Two Lectures on

Black Holes in Loop Quantum Gravity

LQG Summer School, Ft. Lauderdale, 29 April - 3 May 2024

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Goal: To Present a Pedagogical Overview of Quantum Evaporation of Black Holes From a LQG Perspective. NOT meant to be exhaustive! Rather, a summary of some widely accepted views, with emphasis conceptual issues and LQG viewpoints on them, distinguishing between concrete results that lie at the foundation, expectations and hopes. I trust that they will complement other lectures. As we will see, while the general picture is clear, many issues remain open. That makes field attractive to young researchers!! Need careful calculations/analysis as well as new ideas!

Will serve as the background material for talks on Quantum Black Holes in the LQG conference next week. They will cover the ongoing advances.

Based on work by Many Researchers, especially

AA, Beetle, Bianchi, Bojowald, Christodoulou, De Lorenzo, Del Rio, Haggard, Hayward, Krishnan, Lewandowski, Olmedo, Ori, , Pretorius, Pawlowski, Ramazanoglu, Rovelli, Schneider, Singh, Taveras, Varadarajan.

In order not to clutter the slides, generally I will not give references during the presentation. They are collected at the end, divided into topics covered for your convenience. There a few (marked) extra slides that provide supplementary material that was not covered in the talks.

Organization

Lecture 1: General conceptual Framework: Main Issues and the LQG viewpoint on them

Lecture 2: Overview of the semi-classical epoch, singularity resolution in BHs, Current status of the issue of “Information Loss” and Open Issues

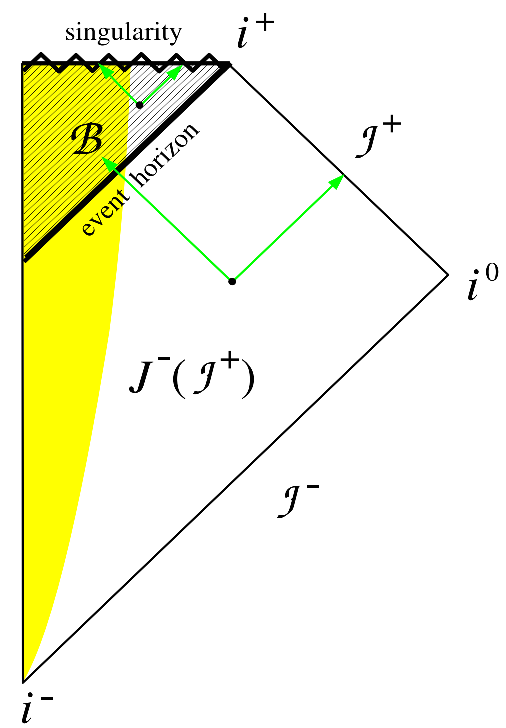
Black Hole Evaporation

Why Information could be be Lost

First, consider the familiar classical gravitational collapse:

While $\mathcal{I}^-$ is a good 'initial data surface', $\mathcal{I}^+$ is not. Part of any incoming field from $\mathcal{I}^-$ falls across the horizon into the singularity and is thus lost for observers in the asymptotic region.

Not directly related to the black hole uniqueness theorems. The loss occurs also in cases where uniqueness fails (Examples: black holes with Yang-Mills or dilatonic hair in 4-d higher dimensions, or higher dimensions.)



Credit: Gourgoulhon & Jaramillo

Quantum Theory

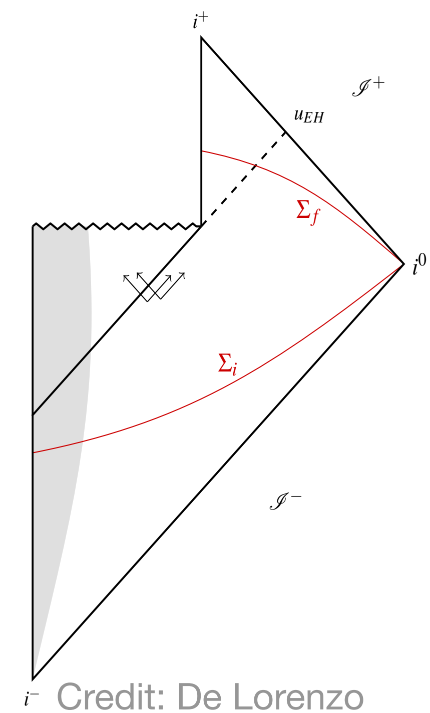
- External field approximation: Hawking Effect In quantum field theory on a black hole background space-time. Approximations: (i) Space-time treated classically: represents a star collapsing to form a black hole. (ii) Test quantum fields; ignore back reaction of the quantum field on the geometry; (iii) Matter field which collapses is classical, distinct from the test quantum field considered. Then:

If the incoming state on $\mathcal{I}^-$ is the vacuum, the outgoing state at $\mathcal{I}^+$ is a mixed state which, at late times, is thermal.

- Inclusion of back reaction

No detailed calculation in 4-d even today. General expectation based on heuristics that led Hawking to propose the space-time diagram shown on the right in 1974. Black hole loses mass and therefore the horizon shrinks to zero. Because the future boundary of space-time again includes a singularity, again information is lost. State at Σ_i determines the state at Σ_f but not vice versa. (Hawking changed his mind more recently, but the
(2016)
original diagram still heavily used.)

LQG Viewpoint: This reasoning is flawed



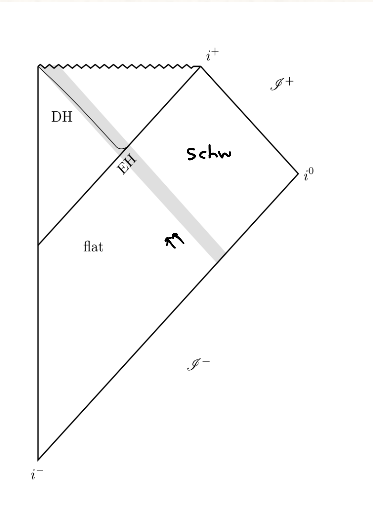
Credit: De Lorenzo

Three drawbacks, which when corrected, leads to very different space-time diagrams

1. The heavy use of event horizons is inappropriate and misleading. Already in classical GR Event Horizons (EHs) have severe limitations. Briefly:

   (i) To know if a space-time admits an EH one needs to know the evolution to infinite future (which is what we are trying to determine evaporating BHs!)

Credit: De Lorenzo

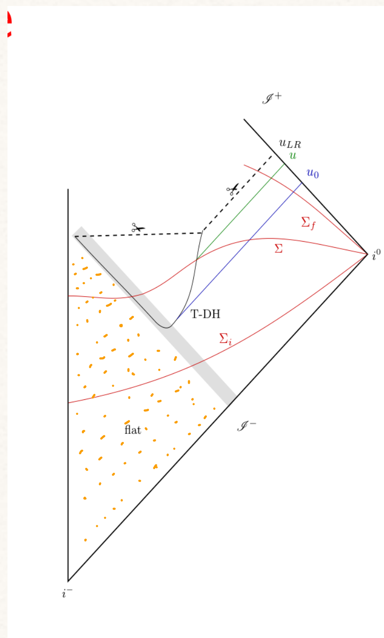


(ii) EHs are teleological and ghostlike: one may be contained and growing in this room right now in anticipation of a gravitational collapse I near us a million years from now. Figure shows a concrete illustration.
Examples with collapse of matter originating at mathematical work (showing that the third law is false!) Keyless-Unger (2402.10190)

More appropriate notion: Quasi-local horizons; in particular the Dynamical Horizon (DH) shown in the figure. They are space-like with increasing area in the classical theory. When the inflow of matter stops they become null, called Isolated Horizons (IHs) (which is also the EH in the figure.)

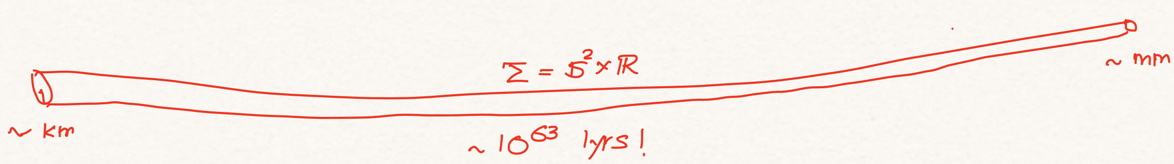
During the BH evaporation process, the DHs are time-like with decreasing area process.
(Definition and further discussion to follow.)

Figure shows the part of space-time where semi-classical description should be an excellent approximation: For example, a solar mass BH is formed by collapse and evaporates till it has lunar mass. During the classical collapse, the DH is space-like and grows in area and during evaporation it becomes time-like and its areas shrinks (from a 3km radius to 0.1mm radius). The two portions enclose a trapped region (where areas of light fronts is contracting). Therefore this piece of the horizon is called Trapping Dynamical Horizon (T-DH).



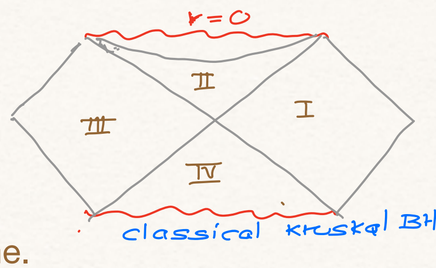
Credit: De Lorenzo

2. Space-time geometry in the trapped region is highly non-trivial and Counter-intuitive, even though the space-time curvature is very low Compared to the Planck scale! Once the back reaction is taken into account, the partial Cauchy surfaces (part of Σ bounded on two sides By T-DH) is increasingly stretched and develops astronomically long necks in a very slow adiabatic process that lasts some 10^{67} Years!

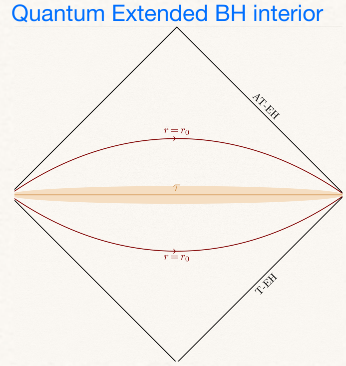


3. Singularity Resolution in LQG: Recall from Kristinas's lectures that area is quantized in LQG & there is area gap $\Delta \approx 5.17\, \ell_P^2$: the smallest non-zero eigenvalue of the area operator. Curvature is defined using holonomies of the gravitational connection around closed loops and then shrinking the loop till the physical area it encloses equals Δ . Thus the curvature operator is fundamentally nonlocal at the Planck scale. Param's lectures on LQC will show that, as a consequence, there is an upper bound to physical observables such as matter density and curvature in LQC: It cannot diverge, whence all space-like strong curvature singularities are resolved in LQC.

