Conformally invariant approach to Einstein spacetimes with non-zero cosmological constant includding scri

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Motivation

Definition of scri of spacetime involves conformal rescaling, so let us look for conformally invariant framework for conformally Einstein spacetimes. It is given by the space of metric tensors that are Bach flat. We can consider the symplectic structure defined on therein, and use it for the Einstein metrics that set a submanifold. This is what we do below.

Normal conformal Cartan connection

Working definition

M **4d spacetime**

$$g = \eta_{ab}\theta^a \otimes \theta^b$$

$$\eta_{ab} = \text{const} \qquad -+++$$

$$\mathrm{Vol} := \frac{1}{4!}\sqrt{|\det\eta|}\varepsilon_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d = \sqrt{|\det\eta|}\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3,$$

$$\epsilon_{abcd} = \sqrt{|\det\eta|}\varepsilon_{abcd}$$

$$\mathrm{d}\theta^a + \Gamma^a{}_b \wedge \theta^b = 0$$

$$\Gamma_{ab} = -\Gamma_{ba}$$

$$\frac{1}{2}R^a{}_{bcd}\theta^a \wedge \theta^d = \mathrm{d}\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b.$$

$$P_a := \left(\frac{1}{12}R\eta_{ab} - \frac{1}{2}R_{ab}\right)\theta^b$$

$$Q = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

G = **So(2,4)** = **So(Q)**

The NCCC:

$$A = \begin{bmatrix} 0 & \theta_b & 0 \\ P^a & \Gamma^a{}_b & \theta^a \\ 0 & P_b & 0 \end{bmatrix}$$

gauge transformations

nice property:

$$A' = h^{-1} A h + h^{-1}\, \mathrm{d}h$$

$$\theta'^a = f\theta^a, \quad f \in C^\infty(M),$$

$$h = \begin{bmatrix} \frac{1}{f} & 0 & 0 \\ \frac{f_{,}{}^{a}}{f^2} & \delta^a{}_b & 0 \\ \frac{f_{,c} f_{,}{}^{c}}{2f^3} & \frac{f_{,b}}{f} & f \end{bmatrix}$$

$$\theta'^a = \Lambda^a{}_b \theta^b,$$

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \Lambda^a{}_b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

there are also:

$$h = \begin{pmatrix} 1 & 0 & 0 \\ b^\mu & \delta^\mu{}_\nu & 0 \\ \frac{1}{2} b_\sigma b^\sigma & b_\nu & 1 \end{pmatrix}$$

however we gauge fixe them:

$$A = \begin{bmatrix} 0 & \theta_b & 0 \\ P^a & \Gamma^a{}_b & \theta^a \\ 0 & P_b & 0 \end{bmatrix}$$

The curvature:

$$A' = h^{-1} A h + h^{-1}\,\mathrm{d}h$$

$$F' = h^{-1} F h$$

$$F = dA + A \wedge A$$

$$F = \begin{bmatrix} 0 & 0 & 0 \\ DP^a & C^a{}_b & 0 \\ 0 & DP_b & 0 \end{bmatrix}$$

$$DP^a = \mathrm{d}P^a + \Gamma^a{}_b \wedge P^b$$

$$C^a{}_b = \frac{1}{2} C^a{}_{bcd}\theta^c \wedge \theta^d,$$

the Weyl tensor

The Bianchi identity:

$$D_A F := \mathrm{d}F + A \wedge F - F \wedge A = 0$$

encodes the differential identities satisfied by the Weyl and the Schouten tensors

The Bach tensor:

$$D_A \star F := \mathrm{d}\star F + A \wedge \star F - \star F \wedge A = \begin{bmatrix} 0 & 0 & 0 \\ B^{ac}\star\theta_c & 0 & 0 \\ 0 & B_b{}^c\star\theta_c & 0 \end{bmatrix}$$

$$B_{ab} = 2\nabla^c \nabla_{[b} P_{c]a} - 2P^{cd} C_{cadb}.$$

$$R_{ab} = \Lambda \eta_{ab} \quad \Longrightarrow \quad B_{ab} = 0 \quad \Longrightarrow \quad D\star F = 0.$$

Examples of reduced holonomy:

