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Symmetry charges in reduced phase space and BMS algebra

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Motivation

- The boundary charges of asymptotically flat spacetime [Barnich and Troessaert (2011), etc.]
- Information paradox [Stephen W. Hawking, Malcolm J. Perry, and Andrew Strominger (2016)]
- Bulk/boundary duality (holographic principle) [Gerard 't Hooft (1999), Leonard Susskind (1995), etc.]

Both are constructed in GR

↓

The bulk is a pure gauge system

↓

Problem of time

Motivation

- Brown-Kuchar Formalism:
  - Introduce a co-moving frame to the spacetime by coupling dust field
  - Introduce gauge fixing scheme to the spacetime
  - Gauge invariant quantities (Dirac Observables) of the phase space functions
  - The Hamiltonian is physical
- Question:
  - Can we reconstruct the results in last slide in Brown-Kuchar formalism?

Motivation

- Construct the asymptotically flat spacetime in Brown-Kuchar formalism
- Construct the symmetry charges in the asymptotically flat spacetime
- Construct the algebra of the symmetry charges

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Preliminary

ADM Formalism of GR

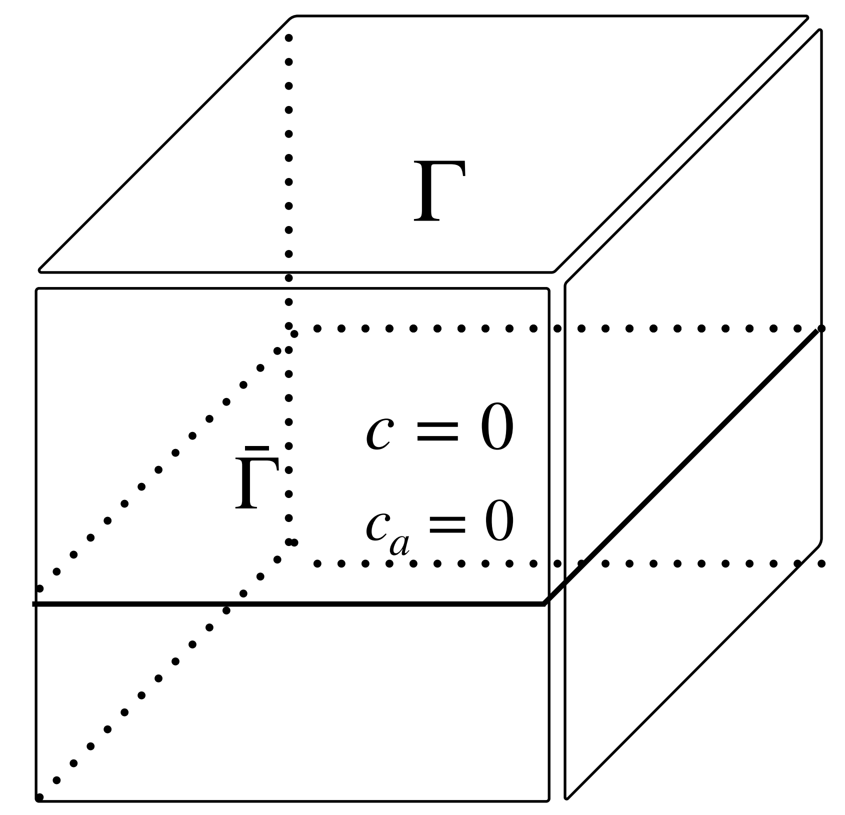
- The Hamiltonian:

$$H = \int_\Sigma d^3x \left(N c + N^a c_a \right)$$

Hamiltonian constraint

Diffeomorphism constraint

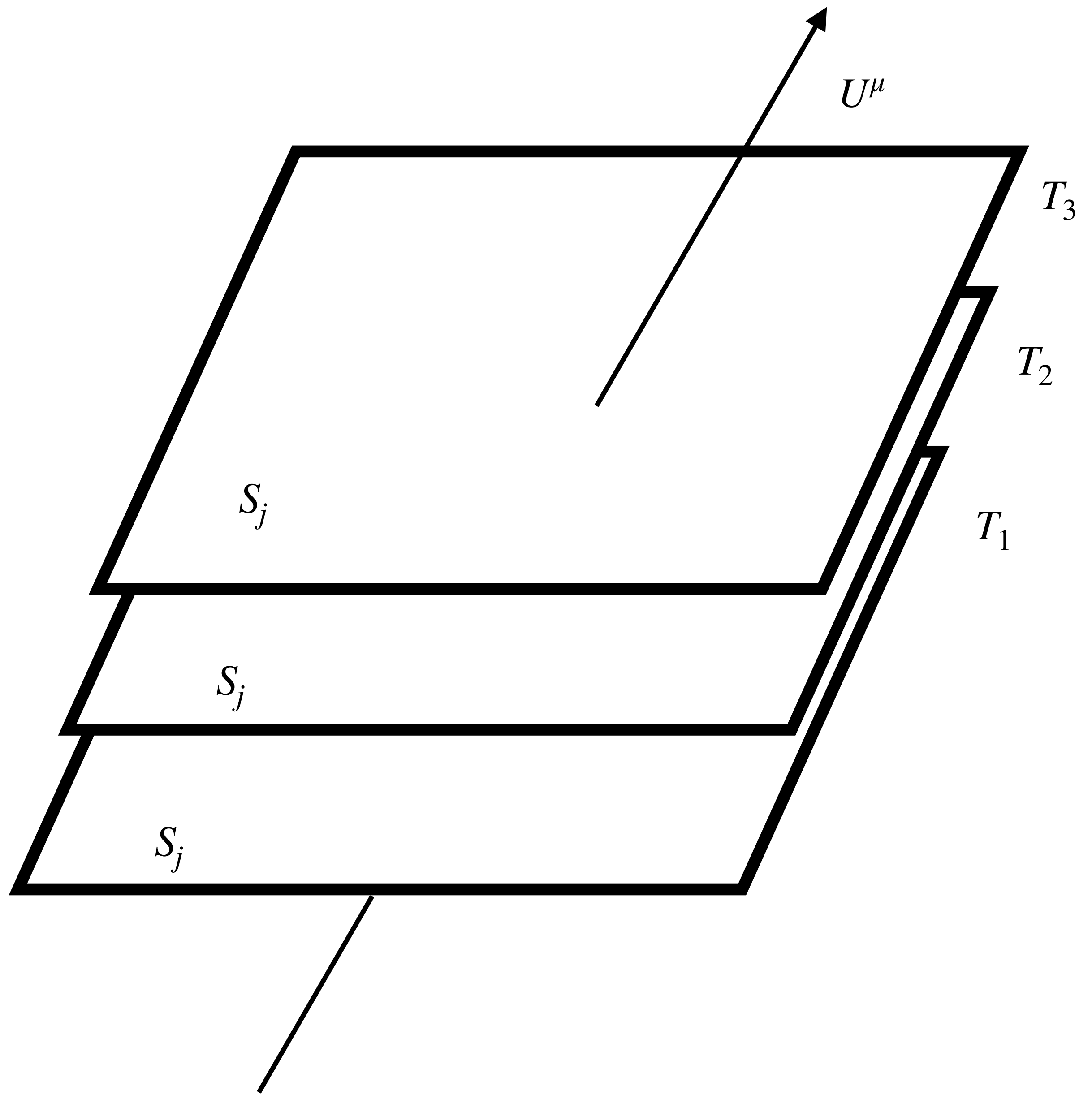
$H = 0$: Time evolution is a gauge transformation (diffeomorphism) [R. Wald 1984, etc]



