Lecture Note on Chern-Simons Theory and Spinfoam Model with Cosmological Constant

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This lecture note is for the mini-lecture on Chern-Simons Theory and Spinfoam Model with Cosmological Constant in Loops' 24 summer school held in April – May 2024 in Fort Lauderdale, Florida, USA. Comments are welcomed.

Contents

I. INTRODUCTION

This series of lectures with a total length of 3 hours aims to give a (hopefully) comprehensive introduction to the 4D Lorentzian spinfoam model with a cosmological constant to the level of frontier research based on this spinfoam model. This lecture note includes all the content in the lectures and provides additional details. We assume the readers have prior knowledge of the canonical Loop Quantum Gravity and Spinfoam model with vanishing cosmological constant, especially the EPRL model, which were introduced during the summer school before this mini-lecture. We refer to [1–4] for an elaborate introduction for these preparing content and excellent review articles [5–8] for an overview on the field of LQG and spinfoam, as well as 2021 Loop Quantum Gravity online Summer School for recorded lectures in a previous Loops' summer school.

Readers may find this lecture note overlaps with [9–13], to which we refer for more details.

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II. PREPARATION: THE IDEA OF SPINFOAM

Before we dig into the spinfoam construction for quantum gravity with $\Lambda \neq 0$, we first briefly review the construction of spinfoam with $\Lambda = 0$, mainly the EPRL spinfoam model, whose formalism will be deformed to include a non-vanishing Λ.

We work on the Lorentzian signature $\eta := \mathrm{diag}(-1,1,1,1)$. The conventions of the completely anti-symmetric symbol is

$$\epsilon^{0123} = 1\,, \quad \epsilon_{0123} = -1\,. \tag{1}$$

The first-order gravity action is in terms of the tetrad e and connection $\mathcal{A}$, which are both $\mathfrak{sl}(2,\mathbb{C})$ valued one-forms. The action on a 4-manifold $\mathcal{M}_4$ takes the form

$$S_{\mathrm{GR}}[e,\mathcal{A}] = \frac{1}{\kappa}\int_{\mathcal{M}_4} \langle e \wedge e \wedge \mathcal{F}(\mathcal{A})\rangle\,, \tag{2}$$

where $\mathcal{F}(\mathcal{A}) = \mathrm{d}\mathcal{A} + [\mathcal{A}\wedge\mathcal{A}]$ is the curvature 2-form of $\mathcal{A}$. The invariant non-degenerate bilinear form over $\mathfrak{sl}(2,\mathbb{C})$ evaluates as $\langle X \wedge Y\rangle := \frac{1}{2}\epsilon_{IJ}{}^{KL}X^{IJ}Y_{KL}$ for two forms X and Y. We will work in units where the reduced gravitational constant $\kappa := 8\pi G \overset{!}{=} 1$ in the rest of the note.

The equations of motion are not altered when one adds the so-called Holst term to the action in terms of an additional constant called the *Barbero-Immirzi parameter* γ, giving rise to the Holst action of general relativity:

$$S_{\mathrm{Holst}} = \int_{\mathcal{M}_4}\left\langle e\wedge e\wedge\mathcal{F}(\mathcal{A}) - \frac{1}{\gamma}\star(e\wedge e)\wedge\mathcal{F}(\mathcal{A})\right\rangle\,, \tag{3}$$

where $\star$ is the Hodge star operator that acts on the internal indices as $(\star X)_{IJ} = \frac{1}{2}\epsilon_{IJ}{}^{KL}X_{KL}$ and satisfies $\star^2 = -1$. Here, we require $\gamma \in \mathbb{R}$.

The actions (2) can be formulated into a constrained $\mathfrak{sl}(2,\mathbb{C})$ BF action

$$S_{\mathrm{BF}}[B,\mathcal{A}] = \int_{\mathcal{M}_4}\langle B\wedge\mathcal{F}(\mathcal{A})\rangle\,, \tag{4}$$

where the B-field is constrained to take a simple form

$$B \overset{!}{=} \pm e\wedge e\,. \tag{5}$$

This is, therefore, called the *simplicity constraint*. Similarly, (3) can also be formulated into BF-type:

$$S_{\mathrm{HBF}}[B,\mathcal{A}] = \int_{\mathcal{M}_4}\left\langle\left(1-\frac{1}{\gamma}\star\right)B\wedge\mathcal{F}(\mathcal{A})\right\rangle = \int_{\mathcal{M}_4}\mathrm{Tr}\left[\left(\star+\frac{1}{\gamma}\right)B\wedge\mathcal{F}(\mathcal{A})\right]\,, \tag{6}$$

where $\mathrm{Tr}(XY) = X^{IJ}Y_{IJ}$ for two forms X, Y. The momentum conjugate to $\mathcal{A}$ is simply

$$\Pi := \star\left(1-\frac{1}{\gamma}\star\right)B \equiv \left(\star+\frac{1}{\gamma}\right)B\,, \quad \text{equivalently} \quad B = \frac{\gamma}{1+\gamma^2}(1-\gamma\star)\,\Pi\,. \tag{7}$$

Apparently, if $\gamma = \pm i$, the relation between B and Π is not invertible. When one separates the action into the self-dual part and anti-self-dual part, $\gamma = i$ (*resp.* $\gamma = -i$) corresponds to projecting out the self-dual part (*resp.* anti-self-dual part) of B.

The constraint $B = \pm e\wedge e$ implies that

$$\frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}B^{IJ}_{\mu\nu}B^{KL}_{\rho\sigma} = \det(e)\epsilon^{IJKL} \quad\Longleftrightarrow\quad \frac{1}{4!}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}B^{IJ}_{\mu\nu}B^{KL}_{\rho\sigma} = -\det(e)\,. \tag{8}$$

When $\gamma \neq i$, which is the case in our setting, integration over B is equivalent to integration over Π as they are linearly dependent. Quantization of the BF theory with action (6) is given by the functional integration

$$\int\mathrm{d}\mathcal{A}\int\mathrm{d}B\, e^{\frac{i}{\ell_{\mathrm{p}}^2}S_{\mathrm{HBF}}[B,\mathcal{A}]} = \int\mathrm{d}\mathcal{A}\int\mathrm{d}\Pi\, e^{\frac{i}{\ell_{\mathrm{p}}^2}\int_{\mathcal{M}_4}\Pi_{IJ}F^{IJ}} = \int\mathrm{d}\mathcal{A}\,\delta(\mathcal{F}(\mathcal{A}))\,, \tag{9}$$

where $\ell_{\mathrm{p}} = \sqrt{8\pi G\hbar}/c^3$ is Planck length.

Given $\mathcal{M}_4$ with boundary $\partial\mathcal{M}_4$, let $\psi[\mathcal{A}_\partial]$ be a gauge-invariant state on $\partial\mathcal{M}_4$ as a function of boundary connection $\mathcal{A}_\partial = \mathcal{A}|_{\partial\mathcal{M}_4}$. We define a BF amplitude of the state as

$$\langle \mathrm{BF}|\psi[\mathcal{A}_\partial]\rangle = \int \mathrm{d}\mathcal{A}\,\delta(\mathcal{F}(\mathcal{A}))\psi[\mathcal{A}_\partial]\,. \tag{10}$$

In spinfoam model, we choose such boundary states, denoted as ψ_Γ that has support only on a graph $\Gamma \in \partial\mathcal{M}_4$ and that depends on $\mathcal{A}_\partial$ through holonomies $G_\ell[\mathcal{A}_\partial]$ along links of Γ. That is, $\psi_\Gamma[\mathcal{A}_\partial] = \psi_\Gamma[G_\ell[\mathcal{A}_\partial]]$. Such states are called the $\mathrm{SL}(2,\mathbb{C})$ *spin network states*.

The building block of the spinfoam amplitude for a general 4-manifold is given by the BF amplitude associated to a 4-ball $\mathcal{M}_4 = \mathcal{B}_4$, whose boundary is a 3-sphere S^3. In this case, it is natural to consider the boundary graph to be Γ_5 which is dual to the boundary of a 4-simplex σ (which is composed by 5 tetrahedra sharing 10 faces pairwise).

To encode gravity in this topological theory, following the EPRL spinfoam model construction, one imposes the simplicity constraints (quantumly) on the BF amplitude $\langle \mathrm{BF}|\psi_{\Gamma_5}\rangle$. In particular, the simplicity constraint (5) is quantized to an operator and it acts on ψ_{Γ_5}, which restricts the validity of boundary states.

Let us look more into how the quantum simplicity constraints are implemented. We first observe that the constraint (5) can be decomposed into 3 parts:

$$\begin{aligned} &\text{diagonal part:} && (\star B)_{\mu\nu}\cdot B_{\mu\nu} = 0\\ &\text{off-diagonal part:} && (\star B)_{\mu\nu}\cdot B_{\mu\rho} = 0\\ &\text{dynamical part:} && (\star B)_{\mu\nu}\cdot B_{\rho\eta} = \tfrac{1}{2}V\epsilon_{\mu\nu\rho\eta} \end{aligned} \tag{11}$$

where the indices μ,ν,ρ,η are all different. Upon triangulation, these constraints are promoted to constraints on a 4-simplex as follows. Define the discretized B-field in the frame of tetrahedron t associated to a boundary triangle f as $B_f^{IJ}(t) = \int_f B^{IJ}(t)$ with $I,J = 0,1,2,3$ being the internal labels, then the discrete versions of (11) are [14–16]

$$\text{diagonal constraints:} \qquad \epsilon_{IJKL}B_f^{IJ}(t)B_f^{KL}(t) = 0\,,\quad \forall f\in t\,, \tag{12a}$$

$$\text{off-diagonal constraints:} \quad \epsilon_{IJKL}B_f^{IJ}(t)B_{f'}^{KL}(t) = 0\,,\quad \forall f,f'\in t\,, f\neq f'\,, \tag{12b}$$

$$\text{dynamical constraints:} \quad \epsilon_{IJKL}B_f^{IJ}(t)B_{f'}^{KL}(t') = \pm 12 V_4(\sigma)\,,\quad \forall f\in t,\ f'\in t'\neq t,\ t,t'\in\sigma\,, \tag{12c}$$

where f, t and σ denote a triangle, a tetrahedron and a 4-simplex respectively. $V_4(\sigma)$ denotes the 4-volume of σ, $f\in t$ denotes that f is on the boundary of t, and $t\in\sigma$ denotes that t is on the boundary of σ. (12c) can be implied from (12a) and (12b) hence is redundant. There are two sets of solutions to the constraints (12):

$$B_f^{IJ}(t) = \pm e^I(t)\wedge e^J(t) \qquad \text{or} \qquad \star B_f^{IJ}(t) = \pm e^I(t)\wedge e^J(t)\,, \tag{13}$$

where $e^I(t)$ is a tetrad 1-form in a Cartesian coordinate patch covering t. The first solution can be viewed as the discretized version of (5). The two quadratic constraints (12a) and (12b) can be strengthened to a single set of linear constraints

$$\text{linear constraints:} \qquad \exists\, N_J \text{ such that } N_J B_f^{IJ}(t) = 0\,,\quad \forall f\in t\,, \tag{14}$$

which selects the first solutions from (13) as wanted. We will treat (14) as the full set of simplicity constraints and generalize it in the new spinfoam model.

The simplicity constraints then imply that the discretized B-field $B_f^{IJ}(t)$ measures the area $\mathfrak{a}_f = |\frac{1}{2}\epsilon_{IJKL}N^J B_f^{KL}(t)|$ of the triangle f. One can gauge fix the vector $N_J = N_0$ to be timelike, then (14) is equivalent to the statement that the tetrahedron t is spacelike. Moreover, not $B_f^{IJ}(t)$'s of all the 4 triangles are independent for a given tetrahedron t but they are subject to the closure constraint that generates the SU(2) gauge symmetry:

$$\sum_{f\in t} B_f^{IJ}(t) = 0 \quad\Longleftrightarrow\quad \sum_{f\in t}\mathfrak{a}_f\mathfrak{n}_f^I = 0\,, \tag{15}$$

where $\mathfrak{n}_f^I$ is the normal vector to f satisfying $|\mathfrak{n}_f| = 1$. By Minkowski's theorem, the simplicity constraint (14) together with the closure condition (15) allows us to identify a convex tetrahedron whose face areas and normals are given by $\mathfrak{a}_f$'s and $\mathfrak{n}_f^I$'s.

In the quantum theory, the constraint (14) will be promoted to constraint operator which acts on the partition function. The above classical description is enough for us to introduce the spinfoam model with a cosmological constant by generalization while the construction of the partition functions in the new spinfoam model is relatively different from the EPRL model. For this reason, we will not describe in detail the quantum theory of the EPRL model here but only mention the necessary ingredients thereinafter. We refer interested readers to the original series of papers [14, 15, 17].

III. FROM 4D GRAVITY WITH Λ TO CHERN-SIMONS PATH INTEGRAL

As above, the starting point of the spinfoam model with a cosmological constant is the Holst action adding a cosmological term. The corresponding topological BF action, denoted as $S_{\Lambda\mathrm{BF}}$, is

$$S_{\Lambda\mathrm{BF}}[B,\mathcal{A}]=\int_{\mathcal{M}_4}\mathrm{Tr}\left[\left(\star+\frac{1}{\gamma}\right)B\wedge\left(\mathcal{F}(\mathcal{A})-\frac{|\Lambda|}{6}B\right)\right]. \tag{16}$$

The trace is taken in the $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra and it evaluates as $\mathrm{Tr}[X\wedge Y]=X^{IJ}Y_{IJ}$. Slightly different from some literature, we let $S_{\Lambda\mathrm{BF}}$ depend on the absolute value of the cosmological constant $|\Lambda|$ so that the sign is encoded in the simplicity constraint:

$$B\cong\mathrm{sgn}(\Lambda)e\wedge e\,, \tag{17}$$

imposing which one recovers the first-order action of general relativity with a cosmological constant Λ, written in terms of the cotetrad e and the connection $\mathcal{A}$

$$S_{\mathrm{GR}}[e,\mathcal{A}]=\int_{\mathcal{M}_4}\mathrm{Tr}\left[\left(\star+\frac{1}{\gamma}\right)(e\wedge e)\wedge\left(\mathcal{F}(\mathcal{A})-\frac{\Lambda}{6}(e\wedge e)\right)\right]. \tag{18}$$

The equations of motion of (16) from varying the B field leads to a linear relation between the $\mathcal{F}$ field and the B field, which transfers to the equation between the curvature and the cotetrad after imposing the simplicity constraints.

$$\frac{\partial S_{\Lambda\mathrm{BF}}}{\partial B^{IJ}}=0\quad\Longrightarrow\quad\mathcal{F}=\frac{|\Lambda|}{3}B\quad\xrightarrow{B\cong\mathrm{sgn}(\Lambda)e\wedge e}\quad\mathcal{F}\cong\frac{\Lambda}{3}e\wedge e\,. \tag{19}$$

The right-most equation above is the simplicity constraint that we will implement to the theory.

The path integral of the action (16) contains a Gaussian integral for the B field, performing which is equivalent to imposing the solution $\mathcal{F}=\frac{|\Lambda|}{3}B$. It leads to

$$\int\mathrm{d}\mathcal{A}\mathrm{d}B\,e^{\frac{i}{\ell_\mathrm{p}^2}S_{\Lambda\mathrm{BF}}}=\int\mathrm{d}\mathcal{A}\,\exp\left(\frac{3i}{2\ell_\mathrm{p}^2|\Lambda|}\int_{\mathcal{M}_4}\mathrm{Tr}\left[\left(\star+\frac{1}{\gamma}\right)\mathcal{F}(\mathcal{A})\wedge\mathcal{F}(\mathcal{A})\right]\right). \tag{20}$$

We separate $\mathcal{F}$ into its self-dual part F and anti-self-dual part $\bar{F}$ *w.r.t.* the $\star$ operation, *i.e.*

$$\mathcal{F}=F+\bar{F}\,,\quad F=\frac{1}{2}\left(1-i\star\right)\mathcal{F}\,,\quad\bar{F}=\frac{1}{2}\left(1+i\star\right)\mathcal{F}\,,\quad\star F=iF,\quad\star\bar{F}=-i\bar{F}\,. \tag{21}$$

As $\star^2=-1$ in the Lorentzian signature, the above was done by first complexifying the $\mathfrak{sl}(2,\mathbb{C})$-valued variables before the separation (see (33)). (20) can then be written as

$$\int\mathrm{d}A\mathrm{d}\bar{A}\,\exp\left(-\frac{3}{2\ell_\mathrm{p}^2|\Lambda|}\int_{\mathcal{M}_4}\left(1-\frac{i}{\gamma}\right)\mathrm{Tr}[F(A)\wedge F(A)]-\left(1+\frac{i}{\gamma}\right)\mathrm{Tr}[\bar{F}(\bar{A})\wedge\bar{F}(\bar{A})]\right), \tag{22}$$

where A and $\bar{A}$ are the self-dual and anti-self-dual parts of $\mathcal{A}$ respectively and ℓ_p is the Planck length. As the exponent is a total derivative term, (22) becomes a path integral of $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons action with complex level on the boundary $\partial\mathcal{M}_4$. When $\mathcal{M}_4$ is topologically trivial, (22) takes the form as

$$\int\mathrm{d}A\mathrm{d}\bar{A}\,e^{-iS_{\mathrm{CS}}[A]-iS_{\mathrm{CS}}[\bar{A}]}=:\int\mathrm{d}\mathcal{A}\,e^{-iS^t_{\mathrm{CS}}[\mathcal{A}]}\,, \tag{23}$$

where

$$S_{\mathrm{CS}}[A]=\frac{t}{8\pi}\int_{\partial\mathcal{M}_4}\mathrm{Tr}\left[A\wedge\mathrm{d}A+\frac{3}{2}A\wedge A\wedge A\right],\quad S_{\mathrm{CS}}[\bar{A}]=\frac{\bar{t}}{8\pi}\int_{\partial\mathcal{M}_4}\mathrm{Tr}\left[\bar{A}\wedge\mathrm{d}\bar{A}+\frac{3}{2}\bar{A}\wedge\bar{A}\wedge\bar{A}\right]. \tag{24}$$

The level t and its complex conjugate $\bar{t}$ can be separated into real and imaginary parts as

$$t=k+is\,,\quad\bar{t}=k-is\,,\quad\text{where }k=\frac{12\pi}{\ell_\mathrm{p}^2\gamma|\Lambda|}\in\mathbb{Z}_+\,,\quad s=\gamma k\in\mathbb{R}\,. \tag{25}$$

$k \in \mathbb{Z}_+$ is required for the gauge invariance of the partition function (23). Therefore, the quantization of gravity on a 4-manifold $\mathcal{M}_4$ with a cosmological constant Λ now relates to quantization of the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory with complex coupling constant on the 3D boundary $\partial\mathcal{M}_4$ of the manifold:

$$S_{\mathrm{CS}}[A,\bar{A}] = \frac{t}{8\pi}\int_{\partial\mathcal{M}_4} \mathrm{Tr}\left[A \wedge \mathrm{d}A + \frac{3}{2} A \wedge A \wedge A\right] + \frac{\bar{t}}{8\pi}\int_{\partial\mathcal{M}_4} \mathrm{Tr}\left[\bar{A} \wedge \mathrm{d}\bar{A} + \frac{3}{2} \bar{A} \wedge \bar{A} \wedge \bar{A}\right] . \tag{26}$$

The connection A (as well as $\bar{A}$) is now restricted to the 3-boundary $\partial\mathcal{M}_4$.

Following the same spirit as in the EPRL model, the spinfoam amplitude can be defined as the inner product of the CS partition function coupled with a gauge-invariant state $\psi[\mathcal{A}]$ on $\partial\mathcal{M}_4$ that encodes the information of quantum geometry, formally written as

$$\mathcal{A}_{\mathrm{SF}} = \langle \mathrm{CS}|\psi[\mathcal{A}]\rangle = \int \mathrm{d}\mathcal{A}\, e^{-iS^t_{\mathrm{CS}}[\mathcal{A}]}\psi[\mathcal{A}]\,, \tag{27}$$

on which we impose the (quantized) simplicity constraint. Since the simplicity constraint requires non-trivial magnetic flux by (19), certain defect has to be introduced to the Chern-Simons theory (otherwise the Chern-Simons theory would imply $\mathcal{F} = 0$ by the equation of motion). Therefore, the construction of spinfoam model with a non-vanishing cosmological constant on a spacetime manifold $\mathcal{M}_4$ relies on the quantization of the $\mathrm{SL}(2,\mathbb{C})$ CS theory with a complex level t on its boundary $\partial\mathcal{M}_4$ with defects, which we describe in detail in the next section.

Interlude: Self-dual and anti-self-dual decomposition for $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$

This part of the note consults mostly Appendix B of [9]. The real generators $\{\mathcal{J}^{IJ}\}_{I<J}(I,J=0,\cdots,3)$ of Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ have components

$$\mathcal{J}^{0i}=K^i,\quad \mathcal{J}^{ij}=\epsilon^{ij}{}_kJ^k\,,\quad i,j,k=1,2,3\,, \tag{28}$$

which satisfy the commutation relations

$$[J^i,J^j]=\epsilon^{ij}{}_kJ^k\,,\quad [K^i,K^j]=-\epsilon^{ij}{}_kJ^k\,,\quad [K^i,J^j]=\epsilon^{ij}{}_kK^k\,. \tag{29}$$

Decompose $X=X_{IJ}\mathcal{J}^{IJ}\in\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$ into its self-dual part $X_+:=\frac{1}{2}(1-i\star)X$ and anti-self-dual part $X_-:=\frac{1}{2}(1+i\star)X$ which satisfy $\star X_\pm=\pm iX_\pm$ whose components are

$$(X_\pm)_{IJ}=\frac{1}{2}\left(X_{IJ}\mp\frac{i}{2}\epsilon_{IJ}{}^{KL}X_{KL}\right)\,. \tag{30}$$

Define the self-dual basis T^k_+ for $\mathfrak{sl}(2,\mathbb{C})$ and anti-self-dual basis T^k_- as

$$T^k_\pm:=\frac{J^k\pm iK^k}{2}\,. \tag{31}$$

Then any (anti-)self-dual component of X can be written under the (anti-)self-dual basis as

$$X_\pm=(X_\pm)_k\,T^k_\pm\,,\quad\text{with}\ \ (X_\pm)_k=\frac{1}{2}\epsilon^{ij}{}_kX_{ij}\mp iX_{0k}\,. \tag{32}$$

The (anti-)self-dual basis satisfy the commutation relation of an $\mathfrak{su}(2)$ Lie algebra separately and they mutually commute:

$$[T^i_\pm,T^j_\pm]=\epsilon^{ij}{}_kT^k_\pm\,,\quad [T^i_\pm,T^j_\mp]=0\,, \tag{33}$$

And their bilinear form is chosen to be

$$\langle T^i_\pm,T^j_\pm\rangle=\pm\frac{i}{2}\delta^{ij}\,,\quad \langle T^i_\pm,T^j_\mp\rangle=0\,, \tag{34}$$

which gives $\langle X,Y\rangle=\frac{1}{2}\epsilon^{IJ}{}_{KL}X_{IJ}Y^{KL}$. The self-dual generators (*resp.* (anti-)self-dual generators) satisfying (33) and (34) can be represented in Weyl's left-handed $(\frac{1}{2},0)$ representation (*resp.* Weyl's right-handed $(0,\frac{1}{2})$ representation) as

$$\begin{aligned}(\frac{1}{2},0):\quad J^k=\frac{\sigma^k}{2i}\,,\quad K^k=-\frac{\sigma^k}{2}\quad\Longrightarrow\quad T^k_+=\frac{\sigma^k}{2i}\,,\\ (0,\frac{1}{2}):\quad J^k=\frac{\sigma^k}{2i}\,,\quad K^k=\frac{\sigma^k}{2}\quad\Longrightarrow\quad T^k_-=\frac{\sigma^k}{2i}\,.\end{aligned} \tag{35}$$

Then the bilinear form (34) is realized by

$$\langle T^i_\pm,T^j_\pm\rangle=\mp i\mathrm{Tr}\left(T^i_\pm T^j_\pm\right)\,. \tag{36}$$

For real-valued X, we have $\bar{X}_-=X_+$ with the bar denoting the complex conjugate.

IV. CHERN-SIMONS PARTITION FUNCTION ON THE TRIANGULATED 3-MANIFOLD

The building block of the spinfoam amplitude is the vertex amplitude $\mathcal{A}_v$. To construct $\mathcal{A}_v$, we consider $\mathcal{M}_4$ to be a 4-simplex which is topologically equivalent to a 4-ball $\mathcal{B}_4$, and quantize the Chern-Simons theory on its boundary which is topologically isomorphic to a 3-sphere S^3.

The triangulation $\mathbf{T}_3$ of S^3 is the boundary of a 4-simplex. It contains 5 tetrahedra sharing 10 triangles. Their duality is summarized in Table I. The dual graph of a 4-simplex σ contains 5 nodes connected by 10 links and is denoted as Γ_5 (See fig.1).

triangulation $\mathbf{T}_3$ of S^3	**Γ_5 graph**
tetrahedron t	node v
triangle f	link e
edge E	face

TABLE I: The one-to-one correspondence of the triangulation of S^3 and its dual graph Γ_5. We will use the same terminologies and notations throughout this note.

Upon triangulation, one should smear the simplicity constraint (19) over the sub-simplices of the 4-simplex, the quantization of which will define an operator on the wave functions on the 4-simplex. As the constraint takes the form of 2-forms, it is natural to smear it over 2-simplices – triangles, then the curvature is smeared as $F_f^{IJ}(t) = \int_f F^{IJ}(t)$. In the dual picture, the violation of flatness occurs *only* on the links of Γ_5. In other words, the simplicity constraint operators are only inserted on the links of Γ_5. Let us now view Γ_5 as a graph embedded in S^3. A key idea of constructing this spinfoam model is to utilize the following equivalent treatment:

operator insertion along a graph
=
remove the graph and its open tubular neighbourhood then impose boundary condition on the graph complement .

Therefore, instead of inserting a simplicity operator to Γ_5, we will remove Γ_5 and define a CS quantum state on the graph complement $S^3\backslash\Gamma_5$ which is the complement of an open tubular neighbourhood of Γ_5 in S^3 and then impose boundary conditions on $\partial(S^3\backslash\Gamma_5)$.

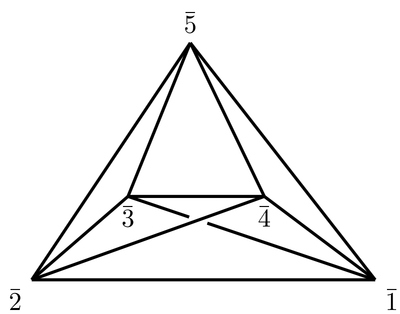


FIG. 1: The Γ_5 graph (projected on $\mathbb{R}^2$) as the dual graph of the triangulation $\mathbf{T}_3$ of S^3.