The BH part of Kruskal space-time is shown as the triangle II In the figure. It is foliated by space-like r= const. 3-surfaces. They are spatially homogeneous. The BH interior is isometric to the Kantowski-Sachs cosmological model and the Schwarzschild singularity at r= 0 is the future, big-crunch singularity of this space-time.



Quantum Extended BH interior



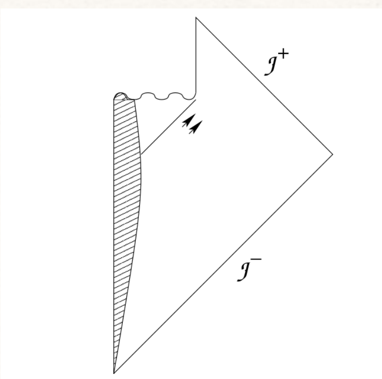
Bottom half: classical Region II

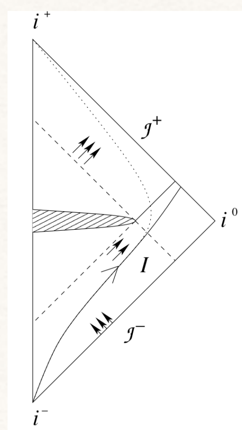
As in LQC, the singularity is resolved because the curvature is bounded above. We have a quantum extension of the space-time in which the singularity is replaced by a transition surface $\mathcal{T}$ to the past of which we have a trapped region (in which the area of 2-spheres decreases along both null normals), and to the future of which there is an anti-trapped region (in which the area of 2-spheres increases along both null normals).

In the rest of these two lectures we will see in some detail how these three novel features

1. Use of Quasi-local Dynamical Horizons in place of event horizons
2. Non-trivial dynamics of space-time geometry in the trapped region that create very long necks during the very slow, adiabatic process over some 10^{67} years, as a solar mass BH shrinks to lunar mass.
3. Singularity resolution and the resulting quantum extension of space-time

introduce a profound revision of the original Hawking paradigm depicted in the figure that seems to be deeply carved in the collective memory. The `LQG paradigm' was proposed 20 years ago (!) (in AA & Bojowald, CQG) suggesting that Hawking's original proposal (Left) should be replaced by one of the two on the right depending on whether evaporation takes finite time or infinite.





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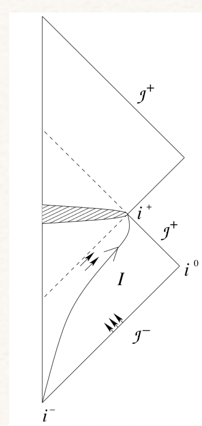
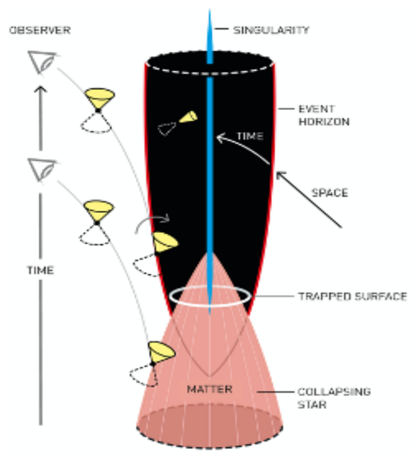


Systematic efforts since then have led to major advances filling various pieces of puzzles, as summarized in lectures on BHs in this School.

1. Event Horizons



• Normally: the outgoing light front expands & ingoing contracts. A compact, large mass 'tilts' light cones towards it. The titling can be so extreme that both both light fonts contract: Light is trapped in a space-time region! An absolute notion; holds for all observers. That region represents a black hole (BH) and its boundary is the event horizon (EH).

(credits:Roger Penrose)

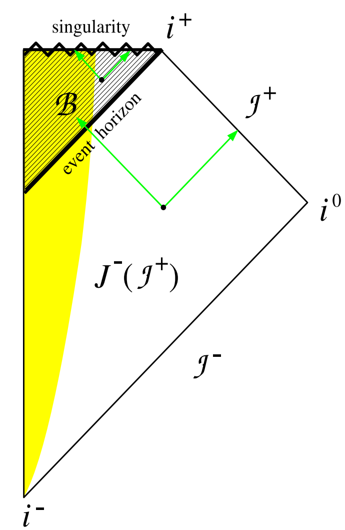
• More precise Definition requires 'Penrose diagrams' that emphasize causal structure: Light cones at 45°.

Consider space-times (M, g_{ab}) that are asymptotically flat with complete $\mathscr{I}^+$.

Black Hole region: $\mathcal{B} := M \setminus J^-(\mathscr{I}^+)$

Captures the Idea: Region from which light cannot escape to infinity.

Event Horizon $E :=$ Future boundary of $J^-(\mathscr{I}^+)$.

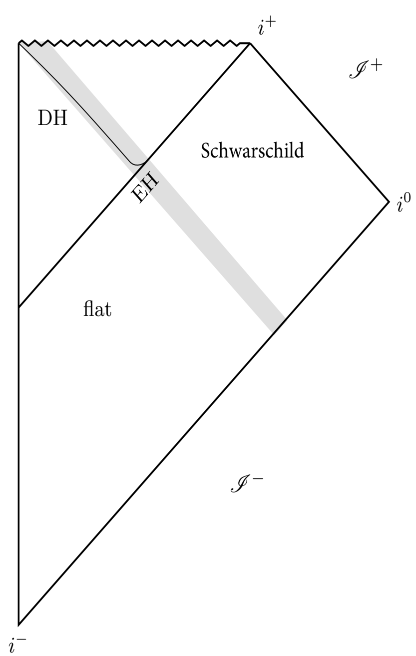


Completeness of $\mathcal{I}^+$ is essential for the notion of event horizons to be meaningful. Otherwise even Minkowski space would admit one!

• EHs have very interesting properties. In presence of matter satisfying 'energy conditions' (as standard classical matter does), Einstein's equations imply Laws of EH Mechanics that are closely analogous to the laws of Thermodynamics.

Thermodynamics: In equilibrium T is constant; In a transition between nearby equilibrium states $\delta E = T\delta S +$ work; and S of a closed system cannot decrease. EHs: In equilibrium, surface-gravity κ is constant on EH; $\delta M = (\kappa/8\pi G)\delta A +$ work; and, A cannot decrease.

Vaidya solution

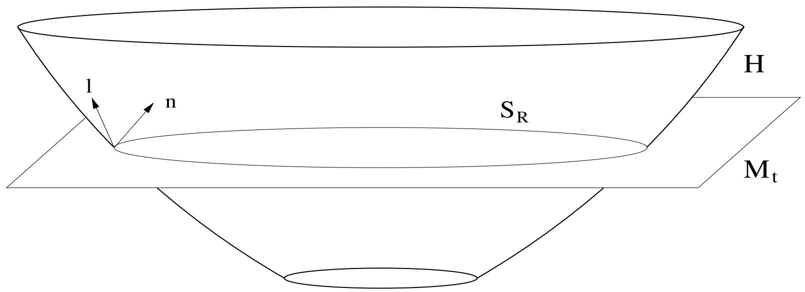


But EHs also have Serious Limitations:

- Teleology! An even horizon may be contained in your room, formed in anticipation of a gravitational collapse in the center of our galaxy in a billion years from now!
- To know if a space-time admits an event horizon, we need to know its entire future evolution & asymptotics. Cannot be used, e.g., in numerical evolution of BBHs!

• The role of $\mathscr{I}^+$ is too global, qualitatively different from its role in the gravitational radiation theory, where it is used just to specify the boundary conditions. We can make it disappear by changing the space-time metric in a small neighborhood of the singularity! (Hajicek; BH evaporation model).

. Quasi-local Horizons



- Main Idea: (Hayward; AA & Krishnan; ...)

⋆ Recall light-fronts ⇝ Marginally Trapped Surfaces (MTSs) (2-sphere S_R in the figure.)

⋆ Quasi-local Horizons H: Stack (MTSs) to form a world-tube. Heavily used in BBH simulations.

The Raychaudhuri equation implies:

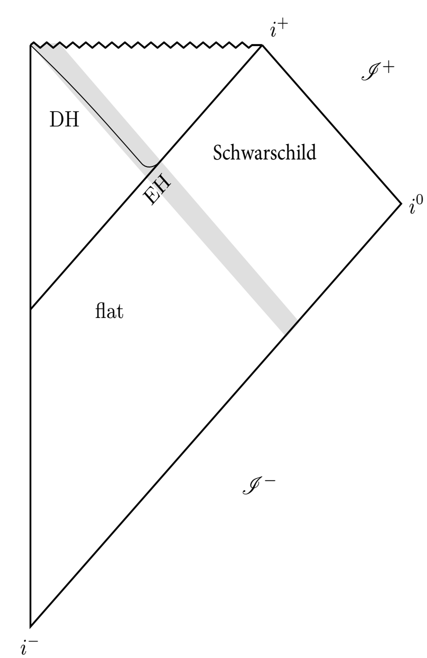
⋆ For the final DH in binary mergers in classical GR: flux into H is positive and the area increases; H is space-like.

⋆ For a BH in equilibrium, the flux into H vanishes, and the area remains constant; H is null.

⋆ During BH evaporation, the flux is negative and the area decreases; H is time-like.

If the area is changing, the QLH is called a Dynamical horizon (DH); if the area is constant, the QLH is called an Isolated Horizon (IH). No teleology : There is no DH or IHs enclosed in the room we are sitting in !!

Vaidya solution



Dynamical Horizons in Classical GR (AA + Krishnan)

• DH H is a space-like 3-dimensional sub-manifold (possibly with boundary) of space-time, foliated by closed 2-surfaces S such that the two null expansions satisfy: $\Theta_{(\ell)} = 0$ & $\Theta_{(n)} < 0$ (and an energy condition holds on H).