Brown-Kuchar Formalism

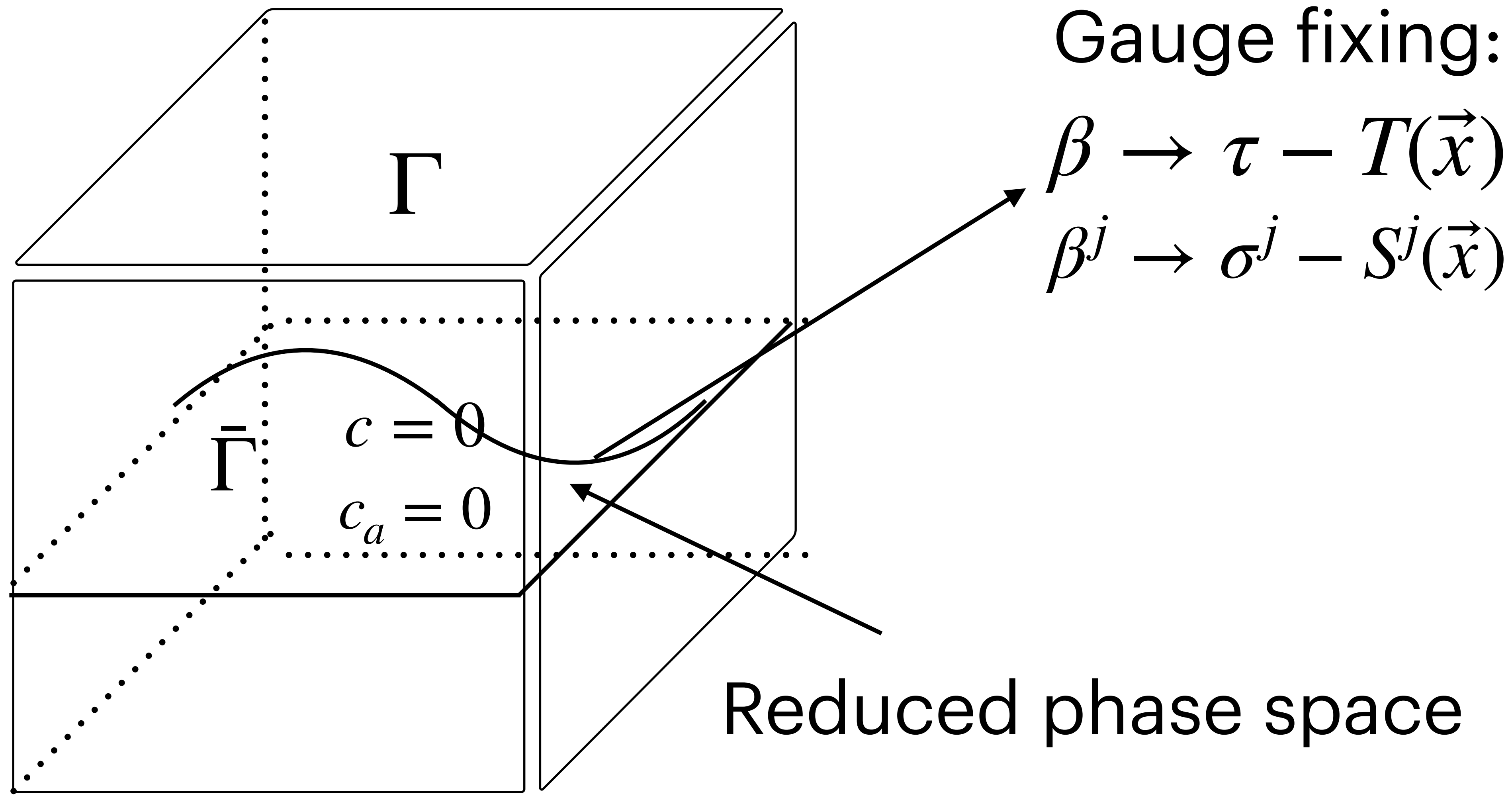
- Brown-Kuchar formalism is one realization of the relational formalism.
  - The action [J. Brown and K. Kuchar (1995)]:
    - $S = S_{\mathrm{EH}} + S_{\mathrm{dust}}$
  - The dust action
    - $S_{\mathrm{dust}} = -\frac{1}{2}\int_M \mathrm{d}^4x\sqrt{|\det(g)|}\,\rho\left[g^{\mu\nu}U_\mu U_\nu + 1\right]$

Equations of motion introduce a natural foliation of the spacetime. Dust fields are the comoving observers.



Dirac Observables of Brown-Kuchar Formalism

- Introduce:
  - $K_\beta \equiv \int_\Sigma \mathrm{d}^3x \left[\beta(\vec{x})\tilde{c}^{\mathrm{tot}}(\vec{x}) + \beta^j(\vec{x})\tilde{c}_j^{\mathrm{tot}}(\vec{x})\right]$
- The Dirac observable of a phase space function f:
  - $O_f[\tau, \sigma] \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\{f, K_\beta\}_{(n)} \begin{smallmatrix} \beta \to \tau - T \\ \beta^j \to \sigma^j - S^j \end{smallmatrix}$



$$H_{\text{phys}} = \sqrt{C^2 - g^{ij}C_iC_j}$$

- The physical Hamiltonian:
  - $\mathbf{H}_{\text{phys}} := \int_{\mathcal{S}} \mathrm{d}^3\sigma H_{\text{phys}}$
- Time evolution of a Dirac observable F:
  - $\frac{\mathrm{d}F}{\mathrm{d}\tau} := \left\{F, \mathbf{H}_{\text{phys}}\right\}$