We first perform the former step – to define a CS partition function on $S^3\backslash\Gamma_5$ applying the method developed in a series of works [18–23]. The quantization of complex Chern-Simons theory uses the *ideal triangulation* of the graph-complement 3-manifold, say Γ-complement of $\mathcal{M}_3$ denoted as $\mathcal{M}_3\backslash\Gamma$. The building blocks of the ideal triangulation are the *ideal tetrahedra* $\triangle$'s, which are tetrahedra with vertices truncated into triangles as shown in fig.2a [1]. The original boundaries of an $\triangle$ before truncation are called the *geodesic boundaries* of $\triangle$ and the truncated vertices are called the *cusp boundaries* (or *disc cusp*) of $\triangle$. The boundaries of $\mathcal{M}_3\backslash\Gamma$ can also separated into two types:

- geodesic boundaries – boundaries created by removing open balls around nodes of Γ, which are holed spheres, and
- cusp boundaries or *annulus cusp* – boundaries created by removing the tubular neighbourhood of links of Γ, which are annuli.

An ideal triangulation decomposes $\mathcal{M}_3\backslash\Gamma$ into a set of ideal tetrahedra such that the geodesic boundaries are triangulated by the geodesic boundaries of $\triangle$'s while the annulus cusps are triangulated by the disc cusps of $\triangle$'s. An example of the ideal triangulation of a four-valent-node-complement of S^3 is illustrated in fig.3. It is part of the ideal triangulation of $S^3\backslash\Gamma_5$.

The triangulation of $S^3\backslash\Gamma_5$ can be decomposed into 5 *ideal octahedra* (see fig.4), then each ideal octahedron can be further decomposed into 4 ideal tetrahedra by adding an internal edge (see fig.2b). As a result, the triangulation

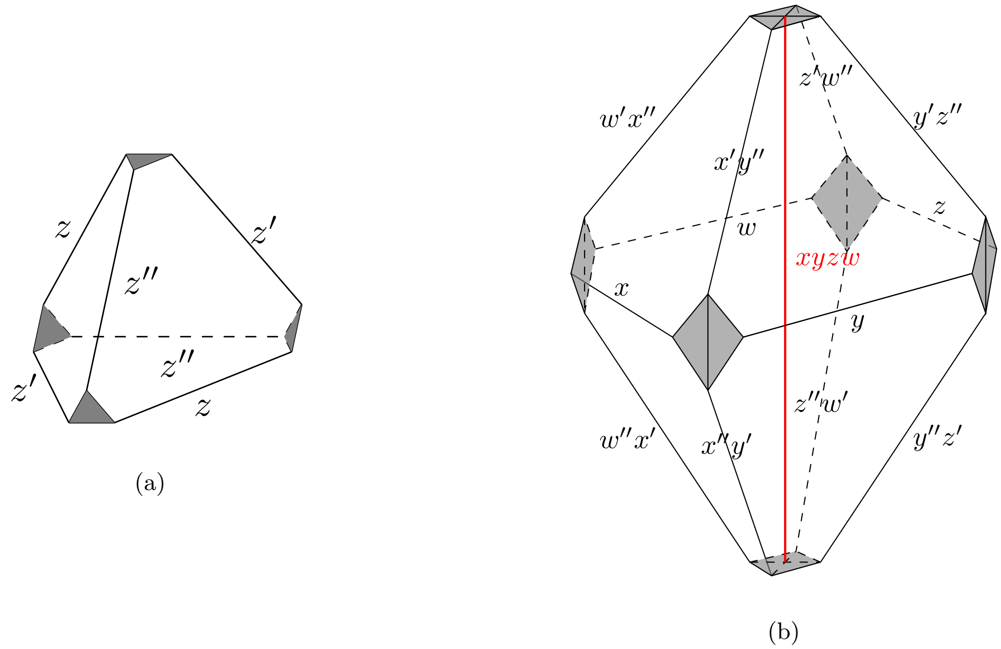


FIG. 2: *(a)* An ideal tetrahedron whose edges are dressed with edge coordinates (z, z', z''). Each pair of opposite edges are dressed with the same coordinate. The disc cusps are filled *in gray*. *(b)* An ideal octahedron. Choose the equator to be edges dressed with x, y, z, w. Adding an internal edge (*in red*) orthogonal to the equator separates the ideal octahedron into four ideal tetrahedra, each of which is dressed with different copies of coordinates $(x, x', x'')\,, (y, y', y'')\,, (z, z', z'')\,, (w, w', w'')$. For edges shared by different ideal tetrahedra, coordinates are multiplied together.

contains 20 ideal tetrahedra in total. (One should not confuse the ideal tetrahedra from triangulating $S^3\backslash\Gamma_5$ with the tetrahedra from triangulating S^3 as the boundary of the 4-simplex.) The boundary $\partial(S^3\backslash\Gamma_5)$ of $S^3\backslash\Gamma_5$ is made of five 4-holed spheres $\{\mathcal{S}_a\}_{a=1}^5$ and 10 annuli $\{(ab)|a<b,\, a,b=1,\cdots,5\}$ connecting these holes. The triangulation of $S^3\backslash\Gamma_5$ induces the ideal triangulation on its boundary $\partial(S^3\backslash\Gamma_5)$. The ideal triangulation of a 4-holed sphere $\mathcal{S}_a$ contains four triangles located at the holes and four hexagons as illustrated in fig.3b. On the other hand, an annulus is triangulated into the boundary of a triangular prism whose two triangles are identified with the cusp discs the annulus connects and the four parallelograms are either split into two triangles or four triangles depending on the choice of equator of each ideal octahedron. Combinatorially, $\partial(S^3\backslash\Gamma_5)$ is triangulated into 20 hexagonal geodesic boundaries and 30 quadrangular cusp boundaries.

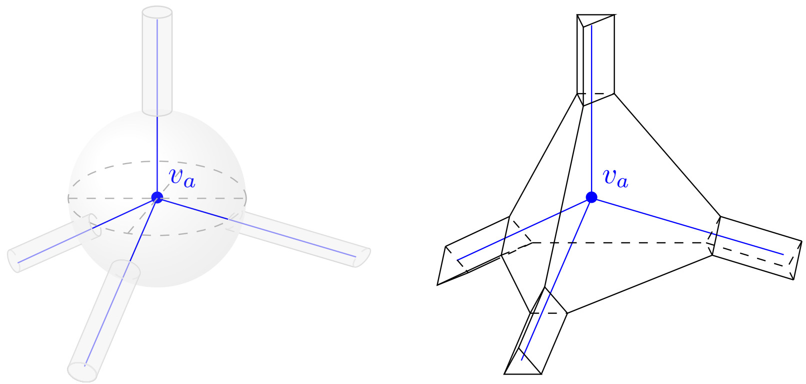
The purpose of such ideal triangulation is to construct the partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$ with the building blocks given by the SL$(2,\mathbb{C})$ Chern-Simons partition function for an ideal tetrahedron $\triangle$, which has been well studied in the literature (see *e.g.* [19, 21, 24]) and we review in the coming subsection.

1. Step 1: Ideal tetrahedron partition function

As is well-known, the phase space of CS theory with gauge group G on a 3-manifold $\mathcal{M}_3$ is the moduli space of flat connection valued in the Lie algebra $\mathfrak{g}$ of G on the boundary $\partial\mathcal{M}_3$ of $\mathcal{M}_3$ which is an oriented surface, denoted as $\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, G)$. It is defined as

$$\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, G) := \{\mathfrak{g}\text{-valued connection } A \text{ on } \partial\mathcal{M}_3 \mid \mathrm{d}A + A\wedge A = 0\}/G\,, \tag{37}$$

[1] An ideal tetrahedron can be lifted to the hyperbolic 3-plane $\mathbb{H}^3$ with all the vertices located at infinity and all faces along geodesic surfaces of $\mathbb{H}^3$, on which one can describe hyperbolic geometry. See more details in *e.g.* [24]. We will, however, not use this picture in our construction.



(a) (b)

FIG. 3: *(a)* Illustration of part of the $S^3\backslash\Gamma_5$. A four-valent node $v_a \in \Gamma_5$ and its neighbourhood is removed from S^3 and generates a part of the boundary as a 4-holed sphere $\mathcal{S}_a$ whose holes are connected to annuli. *(b)* The ideal triangulation of (a). Vertices created by edges of the graph piercing through the sphere are truncated into triangles. Each such triangle is connected to the boundary of a triangular prism which is the ideal triangulation of an annulus in (a). (The triangulation of the parallelograms in triangular prisms is not shown for a clear visual effect.) In the full triangulation of $S^3\backslash\Gamma_5$, each triangular prism is connected to a pair of truncated vertices from two different triangulated 4-holed spheres.

where the quotient is by the conjugate action of G. For $\partial\mathcal{M}_3$, it is isomorphic to the homomorphism from the fundamental group of $\partial\mathcal{M}_3$ to group G up to conjugate action, *i.e.*

$$\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, G) = \text{Hom}(\pi_1(\partial\mathcal{M}_3), G)/G\,. \tag{38}$$

We will take it as the definition of $\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, G)$ in this note. It is a symplectic space endowed with an Atiayh-Bott-Goldmann symplectic structure $\Omega_{\text{CS}} = \int_{\partial\mathcal{M}_3} \text{Tr}\,(\delta A \wedge \delta A)$ (up to constant) for the holomorphic connection with δ denoting the variation on field. Indeed, $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, G)$ is a subspace of $\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, G)$. More importantly, it is a Lagrangian submanifold of the latter. This means we can first construct functions on the phase space $\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, G)$, which is a more natural starting point for quantization, and then restrict to its subspace $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, G)$. This is the strategy we take in constructing this spinfoam model. In particular, we consider $G = \text{PSL}(2,\mathbb{C}) = \text{SL}(2,\mathbb{C})/\mathbb{Z}_2$ and $\mathcal{M}_3$ being an ideal tetrahedron $\triangle$. (We will later lift the gauge group to $\text{SL}(2,\mathbb{C})$ in the quantization step.) To get a non-trivial moduli space for $\triangle$, we add, on top of that, an extra structure – a *framing flag* – to each cusp boundary of $\triangle$[2]. A framing flag s is a flat section in an associated $\mathbb{CP}^1$ bundle over the cusp boundary. It can be viewed as a $\mathbb{C}^2$ vector field on a cusp boundary, defined up to a complex scaling by the flatness equation $\mathrm{d}s = As$. In other words, the vector $s(\mathfrak{p})$ at a point $\mathfrak{p}$ of the cusp boundary is the eigenvector of the holonomy around the cusp boundary based at $\mathfrak{p}$. A flat connection with a choice of framing flags on cusp boundaries is called a *framed flat connection*.

The phase space of $\text{PSL}(2,\mathbb{C})$ Chern-Simons theory on the boundary $\partial\triangle$ of an ideal tetrahedron $\triangle$ is the moduli space of framed flat $\text{PSL}(2,\mathbb{C})$ connection on $\partial\triangle$, which we denote as $\mathcal{P}_{\partial\triangle}$. It can be spanned by the so-called *Fock-Goncharov (FG) coordinates* dressing the edges of the geodesic boundary of $\partial\triangle$, which we now describe [26]. Consider the ideal triangulation of a 4-holed sphere as shown in fig.3b. Label the holes with number 1,2,3,4. Each hole i is triangulated into a *disc cusp* and is associated with a framing flag. Each edge E can be viewed as the diagonal of a quadrilateral as in fig.5. Parallel transport the framing flag from hole i to a common point inside the quadrilateral and denote the parallel transported framing flag as s_i. Referring to the relative locations of the holes and edge E, the FG coordinate x_E associated to E is defined by the cross-ratio of framing flags as

$$x_E = \frac{\langle s_1 \wedge s_2\rangle\langle s_3 \wedge s_4\rangle}{\langle s_1 \wedge s_3\rangle\langle s_2 \wedge s_4\rangle}\,, \quad \text{where } \langle s_i \wedge s_j\rangle := s_i^0 s_j^1 - s_i^1 s_j^0\,, \quad \text{with } s_i = \left(s_i^0, s_i^1\right)^\top\,. \tag{39}$$

[2] The introduction of a framing flag is related to avoiding singularity when the Kähler mass parameters associated to the cusp boundaries are zero. See *e.g.* [21] for a more detailed discussion.

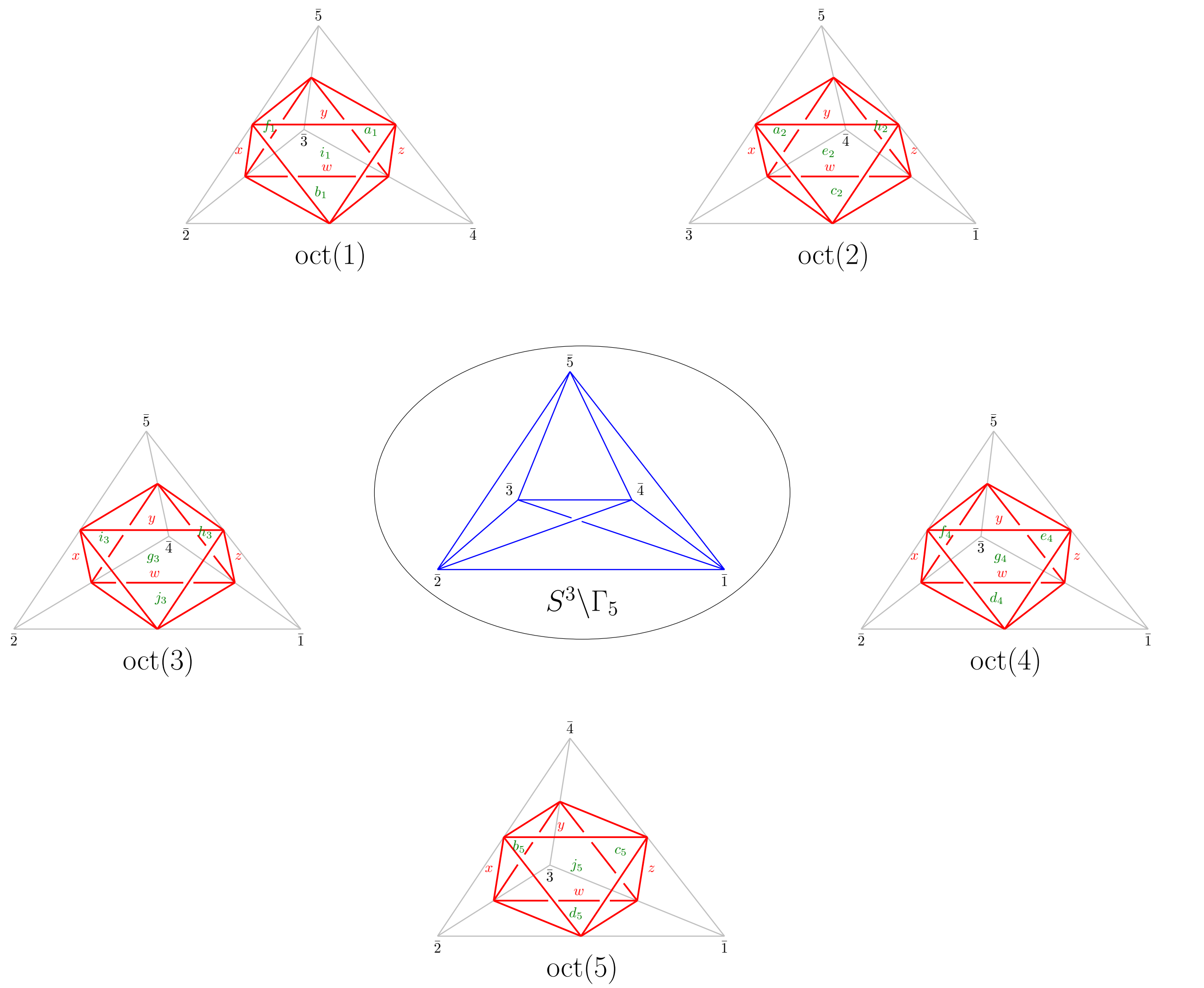


FIG. 4: The decomposition of the ideal triangulation of $M_3 \equiv S^3\backslash\Gamma_5$ into 5 ideal octahedra (*in red*), each of which can be decomposed into 4 ideal tetrahedra. The cusp boundaries of the ideal octahedra are shrunk to vertices in this figure. (See fig.2b) for the ideal octahedron with un-shrunk cusp boundaries.) Numbers $\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$ with bars denote the 4-holed spheres on ∂M_3. The faces $a, b, c, d, e, f, g, h, i, j$ (*labelled in green and each is on a boundary triangle of the tetrahedron in gray*) are the faces where a pair of octahedra are glued. Two ideal octahedra are glued through pairs of faces having the same label (with different subscripts). In each ideal octahedron, x, y, z, w (*labelled in red*) are chosen to form the equator of the octahedron. The same figure appears in [11, 25].

It is apparent that such a definition is invariant under the complex rescaling of any framing flag and $\mathrm{SL}(2,\mathbb{C})$ gauge transformation of all s_i's as the inner product $\langle\cdot,\cdot\rangle$ is $\mathrm{SL}(2,\mathbb{C})$ invariant. The definition (39) can be extended to define the FG coordinates on an n-holed ($n > 4$) sphere.

The $\mathrm{PSL}(2,\mathbb{C})$ holonomies on $\partial\triangle$ can be written as 2×2 matrices whose matrix elements are in terms of the edge coordinates dressing the edges they cross. This is called the "snake rule". There are three rules for transporting a *snake* – an arrow pointing from one vertex of the triangle to another with a *fin* facing inside the triangle, each corresponds to a matrix as follows. (The inverse transportation of each type corresponds to the inverse of the relevant

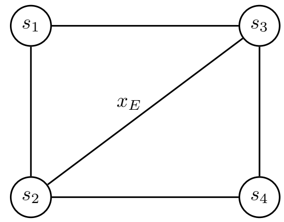
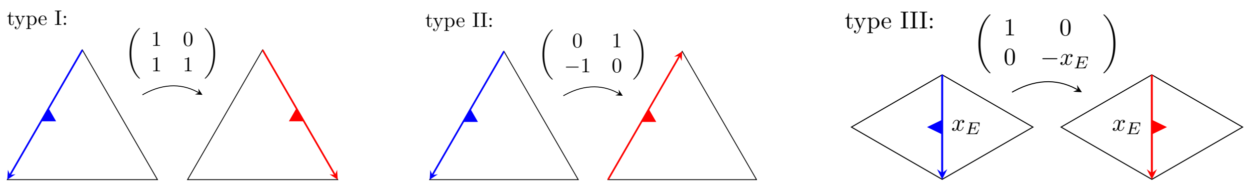


FIG. 5: A quadrilateral in a 2D ideal triangulation to define FG coordinate x_E in terms of the framing flags $\{s_i\}_{i=1,\cdots,4}$ by (39).

matrix.



. (40)

Type I and II correspond to transporting a snake within a triangle and III correspond to moving a snake from one triangle to its adjacent triangle. Any holonomy of a closed loop can be calculated by multiplying the transportation matrices from the left corresponding to moving a snake along the holonomy.

For a holonomy along a disc cusp with eigenvalue $\lambda \equiv e^L$, we use Type I and Type III snake rules to calculate that

$$h = \prod_{E \text{ around disc cusp}} \left[\begin{pmatrix} 1 & 0 \\ 0 & -x_E \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right] \in \mathrm{PGL}(2,\mathbb{C})\,. \tag{41}$$

Its normalization defines a $\mathrm{PSL}(2,\mathbb{C})$ holonomy whose eigenvalue gives

$$\prod_{E \text{ around disc cusp}} (-x_E) = \lambda^2 \quad \Longleftrightarrow \quad \sum_{E \text{ around disc cusp}} (\chi_E - i\pi) = 2L\,, \tag{42}$$

where χ_E is the logarithmic of x_E with a chosen branch. One immediately realizes that the edge coordinates are not sensitive to the sign of the eigenvalue λ. This reflects the fact that the gauge group the FG coordinates describe is $\mathrm{PSL}(2,\mathbb{C})$ rather than $\mathrm{SL}(2,\mathbb{C})$. One can easily choose a lift $\sqrt{-x_E}$ or $-\sqrt{-x_E}$ of the edge coordinates, in which case the gauge group is lifted to $\mathrm{SL}(2,\mathbb{C})$. When the eigenvalues are all fixed for holonomies around the four disc cusps of $\partial\triangle$, the moduli space of flat connection on $\partial\triangle$ is a symplectic space with the Poisson structure given by

$$\{\chi_E, \chi_{E'}\} = \epsilon_{EE'}\,, \tag{43}$$

where $\epsilon_{EE'} = 0, \pm 1$ counts the oriented triangles shared by E, E' and $\epsilon_{EE'} = 1$ if E' occurs to the left of E in the triangle[3].

The FG coordinates on $\partial\triangle$ are obtained from those for a 4-holed sphere by setting the eigenvalue $\lambda = 1$ for holonomy around any of the disc cusp. Consequently, $\mathcal{P}_{\partial\triangle}$ is given by three FG coordinates $\{z, z', z''\} \in \mathbb{C}^*$ each labelling a pair of opposite edges of $\triangle$ as shown in fig.2a and it is defined as

$$\mathcal{P}_{\partial\triangle} = \{z, z', z'' \in \mathbb{C}^* | zz'z'' = -1\} \in (\mathbb{C}^*)^2\,. \tag{44}$$

The corresponding holonomy calculated by the snake rule is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -z' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -z'' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ zz'(z^{-1} + z'' - 1) & -zz'z'' \end{pmatrix}, \tag{45}$$

[3] For degenerate triangles, $\epsilon_{EE'}$ could be ± 2, which we do not encounter here.

is a $\mathrm{PSL}(2,\mathbb{C})$ element as $\lambda=1$ hence $\det(h)=1$. The constraint $zz'z''=-1$ eliminates one edge coordinate, say z', then the holomorphic part of the Atiyah-Bott-Goldman symplectic form can be written as

$$\Omega=\frac{\mathrm{d}z''}{z''}\wedge\frac{\mathrm{d}z}{z}\,. \tag{46}$$

Taking the anti-holomorphic coordinates into account, the symplectic form for the Chern-Simons action (26) is:

$$\omega_{k,s}=\frac{t}{4\pi}\Omega+\frac{\bar{t}}{4\pi}\overline{\Omega}\,. \tag{47}$$

Lift these coordinates to their logarithmic correspondence, $Z:=\log(z), Z':=\log(z'), Z'':=\log(z'')$ and similarly for the anti-holomorphic counterparts, the constraint of the edge coordinates and the Poisson structure induced by (47) are

$$Z+Z'+Z''=i\pi=\overline{Z}+\overline{Z}'+\overline{Z}''\,,\quad \{Z,Z''\}_{\Omega}=1=\{\overline{Z},\overline{Z}''\}_{\overline{\Omega}}\,. \tag{48}$$

Therefore, (Z,Z'') and $(\overline{Z},\overline{Z}'')$ form two canonical pairs. The quantization is based on another equivalent canonical pairs $(\mu,\nu)\in\mathbb{R}^2$ and $(m,n)\in(\mathbb{Z}/k\mathbb{Z})^2$ defined as

$$Z=\frac{2\pi i}{k}\left(-ib\mu-m\right)\,,\quad Z''=\frac{2\pi i}{k}\left(-ib\nu-n\right)\,,\quad \overline{Z}=\frac{2\pi i}{k}\left(-ib^{-1}\mu+m\right)\,,\quad \overline{Z}''=\frac{2\pi i}{k}\left(-ib^{-1}\nu+n\right)\,, \tag{49}$$

where b is a phase parameter related to the Barbero-Immirzi parameter:

$$b^2=\frac{1-i\gamma}{1+i\gamma}\,,\quad \mathrm{Re}(b)>0\,,\quad \mathrm{Im}(b)\neq 0\,,\quad |b|=1\quad\Rightarrow\quad t=\frac{2k}{1+b^2}\,,\quad \bar{t}=\frac{2k}{1+b^{-2}}\,. \tag{50}$$

Conversely, one can express $Z,Z'',\overline{Z},\overline{Z}''$ in terms of (μ,ν,m,n) as

$$\mu=\frac{k}{2\pi Q}\left(Z+\overline{Z}\right)\,,\quad m=\frac{ik}{2\pi bQ}\left(Z-b^2\overline{Z}\right)\,,\quad \nu=\frac{k}{2\pi Q}\left(Z''+\overline{Z}''\right)\,,\quad n=\frac{ik}{2\pi bQ}\left(Z''-b^2\overline{Z}''\right)\,,\quad Q=b+b^{-1}\,. \tag{51}$$

The symplectic form in terms of the new variables and the Poisson brackets it generates are

$$\omega_{k,s}=\frac{2\pi}{k}\left(\mathrm{d}\nu\wedge\mathrm{d}\mu-\mathrm{d}n\wedge\mathrm{d}m\right)\,,\quad \{\mu,\nu\}_{\omega}=\{n,m\}_{\omega}=\frac{k}{2\pi}\,,\quad \{\mu,n\}_{\omega}=\{\nu,m\}_{\omega}=0\,. \tag{52}$$

To promote to the quantum theory, we introduce quantum parameters

$$q=\exp\left(\frac{4\pi i}{t}\right)=\exp\left[\frac{2\pi i}{k}(1+b^2)\right]\equiv e^{h}\,,\quad \tilde{q}=\exp\left(\frac{4\pi i}{\bar{t}}\right)=\exp\left[\frac{2\pi i}{k}(1+b^{-2})\right]\equiv e^{\tilde{h}}\,. \tag{53}$$

