A Chern-Simons approach to self-dual gravity in (2+1)-dimensions and quantisation of Poisson structure

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Outline

Motivation

Holst action in (2+1)-dimensions

Chern-Simons theory for the self-dual variables

Phase space discretisation and Poison structure

The algebra for $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$ on the space of holonomies

Quantisation and quantum symmetries

Motivation

- Implementation of the so called reality conditions in self-dual gravity
- Quantisation Chern-Simons theory with a complex connection

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3d Holst action from 4d action via symmetry reduction

Consider the action for 4d LQG

$$S_{\text{Holst}}[e,\omega] = \frac{1}{4}\int_M \left(\frac{1}{2}\epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} + \frac{1}{\gamma}\delta_{IJK} e^I \wedge e^J \wedge F^{IJ} \right),$$

- perform a space-time reduction without reducing the internal gauge group
- spacial component $\mu = 3$ is singled out and
- view the 4d space-time with topology $M^4 = M^3 \times \mathbb{I}$ where M^3 is a 3d space-time, and $\mathbb{I}$ is a space-like segment with coordinates x^3.
- impose the following conditions

$$\partial_3 = 0, \qquad \omega_3^{IJ} = 0.$$

- The 4d Holst action then reduces to

$$S^{3d}_{\text{Holst}}[\chi, e, \omega] = \int_{M^3} \mathbf{d}^3 x \epsilon^{\mu\nu\rho} \left(\frac{1}{2}\epsilon_{IJKL}\chi^I e^J_\mu F^{KL}_{\nu\rho} + \frac{1}{\gamma}\chi^I e^J_\mu F_{\nu\rho IJ} \right), \ \chi^I \equiv e^I_3$$

Marc Geiller, Karim Noui, Jibril Ben Achour, Chao Yu.

3d Holst action

The dynamical variable of the three dimensional Holst action is now a 1-form E on the 3d space-time with values in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{so}(4)$

$$E^{IJ}_{\mu} = \epsilon^{IJ}{}_{KL}\chi^{K} e^{L}_{\mu}.$$

Consider a decomposition of the 3d Holst action into its self- and anti-self-dual components

$$S^{3d}_{\text{Holst}}[E,A] = \left(\frac{\gamma+1}{\gamma}\right) S_{\mathbb{C}} + \left(\frac{\gamma-1}{\gamma}\right) \bar{S}_{\mathbb{C}},$$

where we have introduced a complex-valued action

$$S_{\mathbb{C}} = \int_{M^3} E \wedge F[A].$$

together with the self-dual variables

$$E^{j}_{\mu} = (\chi^{0} e^{j}_{\mu} - \chi^{j} e^{0}_{\mu}) + i\epsilon^{j}_{kl}\,\chi^{k} e^{l}_{\mu}, \quad A^{j}_{\mu} = \omega^{j}_{\mu} + i\omega^{(0)j}_{\mu},$$

valued in $\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$ or $\mathfrak{su}(2)_{\mathbb{C}}$.

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The Chern-Simons action for the self-dual variables

A CS theory on a $3d$ manifold requires:

- a gauge group
- Ad-invariant, non-degenerate, symmetric bilinear form on the Lie algebra of the gauge group

The Chern-Simons action for the self-dual variables

A CS theory on a $3d$ manifold requires:

- a gauge group
- Ad-invariant, non-degenerate, symmetric bilinear form on the Lie algebra of the gauge group

For a 3d space-time manifold M^3 with topology $\mathbb{R} \times \Sigma$, we consider the gauge group

$$H = SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathfrak{sl}(2,\mathbb{C})^*_{\mathbb{R}} = SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$$

parametrised by

$$(u, \boldsymbol{a}) = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \quad u \in SL(2,\mathbb{C})_{\mathbb{R}}, \boldsymbol{a}, \boldsymbol{j} \in \mathbb{R}^6.$$

Λ	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$(SU(2) \bowtie AN(2)) \ltimes \mathbb{R}^6$	$(SL(2,\mathbb{R}) \bowtie AN(2)) \ltimes \mathbb{R}^6$

Lie algebra for the gauge group

The Lie algebra $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \oplus \mathbb{R}^6$ and satisfy the relation

$$[\mathcal{J}_\alpha, \mathcal{J}_\beta] = f_{\alpha\beta}{}^{\gamma}\mathcal{J}_\gamma, \quad [\mathcal{J}_\alpha, \mathcal{P}^\beta] = -f_{\alpha\gamma}{}^{\beta}\mathcal{P}^\gamma \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0,$$

where $\alpha = 1, ..., 6$.

- $\mathcal{J}_\alpha$ generates $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ and $\mathcal{P}_\alpha$ generates $\mathfrak{sl}(2,\mathbb{C})^*_{\mathbb{R}} \cong \mathbb{R}^6$ viewed as $\mathbb{R}^3 \times \mathbb{R}^3$.

