Spherically-Symmetric Gravity on a Graph

Jorden Roberts

Klaus Liegener & Saeed Rastgoo

Loops '24 – Florida Atlantic University

UNIVERSITY OF ALBERTA

Introduction

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- Continuum Theory
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- Discretized Dynamics & Symmetry Restriction
- Application to FLRW Cosmology

Continuum Phase Space

Full Ashtekar-Barbero phase space[1], $\mathcal{M}_{AB} = \text{span}\left\{\left(A^I_i, E^j_J\right)\right\}$:

- Symplectic form[2]

$$\omega_{AB} = \frac{2}{\kappa\beta}\int_\Sigma dA^I_i \wedge dE^i_I \, d^3x$$

[1] Spatial indices $i, j, \ldots = 1, 2, 3$; $\mathfrak{su}(2)$ indices $I, J, \ldots = 1, 2, 3$

[2] Spacetime $\mathbb{R} \times \Sigma$; $\kappa = 16\pi G$; Immirzi parameter $\beta \in \mathbb{R}\backslash\{0\}$

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Spherically-symmetric subspace $\overline{\mathcal{M}}_{AB} = \text{span}\left\{(a, p_a), (b, p_b)\right\}$:

- Two θ, φ-independent canonical pairs
  - $a = a(r),\ p_a = p_a(r), \ldots$

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$$\overline{\omega}_{AB} = \omega_{AB}\big|_{\overline{\mathcal{M}}_{AB}} = \frac{2}{\kappa\beta}\int \left(da \wedge dp_a + db \wedge dp_b\right) \, dr$$

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Discretization: Gravity on a Graph

Define a set Γ_n^v of $n \in \mathbb{N}$ **vertices** v along each spatial axis

- Separated by coordinate-distances $\varepsilon_i = \varepsilon_i(n)$

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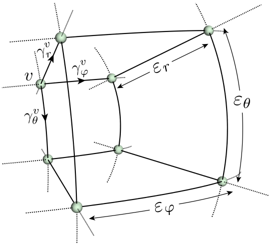
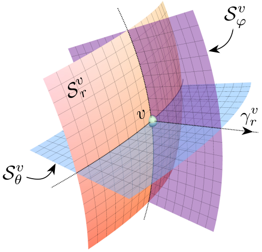


Parameterize **edges** $\gamma : \mathcal{I} \subset \mathbb{R} \to \Sigma$ between successive vertices, and **surfaces** $\mathcal{S}_\gamma : \mathcal{J}^2 \subset \mathbb{R}^2 \to \Sigma$ dual to each edge



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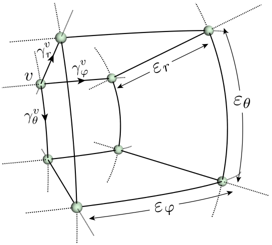

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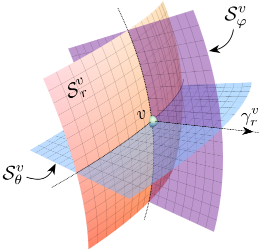
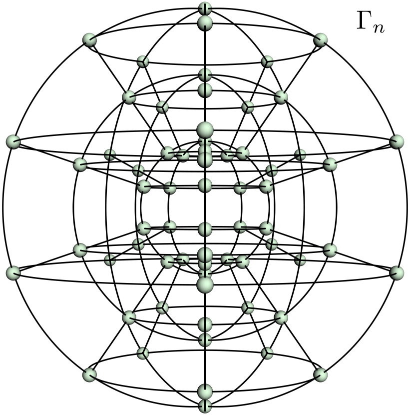
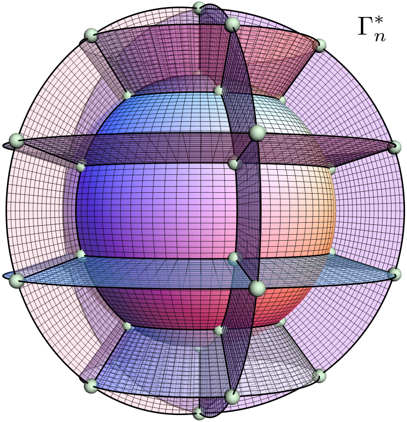
- **Graph** and **dual cell-complex**:

$$\Gamma_n = \coprod_{v \in \Gamma_n^v} \gamma, \qquad \Gamma_n^* = \coprod_{\gamma \in \Gamma_n} \mathcal{S}_\gamma$$



