

# ***Spherically-Symmetric Gravity on a Graph***

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OF ALBERTA**

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- \* Continuum Theory
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- \* Discretized Dynamics & Symmetry Restriction
- \* Application to FLRW Cosmology

# Continuum Phase Space

Full Ashtekar-Barbero phase space<sup>1</sup>,  $\mathcal{M}_{AB} = \text{span} \left\{ \left( A_i^I, E_j^J \right) \right\}$ :

\* Symplectic form<sup>2</sup>

$$\omega_{AB} = \frac{2}{\kappa\beta} \int_{\Sigma} dA_i^I \wedge dE_j^J d^3x$$

---

<sup>1</sup>Spatial indices  $i, j, \dots = 1, 2, 3$ ;  $\mathfrak{su}(2)$  indices  $I, J, \dots = 1, 2, 3$

<sup>2</sup>Spacetime  $\mathbb{R} \times \Sigma$ ;  $\kappa = 16\pi G$ ; Immirzi parameter  $\beta \in \mathbb{R} \setminus \{0\}$

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Spherically-symmetric subspace  $\overline{\mathcal{M}}_{AB} = \text{span} \left\{ (a, p_a), (b, p_b) \right\}$ :

- \* Two  $\theta, \varphi$ -independent canonical pairs
  - $a = a(r), p_a = p_a(r), \dots$

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$$\overline{\omega}_{AB} = \omega_{AB}|_{\overline{\mathcal{M}}_{AB}} = \frac{2}{\kappa\beta} \int (da \wedge dp_a + db \wedge dp_b) dr$$

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## Discretization: Gravity on a Graph

Define a set  $\Gamma_n^v$  of  $n \in \mathbb{N}$  **vertices**  $v$  along each spatial axis

- \* Separated by coordinate-distances

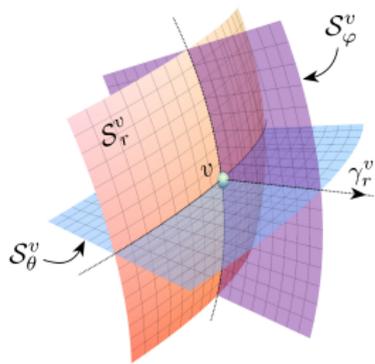
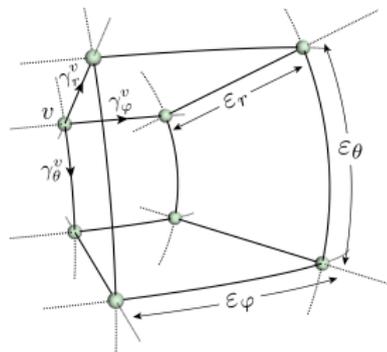
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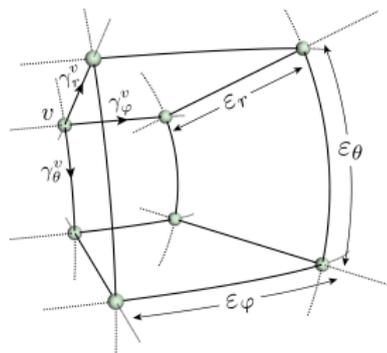
Parameterize **edges**  $\gamma : \mathcal{I} \subset \mathbb{R} \rightarrow \Sigma$   
between successive vertices, and **surfaces**  
 $\mathcal{S}_\gamma : \mathcal{J}^2 \subset \mathbb{R}^2 \rightarrow \Sigma$  dual to each edge



