Higher Order Perturbation Theory in General Relativity with Applications to Black Holes

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FAU

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Motivation

- Exactly solvable models within general relativity are rare
- These models are highly symmetric and characterised by their symmetry group:
  - Cosmology: Isotropy and Homogeneity
  - Non-Rotating Black Holes: Spherical Symmetry
  - Rotating Black Holes: axial symmetry
- Non-Symmetric degrees of freedom cannot be treated exactly $\rightarrow$ perturbation theory

The formalism should address the following issues:

1. **Backreaction**:
   - Background is in general not fixed but dynamical
   - Perturbations influence the background dynamics
2. **Gauge Invariance**:
   - General Relativity is a gauge theory
   - Distinguish between observable (true) and non-observable (gauge) degrees of freedom

The general formalism

The procedure works as follows

1. Fix symmetry group of class of exact solutions
   - Symmetric variables (background): zero modes under the action of the symmetry group
   - Non-symmetric variables (perturbations): non-zero modes
   - Split test functions: symmetric test functions f and non-symmetric test functions g
   - Split constraints: symmetric constraints C and non-symmetric constraints Z
2. Split symmetric and non-symmetric variables into observable (true) and non-observable (gauge) degrees of freedom. Notation:

	True	**Gauge**	
Symmetric	(Q, P)	(q, p)	"background"
Non-Symmetric	(X, Y)	(x, y)	"perturbations"

3. Apply reduced phase space formulation
   - Select gauge fixing conditions, $q = q_*$ and $x = x_*$
   - Solve symmetric constraints C for p and non-symmetric constraints Z for y
   - Determine f_*, g_* through stability condition of the gauge fixing
4. Boundary term analysis
   - Define decay properties of fields
   - Require counter boundary term $B(f, g)$
5. Physical Hamiltonian H: For any function $F(Q, P, X, Y)$ of the true degrees of freedom

$$\{H, F\} = \{C(f) + Z(g) + B(f, g), F\}_{p=p_*, q=q_*, f=f_*, y=y_*, x=x_*, g=g_*}$$

6. Study the physics of H

The general formalism

Classical Theory:

- Disentangle treatment of gauge invariance from perturbation theory
- No need to define notion of n-th order gauge invariance
- Hamiltonian H computable to any order in X, Y: $H = H_0 + H_1 + H_2 + \ldots$

Quantum Theory:

- Non perturbative quantisation of Q, P
- Perturbative treatment of X, Y (c.f. hybrid quantum cosmology)

Remark:

- For Cosmology: Our approach equivalent to hybrid quantum cosmology if only partial reduction is performed [Agullo,Ashtekar,Gomar,Martín-Benito,Mena Marugán,Navascués,Singh]
- Here: Full reduction including symmetric constraints in principle to all orders
- For full reduction, no issues with closure of constraint algebra of remaining symmetric constraints

We apply the formalism to non-rotating black holes in vacuum:

Step 1 - The symmetry group:

- Schwarzschild black hole is spherically symmetric → rotation group $SO(3)$
- Work in ADM formalism (induced metric $m_{\mu\nu}$ and conjugate momentum $W^{\mu\nu}$)
- Expand the variables in terms of spherical scalar, vector and tensor harmonics:

[Freeden,Gervens,Gutting,Schreiner]

$$m_{33} = e^{2\mu} + \sum_{l\geq 1,m} x^v_{lm} L_{lm} \qquad \frac{W^{33}}{\sqrt{\Omega}} = \frac{e^{-2\mu}}{2}\pi_\mu + \sum_{l\geq 1,m} y^v_{lm} L_{lm}$$

$$m_{3A} = \sum_{l\geq 1,m,I} x^I_{lm}[L_{I,lm}]_A \qquad \frac{W^{3A}}{\sqrt{\Omega}} = \frac{1}{2}\sum_{l\geq 1,m,I} y^I_{lm} L^A_{I,lm}$$

$$m_{AB} = e^{2\lambda}\Omega_{AB} + \sum_{l\geq 1,m} x^h_{lm}[L_{h,lm}]_{AB} + \sum_{l\geq 2,m,I} X^I_{lm}[L_{I,lm}]_{AB} \qquad \frac{W^{AB}}{\sqrt{\Omega}} = \frac{e^{-2\lambda}}{4}\Omega^{AB}\pi_\lambda + \frac{1}{2}\sum_{l\geq 1,m} y^h_{lm} L^{AB}_{h,lm} + \sum_{l\geq 2,m,I} Y^I_{lm} L^{AB}_{I,lm}$$

- Background degrees of freedom are spherically symmetric: (μ, π_μ) and (λ, π_λ)
- Perturbation degrees of freedom:(x, y) and (X, Y)
- Symmetric constraints: symmetric Hamiltonian constraint C_v and symmetric radial diffeomorphism constraint C_h
- Non-symmetric constraints: non-symmetric Hamiltonian constraint Z_v, non-symmetric radial diffeomorphism constraint Z_h and angular diffeomorphism constraints $Z_{e/o}$

Application to Black Holes – Steps 2 & 3

Step 2 - Selection of gauge degrees of freedom:

	True	**Gauge**
Symmetric	M	(μ, π_μ), (λ, π_λ)
Non-Symmetric	(X, Y)	(x, y)

Step 3 - The reduced phase formulation:

- Choose Gullstrand Painlevé (GP) gauge

$$m_{33} = 1, \qquad m_{3A} = 0, \qquad \Omega^{AB} m_{AB} = 2r^2 \,,$$

  where $\Omega_{AB} = \mathrm{diag}(1, \sin^2\theta)$ is the metric on the sphere
- Advantage: GP coordinates non-singular at black hole horizon $\rightarrow$ Explore interior of black hole
- Can work with 2 asymptotic ends $\rightarrow$ black to white hole transition
- Symmetric constraints: Solve C_v and C_h for π_μ, π_λ
- Non-symmetric constraints: Solve $Z_v, Z_h, Z_{e/o}$ for $y_v, y_h, y_{e/o}$
- In this step: Iterative solution order by order in the form $\pi_\mu = \pi_\mu^{(0)} + \pi_\mu^{(1)} + \ldots$ and similarly for the other variables

Application to Black Holes – Step 3

Solution of the constraints order by order for the non-rotating black hole:

Zeroth Order:

- Only symmetric constraints C_v and C_h
- The solution depends on an integration constant M:

$$\pi_\mu^{(0)} = 4\sqrt{2Mr}$$
$$\pi_\lambda^{(0)} = 2\sqrt{2Mr}$$

- This is precisely the Schwarzschild solution with mass M in GP coordinates

First Order:

- Only non-symmetric constraints non-vanishing
- Determine $y_v^{(1)}, y_h^{(1)}, y_{e/o}^{(1)}$ as linear functions of X, Y

Second Order:

- For physical Hamiltonian: Only need to consider the second order symmetric constraints
- We obtain a solution for $\pi_\mu^{(2)}, \pi_\lambda^{(2)}$ in terms of X, Y

Step 5 - The physical Hamiltonian (for one asymptotic end):

- Boundary term analysis yields physical Hamiltonian

$$H = \lim_{r\to\infty} \frac{\pi}{2\kappa r}\pi_\mu^2 = \lim_{r\to\infty} \frac{\pi}{2\kappa r}\Big((\pi_\mu^{(0)})^2 + 2\pi_\mu^{(0)}\pi_\mu^{(2)} + O(3)\Big)\,,$$

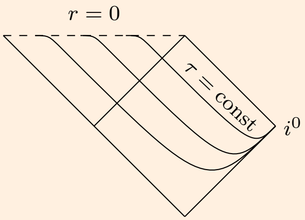
  where $\kappa = 16\pi$ is the gravitational coupling constant
- Zeroth order: $H_0 = M$ (ADM mass)
- Second order:

$$H_2 = \frac{1}{\kappa}\sum_{l\geq 2,m,I}\int_{\mathbb{R}^+} \mathrm{d}r\, N^3 \tilde{Y}^I_{lm}\partial_r \tilde{X}^I_{lm} + \frac{N}{2}\Big((\tilde{Y}^I_{lm})^2 + (\partial_r \tilde{X}^I_{lm})^2 + V_I(\tilde{X}^I_{lm})^2\Big)\,,$$

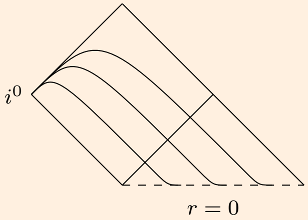
  where $N^3 = \sqrt{\frac{2M}{r}}$, $N = 1$ and $\tilde{X}, \tilde{Y}$ related to X, Y via canonical transformation

- Black Hole perturbation theory well established to second order both in Lagrangian [Regge, Wheeler, Zerilli,...] and Hamiltonian formulation [Moncrief,Brizuela,Martín-García,...]
- Agreement of $H_0 + H_2$ with these works after transforming from GP to Schwarzschild coordinates
- Virtue of this approach: Immediately extensible to higher orders

- Generalisation to Higher Order Perturbations: Interacting gravitational waves $X^2Y, X^3, \ldots$ work in progress
- Extension to Standard Model matter, e.g. electromagnetic field [JN&TT]
- Fock quantisation with respect to Gullstrand Painlevé free-falling observer:
  - GP-time $\tau = \mathrm{const}$ hypersurfaces foliate black hole spacetime
  - **But**: hypersurfaces are **not** Cauchy surfaces
  - Glue outgoing and ingoing GP spacetimes (black hole – white hole transition)
  - Mode system: eigenvalue equation similar to Schrödinger equation for singular potential
  - Possibly regularisation at $r = 0$ (singularity) needed:
  - → New orthonormal basis for singular Schrödinger operators [JN&TT]
  - → Methods of LQC type quantisation of Kantowski-Sachs [Ashtekar,Bodendorfer,Gambini,Haggard,Olmedo,Pullin,Rovelli,Singh,Vidotto]
  - → Methods from dust collapse models [Wilson-Ewing,Hussain]



Ingoing GP coordinates



Outgoing GP coordinates

- Perturbative Expansion and Fock quantisation of the area of apparent horizon

→ Signs of Black hole Evaporation? Decrease of the area?

Thank You!