Here, $h:=4\pi i/t$ (or equivalently $\tilde{h}:=4\pi i/\bar{t}$) is a (non-standard) complex quantum parameter related to the Chern-Simons level. It is called a quantum parameter because it is proportional to the Planck constant $\hbar$ with a complex coefficient (taking convention $G=c=1$):

$$h=\frac{8\pi i|\Lambda|\gamma}{3(1+i\gamma)}\hbar\,. \tag{54}$$

Indeed, the limit $h\to 0$ corresponds to the classical limit. A Poisson bracket $\{x,y\}_{\omega}$ is quantized to a commutator by $[\hat{x},\hat{y}]:=\widehat{\{x,y\}}_{\omega}/i$. We allow the analytic continuation of μ,ν to be in $\mathbb{C}$ by adding imaginary parts, and define $Z,Z'',\widetilde{Z}$ and $\widetilde{Z}''$ in the same way as in (51) with these complex variables. Then $\widetilde{Z}$ (*resp.* $\widetilde{Z}''$) is not necessarily the complex conjugate of Z (*resp.* Z''). The exponential of $\widetilde{Z}$ and $\widetilde{Z}''$ are denoted as $\tilde{z}$ and $\tilde{z}''$ respectively. The quantization of $\mathcal{P}_{\partial\triangle}$ promotes μ,m (*resp.* $Z,\widetilde{Z}$) to be multiplication operators $\boldsymbol{\mu},\mathbf{m}$ (*resp.* $\mathbf{Z},\widetilde{\mathbf{Z}}$) and ν,n (*resp.* $Z'',\widetilde{Z}''$) to be derivative operators $\boldsymbol{\nu},\mathbf{n}$ (*resp.* $\mathbf{Z}'',\widetilde{\mathbf{Z}}''$) with the commutators

$$[\mathbf{Z}'',\mathbf{Z}]=h\,,\quad [\widetilde{\mathbf{Z}}'',\widetilde{\mathbf{Z}}]=\tilde{h}\quad\Longleftrightarrow\quad [\boldsymbol{\mu},\boldsymbol{\nu}]=[\mathbf{n},\mathbf{m}]=\frac{k}{2\pi i}\,,\quad [\boldsymbol{\mu},\mathbf{n}]=[\boldsymbol{\nu},\mathbf{m}]=0\,. \tag{55}$$

Upon quantization, we require the imaginary parts of μ and ν remain to be c-numbers. Projecting the commutators to the exponential operators $\mathbf{z}, \mathbf{z}'', \tilde{\mathbf{z}}, \tilde{\mathbf{z}}''$, one finds q-Weyl and $\tilde{q}$-Weyl algebras

$$\mathbf{z}''\mathbf{z} = q\mathbf{z}\mathbf{z}'' \,, \quad \tilde{\mathbf{z}}''\tilde{\mathbf{z}} = \tilde{q}\tilde{\mathbf{z}}\tilde{\mathbf{z}}'' \,, \quad \tilde{\mathbf{z}}''\mathbf{z} = \mathbf{z}\tilde{\mathbf{z}}'' \,, \quad \mathbf{z}''\tilde{\mathbf{z}} = \tilde{\mathbf{z}}\mathbf{z}'' \,. \tag{56}$$

Due to the discreteness and periodicity of m, n, the spectra of $\mathbf{m}, \mathbf{n}$ are discrete and bounded to be $\mathbb{Z}/k\mathbb{Z}$. On the other hand, the spectra of $\boldsymbol{\mu}, \boldsymbol{\nu}$ are real. The kinematical Hilbert space is hence

$$\mathcal{H}^{\text{kin}}_{k,s} = L^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^k \,, \tag{57}$$

where $\mathbb{C}^k$ is a k-dimensional vector space. The quantum operators $\boldsymbol{\mu}, \boldsymbol{\nu}, \mathbf{m}, \mathbf{n}$ act on a wave function $f(\mu|m) \in \mathcal{H}^{\text{kin}}_{k,s}$ as

$$\boldsymbol{\mu} f(\mu|m) = \mu f(\mu|m) \,, \quad \boldsymbol{\nu} f(\mu|m) = -\frac{k}{2\pi i}\partial_\mu f(\mu|m) \,, \quad e^{\frac{2\pi i}{k}\mathbf{m}} f(\mu|m) = e^{\frac{2\pi i}{k}m} f(\mu|m) \,, \quad e^{\frac{2\pi i}{k}\mathbf{n}} f(\mu|m) = f(\mu|m+1) \,. \tag{58}$$

or a re-parameterized version

$$\mathbf{z} f(z,\tilde{z}) = z f(z,\tilde{z}) \,, \quad \mathbf{z}'' f(z,\tilde{z}) = f(qz,\tilde{z}) \,, \quad \tilde{\mathbf{z}} f(z,\tilde{z}) = \overline{z} f(z,\tilde{z}) \,, \quad \tilde{\mathbf{z}}'' f(z,\tilde{z}) = f(z,\tilde{q}\tilde{z}) \,. \tag{59}$$

Another equivalent way to write it is

$$\mathbf{z} f(\mu|m) = z f(\mu|m) \,, \quad \tilde{\mathbf{z}} f(\mu,m) = \tilde{z} f(\mu|m) \,, \quad \mathbf{z}'' f(\mu|m) = f(\mu+ib|m-1) \,, \quad \tilde{\mathbf{z}}'' f(\mu|m) = f\left(\mu + ib^{-1}|m+1\right) \,. \tag{60}$$

(z, z'') are holomorphic coordinates on $\mathcal{P}_{\partial\triangle}$. The moduli space of flat $\mathrm{PSL}(2,\mathbb{C})$ connection on an ideal tetrahedron, denoted as $\mathcal{L}_\triangle$, is a holomorphic Lagrange submanifold of $\mathcal{P}_{\partial\triangle}$ determined by further requiring the holonomy h defined in (45) to be trivial. In other words, $\mathcal{L}_\triangle$ is an algebraic curve given by

$$\mathcal{L}_\triangle = \{(z, z''; \tilde{z}, \tilde{z}'') \in \mathcal{P}_{\partial\triangle} \mid z^{-1} + z'' - 1 = 0, \tilde{z}^{-1} + \tilde{z}'' - 1 = 0\} \,. \tag{61}$$

Quantization promotes the algebraic curve to the quantum constraints whose solution $\Psi_\triangle(\mu|m)$ satisfying

$$(\mathbf{z}^{-1} + \mathbf{z}'' - 1)\Psi_\triangle = (\tilde{\mathbf{z}}^{-1} + \tilde{\mathbf{z}}'' - 1)\Psi_\triangle(\mu|m) = 0 \tag{62}$$

defines the Chern-Simons partition function. $\Psi_\triangle(\mu|m)$ is the quantum dilogarithm function [22, 27–29] [4]:

$$\Psi_\triangle(\mu|m) = \prod_{j=0}^{\infty} \frac{1 - \tilde{q}^{j+1}\tilde{z}^{-1}}{1 - q^{-j}z^{-1}} \,. \tag{63}$$

The name "quantum dilogarithm" comes from the fact that its classical limit at $q, \tilde{q} \to 1$ reproduce the dilogarithm function $\mathrm{Li}_2(z)$ defined as

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\ln(1-u)}{u}\mathrm{d}u = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \,, \quad z \in \mathbb{C} \,, \tag{64}$$

which is the generalization of the logarithm function whose Taylor expansion around 1 gives

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \,. \tag{65}$$

For each $m \in \mathbb{Z}/k\mathbb{Z}$, $\Psi_\triangle(\mu|m)$ defines a meromorphic function of μ and is analytic. $\Psi_\triangle$ has poles on the real line and in the lower half-plane $\mathrm{Im}(\mu) \leq 0$ but is holomorphic in the upper half-plane $\mathrm{Im}(\mu) > 0$. More precisely, its zeros and poles are at

$$\mu_{\text{zero/pole}} = \{ibu + ib^{-1}v \mid u, v \in \mathbb{Z} \,, \quad u - v = -m + k\mathbb{Z}\} \quad \text{with} \begin{cases} \text{zeros:} & u, v \geq 1 \\ \text{poles:} & u, v \leq 0 \end{cases} . \tag{66}$$

[4] The result (63) is due to the choice of $k \in \mathbb{Z}_+$ hence $\gamma > 0$ and $|q| > 1$. For $k \in Z_-$ and hence $|q| < 1$, the expression of the quantum dilogarithm function is $\Psi_\triangle(\mu|m) = \prod_{j=0}^{\infty} \frac{1-q^{j+1}z^{-1}}{1-\tilde{q}^{-j}\tilde{z}^{-1}}$.

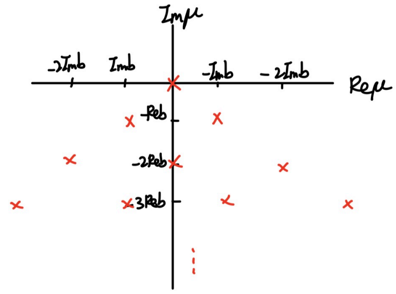


FIG. 6: Distributions of poles (in red) of $\Psi_\triangle(\mu|m)$.

The poles are illustrated in fig.6.

An important aspect of $\Psi_\triangle(\mu|m)$ is its asymptotic behaviour at $\mathrm{Re}(\mu)\to\pm\infty$. Fixing $\mathrm{Im}(\mu)$, $\Psi_\triangle(\mu|m)$ asymptotically behaves as

$$\Psi_\Delta(\mu|m)=\begin{cases} O(1)\,, & \mathrm{Re}(\mu)\to+\infty \\ \exp\left[\frac{i\pi}{k}\left(\mu-\frac{i}{2}Q\right)^2+O(1)\right]\,, & \mathrm{Re}(\mu)\to-\infty \end{cases}\,, \tag{67}$$

where $Q=b+b^{-1}>0$. This follows from the integration expression of the quantum dilogarithm function equivalent to (63) (see *e.g.* [22]). Indeed, $\Psi_\Delta(\mu|m)$ diverges when $\mathrm{Re}(\mu)\to-\infty$ so it is not a squared integrable state in the "naïve" Hilbert space $\mathcal{H}^{\mathrm{kin}}_{k,s}$ defined in (57). However, we can find a functional space where $\Psi_\Delta(\mu|m)$ can be naturally valued and which renders all integrals well-defined.

Consider a $2N$-dimensional symplectic space $(\mathcal{P},\omega)$ with Darboux coordinates $(\mu_i,m_i)_{i=1,\cdots,N}$ and $(\nu_i,n_i)_{i=1,\cdots,N}$ with symplectic structure

$$\omega=\frac{2\pi}{k}\sum_{i=1}^N\left(\mathrm{d}\nu_i\wedge\mathrm{d}\mu_i-\mathrm{d}n_i\wedge m_i\right)\,. \tag{68}$$

This symplectic space is naturally endowed with an "angle space" $(\mathcal{P}_{\mathrm{an}},\omega_{\mathrm{an}})\simeq \mathrm{T}^*\mathbb{R}^N\simeq\mathbb{R}^{2N}$ which is a symplectic space with Darboux coordinate and the symplectic form ω_{an}

$$\alpha_i:=\mathrm{Im}(\mu_i)\,,\quad \beta_i:=\mathrm{Im}(\nu_i)\,,\quad \omega_{\mathrm{an}}=\sum_{i=1}^N\mathrm{d}\beta_i\wedge\mathrm{d}\alpha_i\,. \tag{69}$$

Denote vectors $\vec\alpha=(\alpha_1,\cdots,\alpha_N)^\top$ and $\vec\beta=(\beta_1,\cdots,\beta_N)^\top$. $\mathbb{R}^{2N}$ can be thought of the universal covering of the angle space. We define a $2N$-dimensional *open convex symplectic polytope* (or simply *polytope*) $\mathfrak{P}$ to be an open subset of $(\mathbb{R}^{2N},\omega_{\mathrm{an}})$ cut out of a set of strict linear inequalities, and $\pi(\mathfrak{P})$ to be its projection on the base of $\mathrm{T}^*\mathbb{R}^N$ with coordinates $\vec\alpha$. Also define $\mathrm{strip}(\mathfrak{P})\subset\mathbb{C}^N$ to be

$$\mathrm{strip}(\mathfrak{P}):=\left\{\vec\mu\in\mathbb{C}^N\mid \mathrm{Im}(\vec\mu)\in\pi(\mathfrak{P})\right\}\,. \tag{70}$$

Then we can define the functional space

$$\mathcal{F}_{\mathfrak{P}}:=\{\text{ holomorphic functions } f:\mathrm{strip}(\mathfrak{P})\to\mathbb{C}\ s.t.\ \forall(\vec\alpha,\vec\beta)\in\mathfrak{P},\text{ the function } e^{-\frac{2\pi}{k}\vec\mu\cdot\vec\beta}f(\vec\mu+i\vec\alpha)\in\mathcal{S}\left(\mathbb{R}^N\right)\}\,, \tag{71}$$

where $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwarz class in $\mathbb{R}^N$. This means $e^{-\frac{2\pi}{k}\vec\mu\cdot\vec\beta}f(\vec\mu+i\vec\alpha)$ decays exponentially when $|\mu|\to\infty$. Recall the quantization of $\vec\mu,\vec\nu$ to operator as in (58), $\mathcal{F}_{\mathfrak{P}}$ contains exactly the holomorphic functions f such that

$$e^{\frac{2\pi}{k}\left(\vec\alpha\cdot\boldsymbol{\vec\nu}-\vec\beta\cdot\boldsymbol{\vec\mu}\right)}f(\vec\mu)\in\mathcal{S}\left(\mathbb{R}^N\right)\,,\quad \forall\,(\vec\alpha,\vec\beta)\in\mathfrak{P}\,. \tag{72}$$

as $e^{\frac{2\pi}{k}\left(\vec\alpha\cdot\boldsymbol{\vec\nu}-\vec\beta\cdot\boldsymbol{\vec\mu}\right)}f(\vec\mu)=e^{\frac{2\pi}{k}\left(-\vec\beta\cdot\boldsymbol{\vec\mu}\right)}e^{\frac{2\pi}{k}\left(\vec\alpha\cdot\boldsymbol{\vec\nu}\right)}e^{\left(\frac{2\pi}{k}\right)^{2N}\vec\alpha\cdot\vec\beta\left(\frac{k}{2\pi i}\right)^N}f(\vec\mu)=e^{\left(-\frac{2\pi i}{k}\right)^N\vec\alpha\cdot\vec\beta}e^{\frac{2\pi}{k}\left(-\vec\beta\cdot\vec\mu\right)}f\left(\vec\mu+i\vec\alpha\right)$. In this form, it is easy to see that the Fourier transform of a Schwarz function is also a Schwarz function. In other words, the action of the operator $e^{\frac{2\pi}{k}\left(\vec\alpha\cdot\boldsymbol{\vec\nu}-\vec\beta\cdot\boldsymbol{\vec\mu}\right)}$ simultaneously bounds the the decay of a holomorphic function f and its Fourier transformation. We say that $(\vec\alpha,\vec\beta)\in\mathfrak{P}$ is a *positive angle structure* of f.

To be explicit, consider the case for $\Psi_\triangle$. As mentioned above, $\Psi_\triangle(\mu|m)$ is holomorphic when $\mathrm{Im}(\mu) > 0$, so we first take $\alpha > 0$. Then (the absolute value of) the function $e^{-\frac{2\pi}{k}\beta\mu}\Psi_\triangle(\mu + i\alpha|m)$ with $\mu \in \mathbb{R}$ has limits

$$|e^{-\frac{2\pi}{k}\beta\mu}\Psi_\triangle(\mu + i\alpha|m)| \to \begin{cases} \exp\left[-\frac{2\pi}{k}\beta\mu\right] , & \mu \to +\infty \\ \exp\left[-\frac{2\pi}{k}\mu(\alpha+\beta-Q/2)\right] , & \mu \to -\infty \end{cases} , \tag{73}$$

which can be directly derived from the asymptotic behaviour (67). Therefore, $e^{-\frac{2\pi}{k}\beta\mu}\Psi_\triangle(\mu + i\alpha|m)$ is a Schwartz function when $(\alpha, \beta) \in \mathfrak{P}_\triangle$ satisfy the positive angle structure of $\Psi_\triangle$, or of $\triangle$ for short, defined as

$$\mathfrak{P}_\triangle = \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha, \beta > 0, \alpha + \beta < Q/2\} . \tag{74}$$

Recall that $Q = b + b^{-1} = 2\mathrm{Re}(b) > 0$, $\mathfrak{P}_\triangle$ is an open triangle. Let $\alpha = \mathrm{Im}(\mu), \beta = \mathrm{Im}(\nu)$, then $\int_\mathcal{C} \mathrm{d}\mu \, e^{-\frac{2\pi i}{k}\nu\mu}\Psi(\mu|m)$ is absolutely convergent when the integration contour $\mathcal{C}$ is shift above the real axis while remains in $\mathfrak{P}_\triangle$.

2. Step 2: Ideal octahedron partition function

Now that we have the Chern-Simons partition function $\Psi_\triangle$ on an ideal tetrahedron as the building block, the next step is to construct the partition function on an ideal octahedron. Each ideal octahedron can be decomposed into 4 ideal tetrahedra by adding an internal edge (see fig.2b). We then have 4 copies of edge coordinates $\{x, y, z, w\}$ (or considering the logarithmic coordinates $\{X, Y, Z, W\}$) subject to the constraint

$$\begin{matrix} c = xyzw = 1 \\ \tilde{c} = \tilde{x}\tilde{y}\tilde{z}\tilde{w} = 1 \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} C = X + Y + Z + W = 2\pi i \\ \widetilde{C} = \widetilde{X} + \widetilde{Y} + \widetilde{Z} + \widetilde{W} = 2\pi i \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} \mu_X + \mu_Y + \mu_Z + \mu_W = 0 \\ m_X + m_Y + m_Z + m_W = 0 \end{matrix} . \tag{75}$$

Here, we have chosen a branch for C and $\widetilde{C}$. We define a set of symplectic coordinates $(X, P_X), (Y, P_Y), (Z, P_Z), (C, \Gamma)$ where

$$P_X = X'' - W'' , \quad P_Y = Y'' - W'' , \quad P_Z = Z'' - W'' , \quad \Gamma = W'' , \tag{76}$$

and similarly for the tilde sectors. It is indeed a U-type symplectic transformation with a symplectic matrix

$$\begin{pmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{U}^\top\right)^{-1} \end{pmatrix} , \quad \text{with } \mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} . \tag{77}$$

Interlude: Symplectic matrices and generators A symplectic matrix is an $2N\times 2N$ matrix $\mathbf{M}$ with real entries that satisfies

$$\mathbf{M}^\top\mathbf{\Omega}\mathbf{M}=\mathbf{\Omega}\,,\quad \mathbf{\Omega}=\begin{pmatrix}\mathbf{0} & \mathbb{I}_N\\ -\mathbb{I}_N & \mathbf{0}\end{pmatrix}\,. \tag{78}$$

They are representations of the symplectic group $\mathrm{Sp}(2N,\mathbb{R})$, which is the group of symplectic transformations. The generators of $\mathrm{Sp}(2N,\mathbb{R})$ are given by $\mathbf{\Omega}$ and the set of matrices in the following form

$$\mathbf{D}(\mathbf{U})=\left\{\begin{pmatrix}\mathbf{U} & \mathbf{0}\\ \mathbf{0} & \left(\mathbf{U}^\top\right)^{-1}\end{pmatrix}:\mathbf{U}\in\mathrm{GL}(\mathbf{N};\mathbb{R})\right\}\,,\quad \mathbf{T}(\mathbf{B})=\left\{\begin{pmatrix}\mathbb{I}_N & \mathbf{0}\\ \mathbf{B} & \mathbb{I}_N\end{pmatrix}:\mathbf{B}\in\mathrm{Sym}(\mathbf{N};\mathbb{R})\right\}\,, \tag{79}$$

where $\mathrm{Sym}(N;\mathbb{R})$ is the set of $N\times N$ symmetric matrices. This means any symplectic matrix $\mathbf{M}$ can be written as the multiplication of elements in $\mathbf{D}(\mathbf{U})$, $\mathbf{T}(\mathbf{B})$ and some power of $\mathbf{\Omega}$. The symplectic transformations corresponding to $\mathbf{D}(\mathbf{U})$ can be understood as "rotations" in the position space and momentum space separately, and we call them the U-type transformations. The symplectic transformations corresponding to $\mathbf{T}(\mathbf{B})$ are denoted as T-type transformations as they represent the translations of the momentum space. $\mathbf{\Omega}$ represents an exchange of position and momentum and is an involution. We denote such type of symplectic transformation as the S-type transformation.
Let a symplectic matrix $\mathbf{M}$ be given by a $2N\times 2N$ block matrix

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} \tag{80}$$

with $\mathbf{B}$ being invertible, then $\mathbf{M}$ can be decomposed as follows

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}=\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{DB} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{AB}^T & \mathbf{I}\end{pmatrix}\begin{pmatrix}\left(\mathbf{B}^{-1}\right)^T & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{pmatrix}\,. \tag{81}$$

Performing the symplectic reduction of the four copies of phase space $\mathcal{P}_{\partial\triangle}$ associated to the four ideal tetrahedra by imposing the constraint $C=2\pi i$ as well as quotient out the gauge orbit variable Γ, we obtain the phase space $\mathcal{P}_{\partial\text{oct}}$ of the boundary of the ideal octahedron with the following symplectic form and Poisson structure.

$$\omega^{\text{oct}}_{k,s}=\frac{2\pi}{k}\sum_i(\mathrm{d}\nu_i\wedge\mathrm{d}\mu_i-\mathrm{d}n_i\wedge\mathrm{d}m_i)\,,\quad \left|\begin{array}{l}\{\mu_i,\nu_j\}_\omega=\{n_i,m_j\}_\omega=\frac{k}{2\pi}\delta_{ij}\\ \{\mu_i,n_j\}_\omega=\{\nu_i,m_j\}_\omega=0\end{array}\right.\quad,\quad i,j=X,Y,Z\,. \tag{82}$$

Quantization of the constraint C and $\widetilde{C}$ adds a quantum correction as

$$\begin{array}{lllllll} c=1 & \to & \hat{c}=q & \Longleftrightarrow & C=2\pi i & \to & \hat{C}=2\pi i+h\,,\\ \tilde{c}=1 & \to & \hat{\tilde{c}}=\tilde{q} & \Longleftrightarrow & \widetilde{C}=2\pi i & \to & \hat{\widetilde{C}}=2\pi i+\tilde{h}\,.\end{array} \tag{83}$$

Here, the addition of h or $\tilde{h}$ is necessary for the partition function to be invariant under 3D Pachner moves of ideal triangulation [19], which we want so that the amplitude so-constructed can have *some* ideal triangulation independence.

In terms of $\{\boldsymbol{\mu}_i,\mathbf{m}_i\}_{i=X,Y,Z,W}$ which are the quantization of $\{\mu_i,m_i\}_{i=X,Y,Z,W}$, the quantum constraints read

$$\boldsymbol{\mu}_X+\boldsymbol{\mu}_Y+\boldsymbol{\mu}_Z+\boldsymbol{\mu}_W=iQ\,,\quad \mathbf{m}_X+\mathbf{m}_Y+\mathbf{m}_Z+\mathbf{m}_W=0\,. \tag{84}$$

Each octahedron partition function can hence be written in terms of the position variables $(x,y,z;\tilde{x},\tilde{y},\tilde{z})\equiv\exp[(X,Y,Z;\widetilde{X},\widetilde{Y},\widetilde{Z})]$ as

$$Z_{\text{oct}}(\mu_X,\mu_Y,\mu_Z|m_X,m_Y,m_Z)=\Psi_\Delta(\mu_X|m_X)\Psi_\Delta(\mu_Y|m_Y)\Psi_\Delta(\mu_Z|m_Z)\Psi_\Delta(iQ-\mu_X-\mu_Y-\mu_Z|-m_X-m_Y-m_Z) \tag{85}$$

where we have imposed the constraint (84) to eliminate the variables μ_W and m_W. Equivalently, one can write

$$\mathcal{Z}_{\text{oct}}(x,y,z;\tilde{x},\tilde{y},\tilde{z})=\prod_{i,j,k,l=0}^{\infty}\frac{1-q^{i+1}x^{-1}}{1-\tilde{q}^{-i}\tilde{x}^{-1}}\frac{1-q^{j+1}y^{-1}}{1-\tilde{q}^{-j}\tilde{y}^{-1}}\frac{1-q^{k+1}z^{-1}}{1-\tilde{q}^{-k}\tilde{z}^{-1}}\frac{1-q^{l}xyz}{1-\tilde{q}^{-l-1}\tilde{x}\tilde{y}\tilde{z}}\,. \tag{86}$$

Let us also study its asymptotic behaviour. Denote $\vec{\beta}\cdot\vec{\mu}\equiv\beta_X\mu_X+\beta_Y\mu_Y+\beta_Z\mu_Z$. Then $e^{-\frac{2\pi}{k}\vec{\beta}\cdot\vec{\mu}}\mathcal{Z}_{\text{oct}}(\{\mu_i+i\alpha_i\}|\{m_i\})$ has the following asymptotic behavior

$$|e^{-\frac{2\pi}{k}\vec{\beta}\cdot\vec{\mu}}\mathcal{Z}_{\text{oct}}(\{\mu_i+i\alpha_i\}|\{m_i\})|\sim\left\{\begin{array}{ll}e^{-\frac{2\pi}{k}\mu_i(\alpha_X+\alpha_Y+\alpha_Z+\beta_i-Q/2)}\,, & \mu_i\to+\infty\\ e^{-\frac{2\pi}{k}\mu_i(\alpha_i+\beta_i-Q/2)}\,, & \mu_i\to-\infty\end{array}\right.\,,\quad\forall i=X,Y,Z\,. \tag{87}$$

It is obtained by using both limits in (73) as $\Psi_\triangle(\mu_i|m_i)$ and $\Psi_\triangle(iQ-\mu_X-\mu_Y-\mu_Z|-m_X-m_Y-m_Z)$ always approach the opposite limits. This function is a Schwartz function of μ_X,μ_Y and μ_Z if $(\alpha_X,\alpha_Y,\alpha_Z,\beta_X,\beta_Y,\beta_Z)\in\mathbb{R}^6$ is inside the open polytope $\mathfrak{P}(\text{oct})$ defined by the following inequalities

$$\alpha_i>0\,,\quad \alpha_X+\alpha_Y+\alpha_Z<Q\,,\quad \alpha_i+\beta_i<Q/2\,,\quad \alpha_X+\alpha_Y+\alpha_Z+\beta_i>Q/2\,,\quad \forall i=X,Y,Z\,. \tag{88}$$

$(\vec{\alpha},\vec{\beta})\in\mathfrak{P}(\text{oct})$ is the positive angle of an ideal octahedron. We need to make sure that $\mathfrak{P}(\text{oct})$ is non-empty, otherwise the boundedness of $e^{-\frac{2\pi}{k}\vec{\beta}\cdot\vec{\mu}}\mathcal{Z}_{\text{oct}}(\{\mu_i+i\alpha_i\}|\{m_i\})$ is not guaranteed. We can check this by taking special values for α_i,β_i. Let $\alpha_X=\alpha_Y=\alpha_Z=\alpha$ and $\beta_X=\beta_Y=\beta_Z=\beta$, then (88) is simplified to

$$0<\alpha<Q/3\,,\quad \alpha+\beta<Q/2\,,\quad 3\alpha+\beta>Q/2\,, \tag{89}$$

which is indeed non-empty as illustrated in fig.7. We then conclude that $\mathcal{Z}_{\text{oct}}\in\mathcal{F}^{(k)}_{\mathfrak{P}(\text{oct})}$.