- Four conserved quantities:
  - $\{C_j, \mathbf{H}_{\text{phys}}\} = 0$
  - $\{H_{\text{phys}}, \mathbf{H}_{\text{phys}}\} = 0$

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- Brown-Kuchar formalism:
  - $X^0 \to \tau, \overrightarrow{X} \to \vec{\sigma}, r = \sqrt{\sigma^i \sigma_i}$

- The fall-off conditions:
  - $g_{ij} = \delta_{ij} + \frac{\bar{h}_{ij}}{r} + \frac{\log r}{r^2} h_{ij}^{(\log)} + \frac{h_{ij}^{(2)}}{r^2} + o\left(r^{-2}\right)$
  - $\pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + \frac{\log r}{r^3} \pi^{(\log)ij} + \frac{\pi^{(2)ij}}{r^3} + \frac{\log r}{r^4} \pi^{(ll)ij} + \frac{\pi^{(3)ij}}{r^4} + o\left(r^{-4}\right)$

- In the spherical coordinates:
  - $g_{rr} = 1 + \frac{1}{r}\bar{h}_{rr} + \frac{\log r}{r^2}h^{(\log)}_{rr} + \frac{1}{r^2}h^{(2)}_{rr} + o\left(r^{-2}\right)$
  - $g_{rA} = \bar{h}_{rA} + \frac{\log r}{r}h^{(\log)}_{rA} + \frac{1}{r}h^{(2)}_{rA} + o\left(r^{-1}\right)$
  - $g_{AB} = r^2\bar{\gamma}_{AB} + r\bar{h}_{AB} + \log(r)h^{(\log)}_{AB} + h^{(2)}_{AB} + o\left(r^{0}\right)$
  - $\pi^{rr} = \bar{\pi}^{rr} + \frac{\log r}{r}\pi^{(\log)rr} + \frac{1}{r}\pi^{(2)rr} + \frac{\log r}{r^2}\pi^{(ll)rr} + \frac{\pi^{(3)rr}}{r^2} + o\left(r^{-2}\right)$
  - $\pi^{rA} = \frac{1}{r}\bar{\pi}^{rA} + \frac{\log r}{r^2}\pi^{(\log)rA} + \frac{1}{r^2}\pi^{(2)rA} + \frac{\log r}{r^3}\pi^{(ll)rA} + \frac{\pi^{(3)rA}}{r^3} + o\left(r^{-3}\right)$
  - $\pi^{AB} = \frac{1}{r^2}\bar{\pi}^{AB} + \frac{\log r}{r^3}\pi^{(\log)AB} + \frac{1}{r^3}\pi^{(2)AB} + \frac{\log r}{r^4}\pi^{(ll)AB} + \frac{\pi^{(3)AB}}{r^4} + o\left(r^{-4}\right)$

- The asymptotic behaviors of C_J and C:
  - $C_r = \frac{C_r^{(1)}}{r} + \frac{\log(r)}{r^2} C_r^{(\log)} + \frac{C_r^{(2)}}{r^2} + \frac{\log r}{r^3} C_r^{(ll)} + O\left(r^{-3}\right)$
  - $C_A = C_A^{(0)} + \frac{\log(r)}{r} C_A^{(\log)} + \frac{C_A^{(1)}}{r} + \frac{\log r}{r^2} C_A^{(ll)} + O\left(r^{-2}\right)$
  - $C = \frac{C^{(1)}}{r} + \frac{\log(r)}{r^2} C^{(\log)} + \frac{C^{(2)}}{r^{-2}} + o\left(r^{-2}\right)$
- Follow the notations in [M. Henneaux and C. Troessaert (2018)]:
  - $\bar{\lambda} = \frac{1}{2} h_{rr}, \quad \bar{\tilde{k}}_{AB} = \frac{1}{2} \bar{h}_{AB} + \bar{\lambda} \bar{\gamma}_{AB}, \quad \bar{\lambda}_A = \bar{h}_{rA}$
  - $\bar{p} = 2\left(\bar{\pi}^{rr} - \bar{\pi}^A_A\right), \quad \pi^{AB}_{(k)} = 2\bar{\pi}^{AB}$