• Recall the striking 2nd law of EH Mechanics: Area of its cross-sections never decreases (if energy conditions hold). But one cannot hope to relate the increase in area with a physical process. Ex: Vaidya solution.

⋆ On DHs, the second law again holds. Furthermore, one has a quantitative relation between the increase and flux of energy pouring in!

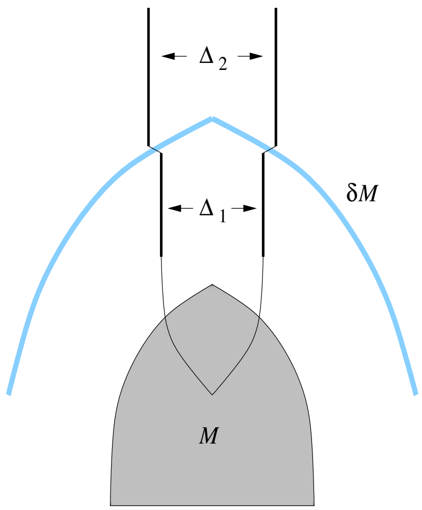
$$\frac{1}{2G}\,(R_2 - R_1) = \underbrace{\int_{\Delta H} N\, T_{ab}\ell^a \hat{\tau}^b\, d^3V}_{\text{Matter energy flux}} + \underbrace{\frac{1}{16\pi G}\int_{\Delta H} N\, \left(|\sigma|^2 + 2|\zeta|^2\right)\, d^3V}_{\text{GW energy flux}}$$

⋆ For the first law, there us an 'active version' involving finite changes in the horizon structure from which the familiar infinitesimal version available for EHs, $(\delta M = \frac{1}{8\pi G}\,\kappa\,\delta A + \Omega\delta J\,)$, follows immediately.

Thus, for DHs, analogs of the 1st and 2nd laws of thermodynamics hold in a more directly physical form.

Isolated Horizons (IHs)

(AA, Beetle, Booth, Fairhurst, Khera, Kolanowski, Krishnan, Lewandowski, Pawlowski, ...)



Stellar collapse + Shell collapse

A Non-Expanding Horizon is a null 3-dimensional submanifold Δ such that the expansion $\Theta_\ell = 0$ for any null normal ℓ^a, and an energy condition holds on Δ.

Raychaudhuri Eq. $\Rightarrow$ Intrinsic metric on Δ is time independent.

Space-time ∇ induces a natural intrinsic derivative operator D on Δ. If D is also time independent, we have an Isolated Horizon (IH). IHs represent horizons in equilibrium.

On every IH Δ, the zeroth and first laws of horizon mechanics hold, even when Δ is distorted, e.g., by matter rings outside.

If black hole is stationary, the EH is an IH. Familiar Examples: Schwarzschild and Kerr space-times. However, IHs can exist in non-stationary space-times. Examples: The Kastor-Traschen multi-BH solutions & Robinson-Trautman radiating solution.

DHs and IHs are widely used in mathematical and especially in numerical GR. This is how black holes are located during simulations. By now there are several thousand simulations representing binary black hole mergers.

Extra slide

Event Horizons Vs Quasi-local Horizons

Event Horizons

⋆ Teleological: One may be forming and growing in the room you are sitting in, due to a gravitational collapse a billion years from now.

⋆ A global notion: Need full evolution of space-time and the notion of $\mathcal{I}$. Not defined without asymptotic flatness if $\Lambda = 0$, for example.

⋆ BH Mechanics: Zeroth and first law: Stationary space-times and transition from one stationary space-time to a nearby one (infinitesimal process)

⋆ Second Law: Powerful but Qualitative: $\delta A_{EH} \geq 0$

⋆ Generically not smooth in dynamical situations; no invariant characterization of geometry or of how it changes as the BH settles down after merger (or collapse). Cannot be used during NR simulations.

Quasi-local Horizons

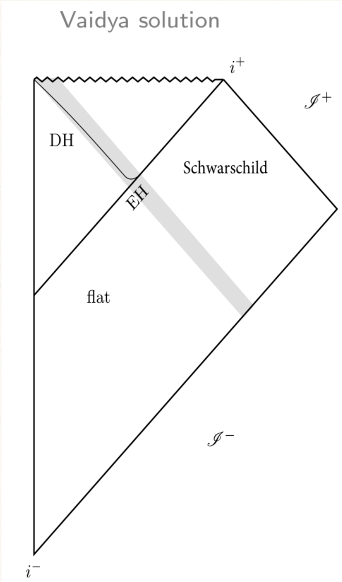
⋆ No teleology. No DH in the room you are sitting in!

⋆ Quasi-local notion. Does not need $\mathcal{I}$.

⋆ Zeroth law requires only NEH; there can be radiation even close to the BH. First law: emerges as the N & S condition for dynamics to be Hamiltonian for space-times with an NEH as an internal boundary. New insights.

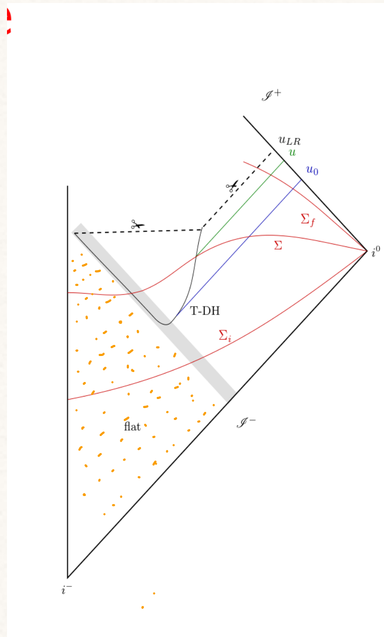
⋆ Second Law: Quantitative: $\delta A_{DH} =$ Infalling Energy Flux. Related to physical processes. (No DHs in Minkowski part of, say Vaidya solution.

⋆ DHs and NEHs characterized invariantly by multipoles. Their evolution provides an invariant characterization of horizon dynamics and shedding of multipoles as it settles down. Quasi-local horizons and multipoles used heavily in NR simulations.



Credit: De Lorenzo

Semiclassical Extension



Summary of Part 1 on Horizon Structure

Left Figure: A Dynamical horizon forms in a null-fluid collapse. It has zero area to begin, is initially space-like. and its area grows. When the collapsing pulse ends, it joins on to the null Isolated horizon which is also the future portion of the Event Horizon. The DH, IH and singularity enclose the Black Hole region.

PART 2: Semi-classical Space-time

Right Figure: We now 'switch on hbar' (i.e. quantum effects) using semi-classical approximation. The growing DH becomes instantaneously null, i.e. IH at the end of the infall. (The Hawking flux starts at the corresponding retarded time u=u_o.) Now the outgoing energy at scri-plus is positive, compensated by a negative energy in-falling flux.

The DH area starts decreasing (very slowly). The figure is cut-off when the DH has shrunk to say a million Planck mass beyond which semi-classical approximation cannot be trusted. The region enclosed by the two parts of DH is trapped (expansions of all future pointing null rays is negative). At the last ray u = u_LR scri-plus is likely to be incomplete. Then the last ray would not the Event Horizon; the semi-classical Space-time would not have an EH!

We will now discuss the inclusion of back reaction in semi-classical gravity i.e., the approximation in which gravity is treated classically, matter quantum mechanically. This can be justified in what is called "large N" approximation in which there is a large number of scalar fields coupled to gravity so the quantum fluctuations in the gravitational sector are negligible compared to the quantum fluctuations in the matter sector. This is a mean field approximation for geometrical variables: Geometric operators are replaced by their expectation values. Matter treated as a quantum field.

We will proceed in two steps:

(i) The CGHS model with a spherical collapse but with simplifications which makes it exactly soluble classically because one can first solve for matter and then for gravitational fields. Semi-classical analysis provides important checks on expectations.

(ii) Spherical collapse of a scalar field in GR; structurally similar to CGHS but in which one has to solve for matter and gravity together as a coupled system. Now the analysis cannot be as detailed.

Classical collapse of a scalar field

• Spherically symmetric collapse of a scalar field f in 4-d: Writing

$^4g_{ab} = g_{ab} + r^2\, s_{ab} \equiv g_{ab} + \frac{e^{-2\phi}}{\kappa^2}\, s_{ab}$, the action reduces to

$$S(g,\phi,f) = \tfrac{1}{2G}\int d^2x\,\sqrt{|g|}\,\left[e^{-2\phi}\,(R + 2\nabla^a\phi\nabla_a\phi + 2e^{-2\phi}\kappa^2) + Ge^{-\phi}\nabla^a f\nabla_a f\right]$$

• The Callen-Giddings-Harvey-Strominger (CGHS) Black hole:

$$S(g,\phi,f) := \tfrac{1}{2G}\int d^2x\,\sqrt{|g|}\,\left[e^{-2\phi}\,(R + 4\nabla^a\phi\nabla_a\phi + 4\kappa^2) + G\,\nabla^a f\nabla_a f\right]$$

f: scalar field; Setting $g^{ab} = \Omega\,\eta^{ab}$, gravitational sector: (ϕ, Ω).

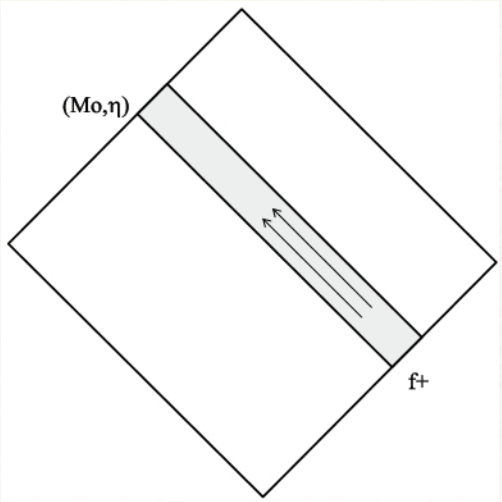
• 4-d and 2-d rather similar but CGHS is technically much simpler because the matter field f now satisfies $\Box_\eta\, f = 0$, and, given any solution f we can write down the solution for ϕ, Ω in a closed form algebraically !

