

Energy density and Hubble rate have upper bound in loop quantum cosmology, but pressure is not bounded.

For highly exotic equations of state, pressure can diverge at a finite value of energy density causing some special singularities

(Barrow, Tsagas (04))

Resolution of all strong singularities in LQC

When is a singularity physically relevant? The singularity at $\tau = \tau_o$ is strong and physically relevant if $\int_0^\tau d\tau' |R_{i\,4j4}|$ diverges as $\tau \to \tau_o$. Else the singularity is weak.

For all the events where curvature invariants diverge in loop quantum gravity, singularities are weak and geodesics can be extended beyond such events. Interestingly, quantum geometry effects ignore weak singularities.

Strong curvature singularities are forbidden in loop quantum gravity at least for isotropic and anisotropic spacetimes.

(PS (09,11); Saini, PS (16-17))

As in LQC, in mLQC-I and mLQC-II scale factor remains finite and non-zero for all finite time evolution. all strong singularities resolved for mLQC-I and mLQC-II **(Saini, PS (18))**

Summary for Part A

- Loop quantum cosmology provides a glimpse on the origin of the Universe in non-perturbative quantum gravity for homogeneous universes. Emerging picture from simple models: Big bang not the beginning, big crunch not the end.
- Singularity resolution achieved in various isotropic and anisotropic models. No need to introduce any exotic matter/ad-hoc assumptions/fine tuning. Existence of bounce tested for extreme conditions using high performance computers.
- Bounce occurs for states in a dense subspace of the physical Hilbert space (not only for those which are semi-classical at late times).
- Discreteness of quantum geometry bounds the energy density, anisotropic shear and curvature scalars.
- Main open question: Is singularity resolution an artifact of symmetry reduced models? Or do these results point towards a generic resolution of singularities in LQG?

Part B

Hamiltonian Framework for Cosmological Perturbations
and How to Explore Quantum Geometry Effects in CMB?

Useful References for Part B

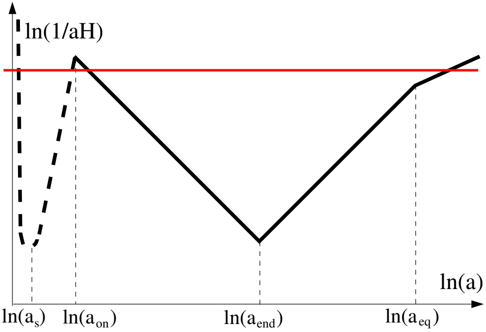
- D. Langlois, Hamiltonian formalism and gauge invariance for linear perturbations in inflation, Class. Quant. Grav. **11** 389 (1994)
- I. Agullo, A. Ashtekar and W. Nelson, "Extension of the quantum theory of cosmological perturbations to the Planck era," Phys. Rev. D **87**, 043507 (2013)
- L. C. Gomar, M. Martín-Benito and G. A. M. Marugán, "Gauge-Invariant Perturbations in Hybrid Quantum Cosmology," JCAP **06**, 045 (2015)
- A. Ashtekar, B. Gupt, D. Jeong and V. Sreenath, "Alleviating the Tension in the Cosmic Microwave Background using Planck-Scale Physics," Phys. Rev. Lett. **125**, 051302 (2020)
- I. Agullo, D. Kranas and V. Sreenath, "Anomalies in the Cosmic Microwave Background and Their Non-Gaussian Origin in Loop Quantum Cosmology," Front. Astron. Space Sci. **8**, 703845 (2021)
- B. F. Li and P. Singh, "Close relationship between the dressed metric and the hybrid approach to perturbations in effective loop quantum cosmology," Phys. Rev. D **106**, 086015 (2022)

Motivation

- The homogeneity assumption allowed us to understand the cosmological models as systems with a finite degrees of freedom which makes their rigorous quantization achievable. But the physical universe is not exactly homogeneous! Astronomical observations show that small inhomogeneities in the early universe serve as the seeds of LSS.
- These inhomogenities arise from the quantum fluctuations of the scalar fields in the very early universe. Quantum fluctuations of the inflaton freeze out on exiting horizon and generate classical density perturbations on re-entry.
- CMB provides an important platform to understand quantum geometric effects using high precision observations.
- While inflationary paradigm predicts remarkable compatability with observations, there are anomalies for largest angular scales which provide a potential window for QG effects.

Motivation

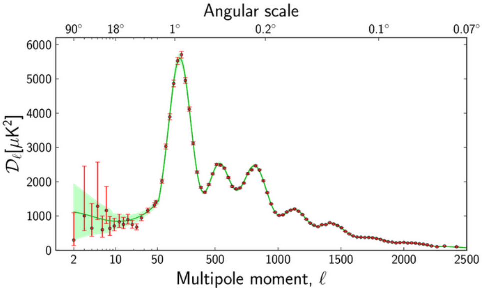
In LQC, due to the bounce comoving horizon has a different behavior in pre-inflation regime.



Modes can cross in and out of horizon even before inflation starts. There can be departures from the vacuum state and the primordial power spectrum. Nature of bounce depends on regularizations (such as mLQC-I, mLQC-II) that can potentially leave an imprint. Similarly, quantum ambiguities can affect the power spectrum too.

CMB anomalies

- Power suppression anomaly: Lack of two point correlations at large angular scales or low multipoles in comparison to ΛCDM model.



- Dipolar modulation anomaly: Evidence of scale dependent dipolar modulation (between multipoles ℓ and $\ell + 1$).
- Parity anomaly: ΛCDM predicts parity neutrality. But there is excess power of odd multipoles for large angular scales.
- Lensing anomaly: CMB undergoes lensing by intervening matter. Incompatibility with ΛCDM model.