Spherical Graphs





Truncated Phase Space

$\mathcal{M}_\Gamma \cong \prod_\gamma T^*SU(2)$ is obtained via a **discretization map**,

$$\mathfrak{D}_\gamma : \mathcal{M}_{AB} \to SU(2) \times \mathfrak{su}(2) : (A, E) \mapsto \Big(\underbrace{\mathcal{P}\exp\left(\int_\gamma A\right)}_{h_\gamma}, \underbrace{\int_{\mathcal{S}_\gamma} \star E}_{P^\gamma} \Big)$$

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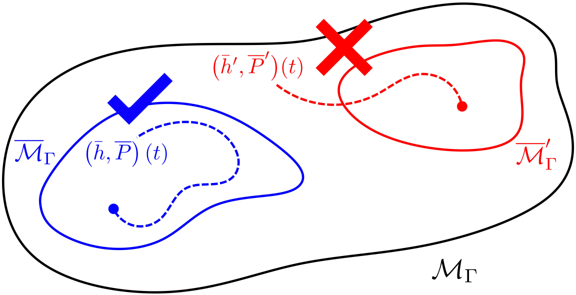
Fixed-graph scalar constraint, $C^n[N] = \sum_{v \in \Gamma_n^v} N(v)\, C^n(v)$:

$$C^n(v) = \frac{1}{16\kappa} \sum_{i,j,k} \epsilon(i,j,k) \Big[\mathcal{F}_K(v, \square_{ij}) - (1+\beta^2)\epsilon_{IJK} \mathcal{K}_i^I(v) \mathcal{K}_j^J(v) \Big] \mathcal{E}_k^K(v)$$

- $\mathcal{E}_\ell \sim \{h_{\gamma_\ell}, V[\Gamma_n]\}\, h^\dagger_{\gamma_\ell}$, $\mathcal{K}_\ell \sim \{h_{\gamma_\ell}, K[\Gamma_n]\}\, h^\dagger_{\gamma_\ell}$

Symmetry Restriction

For certain symmetry groups, symmetric configurations are *guaranteed* to remain symmetric under the flow generated by C^n



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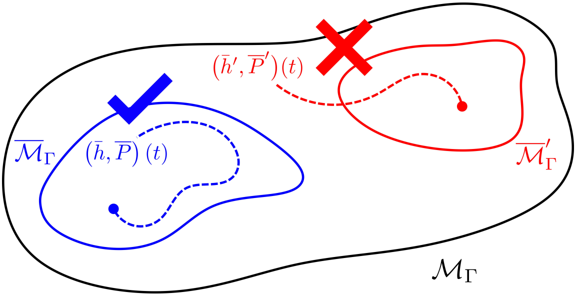

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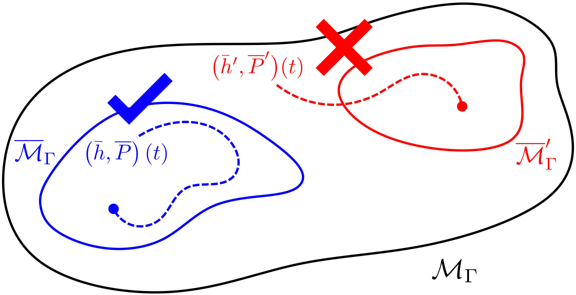
- Poisson brackets can be computed entirely on $\overline{\mathcal{M}}_\Gamma$
- Dynamical calculations are often significantly simplified



Discrete Spherical Symmetry

Group of graph-preserving diffeomorphisms for a spherical graph:

$$\mathbb{D}_\Gamma \cong \mathbb{Z}_2 \times D_n < O(3)$$

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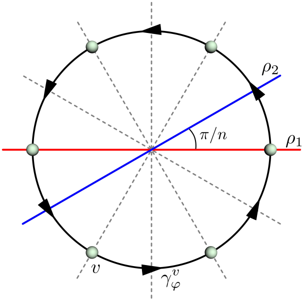

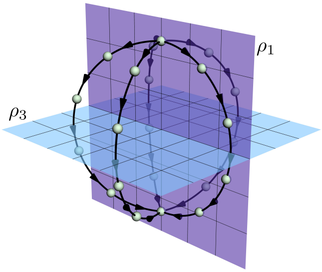

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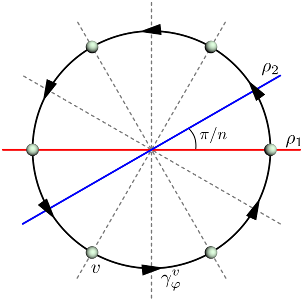
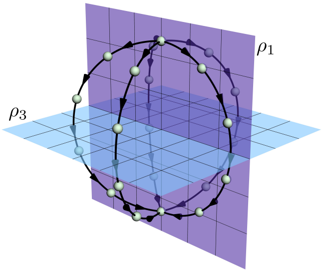
- ✷ $D_n \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_n$ is the symmetry group of a regular n-gon
- ✷ $\mathbb{D}_\Gamma$ is translated into a symmetry group $\Phi_\Gamma < \mathrm{Symp}(\mathcal{M}_\Gamma)$