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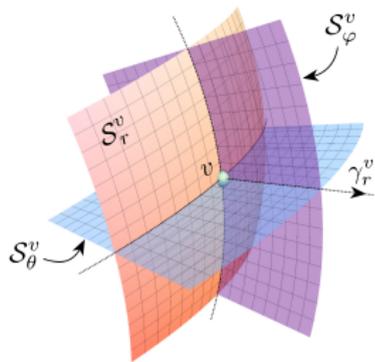
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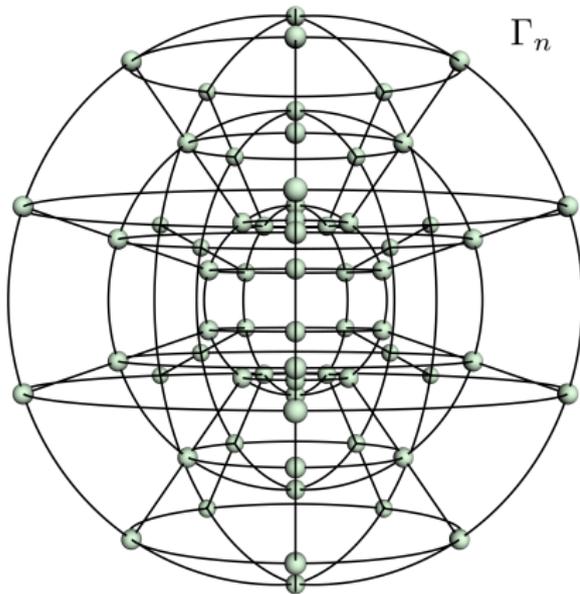
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- \* **Graph** and **dual cell-complex**:

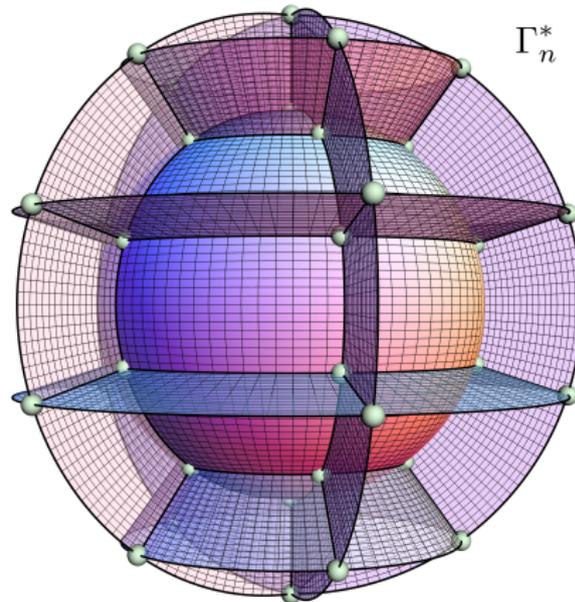
$$\Gamma_n = \coprod_{v \in \Gamma_n^v} \gamma, \quad \Gamma_n^* = \coprod_{\gamma \in \Gamma_n} \mathcal{S}_\gamma$$



# Spherical Graphs



$\Gamma_n$



$\Gamma_n^*$

## Truncated Phase Space

$\mathcal{M}_\Gamma \cong \prod_\gamma T^*SU(2)$  is obtained via a **discretization map**,

$$\mathcal{D}_\gamma : \mathcal{M}_{AB} \rightarrow SU(2) \times \mathfrak{su}(2) : (A, E) \mapsto \left( \underbrace{\mathcal{P} \exp \left( \int_\gamma A \right)}_{h_\gamma}, \underbrace{\int_{\mathcal{S}_\gamma} \star E}_{P_\gamma} \right)$$

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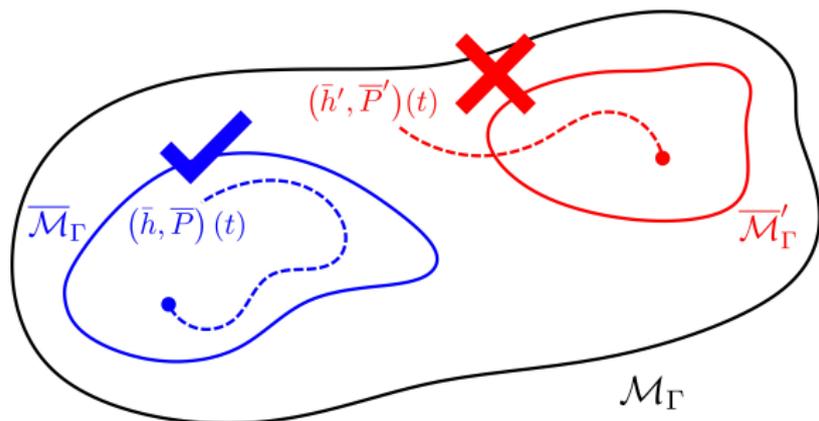
Fixed-graph scalar constraint,  $C^n[M] = \sum_{v \in \Gamma_n^v} N(v) C^n(v)$ :