Brief overview of main approaches

- A pragmatic strategy is to consider Fock quantized perturbations over a loop quantized background. Two main approaches: Dressed metric **(Agullo, Ashtekar, Nelson (12))** and Hybird approach **(Fernandez-Mendez, Mena Marguan, Olmedo (12))**. Both approaches result in modifications to Mukhanov-Sasaki equation albeit following different routes.
- Classical theory of dressed metric approach based on Langlois' work on Hamiltonian method for perturbations **(Langlois (94))**
- Classical theory of hybrid approach based on Halliwell and Hawking's work on cosmological perturbations **(Halliwell, Hawking (85))**
- In classical theory, both the approaches lead to the same Hamiltonian up to the second order in perturbations. At the effective spacetime level, the difference in phenomenological predictions between the two approaches in the Planck regime can be traced to quantum ambiguities **(Li, PS (22))**.

Brief overview of results

Significant advances in last few years to connect quantum geometry effects with precision CMB observations.

- Agreement with observations at ultra-violet scales. Quantum gravity effects encoded in the intermediate regime and infra-red where there is power amplification.
- Special choice of vacuum state **(Ashtekar, Gupt (17))** can potentially alleviate power suppression and lensing anomalies **(Ashtekar, Gupt, Jeong, Sreenath (20))**
- Non-Gaussianities can play a role in alleviating anomalies, including parity asymmetry anomaly **(Agullo, Kranas, Sreenath (20))**
- Other vacuum choices such as non-oscillatory vaccum and states of low energy can also alleviate anomalies **(Martin-Benito, Neves, Olmedo (21))**
- Modified versions of LQC can in principle leave an imprint in CMB too **(Li, PS, Wang (22))**

Hamiltonian framework

(Langlois (94))

Consider a massive scalar field ϕ minimally coupled in GR:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) \qquad (31)$$

In the ADM formalism, we use the lapse function, $N(t, x^i)$, and the shift vector $N^i(t, x^i)$ to foliate the 4-dimensional spacetime into 3-dimensional space-like hypersurfaces labeled Σ_t.

The induced metric on the spatial hypersurfaces, γ_{ij}, and its conjugate momentum π^{ij} are the phase space variables for the gravitational sector. On the other hand, the phase space variables for the scalar field sector are ϕ and π_ϕ. The conjugate variables satisfy

$$\{\gamma_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = \delta^k_{(i}\delta^l_{j)}\,\delta^3(\mathbf{x},\mathbf{y}), \qquad \{\phi, \pi_\phi\} = \delta^3(\mathbf{x},\mathbf{y}) \,. \qquad (32)$$

The action in terms of canonical variables can be written as

$$S = \int \mathrm{d}^4x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_\phi \dot{\phi} - N\mathcal{H} - N^i \mathcal{H}_i \right) \ . \tag{33}$$

It does not contain any time derivatives of lapse and shift. Thus, these act as Lagrange multipliers. Extremization of action with respect to the lapse results in the Hamiltonian constraint

$$\mathcal{H} = \frac{2\kappa}{\sqrt{\gamma}} \left(\pi^{ij}\pi_{ij} - \frac{\pi_i \pi^i}{2} \right) - \frac{\sqrt{\gamma}}{2\kappa} \, {}^{(3)}R + \sqrt{\gamma} \left(\frac{\pi_\phi^2}{2\gamma} + \frac{1}{2}\partial_i\phi\partial^i\phi + U(\phi) \right) \approx 0. \tag{34}$$

Here $\gamma = \det\gamma_{ij}$ and $\pi = \pi_i^i$.

Variation of action with respect to arbitrary shift vector leads to the spatial diffeomorphism constraint

$$\mathcal{H}_i = -2\partial_k(\gamma_{ij}\pi^{jk}) + \pi^{jk}\partial_i\gamma_{jk} + \pi_\phi\partial_i\phi \approx 0. \tag{35}$$

The dynamical evolution of the background and perturbations is determined by the Hamiltonian:

$$\mathbf{H}(\gamma_{ij}, \phi; \pi^{ij}, \pi_\phi) = \int \mathrm{d}^3x (N\mathcal{H} + N^i\mathcal{H}_i) \ . \tag{36}$$

Physical solutions lie on the constrained phase space satisfying $\mathcal{H} \approx 0$ and $\mathcal{H}_i \approx 0$. Since the Hamiltonian is composed entirely of these constraints, which are first class, Hamiltonian evolution preserves constrained dynamics.

Equations of motion (matter):

$$\dot{\phi} = \{\phi, \mathbf{H}\} = \frac{\delta \mathbf{H}}{\delta \pi_\phi} = \frac{N}{\sqrt{\gamma}}\pi_\phi + N^i\partial_i\phi \ , \tag{37}$$

$$\dot{\pi}_\phi = \{\pi_\phi, \mathbf{H}\} = -\frac{\delta \mathbf{H}}{\delta \phi} = -\frac{N}{\sqrt{\gamma}}U_{,\phi} + \partial_i(N^i\pi_\phi) + \partial_i(N\sqrt{\gamma}\gamma^{ij}\partial_j\phi) \ . \tag{38}$$

Equations of motion (gravity):

$$\dot{\gamma}_{ij} = \frac{\delta \mathbf{H}}{\delta \pi^{ij}} = D_i N_j + D_j N_i - \frac{N}{\sqrt{\gamma}}(2\pi_{ij} - \pi^i{}_i \gamma_{ij}) \qquad (39)$$

$$\begin{aligned}
\dot{\pi}^{ij} &= -\frac{\delta \mathbf{H}}{\delta \gamma_{ij}} \\
&= -N\sqrt{\gamma}\left({}^{(3)}R^{ij} - \frac{1}{2}\gamma^{ij\,(3)}R\right) + \frac{N}{2\sqrt{\gamma}}\gamma^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^m{}_m \pi^n{}_n\right) \\
&\quad -\frac{2N}{\sqrt{\gamma}}\left(\pi^{im}\pi^j{}_m - \frac{1}{2}\pi^m{}_m \pi^{ij}\right) + \sqrt{\gamma}\left(D^i D^j N - \gamma^{ij} D_m D^m N\right) \\
&\quad + D_m(\pi^{ij} N^m) - \pi^{im} D_m N^j - \pi^{jm} D_m N^i \ . \qquad (40)
\end{aligned}$$

Here D_a is the covariant derivative on spatial hypersurfaces.