- Useful coefficients:
  - $C_A^{(0)} = -2\left(\bar{\pi}^r_A + \bar{D}_B \bar{\pi}^B_A\right)$
  - $C^{(1)} = -2\sqrt{\bar{\gamma}}\left(\bar{D}_A \bar{D}_B \bar{\tilde{k}}^{AB} - \bar{D}_A \bar{D}^A \bar{\tilde{k}} + \bar{D}_A \bar{\lambda}^A\right)$

↑ Has odd parity with the parity conditions in [M. Henneaux and C. Troessaert (2018)]

- $C = -\sqrt{\det(g)}\rho\left(g^{ij}U_iU_j + 1\right)$ ⟶ Contradiction: ρ can be both physical and phantom in one spacetime.

↓ Relax the parity conditions and introduce a counter term to the symplectic structure [G. Compere and F. Dehouck (2011)]

$$\Omega = \int_{\mathcal{S}_R} \mathrm{d}^3V \delta\pi^{ij} \wedge \delta g_{ij} - \boxed{\lim_{R\to\infty} \log(R) \oint_{S^2} \mathrm{d}^2S \delta\bar{\pi}^{ij} \wedge \delta\bar{h}_{ij}}$$

Boundary Conditions

- $C^{(1)} = C_A^{(0)} = C_A^{(\log)} = C_A^{(1)} = C_r^{(1)} = C_r^{(\log)} = C_r^{(2)} = 0$
- $C^{(\log)} \neq 0$

$$\Downarrow$$

- A finite physical Hamiltonian
- The variation of the physical Hamiltonian is well-defined

Boundary-preserving Symmetry Generators and their Algebra
The Definition

- G is a boundary-preserving symmetry generator if:
  - 1) G commutes with the physical Hamiltonian:
  - $\left\{ G, \mathbf{H}_{\text{phys}} + \mathbf{H}_{\text{bdy}} \right\} = 0;$
  - 2) The Hamiltonian flow of G preserves the boundary conditions of the spacetime.

Boundary-preserving Symmetry Generators

- Introduce the vector field

Killing vector on S^2

- $\xi = f + O\left(r^{-1}\right), \quad \xi^r = W + O\left(r^{-1}\right), \quad \xi^A = Y^A + \frac{1}{r} I^A + O\left(r^{-2}\right)$

- The boundary-preserving symmetry generator

- $G(\xi, \vec{\xi}) := J(\xi, \vec{\xi}) + \mathscr{B}(\xi, \vec{\xi})$ ← Boundary part
- $J(\xi, \vec{\xi}) = \mathscr{T}(\xi) + \mathscr{P}(\vec{\xi})$
- $\mathscr{T}(\xi) = \int_{\mathcal{S}_P} \mathrm{d}^3 V \xi H_{\text{phys}} \qquad \mathscr{P}(\vec{\xi}) = \int_{\mathcal{S}_R} \mathrm{d}^3 V \xi^i C_i$

- With integration by-part:

$$\mathscr{B}(\xi,\vec{\xi}) = \oint_{S^2} \mathrm{d}^2 S Y^A \left(4 \left(\bar{\tilde{k}}_{AB} - \bar{\lambda}\bar{\gamma}_{AB} \right) \bar{\pi}^{rB} + 2\bar{\gamma}_{AB}\pi^{(2)rB} + \bar{\lambda}_A \left(\bar{p} + 2\bar{\pi}^B_B \right) \right)$$

- $$+\oint_{S^2} \mathrm{d}^2 S \left(2I^A\bar{\gamma}_{AB}\bar{\pi}^{rB} + W \left(\bar{p} + 2\bar{\pi}^A_A \right) \pm 2\sqrt{\bar{\gamma}} f \left(2\bar{\lambda} + \bar{D}_A\bar{\lambda}^A \right) \right)$$