$$C^n(v) = \frac{1}{16\kappa} \sum_{i,j,k} \epsilon(i,j,k) \left[ \mathcal{F}_K(v, \square_{ij}) - (1 + \beta^2) \epsilon_{IJK} \mathcal{K}_i^I(v) \mathcal{K}_j^J(v) \right] \mathcal{E}_k^K(v)$$

$$* \quad \mathcal{E}_\ell \sim \{h_{\gamma_\ell}, V[\Gamma_n]\} h_{\gamma_\ell}^\dagger, \quad \mathcal{K}_\ell \sim \{h_{\gamma_\ell}, K[\Gamma_n]\} h_{\gamma_\ell}^\dagger$$

## Symmetry Restriction

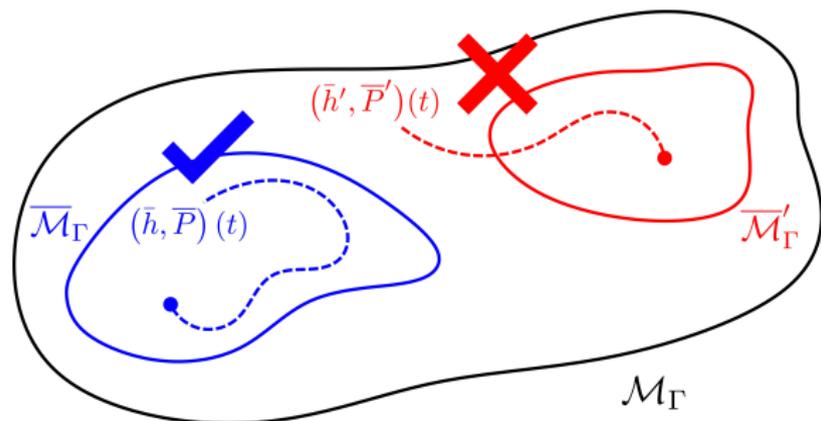
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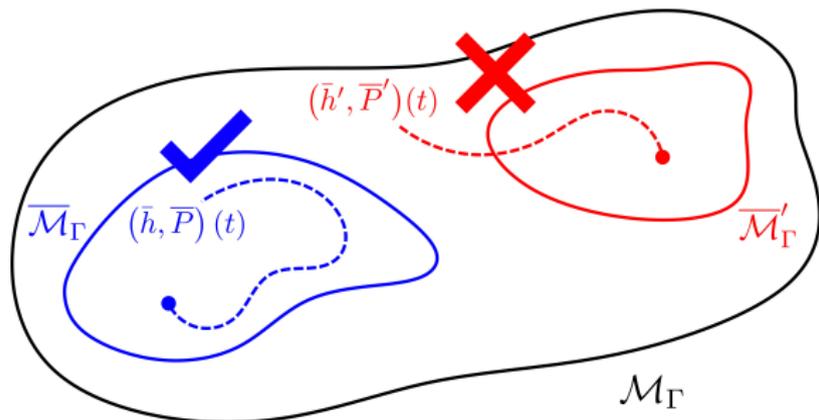
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- Poisson brackets can be computed entirely on  $\overline{\mathcal{M}}_\Gamma$
- Dynamical calculations are often significantly simplified



## Discrete Spherical Symmetry

Group of graph-preserving diffeomorphisms for a spherical graph:

$$\mathbb{D}_\Gamma \cong \mathbb{Z}_2 \times D_n < O(3)$$

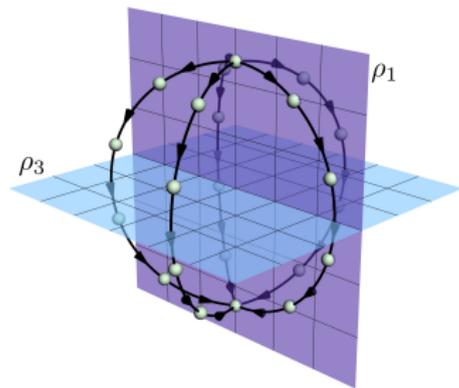
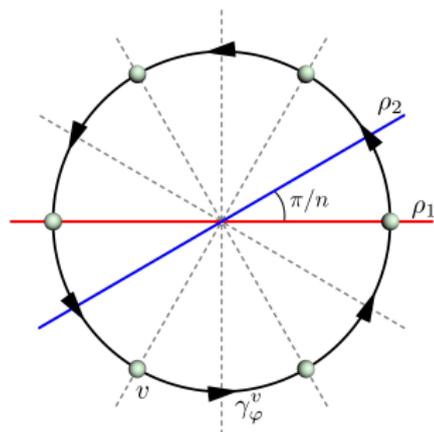
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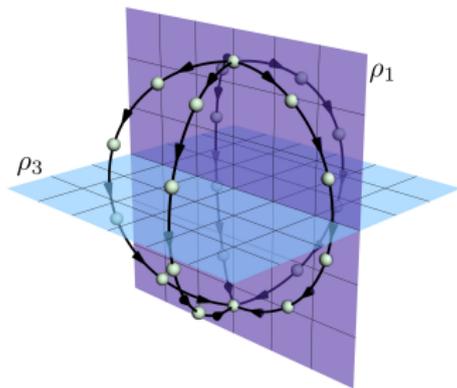
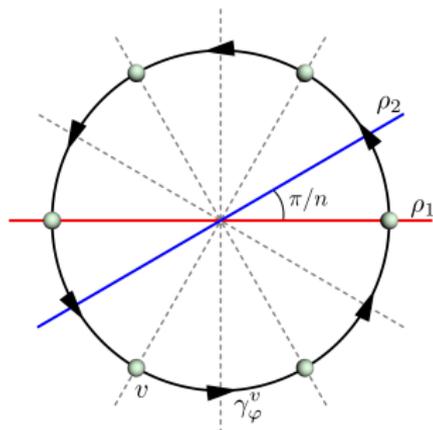


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- \*  $D_n \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_n$  is the symmetry group of a regular  $n$ -gon
- \*  $\mathbb{D}_\Gamma$  is translated into a symmetry group  $\Phi_\Gamma < \text{Symp}(\mathcal{M}_\Gamma)$



## Invariant Subspace

Spherical-graph discretization map applied to  $(\bar{A}, \bar{E}) \in \bar{\mathcal{M}}_{AB}$  produces  $\Phi_\Gamma$ -invariant loop variables  $(\bar{h}_\gamma, \bar{P}_\gamma) \in \bar{\mathcal{M}}_\Gamma \subset \mathcal{M}_\Gamma$

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$$\left\{ \begin{array}{l} \bar{h}_{\gamma_r} = \exp(\varepsilon_r \tilde{a} \tau_1) \\ \bar{h}_{\gamma_\theta} = \exp(\varepsilon_\theta \bar{A}'_\theta \tau_1) \\ \bar{h}_{\gamma_\varphi} = \exp(\varepsilon_\varphi \bar{A}'_\varphi \tau_1) \end{array} \right. \quad \left\{ \begin{array}{l} \bar{P}_{\gamma_r} = \varepsilon_\varphi [\cos \theta - \cos(\theta + \varepsilon_\theta)] p_a \tau_1 \\ \bar{P}_{\gamma_\theta} = \varepsilon_r \varepsilon_\varphi \tilde{p}_b \sin \theta \tau_2 \\ \bar{P}_{\gamma_\varphi} = \varepsilon_r \varepsilon_\theta \tilde{p}_b \tau_3 \end{array} \right.$$

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$$* \quad \tilde{a} = \frac{1}{4\pi} \int_{\mathcal{I}} a(\gamma_r(s)) ds, \quad \tilde{p}_b = \int_{\mathcal{J}} p_b(\gamma_r(s)) ds$$

## Restricted Symplectic Structure

With  $\Phi_\Gamma$ -invariant holonomies  $\bar{h}_\gamma = \exp(f_\gamma^I \tau_I)$ ,

$$\bar{\omega}_\Gamma = \frac{2}{\kappa\beta} \sum_\gamma df_\gamma^J \wedge d \left[ \bar{P}_\gamma^I \pi(\bar{h}_\gamma)_{IJ} \right]$$

$$* \pi(\bar{h}_\gamma)_{IJ} = -2 \text{Tr} \left( \tau_I h_\gamma^\dagger \tau_J h_\gamma \right)$$