The cosmological background in ADM variables

The spacetime metric for FLRW (spatially-flat) model

$$ds^2 = -N^2 dt^2 + e^{2\alpha(t)} \delta_{ij} dx^i dx^j, \tag{41}$$

Background variables:

$$\bar{\gamma}_{ij} = e^{2\alpha}\delta_{ij}, \quad \bar{\pi}^{ij} = \frac{\pi_\alpha}{6}\bar{\gamma}^{ij} \text{ with } \pi_\alpha = -\frac{6\dot{\alpha}e^{3\alpha}}{\kappa N} \tag{42}$$

which satisfy

$$\{\bar{\gamma}_{ij}, \bar{\pi}^{kl}\} = \frac{1}{V_o}\delta^k_{(i}\delta^l_{j)} \tag{43}$$

with V_o is the volume of the fiducial cell. In terms of scale factor, $a = e^\alpha$ and $\pi_a = e^{-\alpha}\pi_\alpha$, this translates to

$$\{a, \pi_a\} = \frac{1}{V_o}. \tag{44}$$

Similarly for matter,

$$\{\bar{\phi}, \bar{\pi}_\phi\} = \frac{1}{V_o} \ . \tag{45}$$

The cosmological background: Hamiltonian

In the homogeneous background the spatial diffeomorphism constraint vanishes, i.e.

$$\mathcal{H}_i^{(0)} = 0 \ . \tag{46}$$

The intrinsic curvature ${}^{(3)}R$ vanishes for the spatially-flat spacetime, hence, the Hamiltonian constraint is

$$\mathcal{H}^{(0)} = -\frac{\kappa}{12}\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{2a^3} + U(\phi)a^3 \approx 0. \tag{47}$$

The zeroth order Hamiltonian becomes

$$\mathbf{H}^{(0)} = \int_{\mathcal{V}} \mathrm{d}^3x\, \mathcal{H}^{(0)} = NV_o\left(-\frac{\kappa}{12}\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{2a^3} + U(\phi)a^3\right) \ . \tag{48}$$

The dynamical evolution of the background phase variables can be found using Hamilton's equations. These are,

$$\dot{a} = \frac{1}{V_o}\frac{\partial \mathbf{H}^{(0)}}{\partial \pi_a} = -N\frac{\kappa}{6}\frac{\pi_a}{a}, \tag{49}$$

$$\dot{\pi}_a = -\frac{1}{V_o}\frac{\partial \mathbf{H}^{(0)}}{\partial \pi_a} = -N\left(\frac{\kappa}{12}\frac{\pi_a^2}{a^2} - \frac{3}{2}\frac{\bar{\pi}_\phi^2}{a^4} + 3a^2 U(\bar{\phi})\right), \tag{50}$$

$$\dot{\bar{\phi}} = \frac{1}{V_o}\frac{\partial \mathbf{H}^{(0)}}{\partial \bar{\pi}_\phi} = N\frac{\bar{\pi}_\phi}{a^3}, \tag{51}$$

$$\dot{\bar{\pi}}_\phi = -\frac{1}{V_o}\frac{\partial \mathbf{H}^{(0)}}{\partial \bar{\phi}} = Na^3 U(\bar{\phi})_{,\bar{\phi}} \ . \tag{52}$$

Classical Friedmann dynamics

Let us fix $N = 1$. Vanishing of zeroth order Hamiltonian gives

$$\frac{\kappa}{12}\frac{\pi_a^2}{a} = \frac{\pi_\phi^2}{2a^3} + U(\phi)a^3 \ . \tag{53}$$

Using the Hamilton's equation for a we get the classical Friedmann equation:

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa}{3}\rho, \quad \text{with} \quad \rho = \frac{\pi_\phi^2}{2a^6} + U(\phi) \tag{54}$$

Hamilton's equation for π_a gives the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3P), \quad \text{with} \quad P = \frac{\pi_\phi^2}{2a^6} - U(\phi) \ . \tag{55}$$

Hamilton's equations for ϕ and π_ϕ yield the Klein-Gordon equation:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + U_{,\phi} = 0 \tag{56}$$

which is same as

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{57}$$

Linear perturbations

Consider the linear perturbations of the spatial metric and its canonical momentum

$$\gamma_{ij}(\vec{x},t) = \bar{\gamma}_{ij}(t) + \delta\gamma_{ij}(\vec{x},t), \qquad \pi^{ij}(\vec{x},t) = \bar{\pi}^{ij}(t) + \delta\pi^{ij}(\vec{x},t) \tag{58}$$

and similarly of the scalar field and its momentum

$$\phi(\vec{x},t) = \bar{\phi}(t) + \delta\phi(\vec{x},t), \qquad \pi_\phi = \bar{\pi}_\phi(t) + \delta\pi_\phi(\vec{x},t) \tag{59}$$

In Fourier space:

$$\delta\phi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k\, \delta\phi(\vec{k})\, e^{i\vec{k}.\vec{x}} \tag{60}$$

with the normalization condition

$$\int \mathrm{d}^3x\, e^{i(\vec{k}+\vec{k}').\vec{x}} = (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}') \ . \tag{61}$$

Real valued perturbations, i.e. $\delta\phi(\vec{k}) = \delta\phi^*(-\vec{k})$ etc.