The spinor representation of the NCCC is the local twistor connection of Penrose. Iff the NCCC can be gauge transformed to the following non-generic form:

$$A = \begin{pmatrix} \psi & \theta^4 & \mathrm{Re}\omega_1 & \mathrm{Im}\omega_1 & -\omega_0 & 0 \\ -\theta^4 & \psi & -\mathrm{Im}\omega_1 & \mathrm{Re}\omega_1 & 0 & -\omega_0 \\ \mathrm{Re}\omega_3 & \mathrm{Im}\omega_3 & 0 & -2\theta^4 & \mathrm{Im}\omega_1 & \mathrm{Re}\omega_1 \\ -\mathrm{Im}\omega_3 & \mathrm{Re}\omega_3 & 2\theta^4 & 0 & -\mathrm{Re}\omega_1 & \mathrm{Im}\omega_1 \\ \omega_4 & 0 & -\mathrm{Im}\omega_3 & -\mathrm{Re}\omega_3 & -\psi & \theta^4 \\ 0 & \omega_4 & \mathrm{Re}\omega_3 & -\mathrm{Im}\omega_3 & -\theta^4 & -\psi \end{pmatrix}$$

then the spacetime conformal geometry admits solutions to the twistor equation:

$$\nabla^{(A}{}_{A'}\omega^{B)} = 0$$

This is the Fefferman family of spacetime metric tensors. Among them are known examples of the Bach flat metric tensors that are not conformal to Einstein.

Symplectic potential densities

The Cartan-Yang-Mills Lagrangian:

$$L_{\text{CYM}}(\theta) = \frac{1}{2} F^I{}_J \wedge \star F^J{}_I$$

$$L_{\text{CYM}}(\theta) = L_{\text{CYM}}(f\theta),$$

$$L_{\text{CYM}}(\theta) = \frac{1}{4} C_{abcd} C^{abcd} \text{Vol},$$

$$\delta = \delta_\theta$$

$$\delta L_{\text{CYM}}(\theta) = \delta A^I{}_J \wedge D_A \star F^J{}_I + \frac{1}{2} F^I{}_J \wedge (\delta \star) F^J{}_I + \mathrm{d}\big(\delta A^I{}_J \wedge \star F^J{}_I\big)$$

$$\star C^a{}_b = \frac{1}{2} \epsilon^a{}_{bc}{}^d C^c{}_d \qquad \Rightarrow \delta(\star) F = 0$$

The symplectic potential density:

$$L_{\mathrm{CYM}}(\theta) = \frac{1}{2} F^I{}_J \wedge \star F^J{}_I$$

$$\delta L_{\mathrm{CYM}}(\theta) = 2\delta\theta^a \wedge B_{ab} \star \theta^b + \mathrm{d}\big(\delta A^I{}_J \wedge \star F^J{}_I\big)$$

$$\Theta_{\mathrm{CYM}}(\theta; \delta\theta) = \delta A^I{}_J \wedge \star F^J{}_I$$

$$\begin{aligned}\big(\delta A^I{}_J \wedge \star F^J{}_I\big)(f\theta; f\delta\theta) &= \Big(\big(h^{-1}\delta A h\big)^I{}_J \wedge \star \big(h^{-1} F h\big)^J{}_I\Big)(\theta; \delta\theta) \\ &= \big(\delta A^I{}_J \wedge \star F^J{}_I\big)(\theta; \delta\theta)\end{aligned}$$

$$\Theta_{\mathrm{CYM}}(\theta; \delta\theta) = 2\delta\theta^a \wedge \star D P_a + \delta\Gamma^a{}_b \wedge \star C^b{}_a$$