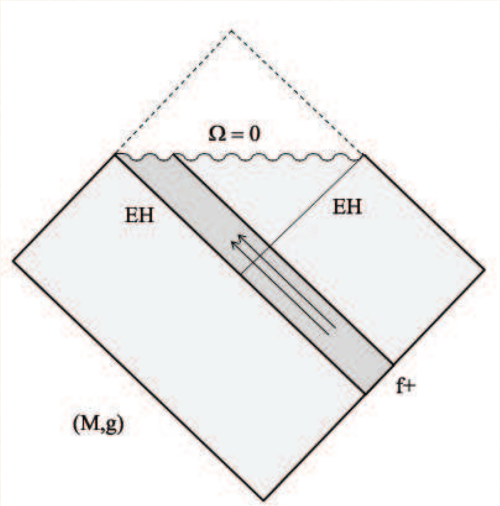
Setting $\eta_{ab} = -\partial_{(a} z^+\,\partial_{b)} z^-$, $\kappa\, x^\pm = \pm e^{\pm\kappa z^\pm}$, $\Phi = e^{-2\phi}$, $\Omega = \Theta^{-1}\Phi$

The solution is: $f = f_+(z^+) + f_-(z^-)$, $\Theta = -\kappa^2 x^+\, x^-$ and

$$\Phi = \Theta - \tfrac{G}{2}\int_0^{x^+} d\bar{x}^+ \int_0^{\bar{x}^+} d\bar{\bar{x}}^+\,(\partial f_+/\partial\bar{\bar{x}}^+)^2 - \tfrac{G}{2}\int_0^{x^-} d\bar{x}^- \int_0^{\bar{x}^-} d\bar{\bar{x}}^-\,(\partial f_-/\partial\bar{\bar{x}}^-)^2$$

Gravitational collapse in 2-d: CGHS solution





• Start with a pulse $f_+(z^+)$ in Minkowski space (M_0, η). It determines a full solution Φ, Θ and $g^{ab} = \Phi\Theta^{-1}\eta^{ab}$. Regular everywhere on M_0.

• How can there be a black hole, then?

• Φ vanishes along a space-like line. Ricci scalar blows up there. So Physical space-time (M, g) is smaller. $\mathcal{I}_R^+$ is complete and its past is not all of M. $\Rightarrow$ Black hole!

• $f_+, \Phi, \Theta, g^{ab}$ smooth fields on all of M_0. But for interpretation: g is the physical metric that determines the space-time geometry.

Same phenomenon for Hawking effect on the BH background (M, g). Past vacuum $|0\rangle_-$ on $\mathcal{I}_L^-$ interpreted as a mixed state because $\mathcal{I}_R^+$ of (M, g) is smaller than the $\mathcal{I}_R^{0\,+}$ of (M_0, η) $\Rightarrow$ must trace over modes in the 'missing part' of $\mathcal{I}_R^+$.

Quantum theory: Mean field approximation

- Framework for full quantum theory exits (ATV). Mean Field Approximation (MFA): Ignore the quantum fluctuations of geometry $(\hat{\Theta}, \hat{\Phi})$ but not of matter $\hat{f}$. Large number N of matter fields $\hat{f}$. PDEs for $\langle \hat{\Phi} \rangle := \Phi$ and $\langle \hat{\Theta} \rangle := \Theta$ but now they include back-reaction.

- Hyperbolic evolution Eqs:

$$\Box_{(\eta)} f = 0 \quad \Leftrightarrow \quad \Box_{(g)} f = 0$$

$$\partial_+ \partial_- \Phi + \kappa^2 \Theta = \frac{G\hbar}{24} \partial_+ \partial_- \ln \Phi \Theta^{-1} \equiv G \langle \hat{T}_{+-} \rangle$$

$$\Phi \partial_+ \partial_- \ln \Theta = -\frac{G\hbar}{24} \partial_+ \partial_- \ln \Phi \Theta^{-1} \equiv -G \langle \hat{T}_{+-} \rangle$$

determined completely by conformal anomaly

- Constraint Eqs imposed at $\mathcal{I}^-$ (and preserved in time):

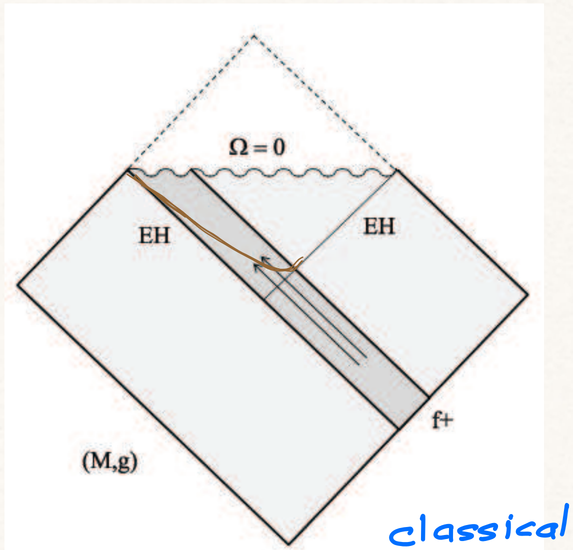
$$-\partial_-^2 \Phi + \partial_- \Phi \partial_- \ln \Theta = G T_{--} \hat{=} 0$$

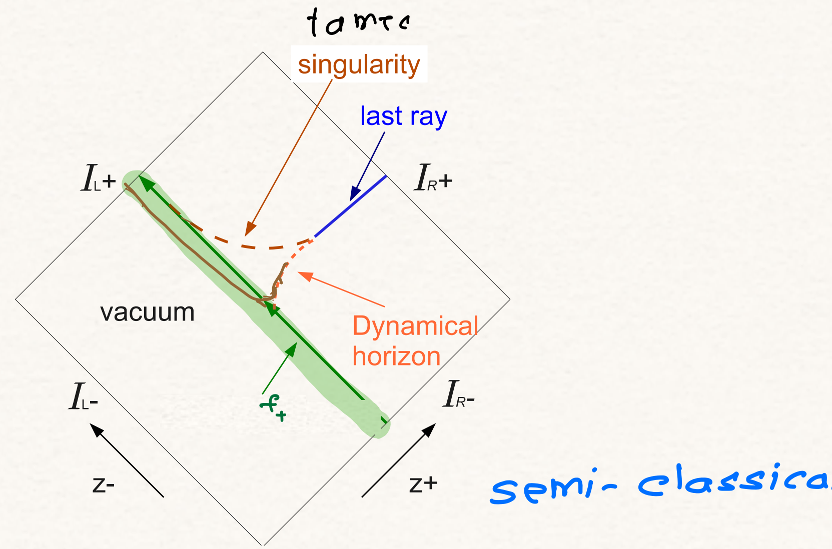
$$-\partial_+^2 \Phi + \partial_+ \Phi \partial_+ \ln \Theta = G \langle \hat{T}_{++} \rangle \hat{=} \Theta(z^{\pm}) - \frac{G}{2} \int_0^{x^+} \mathrm{d}\bar{\mathrm{x}}^+ \int_0^{\bar{\mathrm{x}}^+} \mathrm{d}\bar{\bar{\mathrm{x}}}^+ \, (\partial \mathrm{f}_+ / \partial \bar{\bar{\mathrm{x}}}^+)^2$$

- Physical metric in MFA: $g^{ab} = \Phi \, \Theta^{-1} \eta^{ab}$.

Task: Solve these equations. Global Issues: Do these solutions g^{ab} admit $\mathcal{I}_R^+$? Is there Bondi flux and mass at $\mathcal{I}_R^+$? What is the Bondi mass at the end of MFA space-time? Large? Planck scale? ...

Answers to some long standing issues





• Numerics $\Rightarrow$ g is asymptotically flat at right future null infinity $\mathcal{I}_R^+$.

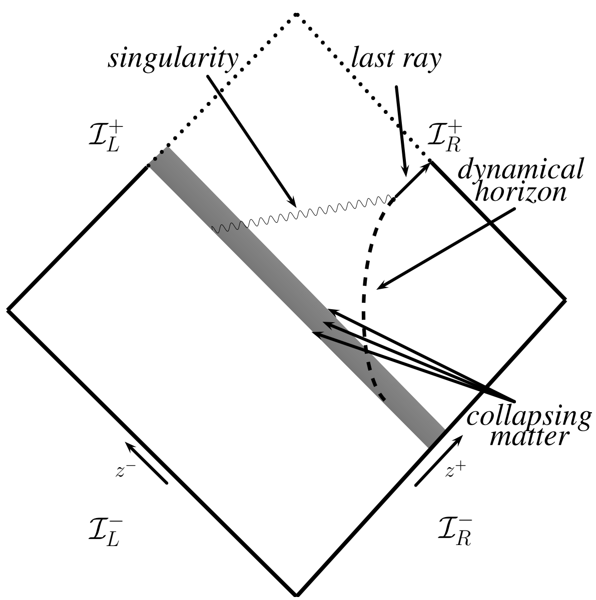
• Space-like singularity persists in MFA. But weak; g is C^0 but not C^1 there. Also ends because of evaporation; does not reach $\mathcal{I}_R^+$ as in the classical space-time. Last ray: future boundary of the MFA space-time.

• How big is $\mathcal{I}_R^+$? Numerics: The affine parameter w.r.t. the physical metric g is finite at the last ray of the MFA space-time, as hoped: NO Event Horizon (Otherwise information would be definitely lost!) Furthermore, Ricci scalar finite at the last ray: the singularity does not propagate out to infinity.

The Callan-Giddings-Harvey-Strominger Model

- Gravitational collapse of a massless scalar field gives rise to a BH. Model is exactly soluble in the classical theory. Hawking effect is realized in the external field approximation –again a thermal flux at late times. Back reaction has been included through detailed calculations using a mixture of analytical and high precision numerical simulations (AA, Pretorius, Ramazanoglu; Ori)

- Examples of Results for the semi-classical space-time:



⋆ The singularity is tamed by back reaction. The physical metric g is continuous there $\rightsquigarrow$ metric can be continued to a larger space-time. Furthermore, the singularity stays well away from $\mathcal{I}^+$.

⋆ There is no thunderbolt singularity. No Firewall. Metric across 'the last ray' is smooth.

⋆ What forms and evaporates is the dynamical horizon H. There is no event horizon in the semi-classical space-time.

Detailed correlations between the decrease in the area of the DH and decrease of (Bondi) mass measured at infinity: back-reaction 'in action'!