Invariant Subspace

Spherical-graph discretization map applied to $(\overline{A}, \overline{E}) \in \overline{\mathcal{M}}_{AB}$ produces Φ_Γ-invariant loop variables $(\overline{h}_\gamma, \overline{P}_\gamma) \in \overline{\mathcal{M}}_\Gamma \subset \mathcal{M}_\Gamma$

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$$\begin{cases} \overline{h}_{\gamma_r} = \exp\left(\varepsilon_r\, \tilde{a}\, \tau_1\right) \\ \overline{h}_{\gamma_\theta} = \exp\left(\varepsilon_\theta\, \overline{A}^I_\theta\, \tau_I\right) \\ \overline{h}_{\gamma_\varphi} = \exp\left(\varepsilon_\varphi\, \overline{A}^I_\varphi\, \tau_I\right) \end{cases} \qquad \begin{cases} \overline{P}_{\gamma_r} = \varepsilon_\varphi \left[\cos\theta - \cos\left(\theta + \varepsilon_\theta\right)\right] p_a\, \tau_1 \\ \overline{P}_{\gamma_\theta} = \varepsilon_r \varepsilon_\varphi\, \tilde{p}_b\, \sin\theta\, \tau_2 \\ \overline{P}_{\gamma_\varphi} = \varepsilon_r \varepsilon_\theta\, \tilde{p}_b\, \tau_3 \end{cases}$$

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✶ $\tilde{a} = \frac{1}{4\pi} \int_{\mathcal{I}} a\left(\gamma_r(s)\right) \, ds$, $\tilde{p}_b = \int_{\mathcal{J}} p_b\left(\gamma_r(s)\right) \, ds$

Restricted Symplectic Structure

With Φ_Γ-invariant holonomies $\bar{h}_\gamma = \exp\left(f_\gamma^I \, \tau_I\right)$,

$$\overline{\omega}_\Gamma = \frac{2}{\kappa\beta} \sum_\gamma df_\gamma^J \wedge d\left[\overline{P}_\gamma^I \, \pi\left(\bar{h}_\gamma\right)_{IJ}\right]$$

- $\pi\left(\bar{h}_\gamma\right)_{IJ} = -2\,\mathrm{Tr}\left(\tau_I \, h_\gamma^\dagger \, \tau_J \, h_\gamma\right)$

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For sufficiently dense graphs,

$$\overline{\omega}_\Gamma \approx \frac{8\pi}{\kappa\beta} \sum_{r_v} \varepsilon_r \left[\cos^2(\varepsilon_\theta/2)\, d\tilde{a} \wedge dp_a + \frac{\varepsilon_\theta}{8\pi} \cot(\varepsilon_\theta/2)\, db \wedge d\tilde{p}_b\right]$$

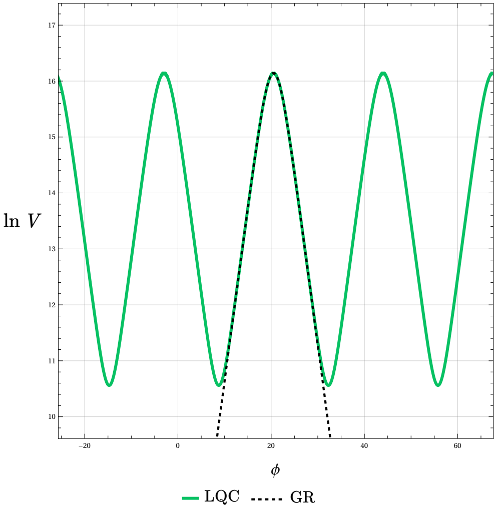
✴ Continuum limit: $\lim_{n\to\infty} \overline{\omega}_\Gamma = \overline{\omega}_{AB}$ ✓

Application to FLRW Cosmology

$k = 1$ Cosmology

- Symmetric models: periodic evolution

$\beta = 0.2375$



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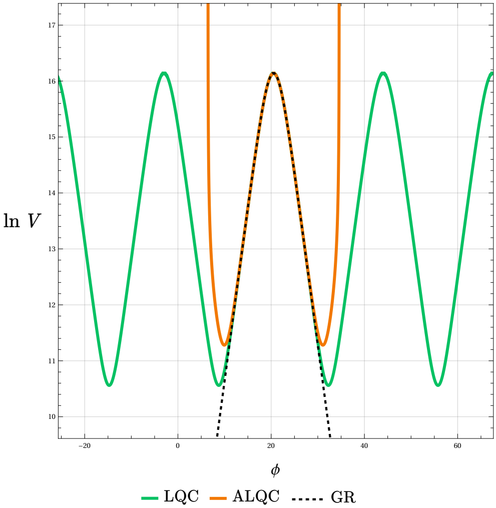