Useful decomposition

$$L_{\mathrm{CYM}} = \frac{1}{4}\mathcal{E} + L_1$$

$$\mathcal{E}(\theta) := \epsilon^{abcd}\mathcal{R}_{ab} \wedge \mathcal{R}_{cd} \qquad \mathcal{R}^a{}_b := \frac{1}{2}R^a{}_{bcd}\theta^a \wedge \theta^d = \mathrm{d}\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b.$$

$$\delta\mathcal{E}(\theta) \;=\; \mathrm{d}\Big(2\epsilon^{abcd}\delta\Gamma_{ab} \wedge \mathcal{R}_{cd}\Big)$$

$$\Theta_{\mathcal{E}}(\theta;\delta\theta) := 2\epsilon^{abcd}\delta\Gamma_{ab} \wedge \mathcal{R}_{cd}$$

$$L_1(\theta) \;:= -4P^{[a}{}_a P^{b]}{}_b \mathrm{Vol} \qquad \delta L_1(\theta) = 2\delta\theta^a \wedge B_{ab}\star\theta^b + \mathrm{d}\Theta_1$$

$$\Theta_1(\theta;\delta\theta) \;= 2\delta\theta^a \wedge \star DP_a + \epsilon^{abcd}\delta\Gamma_{ab} \wedge \theta_c \wedge P_d$$

$$\Theta_{\mathrm{CYM}} = \frac{1}{4}\Theta_{\mathcal{E}} + \Theta_1$$

Einstein ⟶ Bach flat

$$R_{ab} = \Lambda \eta_{ab} \qquad P_a = P_{ab}\theta^b = -\frac{\Lambda}{6}\theta_a$$

$$\Theta_{\mathrm{CYM}}(\theta;\delta\theta) = \delta\Gamma^a{}_b \wedge \star C^b{}_a \qquad \Theta_1(\theta;\delta\theta) = -\frac{\Lambda}{6}\epsilon^{abcd}\theta_a \wedge \theta_b \wedge \delta\Gamma_{cd}$$

We want to compare it with the Einstein theory symplectic potential dencity

$$L_{\mathrm{EH}} = \frac{1}{16\pi G}\left(\frac{1}{2}\epsilon^{abcd}\theta_a \wedge \theta_b \wedge \mathcal{R}_{ab} - 2\Lambda \mathrm{Vol}\right)$$

$$16\pi G \delta L_{\mathrm{EH}}(\theta) = \delta\theta_a \wedge \left(\epsilon^{abcd}\theta_b \wedge \mathcal{R}_{cd} - 2\Lambda \star\theta^a\right) + \mathrm{d}\left(\frac{1}{2}\epsilon^{abcd}\theta_a \wedge \theta_b \wedge \delta\Gamma_{cd}\right)$$

$$\Theta_{\mathrm{EH}}(\theta;\delta\theta) = \frac{1}{32\pi G}\epsilon^{abcd}\theta_a \wedge \theta_b \wedge \delta\Gamma_{cd}$$

$$\Theta_{\mathrm{CYM}} = \frac{1}{4}\Theta_{\mathcal{E}} - \frac{16\pi G\Lambda}{3}\Theta_{\mathrm{EH}}$$

At the scri of asymptotically (A) de Sitter spacetime

The Fefferman-Graham coordinates

$$R_{ab} = \Lambda \eta_{ab} \qquad \Lambda > 0$$

$$g = \frac{\ell^2}{\rho^2}\left(-\,\mathrm{d}\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)}(\rho, x^1, x^2, x^3)\,\mathrm{d}x^i\,\mathrm{d}x^j\right)$$

$$\mathscr{I} \; : \; \rho = 0$$

$$\hat{g} := \frac{\rho^2}{\ell^2} g = -\,\mathrm{d}\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

$$\Theta_{\mathrm{CYM}}(\theta, \delta\theta) = \Theta_{\mathrm{CYM}}(\rho\theta, \rho\delta\theta)$$

$\Rightarrow$ **finiteness at** $\rho = 0$

The pullback of $\Theta_{\text{CYM}}(\theta^a, \delta\theta^a)$ on the scri

$$g = \frac{\ell^2}{\rho^2}\left(-\,\mathrm{d}\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)}(\rho, x^1, x^2, x^3)\,\mathrm{d}x^i\,\mathrm{d}x^j\right)$$

$$g_{ij}^{(1)} = 0, \qquad g_{ij}^{(2)} = \mathring{R}_{ij} - \frac{1}{4}\mathring{g}_{ij}\mathring{R} =: \mathring{S}_{ij} \qquad \mathring{T}_{ij} := g_{ij}^{(3)}$$

$$\Theta_{\text{CYM}}(\theta^a, \delta\theta^a) = \delta\Gamma^b{}_c \wedge *C^c{}_b$$

$$\Gamma^a{}_b \;= 2\rho^{-1}\eta^{ac}\hat{e}^{\rho}_{[c}\hat{e}^{\beta}_{b]}\hat{g}_{\beta\gamma}\,\mathrm{d}x^\gamma + \mathcal{O}(1)$$