Extra slide

Bondi mass and flux at $\mathcal{I}_R^+$

• Traditional definition of Bondi-mass $M_{\rm B}^{\rm T}$ taken from the classical static solutions (Susskind et al, Hayward, ...). Analogy in 4-d: Taking the Bondi mass to be $M_{\rm B} = \oint \mathrm{d}^2\mathrm{V}\, \Psi_2^0$ —rather than $\oint \mathrm{d}^2\mathrm{V}\, (\Psi_2^0 + \bar\sigma^0 \dot\sigma^0)$ — also in dynamic situations. But then $M_{\rm B}$ or its flux would not be positive in dynamical situations.

• Numerics $\Rightarrow$ same thing happens in CGHS! $M_{\rm B}^{\rm T}$ can become arbitrarily negative before the last ray! (Ramazanoglu)

• But Following Bondi, a new expression was already proposed, using the balance law (AA, Taveras, Varadarajan):

Asymptotically, $\Phi = A(z^-)e^{\kappa z^+} + B(z^-) + O(e^{-\kappa z^+})$ and, B satisfies

$$\frac{\mathrm{d}}{\mathrm{dy}^-}\Big[\frac{\mathrm{dB}}{\mathrm{dy}^-} + \kappa B + \frac{N\hbar G}{24}\Big(\frac{\mathrm{d}^2\mathrm{y}^-}{\mathrm{dz}^{-2}}\Big(\frac{\mathrm{dy}^-}{\mathrm{dz}^-}\Big)^{-2}\Big)\Big] = -\frac{N\hbar G}{48}\Big[\frac{\mathrm{d}^2\mathrm{y}^-}{\mathrm{dz}^{-2}}\Big(\frac{\mathrm{dy}^-}{\mathrm{dz}^-}\Big)^{-2}\Big]^2$$

Natural to set: $M_{\rm B}(y^-) = \frac{\mathrm{dB}}{\mathrm{dy}^-} + \kappa B + \frac{N\hbar G}{24}\Big(\frac{\mathrm{d}^2\mathrm{y}^-}{\mathrm{dz}^{-2}}\Big(\frac{\mathrm{dy}^-}{\mathrm{dz}^-}\Big)^{-2}\Big)$

• Then: i) The definition agrees with that in the static case; ii) The flux is manifestly positive; iii) When it vanishes, $\partial/\partial y^- = \partial/\partial z^-$ at $\mathcal{I}_R^+$. Furthermore, iv) Numerics $\Rightarrow$ $M_{\rm B}$ is positive all the way to the last ray!

Scaling symmetry and Universality

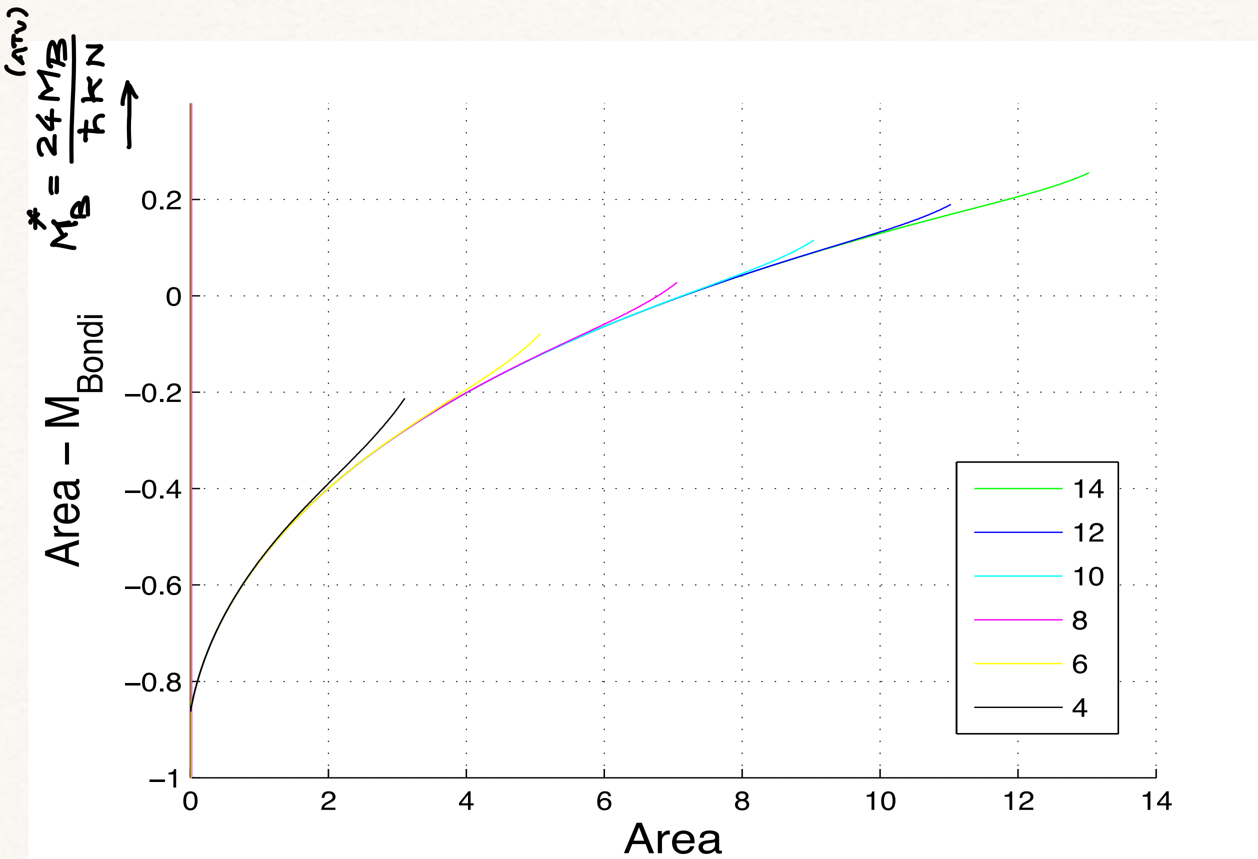
• A New Realization: If $(f, N, \Theta, \Phi,)$ is a solution to MFA equations, so is $(f, \lambda N, \lambda\Theta, \lambda\Phi)$ for any real constant λ.
Under this scaling, $g \to g,\ M_{\rm ADM} \to \lambda M_{\rm ADM}\ \ M_{\rm B} \to \lambda M_{\rm B}$.
So, for geometry, energetics, interpretation at $\mathcal{I}_R^+$, etc what matters are dimensionless quantities, e.g., $M^\star = 24 M_{\rm ADM}/\hbar\kappa N,\ \ m^* = 24 M_{\rm B}/\hbar\kappa N$.

• Numerics $\Rightarrow$ Universal behavior:
i) Global Process: For Macroscopic BHs, at the last ray, i.e. end of the MFA evolution, $m^\star \approx 0.86$ in Planck units.
ii) Dynamics: The ATV-Bondi flux is zero at early times and rises quickly once the trapped surface is formed. After this transient phase, the curve joins a universal curve. Thus for macroscopic BHs, the evolution at $\mathcal{I}_R^+$ is universal.

• However, there is a small but cumulative difference between this MFA evolution (which includes back reaction) and external field approximation of Hawking effect (which does not). $\Rightarrow$ Flux not really thermal. This is important for the recovery of information.

Extra slide

Universality: Dynamics

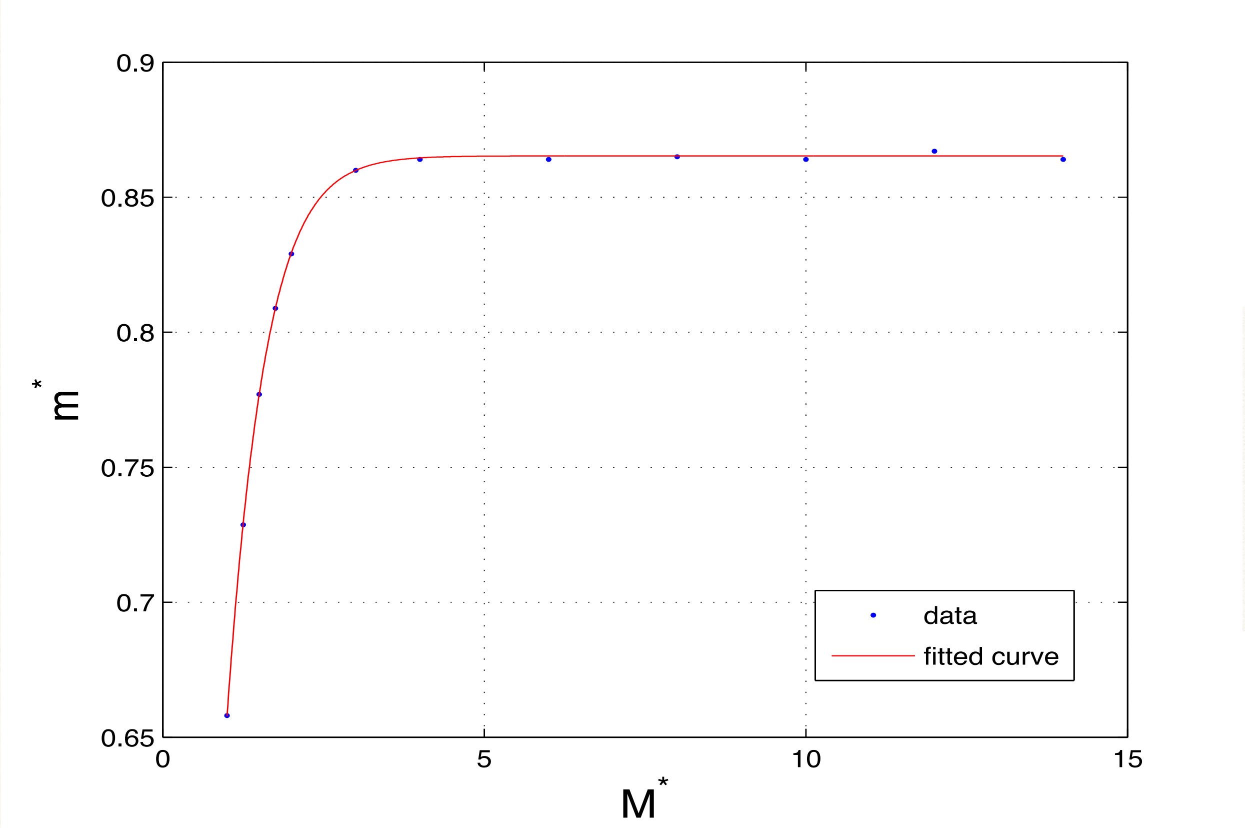


Simulations: Fethi Ramazanoglu

ATV-Bondi mass as a function of of the area of the dynamical horizon. At the end of the transient epoch, the curves for different ADM masses $M^{\star} = \frac{24 M_{ADM}}{\hbar \kappa N}$ join a universal curve.