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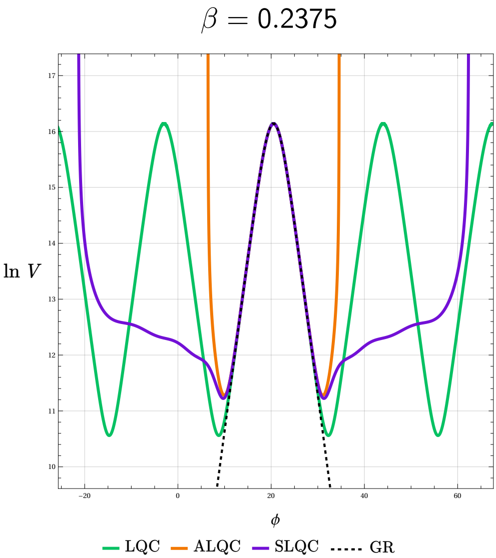
- ✶ Variations among asymmetric models beyond bounces



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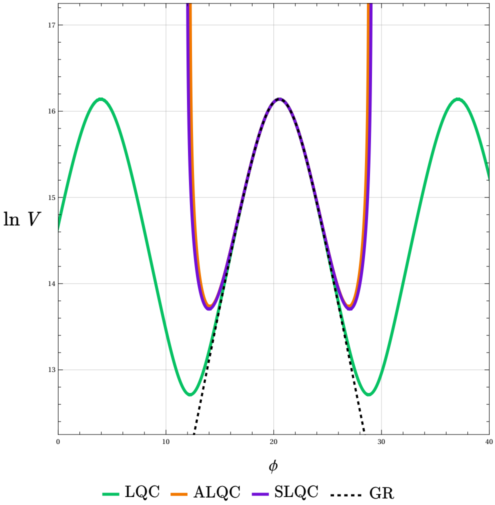
- ✶ Symmetric models: periodic evolution
- ✶ Asymmetric models: non-periodic evolution
- ✶ Variations among asymmetric models beyond bounces
  - Largely influenced by the value of β

$\beta = 1$



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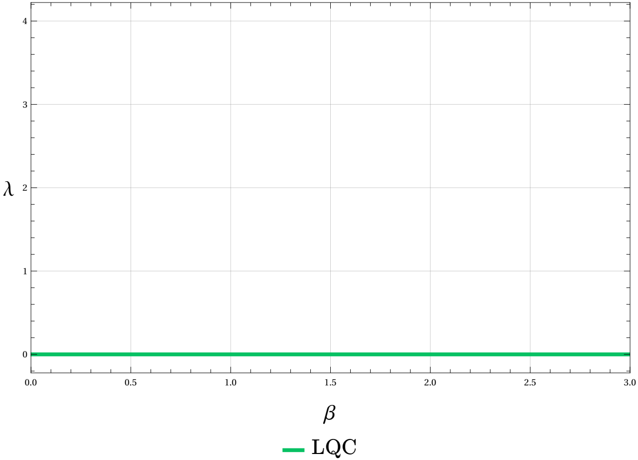
- ✶ Axisymmetric spacetimes
    - Cylindrical graphs
    - Rotating black holes

The End

$k = 1$ Cosmology

- Emergent cosmological constant in pre-bounce universe, $\Lambda_{\text{emerg}} \propto \lambda(\beta)$



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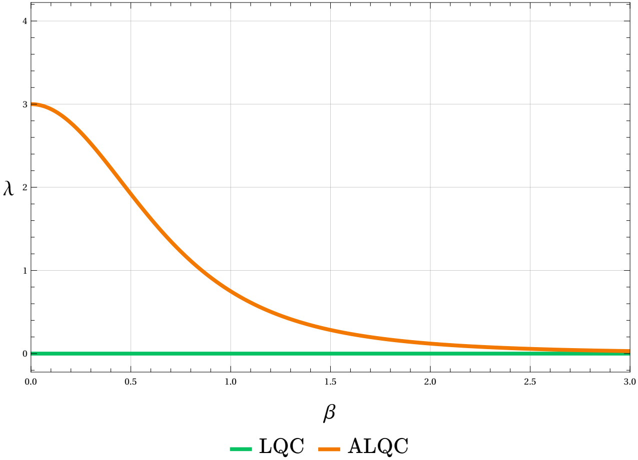

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