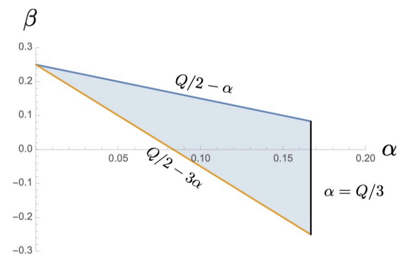


FIG. 7: Positive angle of an ideal octahedron when $\alpha_X=\alpha_Y=\alpha_Z=\alpha$ and $\beta_X=\beta_Y=\beta_Z=\beta$.

3. *Step 3: phase space coordinates of $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$ and partition function on $S^3\backslash\Gamma_5$*

As shown in fig.4, the triangulation of $S^3\backslash\Gamma_5$ contains 5 ideal octahedra with all edges on the boundary $\partial(S^3\backslash\Gamma_5)$. Therefore, the Chern-Simons phase space $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$ is simply the 5 copies of $\mathcal{P}_{\partial\text{oct}}$ with no more constraints to be imposed. Label the octahedra as Oct(i), $i=1,\cdots,5$ (see fig.4). The phase space $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$ has 15 position coordinates $\vec{\Phi}=(X_i,Y_i,Z_i)_{i=1,\cdots,5}$ and 15 momentum coordinates $\vec{\Pi}=(P_{X_i},P_{Y_i},P_{Z_i})_{i=1,\cdots,5}$ where each triple $(P_{X_i},P_{Y_i},P_{Z_i})$ is defined in the same way as (76). Then the partition function on $S^3\backslash\Gamma_5$ is nothing but the product of 5 $\mathcal{Z}_{\text{oct}}$'s as defined in (86). We write

$$\mathcal{Z}_\times(\vec{\mu}|\vec{m}):=\prod_{a=1}^{5}\mathcal{Z}_{\text{oct}}(x_a,y_a,z_a;\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)\,, \tag{90}$$

where $\mu_{1,2,3,\cdots}$ and $m_{1,2,3,\cdots}$ are parameters of $x_1,y_1,z_1,\cdots$ respectively. Indeed, $\mathcal{Z}_\times\in\mathcal{F}^{(k)}_{\mathfrak{P}(\text{oct})^{\times5}}$.

On the other hand, the geodesic boundaries of $S^3\backslash\Gamma_5$ (recall the definition at the beginning of this section) are five 4-holed spheres and the cusp boundaries are 10 annuli. The ideal triangulation of a 4-holed sphere $\mathcal{S}_a$ contains 6 edges on the geodesic boundaries, each shared by two edges from two different ideal octahedra, so the corresponding logarithmic edge coordinate, denoted as $\chi^{(a)}_{ij}$ when it is shared by Oct(i) and Oct(j), is the sum of two edge coordinates on ideal octahedra, which is, in turn, the sum of edge coordinates on ideal tetrahdera (from the set $\{X_i,Y_i,Z_i,W_i,X'_i,Y'_i,Z'_i,W'_i,X''_i,Y''_i,Z''_i,W''_i\}_{i=1,\cdots,5}$). $\chi^{(a)}_{ij}$ is called a (logarithmic) *Fock-Goncharov (FG) coordinate* on $\mathcal{S}_a$ [26]. The precise relations are shown in Table II. Apparently, each $\chi^{(a)}_{ij}$ is also a linear combination of elements in $\vec{\Phi}$ and $\vec{\Pi}$.

These 30 FG coordinates are not mutually independent but are subject to 10 constraints. This is because every two 4-holed spheres, say $\mathcal{S}_a$ and $\mathcal{S}_b$, are connected to an annulus cusp through one hole, say hole p, of $\mathcal{S}_a$ and another hole, say hole q, of $\mathcal{S}_b$ hence the eigenvalue $\lambda^2_{ab}\equiv e^{2L_{ab}}$ of the holonomy around hole p is the same as the inverse of the eigenvalue $\lambda^{-2}_{ba}\equiv e^{-2L_{ba}}$ of the holonomy around hole q. (They are related by an inverse because the holonomies around holes p and q are oriented oppositely relative to the annulus.) The variable $2L_{ab}$ is called the complex (logarithmic) *Fenchel-Nielsen* (FN) length. Recall the result (42) from the snake rule, $2L_{ab}$ is the sum of

$\mathcal{S}_1$:	$\chi^{(1)}_{23}=Z_2+Z_3$	$\chi^{(1)}_{34}=Y''_3+Z'_3+Z''_4+W'_4$	$\chi^{(1)}_{24}=Z''_2+W'_2+Z_4$
	$\chi^{(1)}_{35}=Z''_3+W'_3+Y''_5+Z'_5$	$\chi^{(1)}_{25}=Y''_2+Z'_2+Z_5$	$\chi^{(1)}_{45}=Y''_4+Z'_4+Z''_5+W'_5$
$\mathcal{S}_2$:	$\chi^{(2)}_{13}=X''_1+Y'_1+X_3$	$\chi^{(2)}_{34}=X''_3+Y'_3+W''_4+X'_4$	$\chi^{(2)}_{14}=X_1+X_4$
	$\chi^{(2)}_{35}=W''_3+X'_3+X''_5+Y'_5$	$\chi^{(2)}_{15}=W''_1+X'_1+X_5$	$\chi^{(2)}_{45}=X''_4+Y'_4+W''_5+X'_5$
$\mathcal{S}_3$:	$\chi^{(3)}_{12}=Z'_1+W''_1+X_2$	$\chi^{(3)}_{24}=W''_2+X'_2+Y'_4+Z''_4$	$\chi^{(3)}_{14}=W'_1+X''_1+X'_4+Y''_4$
	$\chi^{(3)}_{25}=X''_2+Y'_2+Z'_5+W''_5$	$\chi^{(3)}_{15}=W_1+W'_5+X''_5$	$\chi^{(3)}_{45}=Y_4+W_5$
$\mathcal{S}_4$:	$\chi^{(4)}_{12}=Z_1+X'_2+Y''_2$	$\chi^{(4)}_{23}=Y'_2+Z''_2+Z_3+W''_3$	$\chi^{(4)}_{13}=Y''_1+Z'_1+W'_3+X''_3$
	$\chi^{(4)}_{25}=Y_2+Y'_5+Z''_5$	$\chi^{(4)}_{15}=Z''_1+W'_1+X'_5+Y''_5$	$\chi^{(4)}_{35}=W_3+Y_5$
$\mathcal{S}_5$:	$\chi^{(5)}_{12}=Y'_1+Z''_1+W'_2+X''_2$	$\chi^{(5)}_{23}=Z'_2+W''_2+Y'_3+Z''_3$	$\chi^{(5)}_{13}=Y_1+X'_3+Y''_3$
	$\chi^{(5)}_{24}=W_2+Z'_4+W''_4$	$\chi^{(5)}_{14}=X'_1+Y''_1+W'_4+X''_4$	$\chi^{(5)}_{34}=Y_3+W_4$

TABLE II: FG coordinates $\chi^{(a)}_{ij}$ of 4-holed spheres in terms of the edge coordinates in $\{\mathrm{Oct}(i)\}$.

three coordinates $\chi^{(a)}_{ij}-i\pi$'s with dressing the three edges connecting to hole p. Similarly for $2L_{ba}$. The precise relations are given as follows.

$$\mathcal{S}_1:\quad 2L_{12}=\chi^{(1)}_{34}+\chi^{(1)}_{35}+\chi^{(1)}_{45}-3i\pi\,,\quad 2L_{13}=\chi^{(1)}_{24}+\chi^{(1)}_{25}+\chi^{(1)}_{45}-3i\pi\,,\tag{91a}$$
$$2L_{14}=\chi^{(1)}_{23}+\chi^{(1)}_{25}+\chi^{(1)}_{35}-3i\pi\,,\quad 2L_{15}=\chi^{(1)}_{23}+\chi^{(1)}_{24}+\chi^{(1)}_{34}-3i\pi\,,\tag{91b}$$
$$\mathcal{S}_2:\quad 2L_{21}=\chi^{(2)}_{34}+\chi^{(2)}_{35}+\chi^{(2)}_{45}-3i\pi\,,\quad 2L_{23}=\chi^{(2)}_{14}+\chi^{(2)}_{15}+\chi^{(2)}_{45}-3i\pi\,,\tag{91c}$$
$$2L_{24}=\chi^{(2)}_{13}+\chi^{(2)}_{15}+\chi^{(2)}_{35}-3i\pi\,,\quad 2L_{25}=\chi^{(2)}_{13}+\chi^{(2)}_{14}+\chi^{(2)}_{34}-3i\pi\,,\tag{91d}$$
$$\mathcal{S}_3:\quad 2L_{31}=\chi^{(3)}_{24}+\chi^{(3)}_{25}+\chi^{(3)}_{45}-3i\pi\,,\quad 2L_{32}=\chi^{(3)}_{14}+\chi^{(3)}_{15}+\chi^{(3)}_{45}-3i\pi\,,\tag{91e}$$
$$2L_{34}=\chi^{(3)}_{12}+\chi^{(3)}_{15}+\chi^{(3)}_{25}-3i\pi\,,\quad 2L_{35}=\chi^{(3)}_{12}+\chi^{(3)}_{14}+\chi^{(3)}_{24}-3i\pi\,,\tag{91f}$$
$$\mathcal{S}_4:\quad 2L_{41}=\chi^{(4)}_{23}+\chi^{(4)}_{25}+\chi^{(4)}_{35}-3i\pi\,,\quad 2L_{42}=\chi^{(4)}_{13}+\chi^{(4)}_{15}+\chi^{(4)}_{35}-3i\pi\,,\tag{91g}$$
$$2L_{43}=\chi^{(4)}_{12}+\chi^{(4)}_{15}+\chi^{(4)}_{25}-3i\pi\,,\quad 2L_{45}=\chi^{(4)}_{12}+\chi^{(4)}_{13}+\chi^{(4)}_{23}-3i\pi\,,\tag{91h}$$
$$\mathcal{S}_5:\quad 2L_{51}=\chi^{(5)}_{23}+\chi^{(5)}_{24}+\chi^{(5)}_{34}-3i\pi\,,\quad 2L_{52}=\chi^{(5)}_{13}+\chi^{(5)}_{14}+\chi^{(5)}_{34}-3i\pi\,,\tag{91i}$$
$$2L_{53}=\chi^{(5)}_{12}+\chi^{(5)}_{14}+\chi^{(5)}_{24}-3i\pi\,,\quad 2L_{54}=\chi^{(5)}_{12}+\chi^{(5)}_{13}+\chi^{(5)}_{23}-3i\pi\,.\tag{91j}$$

It is easy to check, that L_{ab} commute with all the $\chi^{(a)}_{ij}$'s and that, using the relations in Table II, the 10 following constraints are admitted:

$$L_{ab}=-L_{ba}\,,\quad \forall\,(ab)\,.\tag{92}$$

Therefore, one can understand $2L_{ab}(a<b)$ as a coordinate dressing the annulus cusp (ab). We can choose the 10 FN lengths $\{2L_{ab}\}_{a<b}$ to be part of the position variables of the 30-dimensional phase space $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}$. The remaining 5 position variables $\mathcal{X}_a(a=1,\cdots,5)$ are FG coordinates each on one 4-holed sphere $\mathcal{S}_a$ on $\partial(S^3\backslash\Gamma_5)$. We choose these variables as follows.

$$\mathcal{X}_1=\chi^{(1)}_{25}\,,\quad \mathcal{X}_2=\chi^{(2)}_{15}\,,\quad \mathcal{X}_3=\chi^{(3)}_{15}\,,\quad \mathcal{X}_4=\chi^{(4)}_{15}\,,\quad \mathcal{X}_5=\chi^{(5)}_{14}\,.\tag{93}$$

The conjugate variable of $2L_{ab}$, denoted as $\mathcal{T}_{ab}$, is called the (logarithmic) FN twist. We also denote the conjugate variable of $\mathcal{X}_a$ as $\mathcal{Y}_a$. Then a new set of phase space variables equivalent to $(\vec{\Phi},\vec{\Pi})$ is

$$\vec{\mathcal{Q}}=\left(\{2L_{ab}\}_{a<b},\{\mathcal{X}_a\}_{a=1}^5\right)\,,\quad \vec{\mathcal{P}}=\left(\{\mathcal{T}_{ab}\}_{a<b},\{\mathcal{Y}_a\}_{a=1}^5\right)\,,\tag{94}$$

which satisfies the Poisson brackets

$$\{\mathcal{Q}_I,\mathcal{P}_J\}=\delta_{IJ}\,,\quad \forall\,I,J=1,\cdots,15\,.\tag{95}$$

Here, the order of the annuli (ab)'s is fixed to be $\{(12),(13),(14),(15),(23),(24),(25),(34),(35),(45)\}$. We will see later that it is easier to impose the quantum simplicity constraints on the new set of coordinates (94) rather than $(\vec{\Phi},\vec{\Pi})$, which urges us to express the partition function in terms of $(\vec{\mathcal{Q}})$. To do that, we first fix the expression for $\vec{\mathcal{P}}$. As the only requirement is (95), there are freedoms to choose the expression of $\vec{\mathcal{P}}$ in terms of the old coordinates $(\vec{\Phi},\vec{\Pi})$. For simplicity, we choose the symplectic transformation to be the following form.

$$\begin{pmatrix} \vec{\mathcal{Q}} \\ \vec{\mathcal{P}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -(\mathbf{B}^\top)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \vec{\Phi} \\ \vec{\Pi} \end{pmatrix} + \begin{pmatrix} i\pi\vec{t} \\ 0 \end{pmatrix} , \tag{96}$$

where $\mathbf{A}$ and $\mathbf{B}$ are 15×15 matrices with integer entries and $\vec{t}$ is a vector with integer elements. We have taken the advantage that $\mathbf{B}$ is invertible (while $\mathbf{A}$ is not). They are determined by (91) and (93) with the following explicit expressions.

$$\mathbf{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} , \tag{97a}$$

$$\mathbf{B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} , \tag{97b}$$

$$\vec{t} = (3,1,0,0,-4,-3,-2,-1,-2,0,1,-1,3,2,2)^\top . \tag{97c}$$

The fact that $\mathbf{A},\mathbf{B}$ and $\vec{t}$ have only integer entries means that the components of $\vec{\mathcal{Q}}$ are all coordinates of $\mathcal{M}_{\text{flat}}(\partial(S^3\backslash\Gamma_5),\text{PSL}(2,\mathbb{C}))$. However, the matrix $-\left(\mathbf{B}^\top\right)^{-1}$ has half-integer entries. In particular, each $\mathcal{T}_{ab}$ is a linear combination of elements of $\vec{\Phi}$ with half-integer coefficients. This means the FN twist $\tau_{ab} := e^{\mathcal{T}_{ab}}$ is a lift to a coordinate of $\mathcal{M}_{\text{flat}}(\partial(S^3\backslash\Gamma_5),\text{SL}(2,\mathbb{C}))$. Combinatorially, we should view $(\vec{\mathcal{Q}},\vec{\mathcal{P}})$ as a set of coordinates of $\mathcal{M}_{\text{flat}}(\partial(S^3\backslash\Gamma_5),\text{SL}(2,\mathbb{C}))$.

Thanks to our careful choice of $\{\mathcal{X}_a\}$ (93), each momenta $\mathcal{Y}_a$ is also given by an FG coordinate on $\mathcal{S}_a$ up to a sign and $\pm 2\pi i$:

$$\mathcal{Y}_1 = -\chi^{(1)}_{23} , \quad \mathcal{Y}_2 = -\chi^{(2)}_{14} , \quad \mathcal{Y}_3 = -\chi^{(3)}_{45} - 2\pi i , \mathcal{Y}_4 = \chi^{(4)}_{35} - 2\pi i , \quad \mathcal{Y}_5 = -\chi^{(5)}_{34} + 2\pi i . \tag{98}$$

We also parametrize the new set of variables and their tilde sectors in terms of continuous and discrete parameters as before, *i.e.*

$$\vec{\mathcal{Q}} = \frac{2\pi i}{k}\left(-b\vec{\mu} - \vec{m}\right) , \quad \vec{\mathcal{P}} = \frac{2\pi i}{k}\left(-b\vec{\nu} - \vec{n}\right) , \quad \vec{\tilde{\mathcal{Q}}} = \frac{2\pi i}{k}\left(-b^{1}\vec{\mu} + \vec{m}\right) , \quad \vec{\tilde{\mathcal{P}}} = \frac{2\pi i}{k}\left(-b^{-1}\vec{\nu} + \vec{n}\right) , \tag{99}$$

and the inverse relations are

$$\vec{\mu}=\frac{k}{2\pi Q}\left(\vec{\mathcal{Q}}+\vec{\tilde{\mathcal{Q}}}\right)\,,\;\vec{m}=\frac{ik}{2\pi bQ}\left(\vec{\mathcal{Q}}-b^2\vec{\tilde{\mathcal{Q}}}\right)\,,\;\vec{\nu}=\frac{k}{2\pi Q}\left(\vec{\mathcal{P}}+\vec{\tilde{\mathcal{P}}}\right)\,,\;\vec{n}=\frac{ik}{2\pi bQ}\left(\vec{\mathcal{P}}-b^2\vec{\tilde{\mathcal{P}}}\right)\,. \tag{100}$$

We will also use the notations μ_{ab}, m_{ab} (*resp.* ν_{ab}, n_{ab}) to denote the coordinates corresponding to $2L_{ab}$ (*resp.* $\mathcal{T}_{ab}$) and use μ_a, m_a (*resp.* ν_a, n_a) to denote the coordinates corresponding to $\mathcal{X}_a$ (*resp.* $\mathcal{Y}_a$). The Atiyah-Bott-Goldman symplectic form for the Chern-Simons phase space $\mathcal{P}_{\partial(S^3\backslash\Gamma_5)}\equiv\otimes_{i=1}^5\mathcal{P}_{\partial\mathrm{Oct}(i)}$ and the Poisson structure are

$$\Omega=\sum_{I=1}^{15}\mathcal{P}_I\wedge\mathcal{Q}_I\,,\quad\{\mathcal{Q}_I,\mathcal{P}_J\}_\Omega=\delta_{IJ}\,,\quad\{\mathcal{Q}_I,\mathcal{Q}_J\}_\Omega=\{\mathcal{P}_I,\mathcal{P}_J\}_\Omega=0\,,\quad I,J=1,\cdots,15\,. \tag{101}$$

The new coordinate parameters (100) are quantized to operators $\vec{\boldsymbol{\mu}},\vec{\mathbf{m}},\vec{\boldsymbol{\nu}}$ and $\vec{\mathbf{n}}$ respectively (we also use $\boldsymbol{\mu}_{ab},\mathbf{m}_{ab},\boldsymbol{\nu}_{ab},\mathbf{n}_{ab}$ as well as $\boldsymbol{\mu}_a,\mathbf{m}_a,\boldsymbol{\nu}_a,\mathbf{n}_a$ to denote the components).

The symplectic transformation on the phase space coordinates gives rise to a unitary transformation on the wave function $\mathcal{Z}_\times$ (and the operators). To express the unitary transformation clearly, one separates the transformation matrix into generator matrices of the symplectic transformations (recall (81)):

$$\begin{pmatrix}\mathbf{A}&\mathbf{B}\\-(\mathbf{B}^\top)^{-1}&0\end{pmatrix}=\begin{pmatrix}0&-\mathbb{I}\\\mathbb{I}&0\end{pmatrix}\begin{pmatrix}\mathbb{I}&0\\\mathbf{A}\mathbf{B}^\top&\mathbb{I}\end{pmatrix}\begin{pmatrix}-(\mathbf{B}^{-1})^\top&0\\0&-\mathbf{B}\end{pmatrix}\,. \tag{102}$$

The three matrices on the right-hand side correspond to different types of unitary transformations, which we denote as the S-type, T-type and U-type transformations respectively using the terminology in [11, 22]. The addition of $i\pi\vec{t}$ on $\vec{\mathcal{Q}}$ as in (96) corresponds to the affine translation. In general, the unitary transformation of a wave function ψ and an operator $\mathcal{O}$ given by a (time-independent) unitary operator U is defined as

$$\psi\to\psi'=U\triangleright\psi\,,\quad\mathcal{O}\to\mathcal{O}'=U\mathcal{O}U^\dagger\quad\Longrightarrow\quad\mathcal{O}\triangleright\psi\to U\triangleright(\mathcal{O}'\triangleright\psi')\,. \tag{103}$$

importantly, we want the wave function after the unitary transformation to be in functional space (71) with a non-empty positive angle structure so that the boundedness is not lost. Therefore, after each unitary transformation, we will also need to keep track of the positive angle structure for the new variables.

1. U-type transformation:

The U-type transformation is controlled by the matrix

$$\mathbf{U}=\begin{pmatrix}-(\mathbf{B}^{-1})^\top&0\\0&-\mathbf{B}\end{pmatrix}\,, \tag{104}$$

and it corresponds to a "rotation" on the vector of position variables and the vector of momentum variables "in an opposite direction". We only need to express the old variables in the original wave function in terms of the new variables times a scaling factor. That is,

$$\mathcal{Z}_1(\vec{\mu}|\vec{m})=(\mathbf{U}\triangleright\mathcal{Z}_\times)(\vec{\mu}|\vec{m})=\sqrt{\det(-\mathbf{B})}\mathcal{Z}_\times(-\mathbf{B}^\top\vec{\mu}|-\mathbf{B}^\top\vec{m})\,, \tag{105}$$

where $\sqrt{\det(-\mathbf{B})}=4i$. Since $-(\mathbf{B}^{-1})^\top$ has half-integer entries, the new position variables can only be viewed as coordinates of $\mathcal{M}_{\text{flat}}(\partial(S^3\backslash\Gamma_5),\mathrm{SL}(2,\mathbb{C}))$.

In addition, that $\mathcal{Z}_\times\in\mathcal{F}^{(k)}_{\mathfrak{P}(\text{oct})^{\times5}}$ implies that, when $(\vec{\alpha}_0,\vec{\beta}_0)\in\mathfrak{P}(\text{oct})^{\times5}$ and $\vec{\mu}\in\mathbb{R}^{15}$,

$$e^{-\frac{2\pi}{k}(-\mathbf{B}^\top\vec{\mu})\cdot\vec{\beta}_0}\mathcal{Z}_\times(-\mathbf{B}^\top\vec{\mu}+i\vec{\alpha}_0|-\mathbf{B}^\top\vec{m})\equiv e^{-\frac{2\pi}{k}\vec{\mu}\cdot(-\mathbf{B}^{-1}\vec{\beta}_0)}\mathcal{Z}_\times(-\mathbf{B}^\top\left(\vec{\mu}+i\left(-(\mathbf{B}^{-1})^\top\vec{\alpha}_0\right)\right)|-\mathbf{B}^\top\vec{m})\in\mathcal{S}(\mathbb{R}^{15})\,. \tag{106}$$

It is easy to see that the new positive angle structure is

$$\mathfrak{P}_1=\left\{(\vec{\alpha}_1,\vec{\beta}_1)=\left(-(\mathbf{B}^{-1})^\top\vec{\alpha}_0,-\mathbf{B}^{-1}\vec{\beta}_0\right)\mid(\vec{\alpha}_0,\vec{\beta}_0)\in\mathfrak{P}(\text{oct})^{\times5}\right\}\,. \tag{107}$$

We conclude that $\mathcal{Z}_1\in\mathcal{F}^{(k)}_{\mathfrak{P}_1}$.

2. T-type transformation:

the T-type transformation is controlled by the matrix

$$\mathbf{T}=\begin{pmatrix}\mathbb{I}&0\\\mathbf{A}\mathbf{B}^\top&\mathbb{I}\end{pmatrix}\,, \tag{108}$$

where $\mathbf{A}\mathbf{B}^\top$ is a symmetric matrix with integer entries. It corresponds to a change of momenta keeping the position variables unchanged. The partition function after this transformation is

$$\mathcal{Z}_2(\vec{\mu}|\vec{m}) = (\mathbf{T} \triangleright \mathcal{Z}_1)(\vec{\mu}|\vec{m}) = (-1)^{\vec{m}\cdot\mathbf{A}\mathbf{B}^\top\cdot\vec{m}} e^{\frac{\pi i}{k}\left(-\vec{\mu}\cdot\mathbf{A}\mathbf{B}^\top\cdot\vec{\mu}+\vec{m}\cdot\mathbf{A}\mathbf{B}^\top\cdot\vec{m}\right)}\mathcal{Z}_1(\vec{\mu}|\vec{m})\,, \tag{109}$$

where the sign $(-1)^{\vec{m}\cdot\mathbf{A}\mathbf{B}^\top\cdot\vec{m}}$ is there for the cyclic symmetry that $\mathcal{Z}_2$ is unchanged the transformation $m_I \to m_I + k$ for any m_I.

To be convinced that (109) is true. Let us consider a 1-dimensional example of the T-type transformation on the wave function $f(\mu|m)$ given by the unitary operator $U = e^{\frac{\pi i}{k}(-\boldsymbol{\mu}T\boldsymbol{\mu}+(k+1)\mathbf{m}T\mathbf{m})}$ with $T\in\mathbb{Z}$:

$$f(\mu|m) \to f'(\mu|m) = (U \triangleright f)(\mu|m) = e^{\frac{\pi i}{k}(-\mu T\mu+(k+1)mTm)}f(\mu|m)\,. \tag{110}$$

The new operators $\boldsymbol{\mu}', \mathbf{m}'$ are transformed to be multiplicity operators :

$$e^{\boldsymbol{\mu}} \to e^{\boldsymbol{\mu}'} = Ue^{\boldsymbol{\mu}}U^\dagger \equiv e^{\boldsymbol{\mu}}\,, \quad e^{\mathbf{m}} \to e^{\mathbf{m}'} = Ue^{\mathbf{m}}U^\dagger \equiv e^{\mathbf{m}}\,, \tag{111}$$

while $\boldsymbol{\nu}', \mathbf{n}'$ are shift operators:

$$e^{\boldsymbol{\nu}+T\boldsymbol{\mu}} \to e^{\boldsymbol{\nu}'} = Ue^{\boldsymbol{\nu}+T\boldsymbol{\mu}}U^\dagger \equiv e^{\boldsymbol{\nu}}\,, \quad e^{\mathbf{n}+\mathbf{m}} \to e^{\mathbf{n}'} = Ue^{\mathbf{n}+T\mathbf{m}}U^\dagger \equiv e^{\mathbf{n}}\,, \tag{112}$$

as desired. To derive (112), we have used the Baker-Campbell-Hausdorff formula:

$$e^{\mathbf{X}}e^{\mathbf{Y}} = \exp\left[\mathbf{X}+\mathbf{Y}+\frac{1}{2}[\mathbf{X},\mathbf{Y}]+\frac{1}{12}[\mathbf{X},[\mathbf{X},\mathbf{Y}]]-\frac{1}{12}[\mathbf{Y},[\mathbf{X},\mathbf{Y}]]+\cdots\right] \tag{113}$$

Note that although (110) – (112) is still true when the sign factor is $(-1)^{mTm}$ removed from U, but require the cyclic symmetry $f'(\mu|m+k) = f'(\mu|m)$ for the function be true for any integers m and k. Explicitly,

$$f'(\mu|m+k) = (-1)^{T(m+k)^2}e^{\frac{\pi i}{k}\left(-\mu T\mu+T(m+k)^2\right)}f(\mu|m+k) = (-1)^{Tk^2+2Tkm}e^{\pi i(2Tmk+Tk^2)}f'(\mu|m)\,. \tag{114}$$

If the sign $(-1)^{mTm}$ were dropped, the sign of $f'(\mu|m+k)$ in (114) would have changed when k, T are both odd.