Extra slides

Universality: Masses



Bend in the curve:

$a^* = \frac{24 \text{ Initial Area}}{N}$

> 1 Planck unit

iff $M^* > 3$

$M^* \gg 3$

$\Rightarrow$ Initial BH Macroscopic

Simulations: Fethi Ramazanoglu

$m^\star = 24 M_{\rm B}/\kappa\hbar N$ at the last ray as a function of $M^\star = 24 M_{\rm ADM}/\kappa\hbar N$ for $M* = 14, 12, 10, 8, 6, 4, 3, 2, 1.75, 1.50, 1.15, 1$. Data points fitted to the curve $m^\star = \alpha\,(1 - e^{-\beta (M^\star)^\gamma})$. There is a sharp transition around $M^\star = 3$. Larger values: Macroscopic BHs; smaller values: Microscopic BHs.

spherical collapse in GR

Extra slide

Challenges for mathematical relativists

• High precision numerical studies of Fethi Ramazanoglu and Frans Pretorius lead to several interesting conjectures that could be tested analytically for these rather simple PDEs. Examples:

• Start with initial data at $\mathcal{I}^-$ with $f_- = 0$ and regular f_+ with finite energy. Evolve using MFA equations which incorporate back reaction of black hole evaporation. Then

1) The maximal solution is asymptotically flat at right future null infinity.

2) $\mathcal{I}_R^+$ is future incomplete.

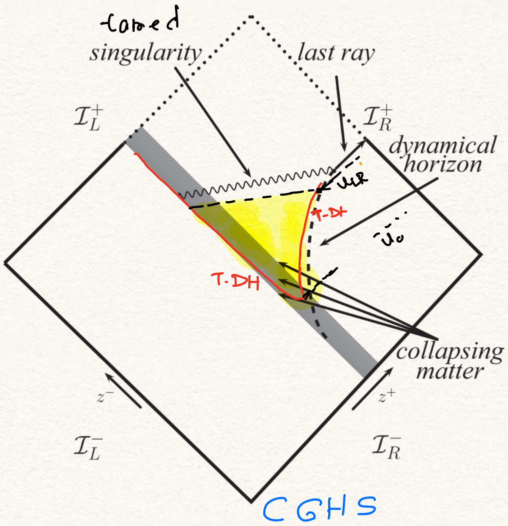
3) Positive mass theorem: The Bondi mass $M_{\rm B}$ is non-negative everywhere on $\mathcal{I}_R^+$

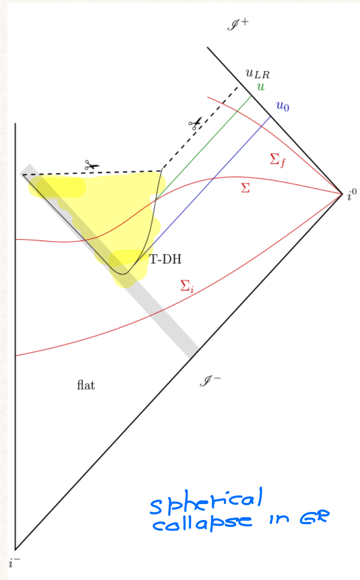
4) The final, rescaled Bondi mass $m^\star = 24 M_{\rm B}/\hbar\kappa N$ satisfies a universal relation such as $m^\star = \alpha(1 - e^{-\beta(M^\star)^\gamma})$, provided $M^\star \geq 4$. Numerics suggest $\alpha \approx 0.85,\ \beta \approx 1.42,$ $\gamma \approx 1.15$

5) There is a universal dynamical curve $F(Ar)$ that the actual ATV-Bondi flux asymptotes to quickly after the formation of the dynamical horizon. This curve departs from the constant value provided by the external field Hawking calculation and the integral of this departure is significant.

PART 2: Continued

Recall: To incorporate the back reaction of Hawking radiation, we proceed in two steps:
(i) The CGHS model with a spherical collapse but with simplifications which makes it exactly soluble classically because one can first solve for matter and then for gravitational fields; and
(ii) Spherical collapse of a scalar field in GR; structurally similar to CGHS but in which one has to solve for matter and gravity together as a coupled system.





The CGHS semi-classical space-time does not admit an event Horizon because its $\mathcal{I}^+$ is incomplete. The semi-classical metric is smooth across the last ray. One expects the same in the more realistic model of gravitational collapse in semi-classical GR.

There is an interesting apparent paradox associated with entanglement already in the semi-classical approximation:

Consider the phase in which $1\,M_\odot$ initial black hole shrinks to Lunar mass $\sim 10^{-7}\,M_\odot$. The process should be well-described by semi-classical gravity. Process takes some 10^{67} years and so a large number $\mathcal{N} \sim 10^{75}$ modes escape to infinity. State is pure because these are correlated with the infalling modes. into T-DH.

Apparent 'information Paradox': The lunar mass BH has radius of ~ 0.1mm! How can such a small ball hold so many modes? Heuristically, even if they all have a wavelength of ~ 0.1 mm, $\mathcal{N}$ modes would have a mass $\sim 10^{22}$ times the lunar mass!

One might think that, since the outer T-DH is time-like, modes that are correlated with the Hawking modes that went out to infinity could escape across this portion of T-DH, leading to restoration of purity all along this long adiabatic process. So by the time the T-DH has shrunk to the lunar mass, there are much fewer modes inside it'. This possibility of addressing the quandary has in fact been proposed (e.g. by Hayward). Unfortunately it is not tenable. Analysis of quantum fields near DH in states of interest shows that there is only ongoing flux across T-DH!

What happens is much more interesting: As I already indicated, the back reaction of the in-falling flux changes the space-time geometry in trapped region in a very interesting and at first astonishing way.

Calculations of the stress-energy tensor on the Schwarzschild space-times confirm the. idea
in semi-classical gravity there is a negative energy flux across the time-
like portion of T-DH such that MT-DH would decrease according to the standard Hawking
formula: $\frac{dM_{T-DH}}{dv} = -\frac{\hbar}{(2M_{T-DH})^2}$ Indeed, this is the basis of the standard view that
the evaporation time goes as ~ M^3. One can then argue that, in the phase of evaporation und
consideration, the form of the space-time metric in the region bounded by the T-DH is well
approximated by the Vaidya metric:

$$ds^2 = -\left(1 - \frac{2GM(v)}{r}\right) dv^2 + 2\,dv\,dr + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

with m(v) = M (v). (This is because during this phase the quantum correction to the
Schwarzschild metric of classical GR are small.) In particular, analogous conclusion is borne o
the detailed semi-classical analysis of evaporation of the CGHS black hole. So, let us work wi
metric. To understand the nature of space-time geometry it bestows on the trapped region, it
convenient to foliate it by some invariantly defined surfaces. It turns out that the most conver
choices (Ori and I found) are : r = const, and $\mathcal{K}$ = const where r is the area radius of the round
spheres and K is the Kretschmann scalar:

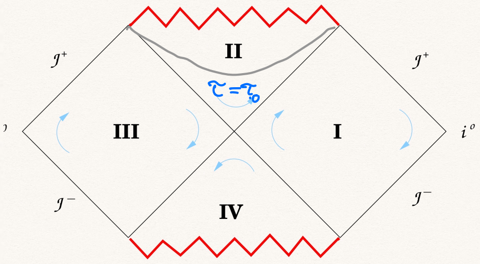
$$\mathcal{K} = R_{abcd}\, R^{abcd} = \frac{48\, G^2 m^2(v)}{r^6}$$

(The third geometrically natural foliation, Tr k =const is not viable with time changing mass: c
not foliate the entire region.) If we were to ignore quantum radiation, m(v) would be a consta
the two foliations coincide. In the semi-classical theory they don't, but are very similar.

Let us set v=0 at the start of quantum evaporation (r = 3 km) and v= v_o at the end of this p
(r = 0.1 mm). For each v, the leaf of the foliation has topology $\mathbb{S}^2 \times \mathbb{R}$ and we can calculate th
radius of the 2-spheres as well as the length of the leaf using the metric. As v= v_o the radiu
constant (10^{-2} cm) for the first foliation and increases (to~10^5 cm) we move from right to left
the length along the R-direction is astronomical 10^{62} lyrs & 10^{64} lyrs respectively!

This expansion of length is analogous to the cosmological expansion during which modes of
perturbations are stretched continuously, now from 3km to ~10^{62} lyrs! Thus during the very
semi-classical phase of the evaporation process, the partner modes that fell into the DH bec
infrared. Hence there is no obstruction to house the ~ `10^{75} quanta of the partner modes' on
v=v surface whose `mouth' has radius of only 0.1mm! What we know about the renormaliz
stress-energy in the Kruskal interior is compatible with this expectation. It should be possible
develop this mechanism in detail.