We identify $\mathcal{J}_\alpha$ with the basis $\mathcal{J} = \{J_0, J_1, J_2, K_0, K_1, K_2\}$ of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ so that first bracket is that of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \simeq \mathfrak{so}(3,1)$ with

$$[J_i, J_j] = \epsilon_{ijk}J^k, \quad [J_i, K_j] = \epsilon_{ijk}K^k, \quad [K_i, K_j] = -\epsilon_{ijk}J^k.$$

Define the generator S_i by

$$S_i = K_i + \epsilon_{ijk}n^j J^k, \quad \boldsymbol{n}^2 < -1.$$

Then the Lie brackets on $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ take the form

$$\left[J_i, J_j\right] = \epsilon_{ijk}J^k, \ \left[J_i, S_j\right] = \epsilon_{ijk}S^k + n_j J_i - \eta_{ij}(n^k J_k), \ \left[S_i, S_j\right] = n_i S_j - n_j S_i.$$

Bilinear form

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \oplus \mathbb{R}^6$ has a symmetric Ad-invariant non-degenerate bilinear form

$$\langle \mathcal{J}_\alpha, \mathcal{J}_\beta \rangle = 0, \quad \langle \mathcal{P}_\alpha, \mathcal{P}_\beta \rangle = 0, \quad \langle \mathcal{J}_\alpha, \mathcal{P}^\beta \rangle = \delta_\alpha^\beta.$$

Denote by $\mathcal{J}_\alpha^L, \mathcal{J}_\alpha^R$ the left-and right-invariant vector fields on $SL(2,\mathbb{C})_{\mathbb{R}}$ associated to the generators $\mathcal{J}_\alpha$ and defined by

$$\mathcal{J}_\alpha^R F(u) := \frac{d}{dt}|_{t=o} F(ue^{t\mathcal{J}_\alpha}), \quad \mathcal{J}_\alpha^L F(u) := \frac{d}{dt}|_{t=o} F(e^{-t\mathcal{J}_\alpha} u),$$

for $u \in SL(2,\mathbb{C})_{\mathbb{R}}$ and $F \in C^\infty(SL(2,\mathbb{C})_{\mathbb{R}})$.

The Chern-Simons action for the self-dual variables

Recall the complex-valued action (3d self-dual gravity action)

$$S_{\mathbb{C}} = \int_{M^3} E \wedge F[A],$$

where

$$E^j_\mu = (\chi^0 e^j_\mu - \chi^j e^0_\mu) + i\epsilon^j_{kl}\,\chi^k e^l_\mu, \quad A^j_\mu = \omega^j_\mu + i\omega^{(0)j}_\mu,$$

are valued in $\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$ or $\mathfrak{su}(2)_{\mathbb{C}}$.

We map the complex variables E_i, A^i to real-valued forms E_α, A^α on $\mathbb{R}^6$ and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ respectively according to

$$E = E_\alpha \mathcal{P}^\alpha = (\chi^0 e^j_\mu - \chi^j e^0_\mu)P_j + \epsilon^j_{kl}\,\chi^k e^l_\mu Q_j = B^j_\mu P_j + \mathcal{B}^j Q_j$$

$$A = A^\alpha \mathcal{J}_\alpha = \omega^j_\mu J_j + \omega^{(0)j}_\mu K_j,$$

where

$$K_j = iJ_j, \qquad Q_j = iP_j.$$

Chern-Simons action for the self-dual variables

The gauge field is locally a 1-form $A \in \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$ which combines the real variables A and E into a Cartan connection

$$\mathcal{A} = A^{\alpha}\mathcal{J}_{\alpha} + E_{\alpha}\mathcal{P}^{\alpha}$$

The curvature of the connection $\mathcal{A}$ is

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = F + T,$$

and combines the curvature for the $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ valued spin connection

$$F^{\gamma} = dA^{\gamma} + \frac{1}{2} f^{\gamma}_{\alpha\beta} A^{\alpha} \wedge A^{\beta}$$

and torsion

$$T = (dE^{\gamma} + f^{\alpha\beta\gamma} A_{\alpha} \wedge E_{\beta})P_{\gamma}.$$

Chern-Simons formulation

The Chern-Simons action for the gauge field $\mathcal{A}$ is

$$S_{CS}[\mathcal{A}] = \frac{1}{2}\int_{M^3} \langle \mathcal{A} \wedge d\mathcal{A}\rangle + \frac{1}{3}\langle \mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]\rangle.$$

The equations of motion which follow amount to the flatness condition on the gauge field

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0.$$

This is equivalent to the condition of vanishing torsion and the equations of motion

$$dE^\gamma + f^{\alpha\beta\gamma} A_\alpha \wedge E_\beta = 0, \quad \mathcal{R}^\gamma = dA^\gamma + \frac{1}{2} f^\gamma_{\alpha\beta} A^\alpha \wedge A^\beta = 0,$$

i.e.