$\mathcal{Z}_1 \in \mathcal{F}^{(k)}_{\mathfrak{P}_1}$ implies that, when $(\vec{\alpha}_1, \vec{\beta}_1) \in \mathfrak{P}_1$ and $\vec{\mu}\in\mathbb{R}^{15}$,

$$e^{-\frac{2\pi}{k}\vec{\mu}\cdot\vec{\beta}_1}\mathcal{Z}_1(\vec{\mu}|\vec{m}) = \text{phase}\cdot e^{-\frac{2\pi}{k}\vec{\mu}\cdot(\vec{\beta}_1+\mathbf{A}\mathbf{B}^\top\vec{\alpha}_1)}\mathcal{Z}_2(\vec{\mu}+i\vec{\alpha}_1|\vec{m}) \in \mathcal{Z}_1 \in \mathcal{F}^{(k)}_{\mathfrak{P}_1}\,. \tag{115}$$

Therefore, the new positive angle structure is

$$\mathfrak{P}_2 = \left\{(\vec{\alpha}_2,\vec{\beta}_2) = \left(\vec{\alpha}_1, \vec{\beta}_1+\mathbf{A}\mathbf{B}^\top\vec{\alpha}_1\right) \mid (\vec{\alpha}_1,\vec{\beta}_1)\in\mathfrak{P}_1\right\}\,, \tag{116}$$

and $\mathcal{Z}_2 \in \mathcal{F}^{(k)}_{\mathfrak{P}_2}$.

3. S-type transformation:

the S-type transformation given by the matrix

$$\mathbf{S} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{117}$$

exchanges the position and momentum operators and it corresponds to the Fourier transform of the partition function. That is,

$$\mathcal{Z}_3(\vec{\mu}|\vec{m}) = \frac{1}{k^{15}}\sum_{\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}}\int_{\mathcal{C}^{\times 15}} \mathrm{d}^{15}\vec{\nu}\, e^{\frac{2\pi i}{k}(-\vec{\nu}\cdot\vec{\mu}+\vec{n}\cdot\vec{m})}\mathcal{Z}_2(\vec{\nu}|\vec{n})\,, \tag{118}$$

where the integration contour is along $\mathbb{R}^{15}+i\vec{\beta}_3$ where $\vec{\beta}_3$ satisfies the new positive angle structure

$$\mathfrak{P}_3 = \left\{(\vec{\alpha}_3,\vec{\beta}_3) = (-\vec{\beta}_2,\vec{\alpha}_3) \mid (\vec{\alpha}_2,\vec{\beta}_2)\in\mathfrak{P}_2\right\}\,. \tag{119}$$

Indeed, $\mathcal{Z}_3 \in \mathcal{F}^{(k)}_{\mathfrak{P}_3}$.

4. Affine shift translation:

Finally, as shown in (96), to arrive at the partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$ for $S^3\backslash\Gamma_5$ in terms of parameters of $\vec{\mathcal{Q}}$ and $\vec{\mathcal{P}}$, one needs to perform an affine shift transformation $\sigma_{\vec{t}}$ given by a vector $\vec{t}$ (97c) on the position variables. It shifts a classical position variables $X \to X + i\pi t$ with $t \in \mathbb{Z}$, and it adds a quantum deformation

$$X \to X + \left(i\pi + \frac{h}{2}\right) t \tag{120}$$

when entering the partition function. It has been argued in the literature (see *e.g.* [19]) that such a deformation is more suited for a state-integral (*e.g.* it is necessary for the invariance of state-integral under 2-3 Pachner move) Indeed, it breaks the periodicity when $X \to X + 2\pi i$. But it can be understood as a quantum effect and the periodicity is recovered at that classical $h \to 0$ limit.

Parametrize $X = \frac{2\pi i}{k}(-ib\mu - m)$ as in (99), the affine shift (120) leads to shifts on the parameters

$$\mu \to \mu - \frac{iQ}{2} t \,, \quad m \to m \,, \quad Q = b + b^{-1} \,. \tag{121}$$

Therefore, the final partition function takes the form

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m}) = \mathcal{Z}_3(\vec{\mu} - i\frac{Q}{2}\vec{t}|\vec{m}) \,. \tag{122}$$

The positive angle structure for $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$ is

$$\mathfrak{P}_{S^3\backslash\Gamma_5} = \left\{ (\vec{\alpha}, \vec{\beta}) = (\vec{\alpha}_3 + \frac{Q}{2}\vec{t}, \vec{\beta}_3) \mid (\vec{\alpha}_3, \vec{\beta}_3) \in \mathfrak{P}_3 \right\} \tag{123}$$

and $\mathcal{Z}^3_S\backslash\Gamma_5 \in \mathcal{F}^{(k)}_{\mathfrak{P}^3_S\backslash\Gamma_5}$.

Combing all the steps above, $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$ can be written as finite sums and convergence integrals in terms of new coordinates (100). The partition takes the following expression.

$$\begin{aligned}\mathcal{Z}'_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m}) &= ((\sigma_{\vec{t}} \circ S \circ T \circ U) \triangleright \mathcal{Z}_\times)(\vec{\mu}|\vec{m}) \\ &= \frac{4i}{k^{15}} \sum_{\vec{n} \in (\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathcal{C}^{\times 15}} \mathrm{d}^{15}\vec{\nu}\, (-1)^{\vec{n}\cdot\mathbf{AB}^\top\cdot\vec{n}} e^{\frac{i\pi}{k}(-\vec{\nu}\cdot\mathbf{AB}^\top\cdot\vec{\nu} + \vec{n}\cdot\mathbf{AB}^\top\cdot\vec{n})} e^{\frac{2\pi i}{k}\left[-\vec{\nu}\cdot(\vec{\mu} - \frac{iQ}{2}\vec{t}) + \vec{n}\cdot\vec{m}\right]} \mathcal{Z}_\times(-\mathbf{B}^\top\vec{\nu}| - \mathbf{B}^\top\vec{n}) \,,\end{aligned} \tag{124}$$

where the integration contour $\mathcal{C}^{\times 15}$ is chosen to be on the plane $\mathbb{R}^{15} + i\vec{\alpha}_2$.

Observe that $\mathbf{AB}^\top$ is a symmetric matrix with integer entries, $(-1)^{\vec{n}\cdot\mathbf{AB}^\top\cdot\vec{n}}$ in (124) can be simplified to be $(-1)^{\vec{D}\cdot\vec{n}}$ where $\vec{D} := \mathrm{diag}(\mathbf{AB}^\top)$ is a vector whose elements are the diagonal elements of $\mathbf{AB}^\top$. The sign $(-1)^{\vec{n}\cdot\mathbf{AB}^\top\cdot\vec{n}}$ depends on the parity of elements in $\vec{D}$ and $\vec{n}$. Also notice that the parity of D_I is the same as the parity of t_I, $\forall I = 1, \cdots, 15$[5]. Combining these facts, we can rewrite the sign factor $(-1)^{\vec{n}\cdot\mathbf{AB}^\top\cdot\vec{n}}$ in (125) to be $(-1)^{\vec{t}\cdot\vec{n}}$ and simplify the expression (124) to

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m}) = \frac{4i}{k^{15}} \sum_{\vec{n} \in (\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathcal{C}^{\times 15}} \mathrm{d}^{15}\vec{\nu}\, (-1)^{\vec{t}\cdot\vec{n}} e^{\frac{i\pi}{k}(-\vec{\nu}\cdot\mathbf{AB}^\top\cdot\vec{\nu} + \vec{n}\cdot\mathbf{AB}^\top\cdot\vec{n})} e^{\frac{2\pi i}{k}\left[-\vec{\nu}\cdot(\vec{\mu} - \frac{iQ}{2}\vec{t}) + \vec{n}\cdot\vec{m}\right]} \mathcal{Z}_\times(-\mathbf{B}^\top\vec{\nu}| - \mathbf{B}^\top\vec{n}) \,. \tag{125}$$

It was checked in [12] that such a change does not alter the equations of motion compared to the ones computed with the original one (124) [11]. The positive angle structure $\mathfrak{P}(S^3\backslash\Gamma_5)$ for $S^3\backslash\Gamma_5$ in terms of the new variables $(\vec{\mu}, \vec{\nu})$ is [11] [6]

$$\begin{aligned}&\mathfrak{P}(S^3\backslash\Gamma_5) = \sigma'_{\vec{t}} \circ S \circ T \circ U \circ \mathfrak{P}(\mathrm{oct})^{\times 5} \\ \Rightarrow \quad &\text{If } (\vec{\alpha}_0, \vec{\beta}_0) \in \mathfrak{P}(\mathrm{oct})^{\times 5} \,, \quad \text{then } (\vec{\alpha}, \vec{\beta}) = (\mathbf{A}\vec{\alpha}_0 + \mathbf{B}\vec{\beta}_0 + \frac{Q}{2}\vec{t}, -(\mathbf{B}^{-1})^\top\vec{\alpha}_0) \in \mathfrak{P}(S^3\backslash\Gamma_5) \,.\end{aligned} \tag{126}$$

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[5] One can check using the explicit expressions (97) of matrices $\mathbf{A}$, $\mathbf{B}$ and vector $\vec{t}$ that the odd elements of $\vec{D}$ and $\vec{t}$ are both the 1th, 2nd, 6th, 8th, 11th, 12th and 13th elements.

Inversely,

$$(\vec{\alpha}_0, \vec{\beta}_0) = (\mathbf{B}^\top \vec{\beta}, \mathbf{B}^{-1}\vec{\alpha} + \mathbf{A}^\top \vec{\beta} - \frac{Q}{2}\vec{t}) \in \mathfrak{P}(\text{oct})^{\times 5}\,. \tag{127}$$

The symplectic transformations ensure that $\mathfrak{P}(S^3\backslash\Gamma_5)$ is non-empty since $\mathfrak{P}(\text{oct})^{\times 5}$ is non-empty, which concludes that $\mathcal{Z}_{S^3\backslash\Gamma_5} \in \mathcal{F}^{(k)}_{\mathfrak{P}(S^3\backslash\Gamma_5)} \equiv \mathcal{F}_{\mathfrak{P}(S^3\backslash\Gamma_5)} \otimes_{\mathbb{C}} (\mathbb{C}^k)^{\otimes 15}$. In other words, $\mathcal{Z}^3_S\backslash\Gamma_5$ is absolute convergent hence the finiteness of the Chern-Simons partition function on $S^3\backslash\Gamma_5$ is manifest. More generally, the Chern-Simons partition function constructed in terms of ideal triangulation converges absolutely as long as the 3-manifold admits a non-empty positive angle structure [22, 30, 31]. This means, given any $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{P}(S^3\backslash\Gamma_5)$ and let $\text{Im}(\vec{\mu}) = \vec{\alpha}$, the integration contours $\mathcal{C}^{\times 15}$ satisfying $\text{Im}(\vec{\nu}) = \vec{\beta}$ renders the finiteness of $\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m})$.

V. FROM CHERN-SIMONS PARTITION FUNCTION TO SPINFOAM AMPLITUDE

The partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$ is for Chern-Simons theory on $S^3\backslash\Gamma_5$ but does not yet encode the quantum gravity information. To define a vertex amplitude associated to a 4-simplex, one needs to impose the quantized version of the simplicity constraint $\mathcal{F} \cong \frac{\Lambda}{3} e \wedge e$ on $\mathcal{Z}_{S^3\backslash\Gamma_5}$. The way to implement this is motivated by the EPRL model. Recall that, in the EPRL model, the simplicity constraint at the classical and discrete level can be implemented by (14) and (15). In other words, the simplicity constraint is to *require the discretized $B^{IJ}_f(t)$-field to encode the geometry of tetrahedra in a 4-simplex* by satisfying two requirements:

(a) For each tetrahedron t in the 4-simplex, there exists a common normal to the four discretized $B^{IJ}_f(t)$-fields each associated to a triangle f;

(b) $B^{IJ}_f(t)$ encodes the area and normal of the triangle by satisfying the closure condition.

This can be generalized to the $\Lambda \neq 0$ case as follows. Consider the (non-ideal) triangulation, denoted as τ_a, of a 4-holed sphere $\Sigma_{0,4}$ such that each hole, denoted by $\mathfrak{p}$, is inside a triangle $f_{\mathfrak{p}}$. See the red lines in fig.8. Define the

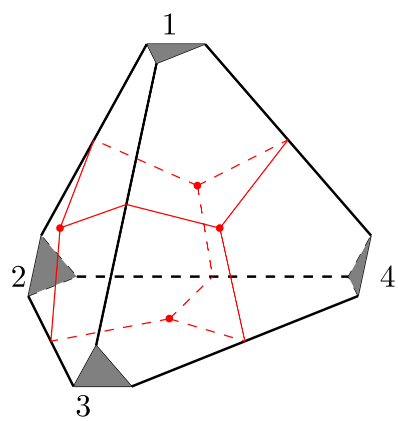


FIG. 8: The ideal triangulation (*in black*) and the (normal) triangulation τ_a (*in red*) of a 4-holed sphere $\Sigma_{0,4}$. Numbers $1, 2, 3, 4$ label the holes of $\Sigma_{0,4}$.

discretized B-field associated to $f_{\mathfrak{p}}$ as in the EPRL model, *i.e.* $B_{f_{\mathfrak{p}}}(\tau_a) = \int_{f_{\mathfrak{p}}} B(\tau_a)$. On the other hand, let us recall the relation $\mathcal{F} = \frac{|\Lambda|}{3} B$ discussed in (19). Consider a local coordinate (x^1, x^2) on one patch of $\mathcal{S}_a$ with the hole $\mathfrak{p}$ at the origin. Then the discretization of this relation gives $\mathcal{F}_{\mathfrak{p}}(\mathcal{S}_a) = \frac{|\Lambda|}{3} B_{f_{\mathfrak{p}}}(\tau_a)\delta^{(2)}(\vec{x})\mathrm{d}x^1 \wedge \mathrm{d}x^2$. This allows us to write

[6] The operator $\vec{\sigma}'_{\vec{t}}$ for the positive angle structure is different from the affine transformation $\vec{\sigma}_{\vec{t}}$ acting on the wave functions. The latter is given in (96) while the former is defined as: $\vec{\sigma}'_{\vec{t}} : (\vec{\alpha}, \vec{\beta}) \mapsto (\vec{\alpha} + \frac{Q}{2}\vec{t}, \vec{\beta})$ [11].

the simplicity constraints in the same form as (14) in terms of the Chern-Simons curvature. That is, for all holes $\mathfrak{p}$'s of $\mathcal{S}_a$,

$$\exists\, N_J \text{ such that } N_J \mathcal{F}_{\mathfrak{p}}^{IJ}(\mathcal{S}_a) = 0\,. \tag{128}$$

By the non-abelian Stokes' theorem, the holonomy around each triangle $f_{\mathfrak{p}}$ of τ_a takes the form $H_{f_{\mathfrak{p}}}(\tau_a) = e^{\frac{|\Lambda|}{3} B_{f_{\mathfrak{p}}}(\tau_a)} \in \mathrm{PSL}(2,\mathbb{C})$. The reason for $H_{f_{\mathfrak{p}}}(\tau_a) \in \mathrm{PSL}(2,\mathbb{C})$ instead of $H_{f_{\mathfrak{p}}}(\tau_a) \in \mathrm{SL}(2,\mathbb{C})$ is because this holonomy can be computed using the FG coordinates on $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSL}(2,\mathbb{C}))$ which we will see later. Eq.(128) can be translated into constraints in terms of $\{H_{f_{\mathfrak{p}}}(\tau_a)\}_{\mathfrak{p}=1}^4$:

$$\exists\, N_J \text{ such that } (H_{f_{\mathfrak{p}}})_I{}^J(\tau_a) N_J = N_I\,, \quad \forall f_{\mathfrak{p}} \in \tau_a\,. \tag{129}$$

Similar to the EPRL case, (128) (or (129)) means that the 4-holed sphere $\mathcal{S}_a$, or its triangulation τ_a, is orthogonal to a common vector $N^J \in \mathbb{R}^4$. Gauge fixing $N_J = (1,0,0,0)$ implements that all the holonomies $\{H_{f_{\mathfrak{p}}}(\tau_a)\}_{\mathfrak{p}=1}^4$ are in a common PSU(2) subgroup of $\mathrm{PSL}(2,\mathbb{C})$. In other words, the simplicity constraints restrict the moduli space $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSL}(2,\mathbb{C}))$ of flat $\mathrm{PSL}(2,\mathbb{C})$ connection to a moduli space $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSU}(2))$ of flat PSU(2) connection.

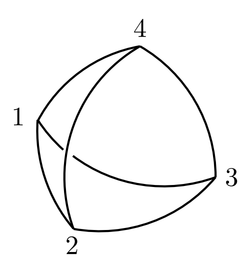
The flat connection in $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSU}(2))$ defines a representation of the fundamental group of $\mathcal{S}_a$ into PSU(2) modulo gauge transformations. Let the holonomies $\{H_{f_{\mathfrak{p}}}(\tau_a)\}$ have the same base point $\mathfrak{b} \in \mathcal{S}_a$. Then they satisfy the non-linear closure condition (we fix the ordering of the holonomies here and for the rest of this note)

$$H_{f_4}(\tau_a) H_{f_3}(\tau_a) H_{f_2}(\tau_a) H_{f_1}(\tau_a) = \mathbb{I}_{\mathrm{PSU}(2)} \tag{130}$$

due to the isomorphism

$$\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSU}(2)) \cong \{H_1, H_2, H_3, H_4 \in \mathrm{PSU}(2) : H_4 H_3 H_2 H_1 = \mathbb{I}_{\mathrm{PSU}(2)}\}/\mathrm{PSU}(2)\,, \tag{131}$$

which is a special case of (38). An interesting fact is that the expression in the bracket on the right-hand side of (131) can determine *uniquely* a (convex) homogeneously curved tetrahedron, whose faces are flatly embedded in a three-sphere S^3 or hyperbolic three-space $\mathbb{H}^3$ (See fig.9). That is, a curved tetrahedron with constant curvature can



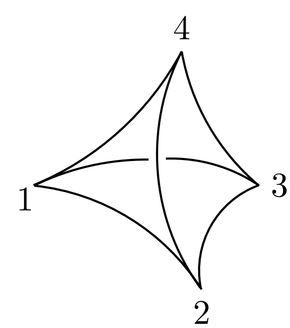


FIG. 9: *(a)* A tetrahedron flatly embedded in S^3. *(b)* A tetrahedron flatly embedded in $\mathbb{H}^3$.

be identified by four PSU(2) holonomies H_1, H_2, H_3, H_4 satisfying the closure condition $H_4 H_3 H_2 H_1 = \mathbb{I}_{\mathrm{PSU}(2)}$ with a few extra restrictions. This is called the "curved Minkowski theorem" for the tetrahedron, proven in [10], which we will briefly summarize below. In the flat limit, it coincides with the well-known Minkowski theorem for flat tetrahedra which was proven in 1897 [32].

A. curved Minkowski theorem for homogeneously curved tetrahedron

Before we describe the curved Minkowski theorem, let us first discuss the geometry of a homogeneously curved tetrahedron, or tetrahedron for short.

We let the sign of the curvature $s \equiv \mathrm{sgn}(\Lambda)$ be identified as the sign of the cosmological constant. To unify the notations, we denote the n-dimensional homogeneously curved space as $\mathbb{E}^{n,s}$ hence $\mathbb{E}^{3,+} = S^3$ and $\mathbb{E}^{3,-} = \mathbb{H}^3$. Each face of a tetrahedron is a triangle flatly embedded in a two-dimensional subspace $\mathbb{E}^{2,s}$ of $\mathbb{E}^{3,s}$. We only focus on the convex tetrahedra. The convexity guarantees that each edge of the triangle is the shortest geodesic on $\mathbb{E}^{2,s}$ connecting the two end vertices of the edge. For each face, we choose a base point p on the boundary and consider the oriented loop ℓ along the boundary starting and ending at p whose orientation is counterclockwise when seen from the outside of the tetrahedron. Such an orientation generates an outward direction normal $\hat{n}_\ell(p)$ to the face at p (and any other point within the face) by the right-hand rule, which is consistent with the topological orientation of the tetrahedron. We also denote the same loop with the opposite orientation as ℓ^{-1}.

Indeed, a vector at p tangent to the face gets rotated after parallel transport along ℓ. The rotation angle is proportional to the area a_ℓ of the face enclosed by ℓ. We denote the holonomy of the Levi-Civita connection along ℓ in the local frame of p as $M_\ell(p)$. It is a group element of SO(3) for both tetrahedra embedded in S^3 and $\mathbb{H}^3$ which can be parametrized as

$$M_\ell(p) = \exp\left[s\frac{|\Lambda|}{3}a_\ell\hat{n}_\ell(p)\cdot\vec{J}\right] \in \mathrm{SO}(3)\,, \qquad \frac{|\Lambda|}{3}a_\ell \in [0,2\pi] \tag{132}$$

where $\vec{J} = \{J_1, J_2, J_3\}$ are the generators of $\mathfrak{so}(3)$ and the sign s determines in which space the tetrahedron is embedded. However, the $M_\ell(p)$ can take another expression

$$M_\ell(p) = \exp\left[s\left(2\pi - \frac{|\Lambda|}{3}a_\ell\right)(-\hat{n}_\ell(p))\cdot\vec{J}\right] \tag{133}$$

as $\mathrm{SO}(3) \cong S^3/\mathbb{Z}_2$. This means $M_\ell(p)$ cannot distinguish the two triangles lying in the same great 2-spheres of S^3 with area and outgoing normal $(\mathfrak{a}_\ell, \mathfrak{n}_\ell) = (a_\ell, \hat{n}_\ell)$ and $(\mathfrak{a}_\ell, \mathfrak{n}_\ell) = (\frac{6\pi}{|\Lambda|} - a_\ell, -\hat{n}_\ell)$ respectively. Due to the isomorphism $\mathrm{SO}(3) \cong \mathrm{PSU}(2) = \mathrm{SU}(2)/\mathbb{Z}_2$, given $M_\ell(p)$ parametrized as (132) or (132), one can identify a PSU(2) group element $H_\ell(p)$:

$$H_\ell(p) = \exp\left[s\frac{|\Lambda|}{3}a_\ell\hat{n}_\ell(p)\cdot\vec{\tau}\right] \equiv \cos\left(s\frac{|\Lambda|}{6}\mathfrak{a}_\ell\right)\mathbb{I} - i\sin\left(s\frac{|\Lambda|}{6}\mathfrak{a}_\ell\right)\hat{n}_\ell\cdot\vec{\sigma} \equiv \exp\left[s\left(2\pi - \frac{|\Lambda|}{3}\mathfrak{a}_\ell\right)(-\hat{n}_\ell(p))\cdot\vec{\tau}\right] \tag{134}$$

where $\frac{|\Lambda|}{6}\mathfrak{a}_\ell \in [0,\pi]$. Here $\vec{\tau} = -\frac{i}{2}\vec{\sigma} \in \mathfrak{su}(2)$ and $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices. Although we can not decide which geometry $H_\ell(p)$ describes for a triangle by looking at this single expression, we will see later that for a convex tetrahedron, either $(a_\ell, \hat{n}_\ell)$ or $(\frac{6\pi}{|\Lambda|} - a_\ell, -\hat{n}_\ell)$ would be single out uniquely.

Changing the base point corresponds to a conjugation action on $H_\ell(p)$ by an PSU(2) group element, say g,

$$H_\ell(p) \quad\longrightarrow\quad H_\ell(p') = gH_\ell(p)g^{-1}\,, \quad g \in \mathrm{PSU}(2)\,. \tag{135}$$

Changing the orientation of ℓ corresponds to changing H_ℓ to its inverse, *i.e.* $H_{\ell^{-1}} = H_\ell^{-1}$. For each curved tetrahedron, there exists a closure condition expressed as

$$H_4H_3H_2H_1 = \mathbb{I}\,, \quad H_\ell \in \mathrm{PSU}(2)\,, \tag{136}$$

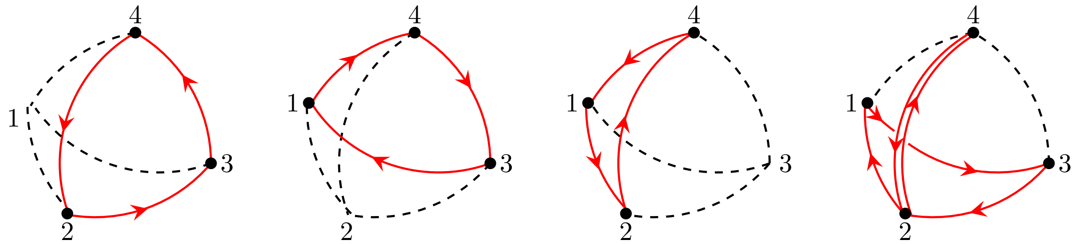
where all four holonomies are defined at the same base point. Indeed, it is easy to find a common point for three of the four holonomies. One then has to parallel transport the base point at least once through a specified path to define all the holonomies properly. As one of the simplest examples, choosing vertex 4 in fig.9 as the base point, $H_1(4), H_2(4), H_3(4)$ can all be defined directly by (134). To define $H_4(4)$, we first define $H_4(2)$ based on vertex 2 by (134) and parallel transport it to vertex 4 through the edge e_{42}.