Pullback on $\mathscr{I}$ $\qquad \mathring{\text{Vol}} := \frac{1}{3!}\mathring{\epsilon}_{ijk}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k.$

$$\hat{\epsilon}_{\rho ijk} = \sqrt{|\det \hat{g}|}\varepsilon_{\rho ijk} = \sqrt{\det \mathring{g}}\varepsilon_{ijk} + \mathcal{O}(\rho^2) = \mathring{\epsilon}_{ijk} + \mathcal{O}(\rho^2),$$

$$\tilde{\Theta}_{\text{CYM}} = \lim_{\rho\to 0} \delta\Gamma^a{}_{bi}\hat{e}^\alpha_a\hat{\theta}^b_\beta\,\mathrm{d}x^i \wedge \left(\frac{1}{2}\hat{C}^\beta{}_\alpha{}^{\gamma\delta}\hat{\epsilon}_{\gamma\delta jk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k\right) \quad = \frac{3}{2}\delta\mathring{g}_{ij}\mathring{T}^{ij}\mathring{\text{Vol}}.$$

Comparison with the standard

$$g = \frac{\ell^2}{\rho^2}\left(-\,\mathrm{d}\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)}(\rho, x^1, x^2, x^3)\,\mathrm{d}x^i\,\mathrm{d}x^j\right)$$

The pullback on $\mathscr{I}$

$$\tilde{\Theta}_{\mathrm{CYM}} = \lim_{\rho\to 0} \delta\Gamma^a{}_{bi}\hat{e}^\alpha_a\hat{\theta}^b_\beta\,\mathrm{d}x^i \wedge \left(\frac{1}{2}\hat{C}^\beta{}_\alpha{}^{\gamma\delta}\hat{\epsilon}_{\gamma\delta jk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k\right) \qquad = \frac{3}{2}\delta\mathring{g}_{ij}\mathring{T}^{ij}\mathring{\mathrm{Vol}}.$$

The standard defnitions: $\quad T_{ij} = \frac{3\ell^2}{16\pi G}\mathring{T}_{ij}$ $\qquad = \frac{8\pi G}{\ell^2}\delta\mathring{g}_{ij}T^{ij}\mathring{\mathrm{Vol}}.$

$$S_{\mathrm{GR}} = \frac{1}{16\pi G}\int_M (R-2\Lambda)\mathrm{Vol} + \frac{1}{16\pi G}\int_{\mathscr{I}}\left(2K + \frac{4}{\ell} - \mathring{R}\right)\mathring{\mathrm{Vol}}$$

$$\tilde{\Theta}_{\mathrm{GR}} = \frac{1}{2}\delta\mathring{g}_{ij}T^{ij}\mathring{\mathrm{Vol}}.$$

$$\tilde{\Theta}_{\mathrm{CYM}} = \frac{16\pi G\Lambda}{3}\tilde{\Theta}_{\mathrm{GR}}.$$

The Noether currents

Diffeomorphisms

ξ **- vector field tangent to the spacetime**

$$J_\xi(\theta) = \mathcal{L}_\xi A^I{}_J \wedge \star F^J{}_I - \xi \lrcorner \left(\frac{1}{2} F^I{}_J \wedge \star F^J{}_I \right)$$

$$J_\xi(\theta) = \mathrm{d}\big((\xi \lrcorner A^I{}_J) \star F^J{}_I \big) - (\xi \lrcorner A^I{}_J) D_A \star F^J{}_I$$

$$Q_\xi(\theta) = (\xi \lrcorner A^I{}_J) \star F^J{}_I$$

l **- generator of the Lotentz rotations or conformal rescalings**

$$J_l = d(l^I{}_J \wedge * F^J{}_I) - l^I{}_J \wedge D_A * F^I{}_J$$

$$= d(l^a{}_b * C^b{}_a)$$ **for the Lorentz**

Summary

Thank you!