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For sufficiently dense graphs,

$$\bar{\omega}_\Gamma \approx \frac{8\pi}{\kappa\beta} \sum_{r_v} \varepsilon_r \left[ \cos^2(\varepsilon_\theta/2) d\tilde{a} \wedge dp_a + \frac{\varepsilon_\theta}{8\pi} \cot(\varepsilon_\theta/2) db \wedge d\tilde{p}_b \right]$$

$$* \text{Continuum limit: } \lim_{n \rightarrow \infty} \bar{\omega}_\Gamma = \bar{\omega}_{AB} \checkmark$$

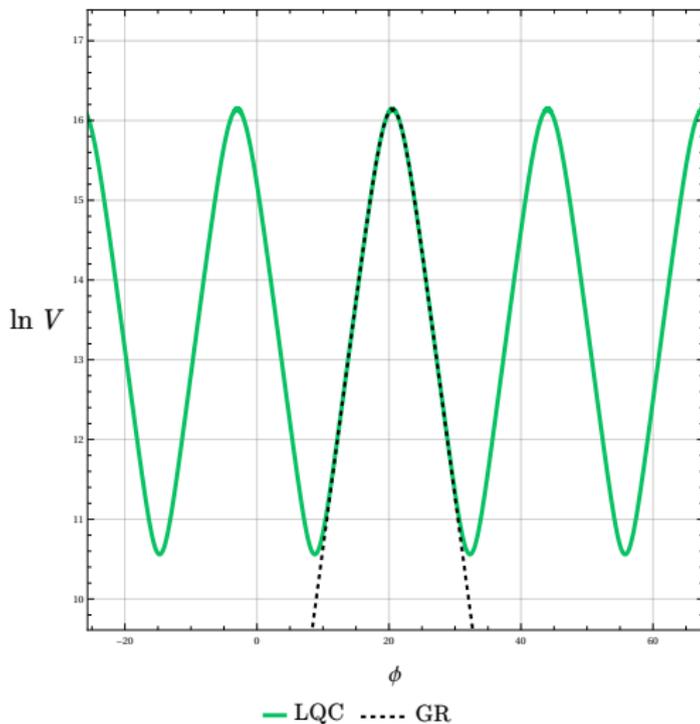
The background features a dark purple and blue grid that recedes into the distance, creating a perspective effect. Scattered throughout are numerous bright, multi-pointed starbursts in shades of purple and blue. In the center, there is a large, glowing inverted triangle. This triangle is composed of three concentric, slightly offset outlines. The outermost and innermost outlines are a vibrant purple, while the middle one is a lighter, more ethereal blue. The overall effect is that of a futuristic or cosmic graphic.

*Application to FLRW  
Cosmology*

# $k = 1$ Cosmology

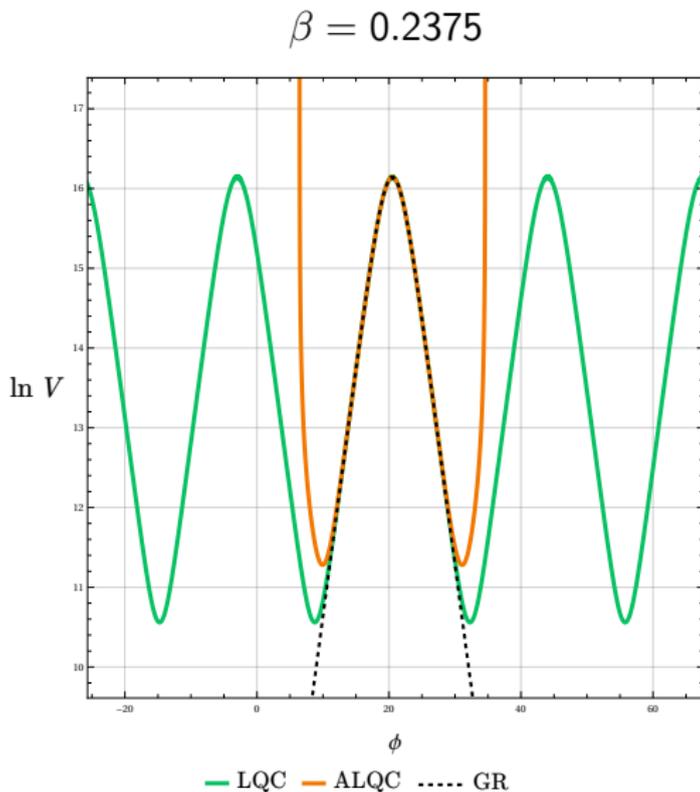
- \* Symmetric models: periodic evolution