Poisson brackets:

$$\{\delta\phi(\vec{k}), \delta\pi_\phi(\vec{k'})\} = \delta^{(3)}(\vec{k}+\vec{k'}). \tag{62}$$

$$\{\delta\gamma_{ij}(\vec{k}), \delta\pi^{kl}(\vec{k'})\} = \delta_i^{(k}\delta_j^{l)}\delta^{(3)}(\vec{k}+\vec{k'}). \tag{63}$$

Orthogonal basis for SVT decomposition

Symmetric matrices $\delta\gamma_{ij}(\vec{k})$ generate $6 \times \infty^3$ dimensional vector space. For each $\vec{k}$ the space can be decomposed into three 2-dimensional subspaces: scalar, vector and tensor subspaces. Introduce an orthogonal basis in the Fourier space,

$$\delta\gamma_{ij}(\vec{k}) = A^{(n)}_{ij}(\vec{k})\,\gamma_{(n)}(\vec{k}) \quad n = 1..6 \tag{64}$$

Let us identify $A^{(1)}_{ij}(\vec{k})$ and $A^{(2)}_{ij}(\vec{k})$ for scalar, $A^{(3)}_{ij}(\vec{k})$ and $A^{(4)}_{ij}(\vec{k})$ for vector, and $A^{(5)}_{ij}(\vec{k})$ and $A^{(6)}_{ij}(\vec{k})$ for tensor.

Consider two orthogonal unit vectors $\hat{\chi}$ and $\hat{\xi}$ which are also orthogonal to $\hat{k}$. Norm defined with respect to the background metric $\bar{\gamma}^{ij}$. If k_i denote components of $\vec{k}$, then $\bar{\gamma}_{ij}k^ik^j = k^2_{\rm phy}$

Scalar:

$$A^{(1)}_{ij}(\vec{k}) = \frac{1}{\sqrt{3}}\bar{\gamma}_{ij}, \quad A^{(2)}_{ij}(\vec{k}) = \sqrt{\frac{3}{2}}\left(\hat{k}_i\hat{k}_j - \frac{\bar{\gamma}_{ij}}{3}\right) . \tag{65}$$

(Similarly for vector and tensor modes).

Metric perturbations in Fourier space

Using $A_{ij}^{(n)}(\vec{k})$ basis, metric perturbations are:

Scalar:

$$\gamma_{(1)}(\vec{k}) = \frac{1}{\sqrt{3}}\bar{\gamma}^{ij}\delta\gamma_{ij}(\vec{k}) \tag{66}$$

and

$$\gamma_{(2)}(\vec{k}) = \sqrt{\frac{3}{2}}\left(\hat{k}^l\hat{k}^m - \frac{\bar{\gamma}^{lm}}{3}\right)\delta\gamma_{ij}(\vec{k}) \ . \tag{67}$$

Vector:

$$\gamma_{(3)}(\vec{k}) = \sqrt{2}\,\hat{k}^i\hat{\chi}^j\delta\gamma_{ij}(\vec{k}), \quad \gamma_{(4)}(\vec{k}) = \sqrt{2}\,\hat{k}^i\,\hat{\xi}^j\delta\gamma_{ij}(\vec{k}) \tag{68}$$

Tensor:

$$\gamma_{(5)}(\vec{k}) = \frac{1}{\sqrt{2}}(\hat{\chi}^i\hat{\chi}^j - \hat{\xi}^i\hat{\xi}^j)\delta\gamma_{ij}(\vec{k}), \ \ \gamma_{(6)}(\vec{k}) = \sqrt{2}\hat{\chi}^i\,\hat{\xi}^j\delta\gamma_{ij}(\vec{k}), \tag{69}$$

Momenta can be found using

$$\pi^{(n)}(\vec{k}) = A_{ij}^{(n)}(\vec{k})\,\delta\pi^{ij}(\vec{k}) \ . \tag{70}$$

Linearized spatial diffeomorphism constraint

Since background is homogeneous, the spatial-diffeomorphim constraint in the configuration space at the linear order becomes:

$$\mathcal{H}_i^{(1)} = -2\bar{\pi}^{jk}\partial_k\delta\gamma_{ij}(\vec{x}) - 2\bar{\gamma}_{ij}\partial_k\delta\pi^{jk}(\vec{x}) + \bar{\pi}^{jk}\partial_i\delta\gamma_{jk}(\vec{x}) + \bar{\pi}_\phi\partial_i\delta\phi(\vec{x}). \tag{71}$$

In Fourier space it gives

$$\mathcal{H}_i^{(1)} = -\frac{a\bar{\pi}_a}{3}k^j\delta\gamma_{ij}(\vec{k}) - 2a^2\delta_{ij}k_k\delta\pi^{jk}(\vec{k}) + \frac{\bar{\pi}_a}{6a}\delta^{jk}k_i\delta\gamma_{jk}(\vec{k}) + \bar{\pi}_\phi k_i\delta\phi(\vec{k}) \approx 0 \tag{72}$$

Leads to three constraints:

$$\mathcal{H}_i^{(1)}\hat{\chi}^i = 0, \quad \mathcal{H}_i^{(1)}\hat{\xi}^i = 0, \quad \mathcal{H}_i^{(1)}\hat{k}^i = 0 \tag{73}$$

Each of these constraints, being first class, require a gauge-fixing condition and eliminate two degrees of freedom in the phase space.

First two constraints fix vector perturbations. Third, along with $\mathcal{H}^{(1)} \approx 0$, eliminates 4 d.o.f. Starting from 10 d.o.f. in scalar and vector subspaces, one is left with only 2 d.o.f. in phase space.

Gauge invariant variables or gauge fixing?

(i) **Gauge-invariant treatment:** Retain all degrees of freedom throughout the calculation and express the final result in the phase space using a gauge-invariant variable, such as the Mukhanov-Sasaki variable Q:

$$Q = \delta\phi + \frac{3\bar{\pi}_\phi}{\kappa\pi_\alpha}\left(\gamma_{(1)} - \frac{\gamma_{(2)}}{3}\right) \tag{74}$$

(ii) **Work in a specific gauge:** In spatially-flat gauge, Q is just $\delta\phi$. Similarly, other gauges lead to natural identification. Eg. In longitudinal gauge or the zero shear gauge which amounts to an isotropic threading of spacetime and a shear free slicing, Bardeen variables are a natural choice as they directly capture the amplitude of metric perturbations.