A solution to (136) can be given by introducing the edge holonomy $h_{v_1v_2}$ for each oriented edge $e_{v_1v_2}$ with $h_{v_1v_2}^{-1} = h_{v_2v_1}$. Then

$$\begin{cases} H_1 = h_{43}h_{32}h_{24} \\ H_2 = h_{41}h_{13}h_{34} \\ H_3 = h_{42}h_{21}h_{14} \\ H_4 = h_{42}H_4(2)h_{24} = h_{42}h_{23}h_{31}h_{12}h_{24} \end{cases} \tag{137}$$

is indeed a solution to (136). The paths for the solution (137) are illustrated in fig.10 for a spherical tetrahedron as an example and are the same for a hyperbolic tetrahedron. These paths are called the *simple paths* as they are the simplest set of paths up to the choice of the base point and the special edge. They are the generators of the fundamental group of a tetrahedron. That is

$$\pi_1(\text{tetra}) = \{\ell_1, \ell_2, \ell_3, \ell_4 | \ell_4 \circ \ell_3 \circ \ell_2 \circ \ell_1 = 1\}\,. \tag{138}$$



(a) ℓ_1. (b) ℓ_2. (c) ℓ_3. (d) ℓ_4.

FIG. 10: The set of simple paths (*in red*) for holonomies $\{H_1, H_2, H_3, H_4\}$ defined in (137) with vertex 4 as the base point and edge (42) as the special edge. They satisfy the closure condition (136).

It can be straightforwardly checked that, given a tetrahedron whose curvature is determined by Λ, the full geometrical information can be described by the four holonomies H_ℓ's explicitly as

$$\cos\left(\frac{\Lambda}{6}\mathfrak{a}_\ell\right) = \frac{1}{2}\mathrm{Tr}(H_\ell)\,, \quad \mathfrak{n}_\ell = \epsilon_\ell \frac{i\mathrm{Tr}(H_\ell\vec{\sigma})}{\sqrt{4-\mathrm{Tr}(H_\ell)^2}}\,, \tag{139}$$

where $\epsilon_\ell = \pm$ with the sign $+$ corresponding to $\mathfrak{n}_\ell = \hat{n}_\ell$ and $-$ corresponding to $\mathfrak{n}_\ell = -\hat{n}_\ell$. Then one can calculate *e.g.* the dihedral angle $\theta_{\ell_1\ell_2}$ between two faces in a tetrahedron by

$$\cos\theta_{\ell_1} := \mathfrak{n}_{\ell_1}(p')\cdot\mathfrak{n}_{\ell_2}(p') = \epsilon_{\ell_1}\epsilon_{\ell_2}\hat{n}_{\ell_1}(p')\cdot\hat{n}_{\ell_2}(p')\,, \tag{140}$$

where p' is any point on the edge shared by the two faces. As the faces of the tetrahedron are flatly embedded in $\mathbb{E}^{3,s}$, these dot products are invariant along the edge shared by two faces and hence the dihedral angles are well defined. For simplicity, one can choose one of the two endpoints of the edge, which is a vertex of the tetrahedron. Note that (139) is valid only when $\hat{n}_\ell$ is defined at the base point of the loops, which is chosen to be the vertex 4 in our convention, *i.e.* $\hat{n}_\ell = \hat{n}_\ell(4)$. Then, to calculate the dihedral angle $\theta_{14}, \theta_{24}, \theta_{34}$, one has to parallel transport n_ℓ to another vertex. For instance, using the simple solution (137),

$$\theta_{24} = \epsilon_2\epsilon_4\hat{n}_2(3)\cdot\hat{n}_4(3) = \epsilon_2\epsilon_4\left[h_{34}\hat{n}_2(4)\right]\cdot\left[h_{32}h_{24}\hat{n}_4(4)\right] = \epsilon_2\epsilon_4\hat{n}_2(4)\cdot H_1\hat{n}_4(4)\,. \tag{141}$$

Another way is to define θ_{24} at vertex 1 which gives an equivalent result $\theta_{24} = \epsilon_2\epsilon_4\hat{n}_2(1)\cdot\hat{n}_4(1) = \epsilon_2\epsilon_4\hat{n}_2(4)\cdot H_3^{-1}\hat{n}_4(4)$.

One can also calculate the triple product of normals of three faces, which is calculated at the vertex where the three faces meet. We require that the tetrahedron be convex. Then the triple product must satisfy

$$\begin{cases} [\mathfrak{n}_1(4)\times\mathfrak{n}_2(4)]\cdot\mathfrak{n}_3(4) > 0\,, & \text{at vertex 4} \\ [\mathfrak{n}_1(2)\times\mathfrak{n}_3(2)]\cdot\mathfrak{n}_4(2) > 0\,, & \text{at vertex 2} \\ [\mathfrak{n}_2(3)\times\mathfrak{n}_1(3)]\cdot\mathfrak{n}_4(3) > 0\,, & \text{at vertex 3} \\ [\mathfrak{n}_3(1)\times\mathfrak{n}_2(1)]\cdot\mathfrak{n}_4(1) > 0\,, & \text{at vertex 1} \end{cases}, \tag{142}$$

which can also be parallel transport to vertex 4 and give

$$\begin{cases} \epsilon_1\epsilon_2\epsilon_3\,(\hat{n}_1\times\hat{n}_2)\cdot\hat{n}_3 & > 0 \\ \epsilon_1\epsilon_3\epsilon_4\,(\hat{n}_1\times\hat{n}_3)\cdot\hat{n}_4 & > 0 \\ \epsilon_2\epsilon_1\epsilon_4\,(\hat{n}_2\times\hat{n}_1)\cdot H_1\hat{n}_4 & > 0 \\ \epsilon_3\epsilon_2\epsilon_4\,(\hat{n}_3\times\hat{n}_2)\cdot H_3^{-1}\hat{n}_4 & > 0 \end{cases}. \tag{143}$$

The four inequalities can uniquely fix the four signs $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and resolve the ambiguity for the areas and outgoing normals of all the faces in a tetrahedron.

Interlude: Explicit expressions for the gauge invariant observables.

The explicit expressions for the gauge invariant observables, namely the areas, dihedral angles and the normal triple products have been given in [10], which we summarize here.
For convenience, let us first introduce the half-traces of the products of one, two and three holonomies, respectively.

$$\langle H_\ell\rangle = \frac{1}{2}\mathrm{Tr}(H_\ell)\,, \tag{144a}$$

$$\langle H_{\ell_1}H_{\ell_2}\rangle = \frac{1}{2}\mathrm{Tr}(H_{\ell_1}H_{\ell_2}) - \frac{1}{4}\mathrm{Tr}(H_{\ell_1})\mathrm{Tr}(H_{\ell_2})\,, \tag{144b}$$

$$\langle H_{\ell_1}H_{\ell_2}H_{\ell_3}\rangle = \frac{1}{2}\mathrm{Tr}(H_{\ell_1}H_{\ell_2}H_{\ell_3}) - \frac{1}{4}\left[\mathrm{Tr}(H_{\ell_1})\mathrm{Tr}(H_{\ell_2}H_{\ell_3}) + \text{cyclic}\right] + \frac{1}{4}\mathrm{Tr}(H_{\ell_1})\mathrm{Tr}(H_{\ell_2})\mathrm{Tr}(H_{\ell_3})\,. \tag{144c}$$

The half-trace (144a) of one holonomy H_ℓ around a face ℓ encodes the area $\mathfrak{a}_\ell$ of the face; the half-trace (144b) of two holonomies H_{ℓ_1} and H_{ℓ_2} encodes the dihedral angle $\theta_{\ell_1\ell_2}$ of the two faces ℓ_1 and ℓ_2; the half-trace (144c) of three holonomies H_{ℓ_1}, H_{ℓ_2} and H_{ℓ_3} encodes the triple product of the normals $(\hat{n}_{\ell_1}\times\hat{n}_{\ell_2})\cdot\hat{n}_{\ell_3}$ to the three faces ℓ_1,ℓ_2,ℓ_3 calculated at the common vertex of the three faces. Explicitly,

$$\cos(s\frac{\Lambda}{6}a_\ell) = \langle H_\ell\rangle\,, \tag{145}$$

$$\cos\theta_{\ell_1\ell_2} := \hat{n}_{\ell_1}\cdot\hat{n}_{\ell_2} = -\frac{\epsilon_{\ell_1}\epsilon_{\ell_2}\langle H_{\ell_1}H_{\ell_2}\rangle}{\sqrt{1-\langle H_{\ell_1}\rangle^2}\sqrt{1-\langle H_{\ell_2}\rangle^2}}\,,\quad \forall\,\{\ell_1,\ell_2\}\neq\{2,4\}\,, \tag{146}$$

$$(\hat{n}_{\ell_1}\times\hat{n}_{\ell_2})\cdot\hat{n}_{\ell_3} = -\frac{\epsilon_{\ell_1}\epsilon_{\ell_2}\epsilon_{\ell_3}\langle H_{\ell_1}H_{\ell_2}H_{\ell_3}\rangle}{\sqrt{1-\langle H_{\ell_1}\rangle^2}\sqrt{1-\langle H_{\ell_2}\rangle^2}\sqrt{1-\langle H_{\ell_3}\rangle^2}}\,,\quad \{\ell_1,\ell_2,\ell_3\}=\{1,2,3\}\text{ or }\{1,3,4\}\,. \tag{147}$$

On the other hand,

$$\cos\theta_{24} := \hat{n}_2\cdot H_1\hat{n}_4 \equiv \hat{n}_2\cdot H_3^{-1}\hat{n}_4 \tag{148}$$

as well as $(\hat{n}_2\times\hat{n}_1)\cdot H_1\hat{n}_4$ and $(\hat{n}_3\times\hat{n}_2)\cdot H_3^{-1}\hat{n}_4$ can also be calculated using the explicit expressions (132) or (133) for all the holonomies but the expressions are more involved and we omit here.

Define the Gram matrix, denoted as $\mathrm{Gram}(H_\ell)$, of the set of four holonomies $\{H_1H_2,H_3,H_4\}$ as $\mathrm{Gram}(H_\ell) := \mathrm{Gram}(\cos\theta_{\ell_1\ell_2})$ with dihedral angles computed in terms of the holonomies using (139) and (140). With these ingredient, the curved Minkowski theorem is stated as follows.

Theorem V.1 (The curved Minkowski theorem for tetrahedron)**.** *[10] Given four* $\mathrm{PSU}(2)$ *holonomies H_ℓ's satisfying the non-degeneracy condition* $\det\mathrm{Gram}(H_\ell)\neq 0$ *and the closure condition $H_4H_3H_2H_1=\mathbb{I}$, one can uniquely determine a non-degenerate homogeneously curved tetrahedron in the following way*[7]*.*

1. *Label the sub-simplices of the tetrahedron as in fig.9. The tetrahedron is flatly embedded in S^3 if* $\mathrm{sgn}(\det\mathrm{Gram}(H_\ell))>0$ *and flatly embedded in $\mathbb{H}^3$ if* $\mathrm{sgn}(\det\mathrm{Gram}(H_\ell))<0$*;*
2. *The holonomies H_ℓ's are associated to a set of simple paths with either the base point at vertex 4 and special edge (42) or the base point at vertex 3 and special edge (31) and the orientation of the paths determine the orientation of the face surrounded by the path;*
3. *Each holonomy H_ℓ encodes the area $\mathfrak{a}_\ell$ of face ℓ and the outward direction normal $\mathfrak{n}_\ell$ (when parallel transported to the base point) in its parametrization* $H_\ell=\exp\left(s\frac{|\Lambda|}{6}\mathfrak{a}_\ell\mathfrak{n}_\ell\cdot\vec{\tau}\right)$ *with* $s:=\mathrm{sgn}(\det\mathrm{Gram}(H_\ell))$*.*

B. Flat connection on 3-manifold and curved 4-simplex geometry

By the definition of $\mathcal{M}_{\text{flat}}(\Sigma_{0,4},\mathrm{PSU}(2))$ (131) and the curved Minkowski theorem V.1, we conclude that, given four $\mathrm{PSU}(2)$ holonomies $\{H_1,H_2,H_3,H_4\}$ satisfying the closure condition $H_4H_3H_2H_1=1$ which define a flat connection

[7] The original theorem in [10] is written in terms of SO(3) holonomies. However, due to the isomorphism $\mathrm{SO}(3)\cong\mathrm{PSU}(2)$, we can write the whole theorem in terms of PSU(2) holonomies, which is more suited for connecting it to $\mathcal{M}_{\text{flat}}(\mathcal{S}_a,\mathrm{PSU}(2))$ and the FG coordinates therein.

in $\mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{PSU}(2))$, one can identify a (non-degenerate convex) tetrahedron. In other words, there is a one-to-one correspondence between the flat connection in $\mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{PSU}(2))$ and the geometry of a tetrahedron. This can be summarized in the following diagram.

$$\begin{array}{ccc} \pi_1(\text{tetra}) & \xrightarrow{X} & \pi_1(\Sigma_{0,4}) \\ \omega_{\text{LC}} \searrow & & \swarrow \omega_{\text{flat}} \\ & \{H_1, H_2, H_3, H_4 \in \text{PSU}(2) | H_4 H_3 H_2 H_1 = \mathbb{I}_{\text{SU}(2)}\}/\text{PSU}(2)\,, & \end{array} \tag{149}$$

where X is an isomorphism, ω_{LC} is the Levi-Civita connection and ω_{flat} is the flat connection, the quotient by the conjugate action of PSU(2).

It is electrifying that such an isomorphism can be generalized to a one-higher dimensional case [33]. To rephrase, (149) relates the fundamental groups of a 3-simplex, *i.e.* a tetrahedron, and of the nodes-complement of its topological boundary S^2 where the nodes are the (3-3=) 0-subcomplexes of dual 2-complex of the boundary of the 3-simplex. Its generalization gives the isomorphism between the fundamental groups of a 4-simplex and of the graph-complement of its topological boundary S^3 where the graph is the (4-3=)1-subcomplex – Γ_5 graph – of the dual 3-complex of the boundary of the 4-simplex.

To write this isomorphism exactly, let us specify the fundamental groups of a 4-simplex and $S^3\backslash\Gamma_5$ separately. The generators of the former are the closed paths based at the same vertex along the 1-skeleton and circling around a triangle. We refer to fig.11 and fix the notations as follows. We use numbers $\bar{1}, \cdots, \bar{5}$ with bars to denote the vertices of the 4-simplex and $(\bar{a}\bar{b})$ to denote the oriented edge that connects (source) $\bar{b}$ to (target) $\bar{a}$. Then $(\bar{b}\bar{a}) = (\bar{a}\bar{b})^{-1}$. tetra_a denotes the tetrahedron that does not contain the vertex $\bar{a}$. Each pair of tetrahedra tetra_a and tetra_b share a triangle f_{ab} (or f_{ba}), which is the one does not contain vertices $\bar{a}$ and $\bar{b}$.

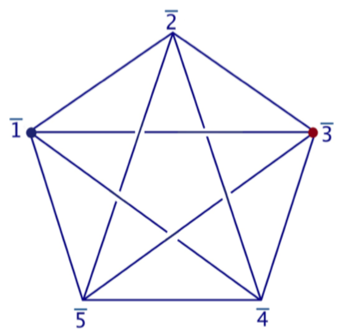


FIG. 11: A 4-simplex projected on $\mathbb{R}^2$. Numbers $\bar{1}, \cdots, \bar{5}$ denote the vertices. tetra_a denotes the tetrahedron that does not contain the vertex $\bar{a}$. f_{ab} or f_{ba} denotes the triangle shared by tetra_a and tetra_b. The over- and under-crossing specify the correct relative positions of vertices in each tetrahedron.

We choose $\bar{1}$ to be the base point and p_{ab} denotes the oriented closed path based at $\bar{1}$ that circles f_{ab} and whose orientation matches the outgoing normal of f_{ab} in tetra_a. To fix the path for triangles not attached to $\bar{1}$, which is the case for all triangles in tetra_1, we need to additionally specify a "special edge" that connects $\bar{1}$ to a vertex on the boundary of the triangle. Two special edges are needed at the minimum. We choose $(\bar{3}\bar{1})$ to be the special edge for triangles f_{12}, f_{14}, f_{15} and choose $(\bar{5}\bar{1})$ to be the special edge for triangle f_{13}. For instance, $p_{12} = (\bar{1}\bar{3}) \circ (\bar{3}\bar{5}) \circ (\bar{5}\bar{4}) \circ (\bar{4}\bar{3}) \circ (\bar{3}\bar{1})$. $p_{ba} = p_{ab}^{-1}$ holds for all $(\bar{a}\bar{b}) \neq (\bar{1}\bar{3})$ or $(\bar{3}\bar{1})$. Specially,

$$\begin{aligned} p_{13} &:= (\bar{1}\bar{3}) \circ (\bar{3}\bar{5}) \circ (\bar{5}\bar{2}) \circ (\bar{2}\bar{4}) \circ (\bar{4}\bar{5}) \circ (\bar{5}\bar{3}) \circ (\bar{3}\bar{1}) \\ p_{31} &:= (\bar{1}\bar{5}) \circ (\bar{5}\bar{4}) \circ (\bar{4}\bar{2}) \circ (\bar{2}\bar{5}) \circ (\bar{5}\bar{1})\,. \end{aligned} \tag{150}$$

Therefore, p_{13} and p_{31} are related by

$$p_{13} = p_{24} \circ p_{31}^{-1} \circ p_{24}^{-1}\,. \tag{151}$$

The generators of the fundamental group $\pi_1(\text{sk}_1(\text{4-simplex}))$ of the 1-skeleton of a 4-simplex are then given by the following 5 relations.

$$\text{tetra}_1: \quad p_{13} \circ p_{12} \circ p_{15} \circ p_{14} = 1\,, \tag{152a}$$

$$\text{tetra}_2: \quad p_{12}^{-1} \circ p_{24} \circ p_{23} \circ p_{25} = 1\,, \tag{152b}$$
$$\text{tetra}_3: \quad p_{31} \circ p_{34} \circ p_{35} \circ p_{23}^{-1} = 1\,, \tag{152c}$$
$$\text{tetra}_4: \quad p_{14}^{-1} \circ p_{45} \circ p_{34}^{-1} \circ p_{24}^{-1} = 1\,, \tag{152d}$$
$$\text{tetra}_5: \quad p_{15}^{-1} \circ p_{12}^{-1} \circ p_{35}^{-1} \circ p_{45}^{-1} = 1\,. \tag{152e}$$

That is, $\pi_1(\text{sk}_1(\text{4-simplex})) = \{\{p_{ab}\}_{a\neq b}|\text{Eqns.}(151) - (152)\}$. On the other hand, the fundamental group of $S^3\backslash\Gamma_5$ can be computed by a generalized Wirtinger representation [34]. It is done in the following steps. Firstly, project Γ_5 onto a plane as in fig.1. Denote the nodes of Γ_5 by numbers $1,\cdots,5$ (with no bars) and the oriented link connecting the target node a and source node b by e_{ab}. There is one crossing that breaks link e_{13} into two links, denoted as $e_{13}^{(1)}$ for the one attached to vertex 1 and $e_{13}^{(3)}$ for the one attached to vertex 3, so there are totally 11 links under this projection, each is associated with a fundamental group generator of $S^3\backslash\Gamma_5$. Choose a base point $\mathfrak{b}$ in $S^3\backslash\Gamma_5$. The generator associated to e_{ab} is given by a non-contractible closed loop $\mathfrak{l}_{ab}$ based at $\mathfrak{b}$ circling e_{ab} whose orientation matches that of e_{ab}. Specifically, the generators associated to $e_{13}^{(1)}$ and $e_{13}^{(3)}$ respectively are denoted as $\mathfrak{l}_{13}^{(1)}$ and $\mathfrak{l}_{13}^{(3)}$ respectively. We associate an orientation to each $\mathfrak{l}_{ab}$ such that it matches the orientation of e_{ab}. Then $\mathfrak{l}_{ba} = \mathfrak{l}_{ab}^{-1}$ for $(a,b)\neq(1,3)$ or $(3,1)$. The 11 generators are subject to the following relations, one for each node or crossing.

$$\text{node } 1: \quad \mathfrak{l}_{13}^{(1)} \circ \mathfrak{l}_{12} \circ \mathfrak{l}_{15} \circ \mathfrak{l}_{14} = 1\,, \tag{153a}$$
$$\text{node } 2: \quad \mathfrak{l}_{12}^{-1} \circ \mathfrak{l}_{24} \circ \mathfrak{l}_{23} \circ \mathfrak{l}_{25} = 1\,, \tag{153b}$$
$$\text{node } 3: \quad \mathfrak{l}_{13}^{(3)\,-1} \circ \mathfrak{l}_{34} \circ \mathfrak{l}_{35} \circ \mathfrak{l}_{23}^{-1} = 1\,, \tag{153c}$$
$$\text{node } 4: \quad \mathfrak{l}_{14}^{-1} \circ \mathfrak{l}_{45} \circ \mathfrak{l}_{34}^{-1} \circ \mathfrak{l}_{24}^{-1} = 1\,, \tag{153d}$$
$$\text{node } 5: \quad \mathfrak{l}_{15}^{-1} \circ \mathfrak{l}_{12}^{-1} \circ \mathfrak{l}_{35}^{-1} \circ \mathfrak{l}_{45}^{-1} = 1\,, \tag{153e}$$
$$\text{crossing}: \quad \mathfrak{l}_{13}^{(1)} = \mathfrak{l}_{24} \circ \mathfrak{l}_{13}^{(3)} \circ \mathfrak{l}_{24}^{-1}\,. \tag{153f}$$

Therefore, $\pi_1(S^3\backslash\Gamma_5) = \{\{\mathfrak{l}_{ab}\}_{a\neq b}|\text{Eqn.}(153)\}$.

Already from the definitions, one can immediately notice an isomorphism $Y: \pi_1(\text{sk}_1(\text{4-simplex})) \to \pi_1(S^3\backslash\Gamma_5)$ that maps $Y(p_{ab}) = \mathfrak{l}_{ab}$ for $(\bar{a}\bar{b}) \neq (\bar{1}\bar{3})$ or $(\bar{3}\bar{1})$ and $Y(p_{13}) = \mathfrak{l}_{13}^{(1)}$, $Y(p_{31}) = \mathfrak{l}_{13}^{(3)\,-1}$.

We are interested in Lorentzian 4-simplex geometry so we represent the fundamental group in $\text{PSL}(2,\mathbb{C}) \cong \text{SO}^+(3,1)$. This corresponds to the case when all the tetrahedra in the 4-simplex are future-pointing [25]. Given a representation $\rho = \text{Hom}(\pi_1(S^3\backslash\Gamma_5), \text{PSL}(2,\mathbb{C}))$ such that $\rho(\mathfrak{l}_{ab}) = \widetilde{H}_{ab}$ and that $\rho(\mathfrak{l}_{ab}^{-1}) = \widetilde{H}_{ab}^{-1}$, (153) gives 5 closure conditions on the holonomies and a conjugate relation to $\widetilde{H}_{13}^{(1)}$ and $\widetilde{H}_{13}^{(3)}$:

$$\widetilde{H}_{13}^{(1)}\widetilde{H}_{12}\widetilde{H}_{15}\widetilde{H}_{14} = 1\,, \tag{154a}$$
$$\widetilde{H}_{12}^{-1}\widetilde{H}_{24}\widetilde{H}_{23}\widetilde{H}_{25} = 1\,, \tag{154b}$$
$$\widetilde{H}_{13}^{(3)\,-1}\widetilde{H}_{34}\widetilde{H}_{35}\widetilde{H}_{23}^{-1} = 1\,, \tag{154c}$$
$$\widetilde{H}_{14}^{-1}\widetilde{H}_{45}\widetilde{H}_{34}^{-1}\widetilde{H}_{24}^{-1} = 1\,, \tag{154d}$$
$$\widetilde{H}_{15}^{-1}\widetilde{H}_{25}^{-1}\widetilde{H}_{35}^{-1}\widetilde{H}_{45}^{-1} = 1\,, \tag{154e}$$
$$\widetilde{H}_{13}^{(1)} = \widetilde{H}_{24}\widetilde{H}_{13}^{(3)}\widetilde{H}_{24}^{-1}\,. \tag{154f}$$

$\widetilde{H}_{ba} = \widetilde{H}_{ab}^{-1}$ for $(a,b) \neq (1,3)$ or $(3,1)$. Representing $\pi_1(\text{4-simplex})$ also in $\text{PSL}(2,\mathbb{C})$ by $\rho' = \text{Hom}(\pi_1(\text{4-simplex}), \text{PSL}(2,\mathbb{C}))$ and identifying $\rho'(p_{ab}) = \rho(\mathfrak{l}_{ab})$ for all $(a,b)\neq(1,3)$ or $(1,3)$ while $\rho'(p_{13}) = \rho(\mathfrak{l}_{13}^{(1)})$ and $\rho'(p_{31}) = \rho(\mathfrak{l}_{13}^{(3)\,-1})$, (154a)–(154e) are nothing but the 5 copies of closure conditions as in (136) but now represented in $\text{PSL}(2,\mathbb{C})$, each corresponds to a tetrahedron on the boundary of the 4-simplex and (154f) relates the holonomy $H_{13}^{(1)}$ around f_{13} as the boundary of tetra_1 and the holonomy $H_{13}^{(3)}$ around the same triangle as the boundary of tetra_3. ρ and ρ' effectively associate flat connection ω_{flat} to $S^3\backslash\Gamma_5$ and Levi-Civita connection ω_{LC} to the 4-simplex. We then have a similar commuting map as (149) but in one higher dimension represented in $\text{PSL}(2,\mathbb{C})$.

$$\begin{array}{ccc} \pi_1(\text{sk}_1(\text{4-simplex})) & \xrightarrow{\;\;Y\;\;} & \pi_1(S^3\backslash\Gamma_5) \\ \omega_{\text{LC}} \searrow & & \swarrow \omega_{\text{flat}} \\ & \{\{\widetilde{H}_{ab}\} \in \text{PSL}(2,\mathbb{C})|\text{Eqn.}(154)\}/\text{PSL}(2,\mathbb{C})\,, & \end{array} \tag{155}$$

where the quotient by the conjugate action of $\mathrm{PSL}(2,\mathbb{C})$.

However, ω_{LC} as a representation of $\pi_1(\mathrm{sk}_1(\text{4-simplex}))$ does not contain enough information about the geometry on the 4-simplex unless there is additional input. We ask that the 4-simplex be embedded in the constant curvature spacetime so that all the triangles are flatly embedded surfaces. The geometry of such a 4-simplex is uniquely determined by 10 areas. Indeed, given 5 tetrahedra of the same constant curvature, their faces can be glued pairwise in an organized pattern (so that no handles are formed) to form a 4-simplex if each of the 10 pairs of faces shares the same area.