$$\beta = 0.2375$$



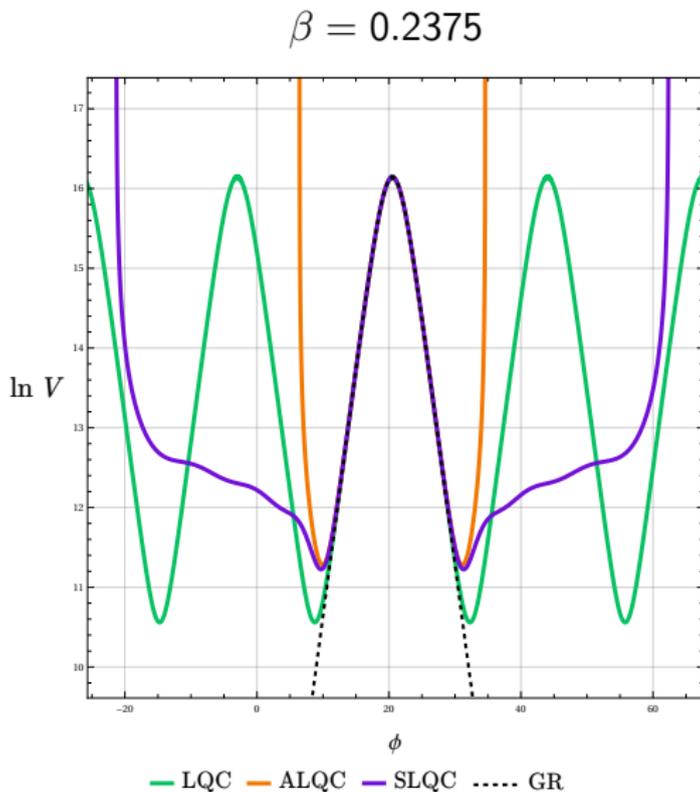
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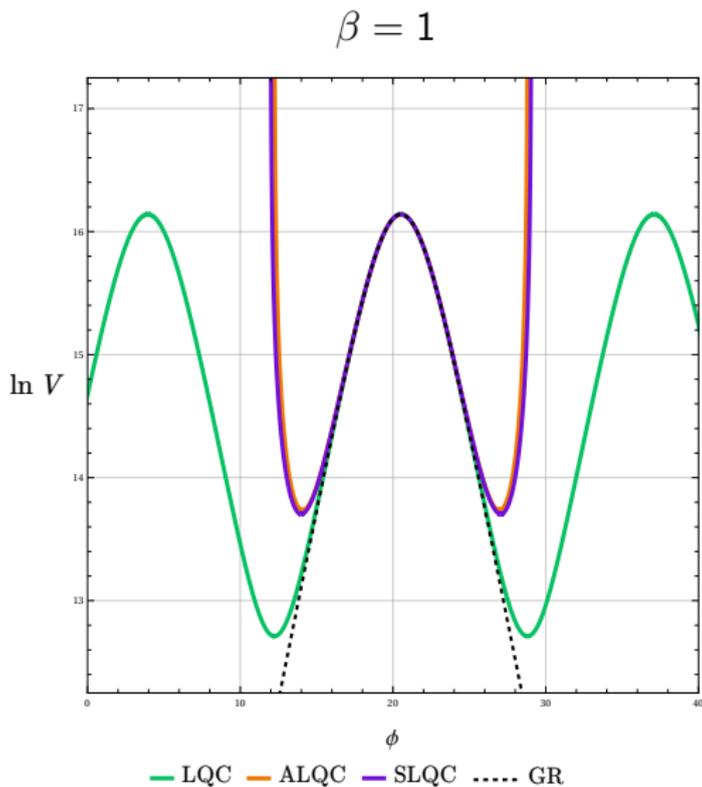
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- \* Variations among asymmetric models beyond bounces



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- \* Asymmetric models: non-periodic evolution
- \* Variations among asymmetric models beyond bounces
  - Largely influenced by the value of  $\beta$