Physical relevance of gauge invariant variables is tied to the gauge fixing conditions. **(Bardeen (88))**

Linearized spatial diffeomorphism constraint with the spatially-flat gauge

Useful gauge for inflationary models is the spatially-flat gauge in which the perturbation of intrinsic curvature is zero, i.e. $\delta\, {}^{(3)}R = 0$ translating to $\gamma_{(1)}(\vec{k}) = \gamma_{(2)}(\vec{k}) = 0$.

We can rewrite the constraint as

$$\begin{aligned} \frac{k^i \mathcal{H}_i^{(1)}}{k_{\mathrm{phy}}^2} &= -\frac{\bar{\pi}_a a}{6\sqrt{3}}\left(\gamma_{(1)}(\vec{k}) - 2\sqrt{2}\gamma_{(2)}(\vec{k})\right) - \frac{2}{\sqrt{3}}\left(\pi^{(1)}(\vec{k}) + \sqrt{2}\pi^{(2)}(\vec{k})\right) \\ &\quad + \bar{\pi}_\phi \delta\phi(\vec{k}) \ \approx\ 0, \end{aligned} \tag{75}$$

In the spatially-flat gauge one can express $\pi^{(1)}(\vec{k})$ and $\pi^{(2)}(\vec{k})$ in terms of perturbations in the scalar field $\delta\phi(\vec{k})$ and its momentum $\delta\pi_\phi(\vec{k})$ using a similar expression obtained from the vanishing of the linearized Hamiltonian constraint.

Linearized Hamiltonian constraint with the spatially-flat gauge

Scalar constraint at the linear order becomes

$$\begin{aligned}\mathcal{H}^{(1)} &= \left[-\frac{\kappa}{72}\frac{\pi_a^2}{a} - \frac{\bar{\pi}_\phi^2}{4a^3} + \frac{a^3}{2}U(\bar{\phi})\right]\bar{\gamma}^{ij}\delta\gamma_{ij} \\ &\quad -\frac{a^3}{2\kappa}\left(\bar{\gamma}^{il}\bar{\gamma}^{jk} - \bar{\gamma}^{ik}\bar{\gamma}^{jl}\right)\partial_j\partial_k\delta\gamma_{ik} \\ &\quad -\frac{\kappa}{3}\frac{\pi_a}{a^2}\bar{\gamma}_{ij}\delta\pi^{ij} + \frac{\bar{\pi}_\phi}{a^3}\delta\pi_\phi + a^3 U(\bar{\phi})_{,\bar{\phi}}\delta\phi \ . \end{aligned} \qquad (76)$$

Consists only of the scalar modes, hence simplifies considerably in the spatially-flat gauge:

$$\mathcal{H}^{(1)} = -\frac{\kappa}{\sqrt{3}}\frac{\pi_a\pi^{(1)}(\vec{k})}{a^2} + \frac{\bar{\pi}_\phi}{a^3}\delta\pi_\phi(\vec{k}) + a^3 U(\bar{\phi})_{,\bar{\phi}}\delta\phi(\vec{k}) \ . \qquad (77)$$

Second order Hamiltonian constraint for scalar perturbations

Using spatially-flat gauge, the perturbation in intrinsic curvature vanishes and we get

$$\begin{aligned}\mathcal{H}^{(2)} &= \frac{2\kappa}{\sqrt{\bar{\gamma}}}\left(\bar{\gamma}_{ik}\bar{\gamma}_{jl}-\frac{1}{2}\bar{\gamma}_{ij}\bar{\gamma}_{kl}\right)\delta\pi^{ij}\delta\pi^{kl}+\frac{1}{2a^3}\delta\pi_\phi^2\\ &\quad+\frac{a^3}{2}U(\bar{\phi})_{,\bar{\phi}\bar{\phi}}\,(\delta\phi)^2+\frac{a^3}{2}\partial_i\delta\phi\,\partial^i\delta\phi\ .\end{aligned} \tag{78}$$

In the spatially-flat gauge, the physical degrees of freedom for scalar modes are $\delta\phi$ and $\delta\pi_\phi$. But we still have scalar degrees of freedom in momenta of metric perturbations. Can be eliminated using linearized constraints, to give

$$\begin{aligned}\int \mathrm{d}^3k\,\mathcal{H}_{\mathrm{s}}^{(2)} = \int \mathrm{d}^3k\Big[&\frac{1}{2a^3}\delta\pi_\phi(\vec{k})\delta\pi_\phi(-\vec{k})-\frac{3\bar{\pi}_\phi^2}{\pi_a a^4}\delta\phi(\vec{k})\delta\pi_\phi(-\vec{k})\\ &+\frac{a^3}{2}\big(U(\bar{\phi})_{,\bar{\phi}\bar{\phi}}+k_{\mathrm{phy}}^2\big)\delta\phi(\vec{k})\delta\phi(-\vec{k})\\ &-\frac{3\bar{\pi}_\phi}{\pi_a a^3}\left(a^5U(\bar{\phi})_{,\bar{\phi}}-\frac{\kappa\pi_a}{4}\bar{\pi}_\phi\right)\delta\phi(\vec{k})\delta\phi(-\vec{k})\Big].\end{aligned} \tag{79}$$

$$\mathbf{H}^{(2)}_{\mathrm{s}\ (\delta\phi,\delta\pi_\phi)} = \int \mathrm{d}^3k \left[\frac{1}{a^3}\delta\pi_\phi(\vec{k})\delta\pi_\phi(-\vec{k}) \right.$$
$$- \frac{3\bar{\pi}_\phi^2}{\pi_a a^4}\left(\delta\phi(\vec{k})\delta\pi_\phi(-\vec{k}) + \delta\phi(-\vec{k})\delta\pi_\phi(\vec{k})\right)$$
$$+ a^3\left(U(\bar{\phi})_{,\bar{\phi}\bar{\phi}} + k_{\mathrm{phy}}^2\right)\delta\phi(\vec{k})\delta\phi(-\vec{k})$$
$$\left. - \frac{6\bar{\pi}_\phi}{\pi_a a^3}\left(a^5 U(\bar{\phi})_{,\bar{\phi}} - \frac{\kappa\pi_a}{4}\bar{\pi}_\phi\right)\delta\phi(\vec{k})\delta\phi(-\vec{k}) \right]. \tag{80}$$