Recall the proposed discrete simplicity constraint (129). It can be implemented as an extra structure on (155) by gauge fixing $\{\widetilde{H}_{ab}\}_b$ that satisfy each of the closure conditions (154) to a common $\mathrm{PSU}(2)$ subgroup of $\mathrm{PSL}(2,\mathbb{C})$. More precisely, let

$$\begin{aligned}&\widetilde{H}_{ab}=g_aH_{ab}g_a^{-1}=g_bH_{ba}^{-1}g_b^{-1}\,,\quad (a,b)\neq(1,3),(3,1)\,,\\&\widetilde{H}_{13}^{(1)}=g_1H_{13}g_1^{-1}\,,\quad \widetilde{H}_{13}^{(3)\,-1}=g_3H_{31}g_3^{-1}\,.\end{aligned}\tag{156}$$

where $g_a,g_b\in\mathrm{PSL}(2,\mathbb{C})$ and $H_{ab},H_{ba}\in\mathrm{PSU}(2)$. g_a can be geometrically interpreted as parallel transport the base point $\mathfrak{b}$ in $S^3\backslash\Gamma_5$ to the base point $\mathfrak{b}_a$ on the 4-holed sphere $\mathcal{S}_a\subset\partial(S^3\backslash\Gamma_5)$. For different a, the gauge fixing group element g_a can be chosen differently. That is, the $\mathrm{PSU}(2)$ closure condition can be written in different $\mathrm{PSU}(2)$ subgroups of $\mathrm{PSL}(2,\mathbb{C})$. In this way, the gauge fixed version of (154a) – (154e)

$$H_{13}H_{12}H_{15}H_{14}=1\,,\tag{157a}$$
$$H_{21}H_{24}H_{23}H_{25}=1\,,\tag{157b}$$
$$H_{31}H_{34}H_{35}H_{32}=1\,,\tag{157c}$$
$$H_{41}H_{45}H_{43}H_{42}=1\,,\tag{157d}$$
$$H_{51}H_{52}H_{53}H_{54}=1\tag{157e}$$

describe 5 tetrahedra, each corresponding to $\mathrm{PSU}(2)$ flat connection on a 4-holed sphere.

These $\mathrm{PSU}(2)$ holonomies be subject to the constraints

$$H_{ab}=G_{ab}H_{ba}^{-1}G_{ba}\,,\quad G_{ba}=G_{ab}^{-1}\in\mathrm{PSL}(2,\mathbb{C})\,,\quad\forall\,(a,b)\tag{158}$$

where

$$\begin{aligned}&G_{ab}:=g_a^{-1}g_b\,,\quad\forall\,(a,b)\neq(1,3),(3,1)\,,\\&G_{13}:=g_1^{-1}\left(g_2H_{24}g_2^{-1}\right)g_3=G_{31}^{-1}\,.\end{aligned}\tag{159}$$

G_{ab} then represents the parallel transport from $\mathfrak{b}_b$ to $\mathfrak{b}_a$ along a path passing through the common base point $\mathfrak{b}$ in $\partial(S^3\backslash\Gamma_5)$. In other words, it changes the local frame from tetra_b to tetra_a and thus we call it a *frame-changing holonomy*. The second line of (159) together with (158) is the constrained version of (154f). Indeed, (158) implies that $\mathrm{Tr}(H_{ab})=\mathrm{Tr}(H_{ba})$, which geometrically means the two triangles these two holonomies surround have the same area if using the parametrization (134), which is exactly what we asked. Therefore, (156) together with (155) describe the geometry of a 4-simplex.

C. Impose the simplicity constraints

With the geometrical discussion above, we *define* the simplicity constraints of the Chern-Simons theory on $S^3\backslash\Gamma_5$ as *restricting the moduli spaces of* $\mathrm{PSL}(2,\mathbb{C})$ *connections on 4-holed spheres to the ones that can be gauge-transformed to* $\mathrm{PSU}(2)$ *flat connections*. This restriction should be imposed on the coordinates $(\vec{\mathcal{Q}},\vec{\mathcal{P}})$ (94) which, in turn, impose constraints on the partition function. We borrow the idea in the EPRL model that the first-class constraints are imposed strongly while the second-class constraints are imposed weakly.

1. The first-class simplicity constraints

We have introduced in Section IV 3 that the 6 FG coordinates $\{\chi_{ij}^{(a)}\}_{i\neq j}$ (see Table II) are the coordinates of $\mathcal{M}_{\mathrm{flat}}(\mathcal{S}_a,\mathrm{PSL}(2,\mathbb{C}))$. Then one can impose restrictions on these coordinates to implement the simplicity constraints. Recall that linear combinations of these FG coordinates, giving rise to the FN coordinates $\{2L_{ab}\}_{b\neq a}$ (91) commute with all the $\{\chi_{ij}^{(a)}\}_{i\neq j}$. The first-class constraints are then given by the function of $2L_{ab}$'s. Each $2L_{ab}$ is the logarithm of the eigenvalue $\lambda_{\mathfrak{p}}^2 \equiv \lambda_{ab}^2 = e^{2L_{ab}}$ of the holonomy H_{ab} around the hole $\mathfrak{p}$ of $\mathcal{S}_a$. $H_{\mathfrak{p}}(\tau_a) \in \mathrm{PSU}(2)$ implies that $\lambda_{\mathfrak{p}}^2 = e^{i2\theta_{ab}}$ with some $\theta_{ab} \in \mathbb{R}$. From the geometrical interpretation in the previous subsection,

$$\theta_{ab} = -\frac{|\Lambda|}{6}\mathfrak{a}_{\mathfrak{p}} \quad \text{or} \quad \theta_{ab} = \frac{|\Lambda|}{6}\mathfrak{a}_{\mathfrak{p}} - \pi\,, \tag{160}$$

where the choice of the expression from the two is selected using the convexity criterion (143) after constructing the PSU(2) holonomies. Therefore, the first-class simplicity constraints can be formulated as

$$2L_{ab} := \frac{2\pi i}{k}\left(-ib\mu_{ab} - m_{ab}\right) \in i\mathbb{R} \quad \Longleftrightarrow \quad \mu_{ab} = 0 \quad \xrightarrow{\text{quantization}} \quad \mathrm{Re}(\boldsymbol{\mu}_{ab})\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}|\vec{m}) = 0\,, \tag{161}$$

where the right-most quantum constraint is written in terms of $\mathrm{Re}(\boldsymbol{\mu}_{ab})$ as the analytic continuation of μ_{ab} to $\mathbb{C}$ is allowed at the quantum level. We allow $\mathrm{Im}(\mu_{ab}) \equiv \alpha_{ab} \neq 0$ as only the real part is quantized. Then $e^{2L_{ab}} \in \mathrm{U}(1)$ is realized only at the classical ($k \to \infty$) level. Define the "spin" j_{ab} such that

$$2j_{ab} = m_{ab} \quad \to \quad j_{ab} = 0, \frac{1}{2}, \cdots, \frac{k-1}{2}\,. \tag{162}$$

j_{ab} encodes the area $\mathfrak{a}_{\mathfrak{p}}$ of the triangle $f_{\mathfrak{p}}$ in a tetrahedron (when we fix the orientation of $f_{\mathfrak{p}}$) by

$$\frac{|\Lambda|}{3}\mathfrak{a}_{\mathfrak{p}} = \frac{4\pi}{k}j_{ab} \quad \text{or} \quad 2\pi - \frac{|\Lambda|}{3}\mathfrak{a}_{\mathfrak{p}} = \frac{4\pi}{k}j_{ab}\,. \tag{163}$$

Whether to choose the first or the second expression depends on the outgoing normals of the faces, which are encoded in FG coordinates on $\mathcal{S}_a$ as we will see later. We label the partition function for $S^3\backslash\Gamma_5$ satisfying the constraint (161) as

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\{i\alpha_{ab}\}_{(ab)}, \{\mu_a\} \mid \{j_{ab}\}_{(ab)}, \{m_a\})\,. \tag{164}$$

Effectively, the first-class simplicity constraints can be seen to be imposed on the FN coordinates on the annulus cusps on the triangulation of $\partial(S^3\backslash\Gamma_5)$. The remaining (second-class) simplicity constraints will be imposed on each $\mathcal{S}_a$.

2. *The second-class simplicity constraints and the Chern-Simons coherent states*

The moduli space $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSL}(2,\mathbb{C}))$ is not a symplectic manifold but a Poisson manifold, due to the presence of Poisson commutative $\{\lambda_{\mathfrak{p}}^2\}_{\mathfrak{p}=1}^4$. Fixing $\{\lambda_{\mathfrak{p}}^2\}_{\mathfrak{p}=1}^4$ by (163) reduces the moduli space $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \mathrm{PSL}(2,\mathbb{C}))$ to a two-complex-dimensional symplectic space $\mathcal{M}_{\vec{\lambda}}$ with coordinates $(\mathcal{X}_a, \mathcal{Y}_a)$, on which we should impose the second-class simplicity constraints.

The implementation of (161) results in the factorization of H_{ab} as follows.

$$H_{ab} = M(\xi_{ab})\mathrm{diag}(\lambda_{ab}, \lambda_{ab}^{-1})M(\xi_{ab})^{-1}\,,\ \lambda_{ab} = e^{-2\pi i\frac{j_{ab}}{k}}\,, \quad M(\xi_{ab}) \in \mathrm{SU}(2)\,, \tag{165}$$

where $j_{ab} = 0, \frac{1}{2}, \cdots, \frac{k-1}{2}$ and $M(\xi_{ab})$ is defined in terms of a spinor $|\xi_{ab}\rangle = (\xi_{ab}^0, \xi_{ab}^1)^\top \in \mathbb{C}^2$ and its dual spinor $|\xi_{ab}] = (-\bar{\xi}_{ab}^1, \bar{\xi}_{ab}^0)^\top$ assigned on the hole of $\mathcal{S}_a$ that connects to $\mathcal{S}_b$ (sometimes it is more convenient to use the notation $\xi_{ab} = |\xi_{ab}\rangle$ and $J\xi_{ab} = |\xi_{ab}]$). $|\xi_{ab}]$ is dual to $|\xi_{ab}\rangle$ in the sense that $[\xi_{ab}|\xi_{ab}\rangle = \langle\xi_{ab}|\xi_{ab}] = 0$ (by definition). They further satisfy the normalization property $\langle\xi_{ab}|\xi_{ab}\rangle := \bar{\xi}_{ab}^0\xi_{ab}^0 + \bar{\xi}_{ab}^1\xi_{ab}^1 = 1 = [\xi_{ab}|\xi_{ab}]$ which guarantees that $M(\xi_{ab}) \in \mathrm{SU}(2)$ by the following definition.

$$M(\xi_{ab}) := \left(\ |\xi_{ab}\rangle\,,\ |\xi_{ab}]\ \right) = \begin{pmatrix} \xi_{ab}^0 & -\bar{\xi}_{ab}^1 \\ \xi_{ab}^1 & \bar{\xi}_{ab}^0 \end{pmatrix}\,. \tag{166}$$

More precisely, $|\xi_{ab}\rangle$ is the normalized eigenvector of H_{ab} at $\mathfrak{b}_a$ so it can be treated as a framing flag of the hole of $\mathcal{S}_a$ connected to $\mathcal{S}_b$ parallel transported to $\mathfrak{b}_a$. Recalling the isomorphism (149) between the moduli space of flat connection on a 4-holed sphere and the geometry of a tetrahedron, the geometry of tetra$_a$ is encoded in $\{H_{ab}\}_{b\neq a}$. More precisely, in the decomposition (165), λ_{ab} encodes the area $\mathfrak{a}_{ab} = \mathfrak{a}_{ba}$ of f_{ab} by (163) and $|\xi_{ab}\rangle$ encodes the 3D normal vector to f_{ab} in the local frame of tetra$_a$ by

$$\hat{n}_{ab} = \langle\xi_{ab}|\vec{\sigma}|\xi_{ab}\rangle \quad \text{or} \quad \hat{n}_{ab} = -\langle\xi_{ab}|\vec{\sigma}|\xi_{ab}\rangle \tag{167}$$

where $\vec{\sigma}=(\sigma^1,\sigma^2,\sigma^3)$ is a vector of Pauli matrices. The outward-pointing normal $\hat{\mathfrak{n}}_{ab}$ to f_{ab} is different from $\hat{n}_{ab}$ by a sign factor $s=\mathrm{sgn}(\Lambda)$, namely

$$\hat{\mathfrak{n}}_{ab}=s\hat{n}_{ab}\,. \tag{168}$$

This is because the normalized eigenvector $|\xi_{ab}\rangle$ is the same for holonomies around a spherical triangle (corresponding to $s=+$) with eigenvalue, say λ, and a hyperbolic one (corresponding to $s=-$) with eigenvalue λ^{-1} (recall $H_\ell=\exp\left(s\frac{|\Lambda|}{6}\mathfrak{a}_\ell\mathfrak{n}_\ell\cdot\vec{\tau}\right)$). For each of all four triangles in a tetrahedron, either the area is related to j_{ab} in the first or the second option in (163) is determined by the triple product $(\hat{n}_i\times\hat{n}_j)\cdot\hat{n}_k\overset{!}{=}s$ for any set of three triangles in a tetrahedron. On the other hand, $(\hat{\mathfrak{n}}_i\times\hat{\mathfrak{n}}_j)\cdot\hat{\mathfrak{n}}_k>0$ for either s.

A similar factorization for H_{ba} gives

$$H_{ba}=M(\xi_{ba})\mathrm{diag}(\lambda_{ba},\lambda_{ba}^{-1})M(\xi_{ba})^{-1}\,,\ \lambda_{ba}=e^{2\pi i\frac{j_{ba}}{k}}\,, \tag{169}$$

where $j_{ba}=\frac{k}{2}-j_{ab}$ (hence $\lambda_{ba}^2=\lambda_{ab}^{-2}$) and $M(\xi_{ba})$ is defined in the same way as $M(\xi_{ab})$ but with spinors $|\xi_{ba}\rangle$ and its dual $|\xi_{ba}\rangle$ on as eigenvector of H_{ba} at $\mathfrak{b}_b$ on $\mathcal{S}_b$. Importantly, the 3D normal vector to f_{ab} in the local frame of tetra$_b$ defined as

$$\hat{n}_{ba}=\langle\xi_{ba}|\vec{\sigma}|\xi_{ba}\rangle\quad\text{or}\quad\hat{n}_{ba}=-\langle\xi_{ba}|\vec{\sigma}|\xi_{ba}\rangle \tag{170}$$

is different from $\hat{n}_{ab}$ in general as the two spinors are different. Indeed, $\hat{n}_{ab}$ and $\hat{n}_{ab}$ are related by the dihedral angle, denoted as Θ_{ab} of tetra$_a$ and tetra$_b$ hinged by f_{ab}. Θ_{ab} is encoded in the frame-changing holonomy G_{ab} and the pair of spinors $(|\xi_{ab}\rangle,|\xi_{ba}\rangle)$ (or $(|\xi_{ab}],|\xi_{ba}])$):

$$G_{ab}=M(\xi_{ab})\begin{pmatrix}\gamma_{ab}&0\\0&\gamma_{ab}^{-1}\end{pmatrix}M(\xi_{ba})^{-1}\,,\ \gamma_{ab}=e^{-s\,\mathrm{sgn}(V_4)\frac{\Theta_{ab}}{2}+i\theta_{ab}}\,, \tag{171}$$

where $\Theta_{ab},\theta_{ab}\in\mathbb{R}$ [9]. The calculation calculating the amplitude of $\gamma_{ab}=e^{-s\,\mathrm{sgn}(V_4)\frac{\Theta_{ab}}{2}}$ is rather lengthy and we omit here. See [9], or Appendix B of [13]. Therefore, given the 4-simplex geometry, including the areas and normals of all triangles in different tetrahedron frames and the dihedral angles hinged by the triangles, one can reconstruct all the G_{ab}'s up to some phases $\{\theta_{ab}\}_{a\neq b}$ determined by the boundary condition (as all edges of a 4-simplex are on the boundary). Further, flat connection holonomies $\{\widetilde{H}_{ab}\}$ on $S^3\backslash\Gamma_5$ can be determined by $\{G_{ab}\}$ through (156) up to a $\mathrm{PSL}(2,\mathbb{C})$ gauge as G_{ab} is invariant under the gauge transformation from the left $g_a\to hg_a$, $\forall h\in\mathrm{PSL}(2,\mathbb{C})$ (*r.f.* (159)). Such a gauge transformation corresponds to changing the common base point for defining $\{\widetilde{H}_{ab}\}$.

As the spinor ξ_{ab} is the eigenvector of $\mathrm{PSU}(2)$ holonomy H_{ab}, it can be treated as a normalized framing flag $\xi_{ab}=\frac{s_i}{||s_i||}$ of, say, hole i parallel transported to the base point $\mathfrak{p}_a$, which is a coordinate in $\mathcal{M}_{\text{flat}}(\mathcal{S}_a,\mathrm{PSU}(2))$. Recall the definition (39) of an FG coordinate in terms of framing flag, the constrained coordinates, denoted as $\hat{x}_a\equiv e^{\widehat{\mathcal{X}}_a}$ and $\hat{y}_a\equiv e^{\widehat{\mathcal{Y}}_a}$, referring to the labelling of holes in fig.12, can be defined as

$$\hat{x}_a=\frac{[\xi_1|\xi_2\rangle[\xi_3|\xi_4\rangle}{[\xi_1|\xi_3\rangle[\xi_2|\xi_4\rangle}\,,\quad\hat{y}_a^{-1}=\frac{[\xi_4|H_2\xi_2\rangle[\xi_1|\xi_3\rangle}{[\xi_4|H_2\xi_1\rangle[\xi_2|\xi_3\rangle}\,, \tag{172}$$

where H_2 is the $\mathrm{PSU}(2)$ holonomy around hole 2 given by

$$H_2=M(\xi_2)\mathrm{diag}(\lambda_2,\lambda_2^{-1})M(\xi_2)^{-1}\,,\quad\lambda_2=\exp\left[\frac{\pi i}{k}\left(b\alpha_{ab}-2j_{ab}\right)\right]\,,\ \text{with some }\ b\neq a\,. \tag{173}$$

Parametrize $\left(\widehat{\mathcal{X}}_a,\widehat{\mathcal{Y}}_a\right)$ as

$$\widehat{\mathcal{X}}_a=\frac{2\pi i}{k}\left(-ib\hat{\mu}_a-\hat{m}_a\right)\,,\quad\widehat{\mathcal{Y}}_a=\frac{2\pi i}{k}\left(-ib\hat{\nu}_a-\hat{n}_a\right)\,, \tag{174}$$

where $(\hat{\mu}_a,\hat{\nu}_a)\in\mathbb{R}^2$. $\left(\widehat{\mathcal{X}}_a,\widehat{\mathcal{Y}}_a\right)$ live in the two-real-dimensional *compact* symplectic space $\overline{\mathcal{M}}_{\vec{\lambda}}$ and there is a pair of Darboux coordinate $(\theta_a,\phi_a)\in[0,\pi)^2$ spanning the space $\overline{\mathcal{M}}_{\vec{\lambda}}$ [35]. Then one can define the integral of any function f on $\overline{\mathcal{M}}_{\vec{\lambda}}$ as

$$\int_{\overline{\mathcal{M}}_{\vec{\lambda}}}\mathrm{d}\xi\,f:=\int_{\overline{\mathcal{M}}_{\vec{\lambda}}}\mathrm{d}\theta_a\wedge\phi_a\,f\,. \tag{175}$$

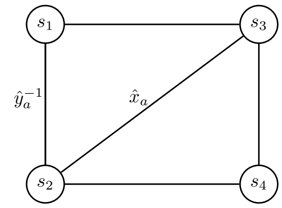


FIG. 12: Constrained FG coordinates on $\mathcal{S}_a$ corresponding to the definitions (172).

Any integration on $\overline{\mathcal{M}}_{\vec{j}}$ is finite as long as the integrand is bounded.

As second-class constraints, we will impose them weakly by using Chern-Simons coherent states, which we define in the following. By definition, coherent states are peaked at the classical phase space points hence the labels of coherent states are given by both the position variables $\{\mathcal{X}_a\}$ and the momentum variables $\{\mathcal{Y}_a\}$. Recall the notations

$$\mathcal{X}_a = \frac{2\pi i}{k}\left(-ib\mu_a - m_a\right)\,, \quad \mathcal{Y}_a = \frac{2\pi i}{k}\left(-ib\nu_a - n_a\right)\,. \tag{176}$$

Chern-Simons coherent states on $\mathcal{S}_a$. After fixing the FN coordinates $\{L_{ab}\}_{(ab)}$ to be given by the spins $\{j_{ab}\}_{(ab)}$, the Hilbert space of each 4-holed sphere $\mathcal{S}_a$ is locally $\mathbb{C}^2$. We also fix $\mathrm{Im}(\mu_a) = \alpha_a$ and consider the degrees of freedom $\mathrm{Re}(\mu_a) \in \mathbb{R}$ and $m_a \in \mathbb{Z}/k\mathbb{Z}$. To simplify the notation, we will denote $\mathrm{Re}(\mu_a)$ by $\mu_a \in \mathbb{R}$ in the rest of this subsection. The Hilbert space for $\mathcal{S}_a$ is

$$\mathcal{H}_{\mathcal{S}_a} = L^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^k\,.$$

Firstly, the coherent state $\psi^0_{z_a}(\mu)$ on $L^2(\mathbb{R})$ is defined as the solution to

$$\frac{1}{\sqrt{2}}\left(\sqrt{\frac{2\pi}{k}}\boldsymbol{\mu} + i\sqrt{\frac{2\pi}{k}}\boldsymbol{\nu}\right)\psi^0_{z_a}(\mu) = \sqrt{\frac{k}{2\pi}}z_a\psi^0_{z_a}(\mu)\,, \tag{177}$$

which solves

$$\psi^0_{z_a}(\mu) = \left(\frac{2}{k}\right)^{1/4} e^{-\frac{\pi}{k}\left(\mu - \frac{k}{\pi\sqrt{2}}\mathrm{Re}(z_a)\right)^2} e^{-i\sqrt{2}\mu\mathrm{Im}(z_a)}\,, \tag{178}$$

with the over-completeness property

$$\frac{k}{2\pi^2}\int_{\mathbb{C}} \mathrm{dRe}(z_a)\mathrm{dIm}(z_a)\,\psi^0_{z_a}(\mu)\bar{\psi}^0_{z_a}(\mu') = \delta_{\mu,\mu'}\,. \tag{179}$$

The coherent state label $z_a \in \mathbb{C}$ parameterizing a complex plane is related to the constrained coordinates (174) by

$$z_a = \frac{\sqrt{2}\pi}{k}(\hat{\mu}_a + i\hat{\nu}_a)\,. \tag{180}$$

For the finiteness of the vertex amplitude given any boundary condition, we add a prefactor to this coherent state and define

$$\psi_{z_a}(\mu) = e^{-\sqrt{2}\beta_a\,\mathrm{Re}(z_a)}\psi^0_{z_a}(\mu)\,, \tag{181}$$

where β_a is the component in $(\vec{\alpha},\vec{\beta}) \in \mathfrak{P}_{S^3\backslash\Gamma_5}$. The prefactor is subdominant at large k so it does not affect the semiclassical behaviour of ψ_{z_a}.

Secondly, the coherent state $\zeta_{(x_a,y_a)}(m)$ on $\mathbb{C}^k$ is labelled by $(x_a, y_a) \in [0,2\pi)\times[0,2\pi)$, which can be viewed as the angle coordinates on a torus $\mathbb{T}^2$. It is defined as [36]

$$\zeta_{(x_a,y_a)}(m) = \left(\frac{2}{k}\right)^{1/4} e^{\frac{ikx_ay_a}{4\pi}} \sum_{p_a\in\mathbb{Z}} e^{-\frac{k}{4\pi}\left(\frac{2\pi m}{k} - 2\pi p_a - x_a\right)^2} e^{\frac{ik}{2\pi}y_a\left(\frac{2\pi m}{k} - 2\pi p_a - x_a\right)}\,. \tag{182}$$

x_a, y_a are related to the constrained coordinates (174) by

$$x_a = \mathrm{mod}(\frac{2\pi}{k}\hat{m}_a, 2\pi)\,, \quad y_a = \mathrm{mod}(\frac{2\pi}{k}\hat{n}_a, 2\pi)\,. \tag{183}$$

The over-completeness property of $\zeta_{(x_a,y_a)}(m)$ reads

$$\frac{k}{4\pi^2}\int_{\mathbb{T}^2} \mathrm{d}x_a \mathrm{d}y_a\, \zeta_{(x_a,y_a)}(m)\bar{\zeta}_{(x_a,y_a)}(m') = \delta_{e^{\frac{2\pi i}{k}(m-m')},1}\,. \tag{184}$$

The coherent state in $\mathcal{H}_{\mathcal{S}_a}$ is the tensor product of these two coherent states

$$\Psi_{\rho_a}(\mu|m) := \psi_{z_a} \otimes \zeta_{(x_a,y_a)} \in \mathcal{H}_{\mathcal{S}_a}\,, \quad \rho_a = (z_a, x_a, y_a)\,. \tag{185}$$

It is easy to confirm that the expectation values of the operators $\boldsymbol{\mu}, \boldsymbol{\nu}, \mathbf{m}, \mathbf{n}$ calculated by the coherent state $\Psi_{\rho_a}(\mu|m)$ are given by the coherent state labels, or the classical phase space coordinate at large k limit, *i.e.*

$$\langle \boldsymbol{\mu} \rangle \xrightarrow{k\to\infty} \hat{\mu}_a\,, \quad \langle \boldsymbol{\nu} \rangle \xrightarrow{k\to\infty} \hat{\nu}_a\,, \quad \langle \exp(\frac{2\pi i}{k}\mathbf{m})\rangle \xrightarrow{k\to\infty} \exp(\frac{2\pi i}{k}\hat{m}_a)\,, \quad \langle \exp(\frac{2\pi i}{k}\mathbf{n})\rangle = \exp(\frac{2\pi i}{k}\hat{n}_a)\,. \tag{186}$$

It is only valid at the large k limit since the torus part of the coherent state $\zeta_{(x,y)}(m)$ is normalized only at this limit.