Part 3. Singularity Resolution



Only the Part II of the Kruskal space-time is directly relevant for singularity resolution. As discussed before, it is isometric to the spatially homogeneous Kantwoski-sachs model where r plays the role of time and the translational Killing field is now a space-like tangential to the constant time surfaces. Let us to the replacement $r \to \tau$ & $t \to x$, so $\partial/\partial x$ spatial KVF

In the phase space framework there are trivial infrared divergences because the range of x is the full real line. So one introduces an infrared cutoff $-L_o \le x \le L_o$. In the intermediate calculations. One has two make sure that the final physical results don't depend on this cut-off. Rather delicate issue!

$$ds^2 = -\frac{d\tau^2}{(2m/\tau - 1)} + \underbrace{\left(\frac{2m}{\tau} - 1\right) dr^2 + \tau^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)}_{\text{Homogeneous spatial metric}} \qquad (m = GM)$$

Recall : In LQG phase space variables are A_a^i and E^a_i

$$A_a^i \,\sigma_i \longleftrightarrow c/L_o \;\&\; b\,; \qquad E^a{}_i \sigma^i \longleftrightarrow \underset{\sim\,\tau^2}{p_c}\,, \;\; \underset{\sim\,\tau\cdot\left(\frac{2m}{\tau}-1\right)^{1/2}}{p_b/L_o}$$

Horizon : $b = 0$, $\frac{c}{L_o} \to \frac{\gamma}{4m}$, $p_b/L_o = 0$, $p_c = 4m^2$

singularity : $b \to \infty$ $\quad c \to \infty$, $p_c = 0$, $p_b = 0$

Since the singularity is a Big Crunch type cosmological, space-like singularity one can mimic ideas from Loop Quantum Cosmology. (Param's Lectures).

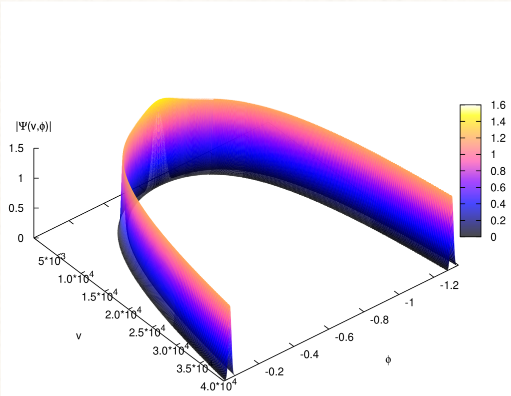
Key point in LQG: (From Kristina's Lectures) holonomies (Wilson loops) of connections are well-defined operators; the connection by itself is not. The curvature operator is replaced by the calculating the holomomy around a closed loop, dividing by its area and taking the limit as the area shrinks to the area gap; the minimum non-zero eigenvalue of the area operator. (In LQC two schemes were considered, so called the μ_o- Scheme in which the area refers to FLRW coordinates and the $\bar{\mu}$ - Scheme in which it refers to the physical area in the state under consideration.)

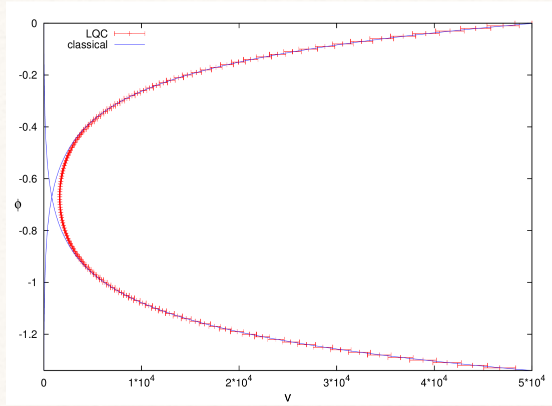
Scheme: Curvature Operator $\quad \hat{F}_{ab}^{\,k} \cdot \Psi(\mu) = \underbrace{\epsilon_{ij}^{\;\;k}\, L_o^{-2}\, {}^{o}\omega_a^i \, {}^{o}\omega_b^j}_{\text{kinematical factors}} \, \frac{\sin^2(\bar{\mu} c)}{\bar{\mu}^2} \, \Psi(\mu).$

$\bar{\mu}$ is a new phase space function \`\`quantum parameter'' of LQG $\quad \bar{\mu}^2 = 3\sqrt{3}/\mu$
($\bar{\mu} L_o$ = length of the edge of the square that encloses the area Δ in the given state $|\mu\rangle$). In the classical limit Δ and hence $\bar{\mu}$ goes to zero and curvature diverges as c^2 at the Big Bang. In LQG, it is replaced by a bounded trigonometric function.

Using this LQG curvature operator in the Quantum Hamiltonian Constraint in the Connection Variables, one finds that the quantum evolution does not break down: The Big Bang is replaced by a Big Bounce. The singularity is replaced by a regular surface at which all physical observables remain finite. Curvature and matter density that diverges in the classical theory attain their maximum values that are universal: $\rho_{sup} = \frac{\text{const}}{\Delta^3} \sim 0.41\, \rho_{Pl}$ $(\neq \infty)$ $\qquad (\text{curv rad})_{min} \approx 1.2\, \ell_P$ $(\neq 0)$

The LQC Bounce





• Effective equations that capture the motion of the peak of the wave function have been obtained using geometric quantum mechanics. The Friedmann equation is replaced by: $(\dot{a}/a)^2 = (8\pi G\rho/3)\,[1 - \rho/\rho_{\rm sup}]$ where $\rho_{\rm sup} \sim 0.41\rho_{\rm Pl}$.
Big Bang replaced by a quantum bounce.

• The matter density operator $\hat{\rho} = \frac{1}{2}\,(\hat{V}_\phi)^{-1}\,\hat{p}^2_{(\phi)}\,(\hat{V}_\phi)^{-1}$ has an absolute upper bound on the physical Hilbert space (AA, Corichi, Singh):

$$\rho_{\rm sup} = 18\pi/(G^2\hbar\Delta_o^3) \approx 0.41\rho_{\rm Pl}\ !$$

Provides a precise sense in which the singularity is resolved. Note: as $\Delta_o \to 0$, we have $\rho_{\rm sup} \to \infty$, i.e., classical and WDW result recovered.

Let us return to Region II (the BH Region). of the Kruskal space-time.
Situation is technically more complicated than in the FLRW and Bianchi models extensively studied in LQC. The quantum Hamiltonian constraint has been written down and singularity has been shown to be resolved. But details of full quantum dynamics have not been worked out: Open Problem!

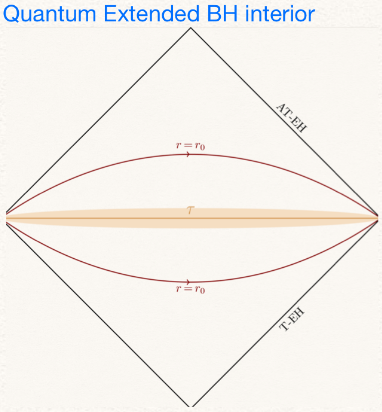
So far the focus has been on effective equations that capture key quantum corrections extremely well in the well-studied models. Even this analysis is surprisingly subtle. The analog of the μ_o and $\bar{\mu}$ schemes have been worked out and singularity is resolved in both. However, detailed examination revealed important limitations;

(i) In the μ_o scheme physical results depend on the infrared cutoff (not surprising);

(ii) In both schemes there are large quantum effects in the low curvature region which are physically unacceptable;

(iii) In the $\bar{\mu}$ scheme one encounters self-inconsistencies (evolution beyond the bounce leads to a geometry with 2-spheres whose area is less than the area gap)

Quantum Extended BH interior



Bottom half: classical Region II

The two schemes are distinguished by the way quantum parameters δ_b and δ_c are selected. But one can select them differently — phase space functions as in the $\bar{\mu}$-scheme, which however are constants along the effective trajectories, i.e., Dirac observables. Then all known limitations are overcome! The geometry of the quantum corrected space-time has been worked out in detail. The singularity is replaced by a regular 3-manifold called the transition surface, denoted $\mathcal{T}$, to the past of which one has a trapped region and to the future of which there is an anti-trapped region. This type of geometry never arises in classical space-times. Some of the Key features:

1. Universal upper bounds on curvature scalars, reached on $\mathcal{T}$

$$R^2\Big|_{\mathcal{T}} = \frac{256\pi^2}{\gamma^4\Delta^2} + \cdots \qquad R^{ab}R_{ab} = \frac{256\pi^2}{\gamma^4\Delta^2} + \cdots \qquad C_{abcd}C^{abcd} = \frac{1024}{3\gamma^4\Delta^2} + \cdots$$

Sub-leading corrections .... have the same form $\mathcal{O}\left(\frac{\Delta}{m^2}\right)^{1/3} \ln\left(\frac{m^2}{\Delta}\right)$

(In the classical limit $\Delta \to 0$ and all curvature invariants diverge.)

$(\text{curv rad})_{min} \sim 0.057\,\ell_P$

$\sim \frac{1}{2}$ of FLRW value.

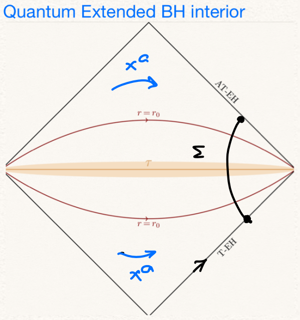
2. As one moves away from $\mathcal{T}$, Curvature rapidly approaches the classical values. Even for ADM mass $10^4 m_P$, the relative correction to the horizon radius is 10^{-15}. For a solar mass, it is 10^{-115}!

3. Even though the bounce is not exactly symmetric, the radii of trapping and anti-trapping horizons are almost the same. For $r_T = 3\,\text{km}$, $r_{AT} = 3 + O(10^{-25})\,\text{km}$, as one would expect since there is no physical mechanism for macroscopic mass inflation or deflation. Surprisingly, this is quite non-trivial to achieve.