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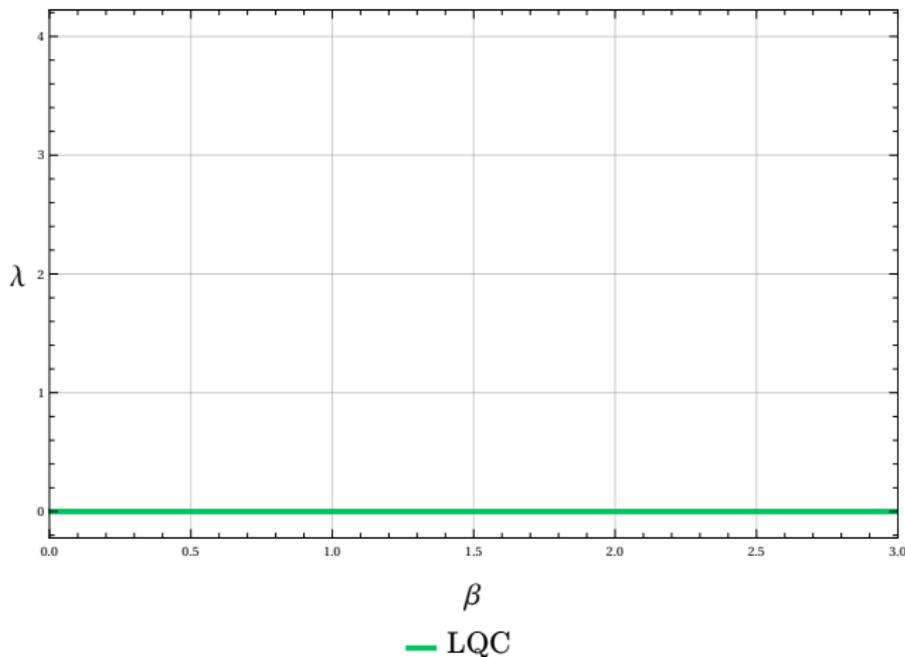
- \* Discretized black holes
  - Nature of horizon(s)
  - Quantum corrections to BH shadows
- \* Axisymmetric spacetimes
  - Cylindrical graphs
  - Rotating black holes

The image features a vibrant, futuristic background with a purple and blue grid pattern that recedes into the distance. In the center, there is a large, glowing inverted triangle composed of three nested, luminous outlines. Below this graphic, a white rectangular box with a thin red border contains the text "The End" in a bold, italicized, purple font.

***The End***

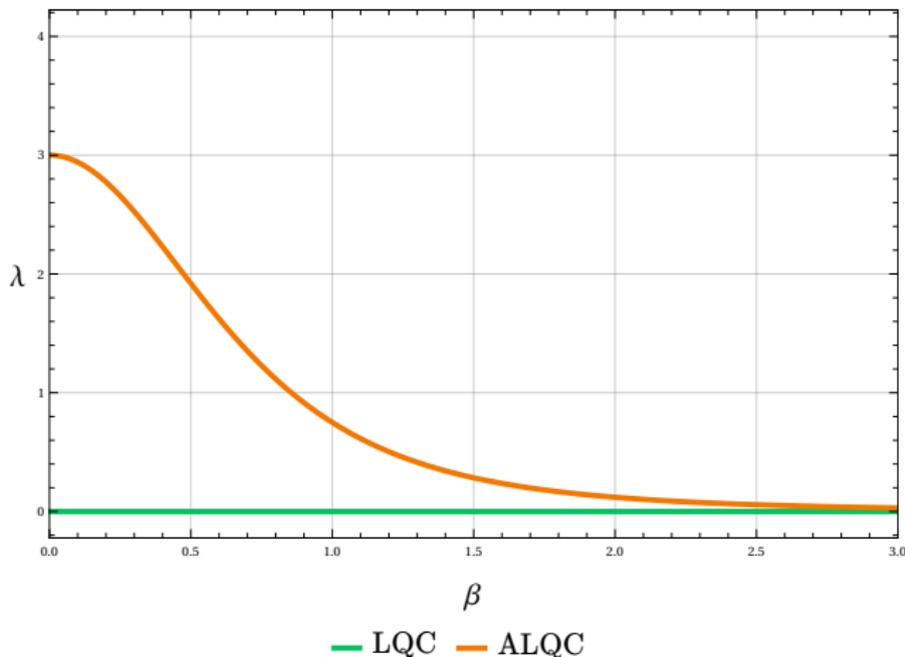
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