With the second-class simplicity constraints imposed on the coherent state labels, one can define the vertex amplitude by the inner product of partition function (164) and five coherent states (185), each associated to one $\mathcal{S}_a$. That is

$$\mathcal{A}_v(\iota) := \langle \prod_{a=1}^5 \bar{\Psi}_{\rho_a} | \mathcal{Z}_{S^3\backslash\Gamma_5}\rangle = \sum_{\{m_a\}\in(\mathbb{Z}/k\mathbb{Z})^5} \int_{\mathbb{R}^5} \mathrm{d}^5\mu_a\, \mathcal{Z}_{S^3\backslash\Gamma_5}(\{i\alpha_{ab}\}_{a<b}, \{\mu_a + i\alpha_a\} \mid \{j_{ab}\}_{a<b}, \{m_a\}) \prod_{a=1}^5 \Psi_{\rho_a}(\mu_a|m_a)\,, \tag{187}$$

where $\iota = (\{\alpha_{ab}, j_{ab}\}_{a<b}, \{\rho_a\}_{a=1}^5, \{\alpha_a, \beta_a\}_{a=1}^5)$. It will also be convenient to denote the measure (175) in terms of the coherent sate label:

$$\int_{\mathcal{M}_{\vec{\chi}}} \mathrm{d}\rho\, f \equiv \int_{\mathcal{M}_{\vec{\chi}}} \mathrm{d}\xi\, f\,. \tag{188}$$

One of the most important features of the vertex amplitude $\mathcal{A}_v(\iota)$ is boundedness for any $\{\hat{\rho}_a\}_{a=1}^5$, which we now explain. Firstly, the partition function $\mathcal{Z}_{S^3\backslash\Gamma_5} \in \mathcal{F}^{(k)}_{\mathfrak{P}_{S^3\backslash\Gamma_5}}$ so, hence

$$\prod_{a=1}^5 e^{-\frac{2\pi}{k}\beta_a\mu_a}\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}+i\vec{\alpha}|\vec{m}) \in \mathcal{S}(\mathbb{R}^5) \quad \Longrightarrow \quad \left|\prod_{a=1}^5 e^{-\frac{2\pi}{k}\beta_a\mu_a}\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu}+i\vec{\alpha}|\vec{m})\right| \leq C_1 \quad \text{with some } 0<C_1<\infty\,. \tag{189}$$

Secondly, $\zeta_{(x_a,y_a)}$ is bounded

$$\left|\sum_{m_a}\xi_{(x_a,y_a)}(m_a)\right| \leq \sum_{m_a}\left|\zeta_{(x_a,y_a)}(m_a)\right| \leq C_2 \quad \text{with some } 0<C_2<\infty\,. \tag{190}$$

Lastly, we need to evaluate the integration over the bounded function $\psi_{z_a}(\mu_a)$, which is a Gaussian integral:

$$\begin{aligned}\left|\int_{\mathbb{R}}\mathrm{d}\mu_a\, e^{\frac{2\pi}{k}\beta_a\mu_a}\psi_{z_a}(\mu_a)\right| \leq \int_{\mathbb{R}}\mathrm{d}\mu_a\left|e^{\frac{2\pi}{k}\beta_a\mu_a}\psi_{z_a}(\mu_a)\right| &= \left(\frac{2}{k}\right)^{\frac{1}{4}}\int_{\mathbb{R}}\mathrm{d}\mu_a\, e^{-\frac{\pi}{k}\left(\mu_a - \frac{k}{\pi\sqrt{2}}\mathrm{Re}(z_a)\right)^2 + \frac{2\pi}{k}\beta_a\left(\mu_a - \frac{k}{\pi\sqrt{2}}\mathrm{Re}(z_a)\right)} \\ &= (2k)^{\frac{1}{4}}e^{\frac{\pi}{k}\beta_a^2}\,.\end{aligned} \tag{191}$$

Combing the results (189) – (191), we conclude that the vertex amplitude $\mathcal{A}_v(\iota)$ is finite. From the expression (191), we have also seen the reason for the introduction of the prefactor $e^{-\sqrt{2}\beta_a\mathrm{Re}(z_a)}$ in defining ψ_{z_a} (181). If this term is missing, we can only conclude that $\mathcal{A}_v(\iota)$ is finite given finite $\{\mathrm{Re}(z_a)\}$.

VI. SEMICLASSICAL ANALYSIS OF THE VERTEX AMPLITUDE

To see that $\mathcal{A}_v(\iota)$ is a good definition of a spinfoam amplitude for a 4-simplex, one should extract the geometry encoded in the vertex amplitude. This can be done by looking at the semiclassical limit of $\mathcal{A}_v(\iota)$. The semiclassical limit here refers to taking $k \to \infty, j_{ab} \to \infty$ while keeping their ratio j_{ab}/k fixed. Recalling that $k = \frac{12\pi}{\ell_{\rm p}^2 \gamma |\Lambda|}$ and that $\mathfrak{a} = \frac{12\pi}{|\Lambda|}\frac{j_{ab}}{k}$ (or the other option of (163)), this means $\ell_{\rm p} \to 0$ while the geometrical quantities *e.g.* $\mathfrak{a}, \mathfrak{n}, \Lambda$ as well as the Barbero-Immirzi parameter γ are kept fixed. It turns out that the Regge action with a cosmological constant term is reproduced at the semiclassical limit of $\mathcal{A}_v$. We only summarize the idea of how to get it and refer to [9, 11] for details.

The large-k asymptotics of the vertex amplitude can be analyzed by the stationary phase approximation. That is to express $\mathcal{A}_v(\iota)$ into an integral of an exponentiated action:

$$\mathcal{A}_v(\iota) \xrightarrow{k\to\infty} \int \mathrm{d}\mu(X)\, e^{kS(X)}\,, \tag{192}$$

where X is a set of k-invariant variables, $\mu(X)$ is its measure and $S(X)$ is the action. Let $X = X_0^\alpha$ be the α-th stationary point of $S(X)$ such that

$$\left.\frac{\partial S}{\partial X}\right|_{X_0^\alpha} = 0\,, \tag{193}$$

then the integral (192) can be approximated as

$$\int \mathrm{d}\mu(X)\, e^{kS(X)} \sim \sum_\alpha \frac{1}{\sqrt{\det(-H_\alpha/2\pi)}} e^{kS(X_0^\alpha)}\,, \tag{194}$$

where $H_\alpha = \left.\frac{\partial^2(kS)}{\partial X^2}\right|_{X_0^\alpha}$ is the Hessian matrix evaluated at the α-th critical point.

Under this scheme, we first express the integration and summation variables $\vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}$ back to $\vec{\mathcal{Q}}, \vec{\widetilde{\mathcal{Q}}}, \vec{\mathcal{P}}, \vec{\widetilde{\mathcal{P}}}$ as they are j_{ab}/k-dependent hence are scaleless:

$$\mu_I = \frac{kb}{2\pi(b^2+1)}\left(\mathfrak{Q}_I + \widetilde{\mathfrak{Q}}_I\right)\,, \quad m_I = \frac{ik}{2\pi(b^2+1)}\left(\mathfrak{Q}_I - b^2\widetilde{\mathfrak{Q}}_I\right)\,, \tag{195a}$$

$$\nu_I = \frac{kb}{2\pi(b^2+1)}\left(\mathfrak{P}_I + \widetilde{\mathfrak{P}}_I\right)\,, \quad n_I = \frac{ik}{2\pi(b^2+1)}\left(\mathfrak{P}_I - b^2\widetilde{\mathfrak{P}}_I\right)\,, \tag{195b}$$

The summations over m_a and n_a also need to be altered to integrals for the stationary analysis. The trick is to use the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}/k\mathbb{Z}} f(n) = \sum_{n=0}^{k-1} f(n) = \sum_{p\in\mathbb{Z}}\int_{-\delta}^{k-\delta} \mathrm{d}n\, f(n) e^{2\pi i p n} = \frac{k}{2\pi}\sum_{p\in\mathbb{Z}}\int_{-\delta/k}^{2\pi-\delta/k} \mathrm{d}\mathcal{J}\, f\left(\frac{k}{2\pi}\mathcal{J}\right) e^{ikp\mathcal{J}}\,. \tag{196}$$

Putting these ingredients together, $\mathcal{A}_v(\iota)$ can be written into the form of (192) with

$$\int \mathrm{d}\mu(X) = \mathcal{N}\sum_{\vec{p}\in\mathbb{Z}^{15}}\sum_{\vec{u}\in\mathbb{Z}^5}\int_{\mathcal{C}^{\times 30}}\bigwedge_{I=1}^{15}\left(-i\,\mathrm{d}\mathcal{P}_I\wedge\mathrm{d}\widetilde{\mathcal{P}}_I\right)\int_{\mathcal{C}^{\times 10}}\bigwedge_{a=1}^{5}\left(-i\,\mathrm{d}\mathcal{Q}_a\wedge\mathrm{d}\widetilde{\mathcal{Q}}_a\right)\,,\quad S(X) = S(\vec{\mathcal{Q}}, \vec{\widetilde{\mathcal{Q}}}, \vec{\mathcal{P}}, \vec{\widetilde{\mathcal{P}}})\,, \tag{197}$$

where $\mathcal{N}$ is some k-dependent numerical factor, $\vec{p}$ comes from the Poisson resummation of $\vec{n}$ and $\vec{u}$ comes from the Poisson resummation of m_a. The positive angle $\vec{\alpha}$ does not scale with k so it is not seen at the large-k limit of the action.

The critical equations are given by the partial derivative with respect to the integration variables $\vec{\mathcal{P}}, \vec{\widetilde{\mathcal{P}}}$ and $\mathcal{Q}_a, \widetilde{\mathcal{Q}}_a$. A direct calculation (see more details in [11]) shows that $\partial S/\partial\mathcal{P}_I = 0$ and $\partial S/\partial\widetilde{\mathcal{P}}_I = 0$ are nothing but the reformulation of the algebraic curve equations $z^{-1} + z'' - 1 = 0$ and $\tilde{z}^{-1} + \tilde{z}'' - 1 = 0$ respectively in terms of the new coordinates, which solves $\vec{\mathcal{P}}^\alpha(\vec{\mathcal{Q}}), \vec{\widetilde{\mathcal{P}}}^\alpha(\vec{\widetilde{\mathcal{Q}}})$ in terms of $\vec{\mathcal{Q}}$ and $\vec{\widetilde{\mathcal{Q}}}$ with α labelling the branches of $\mathcal{L}_{S^3\backslash\Gamma_5}$.

On the other hand, $\partial S/\partial\mathcal{Q}_a = 0$ and $\partial S/\partial\widetilde{\mathcal{Q}}_a = 0$ relates the FG coordinates on the 4-holed sphere $\mathcal{S}_a$ to the labels of the coherent state Ψ_{ρ_a} in a natural way:

$$\mu_a = \hat{\mu}_a\,,\quad \nu_a = \hat{\nu}_a\,,\quad m_a = \hat{m}_a\,,\quad n_a = \hat{n}_a\,. \tag{198}$$

Note that after imposing the first-class constraints, $2L_{ab} = -\frac{4\pi i}{k} j_{ab} = -2\widetilde{L}_{ab}$. The action evaluated at the critical point is then

$$S(X_0^\alpha) = S(\{2L_{ab}, \mathcal{T}_{ab}^\alpha(\{2L_{ab}\}), \widetilde{\mathcal{T}}_{ab}^\alpha(\{2L_{ab}\})\}, \{\rho_a\}) \tag{199}$$

The derivative of S with respect to $2L_{ab}$ is

$$\frac{\partial S}{\partial(2L_{ab})} = -\frac{ik}{2\pi(1+b^2)}\left(\mathcal{T}_{ab}^\alpha(\{2L_{ab}\}) - b^2\widetilde{\mathcal{T}}_{ab}^\alpha(\{2L_{ab}\})\right), \tag{200}$$

which implies that

$$S(X_0^\alpha) = -\frac{ik}{2\pi(1+b^2)}\sum_{a<b}\int^{2L_{ab}} \mathcal{T}_{ab}^\alpha \, \mathrm{d}(2L'_{ab}) + c.c. + C^\alpha, \tag{201}$$

where C^α is an integration constant. $\mathcal{T}_{ab}^\alpha$ can be calculated using framing flags [33] and it gives

$$\mathcal{T}_{ab}^\alpha = -\frac{1}{2}s\,\mathrm{sgn}(V_4)\Theta_{ab} + i\pi N_{ab} + f(\theta_{ab}, \{\xi\}, L_{ab}), \quad N_{ab}\in\mathbb{Z}, \tag{202}$$

where $\{\xi\}$ denotes a set of spinors, N_{ab} corresponds to a lift of $\mathcal{T}_{ab}^\alpha$ from $\tau_{ab}^\alpha = e^{\mathcal{T}_{ab}^\alpha}$, and $f(\theta_{ab}, \{\xi\}, L_{ab})$ is a function depending on the boundary condition.

The result (201) can be improved by taking into account another critical solution as follows. Given boundary conditions corresponding to boundary tetrahedra of a non-degenerate 4-simplex, there are exactly two critical points $\mathcal{A}$ and $\overline{\mathcal{A}}$. They are called the parity pair, which also exist in the EPRL model [37]. Intuitively, the Chern-Simons action (26) involves the self-dual and the anti-self-dual parts of the $\mathrm{SL}(2,\mathbb{C})$ connection in the same footing, it is not hard to realize that the transformation

$$P: \mathcal{A} = (A, \bar{A}) \quad \longrightarrow \quad \overline{\mathcal{A}} = (\bar{A}, A) \tag{203}$$

is a symmetry of the equations of motion. The main consequence of such transformation is that the FN twist

$$P: \mathcal{T}_{ab}^\alpha|_{\mathcal{A}} = -\frac{1}{2}s\,\mathrm{sgn}(V_4)\Theta_{ab} + i\pi\; N_{ab}^\alpha|_{\mathcal{A}} + f \quad \longrightarrow \quad \mathcal{T}_{ab}^\alpha|_{\overline{\mathcal{A}}} = \frac{1}{2}s\,\mathrm{sgn}(V_4)\Theta_{ab} + i\pi\; N_{ab}^\alpha|_{\overline{\mathcal{A}}} + f\,. \tag{204}$$

This means the two solutions of the parity pair correspond to opposite 4D orientation of the 4-simplex. The difference between the two solutions is

$$\mathcal{T}_{ab}^\alpha|_{\mathcal{A}} - \mathcal{T}_{ab}^\alpha|_{\overline{\mathcal{A}}} = \mathrm{sgn}(V_4)\left(-s\,\Theta_{ab} + 2\pi i N_{ab}\right), \quad 2N_{ab} = \mathrm{sgn}(V_4)\left(N_{ab}^\alpha|_{\mathcal{A}} - N_{ab}^\alpha|_{\overline{\mathcal{A}}}\right) \in 2\mathbb{Z}\,. \tag{205}$$

Taking into the parity pair, we rewrite (201) and calculate its variation

$$\begin{aligned}\delta S &= -\frac{ik}{2\pi(1+b^2)}\mathrm{sgn}(V_4)\sum_{a<b}\left(-s\,\Theta_{ab} + 2\pi i N_{ab}\right)\,\delta(2L_{ab}) + c.c.\\ &= -\frac{k\Lambda}{6\pi(1+b^2)}\mathrm{sgn}(V_4)\sum_{a<b}\left(\Theta_{ab} - 2\pi i N_{ab}\right)\delta\mathfrak{a}_{ab} + c.c.\\ &= -\frac{k\gamma\Lambda i}{6\pi}\mathrm{sgn}(V_4)\sum_{a<b}\Theta_{ab}\delta\mathfrak{a}_{ab} - \frac{i\Lambda k}{3}\sum_{a<b}N_{ab}\delta\mathfrak{a}_{ab}\,.\end{aligned} \tag{206}$$

By the Schläfli identity of constant curvature 4-simplex [38]

$$\sum_{a<B}\delta\Theta_{ab}\mathfrak{a}_{ab} = \Lambda|V_4|\,, \tag{207}$$

δS can be integrated and gives

$$S = -\frac{ik\gamma\Lambda}{6\pi}\mathrm{sgn}(V_4)\left(\sum_{a<b}\Theta_{ab}\mathfrak{a}_{ab} - \Lambda|V_4|\right) - \frac{i\Lambda k}{3}\sum_{a<b}N_{ab}\mathfrak{a}_{ab}\,. \tag{208}$$

Recalling that $\mathfrak{a}_{ab} = \frac{12\pi j_{ab}}{k|\Lambda|}$, the second term can be ignored in the exponential. We then conclude that the large-k limit of the vertex amplitude does give the Regge action with a cosmological constant term:

$$\mathcal{A}_v \xrightarrow{k\to\infty} = \mathcal{N}_+ e^{ikS_{\text{Regge}}+C} + \mathcal{N}_- e^{-ikS_{\text{Regge}}-C}\,, \tag{209}$$

where $\mathcal{N}_\pm$ encode the Hessian, C is a geometry independent integration constant, and

$$S_{\text{Regge}} = \frac{\gamma\Lambda}{12\pi}\left(\sum_{a<b}\Theta_{ab}\mathfrak{a}_{ab} - \Lambda|V_4|\right)\,. \tag{210}$$

Now, we see the second main feature of the spinfoam vertex amplitude – reproducing the 4D Lorentzian Regge action with Λ for a homogeneously curved 4-simplex.

A. Edge amplitude, face amplitude and the full amplitude

After a long journey, we have only defined and analyzed the spinfoam amplitude for a single 4-simplex. What about a 4-complex? A straightforward way is to define edge amplitudes $\mathcal{A}_e$'s and face amplitudes $\mathcal{A}_f$'s for spinfoam edges and spinfoam faces respectively, and then the amplitude for a 4-complex can be formally written as an integral and sum of the product of vertex, edge and face amplitudes, *i.e.*

$$\mathcal{A}_{\text{4-complex}} = \sum_{\{j\}}\int \mathrm{d}\mu(Y)\prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v\,, \tag{211}$$

where Y is a set of internal configurations, so as the spins $\{j\}$. The edge amplitude describes the gluing of 4-simplices through their 3D boundaries. As the vertex amplitude is defined relying on the 3-manifold $S^3\backslash\Gamma_5$, such gluing can be represented by gluing $S^3\backslash\Gamma_5$'s through their 2D boundaries. On the other hand, the face amplitude is related to the boundary Hilbert space [39]. We refer to [12, 13] for recent proposals of edge and face amplitudes and only formally write the general spinfoam amplitude as follows.

The spinfoam amplitude for a spinfoam 2-complex consisting of V spinfoam vertices, E_{in} internal spinfoam edges and F_{in} internal spinfoam faces takes the form

$$\mathcal{Z}_{\vec{\rho}_\partial}(\vec{\alpha}|\vec{j}_b) = \sum_{j_f=0}^{(k-1)/2}\int_{\mathcal{M}_{\vec{j}^v_a}}\mathrm{d}\rho_a^{v\in e}\int_{\mathcal{M}_{\vec{j}^{v'}_b}}\mathrm{d}\hat{\rho}_b^{v'\in e}\left[\prod_{f=1}^{F_{\text{in}}}\mathcal{A}_f(2j_f)\right]\left[\prod_{e=1}^{E_{\text{in}}}\mathcal{A}_e(\rho_a^{v\in e},\rho_b^{v'\in e}|\{j_{ac}^{v\in e},j_{bd}^{v'\in e}\}_{c,d})\right]\left[\prod_{v=1}^{V}\mathcal{A}_v(\vec{\alpha}^v,\vec{j}^v,\vec{\rho}^v)\right]\,, \tag{212}$$

where $v\in e$ denotes that v is at the (source or target) end of e, $\vec{\alpha}$ contains all the positive angles, $\vec{\rho}_\partial$ contains all the coherent state labels on the boundary, the summations in j_f are for all the internal spinfoam faces and the integrations over coherent state labels are for all the internal spinfoam edges.

We require that the vertex amplitudes, edge amplitudes and face amplitudes are all bounded and that the integrations over the coherent state labels are over compact domains, then the spinfoam amplitude defined in (212) for *any* spinfoam 2-complex is *finite* given finite boundary spins $\vec{j}_b$ and finite Chern-Simons level k.

VII. DISCUSSION, CURRENT STATUS AND FUTURE DEVELOPMENTS

In this note, we have illustrated the construction of the spinfoam amplitude for 4D Lorentzian quantum gravity with a non-vanishing Λ, which combines the techniques from the existing spinfoam model with $\Lambda = 0$ (mainly the EPRL model) and the geometrical quantization of the Chern-Simons theory on a graph-complement 3-manifold. The spinfoam amplitude is finite by construction and it reproduces Regge action with a Λ term as desired. These robust features enhance the capability of this spinfoam model to address Lorentzian quantum gravity problems at hand.

- A "moduli-space field theory" (MFT) formalism of the spinfoam model – a possible UV complete, triangulation-independent quantum gravity theory with order-by-order finite amplitudes.

  A MFT is an analogy of the Group Field Theory (GFT). The rough idea of group field theory is to express quantum gravity in the action of a group-dependent field $\phi: G^d \to \mathbb{C}$ and compute the amplitudes, or correlation functions, for different spinfoam graphs using the path-integral method.

Here, we can use the coherent state defined in Section V C 2. Ψ_ρ which is a field on the coherent state label ρ. Recall that $\rho=(\mathrm{Re}(z),\mathrm{Im}(z),x,y)$ labels the phase space coordinates $\mathcal{X}=\frac{2\pi i}{k}\left(-ib\mu-m\right)\,,\quad \mathcal{Y}=\frac{2\pi i}{k}\left(-ib\nu-n\right)$ with the relation (treating $\mu,\nu\in\mathbb{R}$ as the positive angles do not encode the geometrical information of the 4-simplex)

$$\mu=\frac{k}{\sqrt{2}\pi}\mathrm{Re}(z)\,,\quad \nu=\frac{k}{\sqrt{2}\pi}\mathrm{Im}(z)\,,\quad m=\frac{k}{2\pi}x\,,\quad n=\frac{k}{2\pi}y\,. \tag{213}$$

Group them into two complex variables

$$u=\frac{2\pi}{k}\left(\mu+i\nu\right)\,,\quad v=\frac{2\pi}{k}\left(m+in\right)\,, \tag{214}$$

and define the configuration for a tetrahedron (or equivalently a 4-holed sphere)

$$\iota=(\{j_{\mathfrak{p}}\}_{\mathfrak{p}=1}^4,u,v)\,,\quad \iota^*=(\{j_{\mathfrak{p}}\}_{\mathfrak{p}=1}^4,\bar{u},\bar{v})\,. \tag{215}$$

For notation consistency, the measure (188) can now be denoted as

$$\int_{\mathcal{M}_{\vec{\chi}}}\mathrm{d}\iota\equiv\int_{\mathcal{M}_{\vec{\chi}}}\mathrm{d}\rho\,. \tag{216}$$

The coherent state $\Psi(\iota)$ is then viewed as a function on the moduli space $\mathcal{M}_{\text{flat}}(\Sigma_{0,4},\mathrm{PSU}(2))$ of PSU(2) flat connection. Require that $\Psi(\iota)$ satisfies the reality condition

$$\overline{\Psi(\iota)}=\Psi(\iota^*)\,. \tag{217}$$

The generalized moduli-space field action

$$S[\Psi]=K[\Psi]+V[\Psi] \tag{218}$$

contains a kinetic term $K[\Psi]\sim\Psi^2$ and an interaction term $V[\Psi]\sim\Psi^5$. The kinetic term is defined as

$$K[\Psi]=\sum_{\{j_{\mathfrak{p}}\}\in(\mathbb{Z}/k\mathbb{Z})^4}\int_{\mathcal{M}_{\vec{\chi}}}\mathrm{d}\iota\,\Psi[\iota^*]\Psi[\iota]\,, \tag{219}$$

where the sum is only for the admissible spins satisfying the triangular inequalities. The potential term should reproduce the vertex amplitude hence we define it to be

$$V[\Psi]=\frac{g}{5!}\sum_{\{j_{ab}\}_{a<b}}\prod_{a=1}^{5}\int_{\mathcal{M}_{\vec{\chi}_a}}\mathrm{d}\iota_a\,\mathcal{A}_v(\{\iota_a\}_{a=1}^5)\prod_{a=1}^{5}\Psi[\iota_a]\,, \tag{220}$$

where g is the coupling constant. Since the measure is over a compact space and the integrands we encounter are all bounded functions, it is not hard to expect that the path integral of the action (218) gives finite amplitude order-by-order. Such a formalism includes a sum over all the triangulation hence it is triangulation-independent. The cutoff on spins renders UV completeness of the theory under this formalism.

- We have seen in Section V C 2 that the framing flags on $\Sigma_{0,4}$ can be replaced by the spinors when $\mathcal{M}_{\text{flat}}(\Sigma_{0,4},\mathrm{PSU}(2))$ is concerned, *i.e.* when the simplicity constraints are imposed to implement 4-simplex geometry on the Chern-Simons partition function. These spinors (together with the spins $\{j_{ab}\}$) can be used to reconstruct the phase space coordinates (including the FN lengths and FG coordinates on 4-holed spheres) that correspond to a (real) critical point of the spinfoam amplitude. This provides an algorithm, which is ready to be used for numerical development, to compute the critical behaviour of the spinfoam model, starting from which one can study *e.g.* the complex critical points and perturbation theory of the spinfoam model. This is similar to the strategy of the numerical study of the EPRL model [40–42, 42].
- One can also potentially tackle physical questions with this spinfoam model: What is the physical Hamiltonian corresponding to the spinfoam model? How do we couple matter field to the spinfoam model to define physical time? How does it apply to reduced model *e.g.* cosmology and black holes? What are the boundary symmetries and charges encoded in the spinfoam model for a general 4-complex?

Apart from the above plausible directions of development, the spinfoam itself could still be modified or improved.

1. Firstly, the spinfoam amplitude for a general 4-complex is so far constructed by gluing spinfoam vertices together with edge and face amplitudes using the spinfoam ansatz (211). Chern-Simons partition functions are only used to define the vertex amplitude. However, when a more complicated 4-complex is concerned, one can in principle construct the Chern-Simons partition function for the graph-compliment of the boundary of the 4-complex in the same way as how we construct the Chern-Simons partition function on a single 4-simplex. One can then impose simplicity constraints all at once on such a more-volumed partition function, which could potentially simplify the construction. We also need to check if such construction gives the same result as gluing vertex amplitudes.

2. The way to impose the second-class simplicity constraints is flexible as we only require weak imposition. It is also possible to choose another coherent state that peaks at the same phase space configuration but lives in a different Hilbert space, which would potentially change the construction of the GFT.

3. The simplest way to glue vertex amplitude is to identify FG coordinates $\mathcal{X}_a^v = \mathcal{Y}_b^{v'}$, $\mathcal{Y}_a^v = \mathcal{X}_b^{v'}$ if $\mathcal{S}_a$ from 3-manifold corresponding to spinfoam vertex v is glued to $\mathcal{S}_b$ from 3-manifold corresponding to spinfoam vertex v' (this way of gluing was used in [13]). It potentially imposes constraints on the topology of the 4-complex, because such requirement is rather strong. It is better to define a gluing for any pair of edges on the ideal triangulation of the 4-holed spheres. This would involve some local symplectic transformation which leads to unitary transformation of the vertex Chern-Simons partition function. There is evidence that one may have to lift $\mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{PSL}(2,\mathbb{C}))$ to $\mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{SL}(2,\mathbb{C}))$. Nevertheless, this allows us to generalize the construction of the spinfoam model to be more adaptive for different 4-complexes.

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