4. Puzzle about Komar masses of trapping and anti-trapping horizons:

$$M_K^{T} - M_K^{AT} = 2\int_{\Sigma} \left(T^{eff}_{ab} - \tfrac{1}{2} T^{eff} g_{ab}\right) x^a d\Sigma^b$$

Quantum Extended BH interior



Bottom half: classical Region II

where T^{eff}_{ab} is the quantum stress-energy tensor, defined simply as

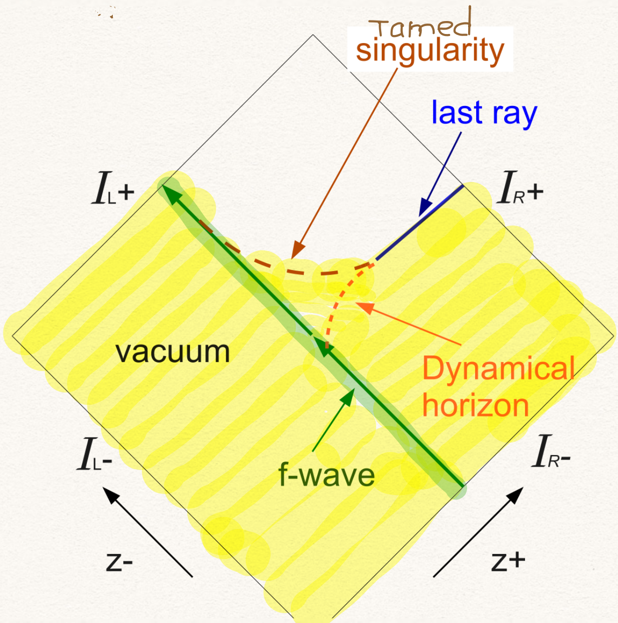
$$T^{eff}_{ab} := 8\pi G\left(R_{ab} - \tfrac{1}{2} R g_{ab}\right).$$

For macroscopic masses, the integrand on the right is large and negative near $\mathcal{T}$. This is why the classical singularity could be resolved. But then how can. the two Komar masses be approximately the same? Answer: The integrand is indeed large and negative, but in this particular effective description (AA, Olmedo, Singh), its value of the integral is very close to $-2M_K^T$! So $M_K^{AT} = -M_K^T$ and the minus sign is right because the Killing vector is future directed on. the Trapping Horizon and past directed on the anti-Trapping Horizon!

These are some illustrations of subtle features that one has to explore to have confidence in the model. It is not so difficult to resolve the singularity in LQG but very non-trivial to do so without triggering unintended, spurious effects.

Singularity Resolution:
Quantum Extension of space-time beyond semi-classical approximation

Quantum Extension of semi-classical space-time



Semi-classical Region in Yellow

The CGHS Model

Singularity already tamed semi-classically; metric is rendered C^0 (but not C^1) by quantum corrections. LQG provides strong evidence that it would be a well-defined operator (valued distribution) $\hat{g}$ in the full quantum theory. Two rather tame assumptions (made in all approaches) :

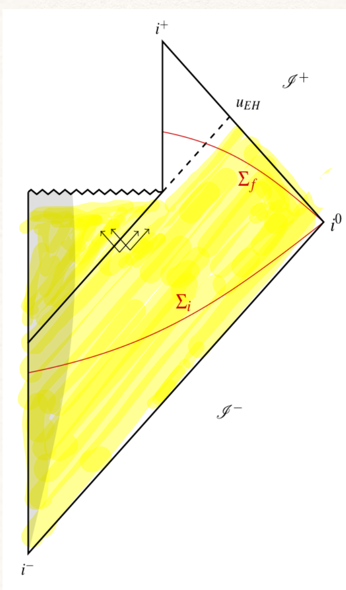
(i) $\langle \hat{g} \rangle$ admits $\mathscr{I}^+_R$ and MFA is excellent near $\mathscr{I}^+_R$
(ii) Flux of quantum radiation ends at some finite time
(this can be weakened)

imply: $\mathscr{I}^+_R$ is infinitely long as in Minkowski space and in the physical geometry given by $\langle \hat{g} \rangle$ the Heisenberg quantum state $|0\rangle_-$ is a pure state in the Fock space at $\mathscr{I}^+_R$ of $\langle \hat{g} \rangle$. In the physical geometry , it has the interpretation that populated by pairs with correlations between those emitted early (Hawking quanta) and those emitted at late times. (AA,Taveras,varadarajan)

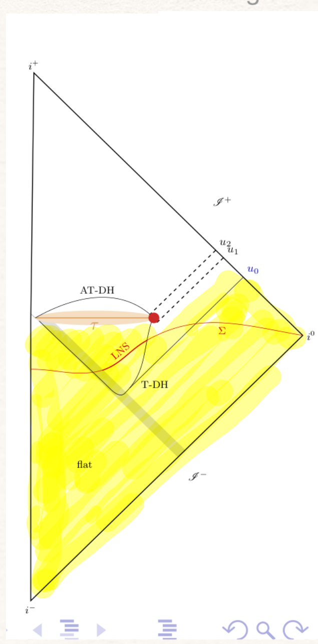
Even in this model, there is scope for further work. Results could be made sharper using techniques introduced by Deutsch and Fredenhagen in rigorous QFT. Also, further detailed calculations will provide valuable insight on the purification process. But already it bears out the LQG paradigm.

For the more realistic spherical collapse in GR

Hawking's first proposal.



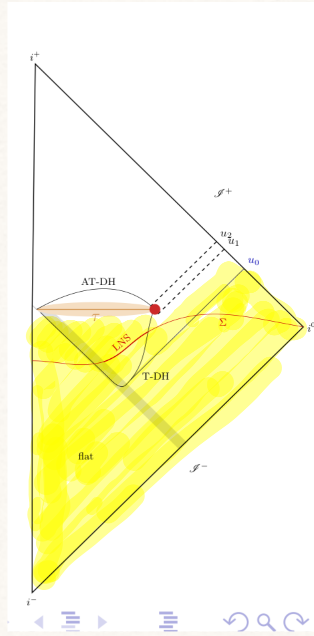
ɜ Paradigm



Semi-classical Region in Yellow

1. Even in the semi-classical regions, there are some conceptually important differences:
   In LQG what forms and evaporates is a DH; there is no EH.

2. In LQG there is no singularity. As the LQG analysis of the Kruskal interior suggests, it is replaced by a space-like transition surface to the past of which we have a trapped region but now bounded by the DH (rather than an EH and the singularity).



3. In both cases, when curvature reaches Planck scale, the semi-classical analysis cannot be trusted. In the Hawking picture, singularity is just maintained as a future boundary. The fact that curvature attains Planck scale near it is ignored. In LQG the pink region depicts potential large quantum gravity corrections.

4. In the pink region there are two independent QG effects:
(i) those that come from the negative energy in-falling Hawking quanta (that are entangled with the outgoing ones) . This effect is completely negligible at the left end of the pink region because it corresponds to the Hawking radiation Emitted by a solar mass BH. The effect grows very very slowly as we move to to the right.
(ii) Negative energy from the LQG induced effective stress-energy tensor we found in Kruskal space-time which is independent of the Hawking radiation. It is strong throughout the pink region because the leading term is independent of the mass of the (now shrinking) BH. So it is time independent. This is a brand new LQG effect, at the heart of singularity resolution.

5. Throughout the pink region, except for the right red blob, both effects can be analyzed using what is known as the “effective dressed metric”. It is a smooth metric whose coefficients, however, depend on hbar thereby incorporating the leading quantum effects. Therefore I believe that this phase can be analyzed regarding the Hawking partner modes as quantum fields propagating on the dressed effective metric. This analysis is an important open problem; it is feasible and will reveal interesting physics. Once we are on the other side of the pink region evolution across the anti-trapping horizon would be easy. The late time modes would be correlated with the early Hawking radiation, restoring purity, just as in the case of Page’s charcoal!

6. The red blob is conceptually and technically difficult because physics there is highly dynamical (last stage of the evaporation process) —so we cannot made adiabatic approximation— and at the same time curvature is Planck scale. The CGHS analysis does provide ideas but on the whole this, in my view, is the most difficult of open issues.

Two Lectures on

Black Holes in Loop Quantum Gravity

LQG Summer School, Ft. Lauderdale, 29 April - 3 May 2024

Abhay Ashtekar
Physics Department & Institute for Gravitation and the Cosmos, Penn State

Goal: To Present a Pedagogical Overview of Quantum Evaporation of Black Holes From a LQG Perspective. NOT meant to be exhaustive! Rather, a summary of some widely accepted views, with emphasis conceptual issues and LQG viewpoints on them, distinguishing between concrete results that lie at the foundation, expectations and hopes. My hope is that they will complement Carlo's lectures. As we will see, while the general picture is clear, many issues remain open. That makes field attractive to young researchers!! Need careful calculations/analysis as well as new ideas!

Will serve as the background material for talks on Quantum Black Holes in the LQG conference next week. They will cover the ongoing advances.

Based on work by Many Researchers, especially

AA, Beetle, Bianchi, Bojowald, Christodoulou, De Lorenzo, Del Rio, Haggard, Hayward, Krishnan, Lewandowski, Olmedo, Ori, , Pretorius, Pawlowski, Ramazanoglu, Rovelli, Schneider, Singh, Taveras, Varadarajan.

In order not to clutter the slides, generally I will not give references during the presentation. They are collected at the end, divided into topics covered for your convenience.

Organization

Lecture 1: General conceptual Framework: Main Issues and the LQG viewpoint on them

Lecture 2: Overview of the semi-classical epoch, singularity resolution in BHs, Current status of the issue of "Information Loss" and Open Issues

References where further details of the topics covered can be found

Reviews

1. A. Ashtekar, Black Hole Evaporation: A Perspective from Loop Quantum Gravity, 2002.08833
2. A. Ashtekar, J. Olmedo and P. Singh, Regular Black Holes from Loop Quantum Gravity, 2301.01309
3. A. Ori, Firewall or Smooth Horizon, Gen. Relativity. Gravit. 48, 9-24 (2016)
4. A. Ashtekar and B. Krishnan, Isolated and Dynamical Horizons and Their Applications, gr/qc0407042
5. A. Ashtekar and P. Singh Loop Quantum Cosmology: A Status Report, 1108.0893 (Section V for conceptual foundation of Effective Equations in LQG)

A few research articles that would be helpful to deepen your understanding

1. D. Page, Information in Black Hole Radiation, Phys. Rev. Lett. 71, 3743-46 (1993)
2. A. Ashtekar and M. Bojowald, Black Hole Evaporation: A Paradigm; gr-c/0504029
3. A. Ashtekar and M. Varadarajan, Information is NOT Lost in the Evaporation of 2-dimensional Black Holes, 0801.1811
4. A. Ashtekar, F. Pretorius and F. Ramazanoglu, Surprises in the Evaporation of 2-dimensional Black Holes, 1011.6442 (PRL), and 1012.0077 (Detailed paper)
5. D. Lebanon's and A. Ori, Interior Design of a 2-dimensional Semi-Classical Black Hole, 1005.2740
6. A. Ashtekar, J. Olmedo and P. Singh, Quantum Transfiguration of Kruskal Black Holes 1806.00648 (PRL) and 1806.02406 (detailed paper)
7. M. Christodoulou and T. De Lorenzo, On the Volume Inside an Old Black Hole, Phys. Rev. D94, 104002 (2016)