Algebra of the Boundary-preserving Symmetry Generators

- The algebra is construct with Poisson bracket:
- $$\left\{G\left(\xi_1,\vec{\xi}_1\right),G\left(\xi_2,\vec{\xi}_2\right)\right\}=G(\hat{\xi},\hat{\vec{\xi}})+C(\hat{\xi},\hat{\vec{\xi}})$$
- $(\hat{\xi},\hat{\vec{\xi}})=(\hat{\xi},\hat{\vec{\xi}})(\hat{Y},\hat{I},\hat{f},\widehat{W})$

$$\hat{Y}^A=Y_1^B\bar{D}_BY_2^A-(1\leftrightarrow 2)$$

$$\hat{f}=Y_1^A\partial_Af_2-(1\leftrightarrow 2)$$

- $$\widehat{W}=Y_1^A\partial_AW_2-(1\leftrightarrow 2)$$

$$\hat{I}^A=Y_1^B\bar{D}_BI_2^A+I_1^B\bar{D}_BY_2^A-(1\leftrightarrow 2)$$

- $G\in\mathscr{A}_G$

Central charges:

$$C(\hat{\xi},\hat{\vec{\xi}})=\mp 2\oint_{S^2}\mathrm{d}^2S\left(\sqrt{\bar{\gamma}}f_2\left(\bar{D}^A\bar{D}_AW_1-\bar{D}_AI_1^A\right)-2\sqrt{\bar{\gamma}}f_1\left(\bar{D}^A\bar{D}_AW_2-\bar{D}_AI_2^A\right)\right)$$

Compare with the BMS algebra

- Bulk terms of the charge:
- $\tilde{G}(v,\vec{v}) := \int_{\mathcal{S}} \mathrm{d}^3V\, v H_{\text{phys}} + \int_{\mathcal{S}} \mathrm{d}^3V\, v^j C_j$

$f = W = 0$ $Y^A = I^A = 0$

- $\tilde{G} \in \tilde{\mathscr{A}}_G$
- $\tilde{\mathscr{A}}_G$ is an ideal of $\mathscr{A}_G$
- Quotient algebra: $\hat{\mathscr{A}}_G := \mathscr{A}_G / \tilde{\mathscr{A}}_G$
- $\hat{\mathscr{A}}_G$ is analogous to $\mathscr{A}_{BMS}$

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Conclusions and Outlooks

Conclusions

- Introduce asymptotic flat spacetime in BK formalism.
- Relax the parity conditions and introduce a counter term to the symplectic form.
- Introduce additional boundary conditions to make the physical Hamiltonian finite and its variation well-defined.
- Construct the boundary-preserving symmetry generators, whose boundary terms are construct by integration by part.
- construct the algebra of the generators by Poisson bracket, which contains a central extension.

Outlooks

- Apply our work to Ashtekar variables
- Generalize our work to asymptotically AdS spacetimes
- Investigate more generic boundary-preserving symmetry generators in Brown-Kuchar formalism

THANK YOU!

Deparameterization and Dirac Observables

- Deparameterizable system:
  - For a constraint system, if ∃ solvable first-class constraints c_I, P_I can be defined:
    - $c_I = 0 \to P_I = h_I\left(q^a, p_a\right)$
  - The canonical pairs are divided into two categories: $\left(q^a, p_a\right)$ and $\left(T^I, P_I\right)$.

- The Dirac observables (gauge invariant) are constructed from the canonical variables.
- h_I play the roles as the real (physical) Hamiltonians.

Deparameterizing Brown-Kuchar Formalism

- Constraint functions:

$$\tilde{c}^{\text{tot}} = P + h, \quad h = \sqrt{c^2 - q^{ab} c_a c_b}$$

- $\tilde{c}_j^{\text{tot}} = P_j + h_j, \quad h_j = S_j^a \left(-hT_{,a} + c_a\right)$

$\downarrow$ $\tilde{c} = 0$, $\tilde{c}_j = 0$

$$P = \text{sgn}(P) h$$

$$P_j = - h_j$$

h is the physical Hamiltonian density

- Introduce:
  - $K_\beta \equiv \int_{\mathcal{X}} \mathrm{d}^3x \left[\beta(\vec{x})\tilde{c}^{\text{tot}}(\vec{x}) + \beta^j(\vec{x})\tilde{c}_j^{\text{tot}}(\vec{x})\right]$
- The Dirac observable of a phase space function f:
  - $O_f[\tau, \sigma] \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \{f, K_\beta\}_{(n)} \Big|_{\substack{\beta \to \tau - T \\ \beta^j \to \sigma^j - S^j}}$