The cross-terms can be eliminated by going to new variables:

$$\delta\tilde{\phi}(\vec{k}) = \delta\phi(\vec{k}), \qquad \delta\tilde{\pi}_\phi(\vec{k}) = \delta\pi_\phi(\vec{k}) - \frac{3\bar{\pi}_\phi^2}{\pi_a a}\delta\phi(\vec{k}) \ . \tag{81}$$

Second order Hamiltonian for scalar perturbations

Identifying $Q = \delta\phi$ in the spatially-flat gauge, the Hamiltonian becomes

$$\mathbf{H}_{\mathrm{s}}^{(2)}{}_{(Q,P_Q)} = \int \mathrm{d}^3k \left[\frac{1}{a^3} P_Q(\vec{k}) P_Q(-\vec{k}) + a(k^2 + \Omega_Q^2) Q(\vec{k}) Q(-\vec{k})\right] \tag{82}$$

where $k = ak_{\mathrm{phy}}$ and

$$\Omega_Q^2 = 3\kappa \frac{\bar{\pi}_\phi^2}{a^4} - 18 \frac{\bar{\pi}_\phi^4}{a^6 \pi_a^2} - 12a \frac{\bar{\pi}_\phi}{\pi_a} U_{,\phi} + a^2 U_{,\phi\phi}. \tag{83}$$

Hamilton's equations:

$$\dot{Q}(\vec{k}) = \{Q((\vec{k})), \mathbf{H}_{\mathrm{s}}^{(2)}{}_{(Q,P_Q)}\} = \frac{1}{a^3} P_Q(\vec{k}) \tag{84}$$

and

$$\dot{P}_Q(\vec{k}) = \{P_Q((\vec{k})), \mathbf{H}_{\mathrm{s}}^{(2)}{}_{(Q,P_Q)}\} = a(k^2 + \Omega_Q^2) Q(\vec{k}) \ . \tag{85}$$

Mukhanov-Sasaki equation

Taking the time derivative of (84) and using (85) one obtains

$$\ddot{Q}(\vec{k}) + 3H\dot{Q}(\vec{k}) + \frac{1}{a^2}(k^2 + \Omega_Q^2)Q(\vec{k}) = 0 \qquad (86)$$

In the effective description of LQC, this is the starting point to understand the quantum geometric effects on perturbations by introducing quantum gravitational modifications in the potential Ω_Q via polymerization of relevant background quantities.

If we simplify the terms in Ω_Q except $U(\bar{\phi})_{,\bar{\phi}\bar{\phi}}$ term, above eq. can be written in original form (Mukhanov (88))

$$\ddot{Q}(\vec{k})+3H\dot{Q}(\vec{k})+\left(\frac{4\dot{H}}{H}\frac{\ddot{\phi}}{\dot{\phi}} - 2\frac{\dot{H}^2}{H^2} + 6\dot{H} + U(\bar{\phi})_{,\bar{\phi}\bar{\phi}} + k_{\rm phy}^2\right)Q(\vec{k}) = 0\,. \qquad (87)$$

Not constrained by the linearized constraints. Are gauge-invariant.

A similar calculation as in the case of scalar perturbations leads to:

$$\mathbf{H}^{(2)}_{\mathrm{T}\ (\tilde{\gamma}_{(n)},\tilde{\pi}^{(n)})} = \int \mathrm{d}^3k \left[\frac{4\kappa}{a^3}\pi^{(n)}(-\vec{k})\pi^{(n)}(\vec{k})+\frac{a^3}{4\kappa}k^2_{\mathrm{phy}}\gamma_{(n)}(-\vec{k})\gamma_{(n)}(\vec{k})\right] . \tag{88}$$

which using Hamilton's equations gives:

$$\ddot{\tilde{\gamma}}_{(n)}(\vec{k}) + 3H\dot{\tilde{\gamma}}_{(n)}(\vec{k}) + \frac{k^2}{a^2}\tilde{\gamma}_{(n)}(\vec{k}) = 0 \ . \tag{89}$$

Is similar to the one staisfied by scalar perturbations except for the absence of the effective potential Ω_Q.

Back to Mukhanov-Sasaki equation: mass function

Perturbation of the spatial curvature in the comoving gauge (in which covariant velocity perturbations vanish)– the comoving curvature perturbation $\mathcal{R} = \nu/z$ where $z = a\dot{\bar{\phi}}/H$ is conserved at the super-horizon scales $k \ll aH$. Here ν is the rescaled Mukhanov-Sasaki variable: $\nu = aQ$.

Substituting $\nu(\vec{k}) = aQ(\vec{k})$ in (86) one easily obtains

$$\nu(\vec{k})'' + \left(k^2 + \Omega_Q^2 - \frac{a''}{a}\right)\nu(\vec{k}) = 0\ . \tag{90}$$

Here a 'prime' denotes a derivative with respect to conformal time $\eta = \int \frac{\mathrm{d}t}{a}$. Resembles harmonic oscillator with a time dependent mass term:

$$\nu(\vec{k})'' + \left(k^2 + m^2\right)\nu(\vec{k}) = 0 \tag{91}$$

with

$$m^2 = \Omega_Q^2 - \frac{a''}{a} = \frac{3\kappa\bar{\pi}_\phi^2}{a^4} - 18\frac{\bar{\pi}_\phi^4}{\pi_a^2 a^6} - 12a\frac{\bar{\pi}_\phi U_{,\bar{\phi}}}{\pi_a} + a^2 U_{,\bar{\phi}\bar{\phi}} - \frac{a''}{a}. \tag{92}$$

One can directly find the second order Hamiltonian for the variable $\nu_{\vec{k}}$ and its conjugate momentum using:

$$\nu_{\vec{k}} = a\delta\phi_{\vec{k}}, \qquad \pi_{\nu_{\vec{k}}} = \frac{\delta\pi_{\phi_{\vec{k}}}}{a} - \frac{3\bar{\pi}_{\phi}^{2}}{\pi_{a}a^{2}}\delta\phi_{\vec{k}} - \frac{a}{6}\kappa\pi_{a}\delta\phi_{\vec{k}}. \tag{93}$$

The resulting Hamiltonian leads to

$$\nu_{\vec{k}}'' + \left(k^{2} + \tilde{m}_{\rm SF}^{2}\right)\nu_{\vec{k}} = 0. \tag{94}$$

with

$$\tilde{m}^{2} = -\frac{4\pi G}{3}a^{2}\left(\rho - 3P\right) + \mathfrak{U}, \tag{95}$$

and

$$\mathfrak{U} = a^{2}\left(U_{,\bar{\phi}\bar{\phi}} + 48\pi G U + 6H\frac{\dot{\bar{\phi}}}{\rho}U_{,\bar{\phi}} - \frac{48\pi G}{\rho}U^{2}\right). \tag{96}$$

Are m^2 and $\tilde{m}^2$ the same?

Using the classical Raychaudhuri equation

$$\delta m^2 = m^2_{\rm SF} - \tilde{m}^2_{\rm SF} = -\left(\frac{9\bar{\pi}_\phi^2}{a^3\pi_a^2} + \frac{\kappa}{6a}\right)\mathcal{H}^{(0)} \approx 0. \qquad (97)$$

Two mass functions are equivalent on the physical solutions of the classical background dynamics (Li, PS (22))

One can obtain similar mass functions in other gauges and they are all equivalent on physical solutions of the classical theory.

Inclusion of quantum geometric effects

To explore effects of quantum geometry in CMB, a strategy is to replace the classical background quantities in the Mukhanov-Sasaki equation with those from effective LQC.

Expressions of classical mass function depend on $1/\pi_a^2$ and $1/\pi_a$. These need to be **consistently polymerized.**

- Use the same polymerization for variables in the propagation equation for perturbations as in the background Hamiltonian constraint. Otherwise results are gauge dependent!
- Care should be taken to avoid any discontinuous behavior of potentials Ω_Q or Ω_ν. For some choices this can happen at the bounce and one can not either set initial conditions before the bounce or propagate perturbations across the bounce.

(Li, PS (22))

Inclusion of quantum geometric effects

From zeroth order classical Hamiltonian constraint:
$\pi_a = -6a^2 b/\kappa\gamma$ which on using

$$b^2 \to \frac{\sin^2(\lambda b)}{\lambda^2} \tag{98}$$

results in

$$\frac{1}{\pi_a^2} = \frac{\kappa^2\gamma^2}{36a^4b^2} \to \frac{\kappa^2\gamma^2\lambda^2}{36v^{4/3}\sin^2(\lambda b)} = \frac{\kappa}{12v^{4/3}\rho} \tag{99}$$

For $1/\pi_a$, one may simply consider a square root of the above equation but this is problematic since the resulting effective potential Ω_Q turns out to be discontinuous at the bounce.

Insights from the behavior of background quantum operator (superselction sectors) results in

$$\frac{1}{\pi_a} \to -\frac{H}{2v^{2/3}\rho}. \tag{100}$$

(Gomar et al, JCAP 06, 045 (15); Li, PS, Wang (20))

Trace of quantum geometry in mass functions

Although two mass functions are equivalent in the classical theory, they are no longer so at the level of the effective dynamics.

$$\delta m_{\text{eff}}^2 = m_{\text{eff}}^2 - \tilde{m}_{\text{eff}}^2 = -\frac{8\pi G}{3} a^2 \left(\rho + 3P\right) \frac{\rho}{\rho_c}, \tag{101}$$

RHS does not vanish on the physical solutions of the effective background dynamics, especially in the Planck regime where the energy density becomes comparable with the maximum energy density in LQC. This happens because:

- Different approaches to include quantum geometric efffects may bring them at different steps. As an example, the polymerization of the classical equation of motion of a''/a is not equal to the equation of motion of a''/a from the effective dynamics.

Can there be quantum geometric signatures in CMB?

Underlying quantum geometry in LQC which is responsible for a non-singular bounce also results in modifications to the Mukhanov-Sasaki equation for cosmological perturbations. An important question is whether these quantum geometric effects leave an imprint on the modes we can observe today in CMB.

Given an inflationary potential and suitable initial conditions, there can exist an observational window which can potentially lead to observable imprint of the pre-inflationary stage.

Question is whether physical wavelength of the modes experience a curvature scale. If this is the case then such modes get excited curvature leaves no effect **(Parker, Fulling, Phys. Rev. D, 9, 341 (1974))**

In LQC, this curvature scale arises from the underlying quantum geometry **(Agullo, Ashtekar, Nelson, Class. Quant. Grav. 30, 085014 (2013))**

Can there be quantum geometric signatures in CMB?

Typically in LQC in the presence of inflationary potentials, the big bounce is dominated by the kinetic energy of the scalar field. Then $\Omega^2 \ll a''/a$ in the bounce regime and the Mukhanov-Sasaki eq. becomes

$$\nu_k'' + a^2 \left(\frac{k^2}{a^2} - \frac{R}{6} \right) = 0 \, . \tag{102}$$

The four-dimensional Ricci scalar defines a curvature scale in the bounce regime

$$\lambda_{\mathrm{R_B}} := 2\pi \sqrt{\frac{6}{|R_{\mathrm{B}}|}} \approx 1.96 l_{\mathrm{Pl}} \tag{103}$$

If $\lambda_{\mathrm{phy}} \ll \lambda_{\mathrm{R_B}}$ then the curvature term in (102) is negligible, and they propagate as if in a Minkowski spacetime. These modes are unaffected by the underlying quantum geometry.

Modes with $\lambda_{\mathrm{phy}} \gg \lambda_{\mathrm{R_B}}$ feel curvature of quantum geometry and get excited. Their departure from the Bunch-Davies vacuum state can carry the quantum gravitational signature of the bounce.

The observational window

The power amplitude of perturbations in the CMB is parameterized as

$$A = A_s \left(\frac{k}{k_*} \right)^{n_s - 1} \tag{104}$$

where A_s is the initial super-horizon amplitude of the curvature perturbation and n_s denotes the spectral index.

To find the window for potential QG effects use the Planck 2018 data in which the pivot mode is taken to be $k_* = 0.05 Mpc^{-1}$ and the values of A_s and n_s are

$$A_s = 2.099 \times 10^{-9}, \qquad n_s = 0.9649 \pm 0.0042. \tag{105}$$

Find the mode with minimum physical wavenumber at the bounce which is observable today. If $k^{\mathrm{phy}}_{\mathrm{min}} < k^{\mathrm{phy}}_{\mathrm{R_B}}$ we have a desired window of wavenumbers which can leave a signature of pre-inflationary branch in CMB.

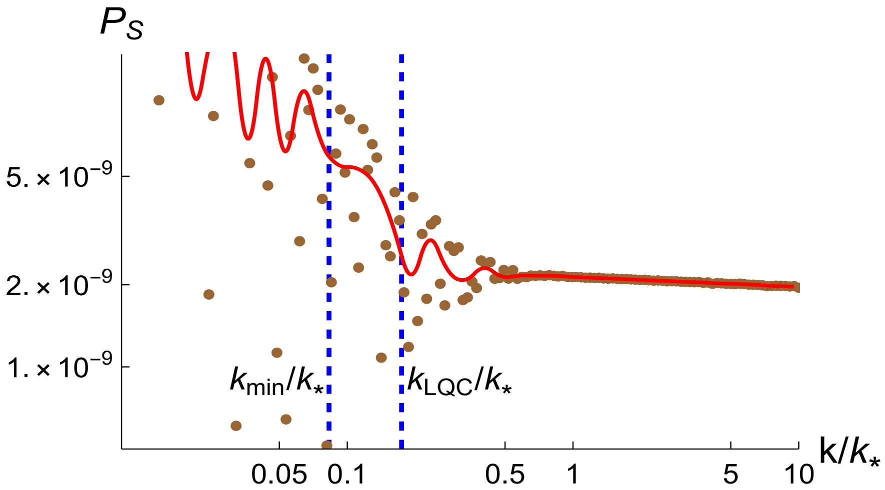
The observational window: Example

Consider ϕ^2 inflationary potential with $\phi_{\mathrm{B}} = 1.0\, m_{\mathrm{Pl}}$. The physical wavenumber of the pivot mode at the Hubble radius crossing during slow-roll inflation $k_*^{\mathrm{phy}} = 4.72\, l_{\mathrm{Pl}}^{-1}$, which results in $k_{\mathrm{min}}^{\mathrm{phy}} = 0.02 < k_{\mathrm{LQC}}^{\mathrm{phy}} = k_{\mathrm{R_B}}^{\mathrm{phy}}$.

If ϕ_{B} is increased (decreased) the minimum physical wavenumber at the bounce which is observable today increases (decreases) . There is only a small range of the values of ϕ_{B} which is observationally viable.

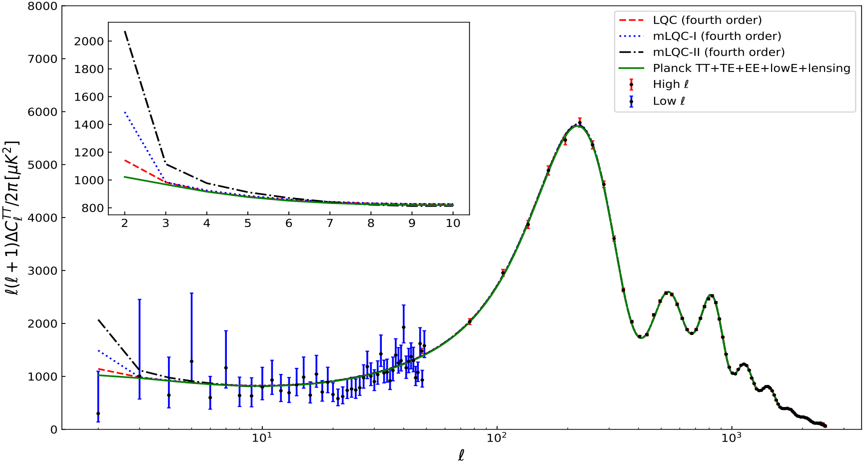
If bounce occurs for $\phi_{\mathrm{B}} \gtrsim 1.13$, there is no window. On the other hand, for $\phi_{\mathrm{B}} \lesssim 0.95$, there is a large window of physical wavenumbers potentially leaving quantum gravity signatures. However, the window of wavenumbers needs to be also consistent with almost scale invariant spectrum of perturbations.

Primordial power spectrum



$$P_S = \frac{k^3}{2\pi^2}\frac{|\nu_k|^2}{z_s^2}$$

The exercise can be repeated for dressed/hybrid approaches, different potentials and different regularizations to explore quantum geometric effects in CMB.



Summary for Part B

- Quantum geometric effects in LQC have been extensively explored in CMB. Dressed/Hybrid approaches use Fock quantized fluctuations over loop quantized background.
- At an effective level, Mukhanov-Sasaki equation plays an important role where background quantities from LQC are imported.
- Power spectrum depends on the choice of states. For certain choices, it is possible to potentially explain CMB anomalies.
- Different regularizations can potentially leave different signatures.
- Open question: Is it possible to identify unique observable signatures in CMB from different LQC models and in contrast to other models in GR/modified gravity?