$$dB^i - \epsilon_{ij}{}^k(\omega^i + \omega^{(0)i}) \wedge B^j = 0, \quad d\mathcal{B}^i - \epsilon_{ij}{}^k(\omega^i + \omega^{(0)i}) \wedge \mathcal{B}^j = 0$$

$$d\omega^i + \frac{1}{2}\epsilon^i{}_{jk}(\omega^j \wedge \omega^k - \omega^{(0)j} \wedge \omega^{(0)k}) = 0,$$

$$d\omega^{(0)k} + \frac{1}{2}\epsilon_{ijk}(\omega^j \wedge \omega^{(0)k} - \omega^{(0)j} \wedge \omega^k) = 0$$

Chern-Simons formulation

$M^3 = \mathbb{R} \times \Sigma$ enables one to decompose $\mathcal{A}$ with respect to the coordinate x^0 on $\mathbb{R}$ according to

$$\mathcal{A} = \mathcal{A}_0 dx^0 + \mathcal{A}_\Sigma,$$

where the spacial gauge field $\mathcal{A}_\Sigma = \mathcal{A}_a dx^a$ is an x^0-dependent $1-$form on Σ and $\mathcal{A}_0$ is a Lie algebra valued function on $\mathbb{R} \times \Sigma$. The Chern-Simons action become

$$S_{SC}[\mathcal{A}_0, \mathcal{A}_\Sigma] = \int_{\mathbb{R}} dx^0 \int_\Sigma dx^2 \left(-\langle \mathcal{A}_\Sigma, \partial_0 \mathcal{A}_\Sigma \rangle + \langle A_0, \mathcal{F}_\Sigma \rangle\right).$$

Thus the phase space variables are the components of the $\mathcal{A}_\Sigma$ with canonical Poisson brackets

$$\{\mathcal{A}(x)^a_I, \mathcal{A}(y)^b_J\} = \epsilon^{ab} \delta^2(x-y) \langle X_\alpha, X_\alpha \rangle$$

where the $X_\alpha = \{\mathcal{J}_\alpha, \mathcal{P}_\beta\}_{\alpha,\beta=1,\ldots,6}$ are generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \ltimes \mathbb{R}^6$ and $\delta^2(x-y)$ is the Dirac delta function on Σ.

Chern-Simons formulation

To better understand the of the CS action to the original complex action, we use the decomposition

$$E = E_\alpha \mathcal{P}^\alpha = (\chi^0 e^j_\mu - \chi^j e^0_\mu) P_j + \epsilon^j_{kl} \chi^k e^l_\mu Q_j = B^j_\mu P_j + \mathcal{B}^j Q_j$$

$$A = A^\alpha \mathcal{J}_\alpha = \omega^j_\mu J_j + \omega^{(0)j}_\mu K_j,$$

After performing integration by parts and dropping the boundary terms, the action CS action can be expanded as

$$S_{\mathbb{R}}[E, A] = \int_{M^3} E_\alpha \wedge F^\alpha.$$

We refer to this as the real form of the complex self-dual action. One recovers the complex action using the decomposition.

Chern-Simons action for 3d gravity with Barbero-Immirzi parameter

To include the Barbero-Immirzi parameter in the Chern-Simons theory, one could consider a more general Chern-Simons theory

$$S^{\gamma}_{SC} = \left(\frac{\gamma + 1}{\gamma}\right) S_{SC}[A] + \left(\frac{\gamma - 1}{\gamma}\right) \bar{S}_{SC}[\bar{A}],$$

which maps both the self-dual and anti-self-dual connections to reals $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-valued variables.

Phase space

- For M^3 with topology $\mathbb{R} \times \Sigma$, the phase space is the moduli space of flat $SL(2, \mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-connections on Σ equipped with the Atiyah-Bott symplectic structure defined in terms of $\langle \cdot, \cdot \rangle$
- A theory of quantum gravity amounts to quantising the moduli space of flat $SL(2, \mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-connections on Σ, with a symplectic structure induced by $\langle \cdot, \cdot \rangle$

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Graph representation moduli space of flat connections

- The oriented surface Σ is represented by a directed graph Γ embedded in the surface Σ.
- The description of the moduli space and its Poisson structure in terms of $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-valued holonomies applies Fock-Rosly construction
- which encodes the information about the inner product used in defining the CS action in a compatible way

Classical r-matrix and Fock-Rosly Poisson structure

Starting point: a description of the Poisson structure on the classical phase space in terms of a classical r-matrix.

DEFINITION:

A classical $r-$matrix is said to a CS action compatible if:

- its symmetric part is equal to the Casimir associated to the $\mathrm{Ad}-$invariant, non-degenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$ used in the CS action.
  V. V. Fock , A. A. Rosly

Classical r-matrix and Fock-Rosly Poisson structure

In the case of the $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-Chern-Simons theory under consideration, the relevant Casimir operator for the bilinear form is

$$K = \mathcal{J}_\alpha \otimes \mathcal{P}^\alpha + \mathcal{P}^\alpha \otimes \mathcal{J}_\alpha$$

which takes the form

$$K = J_i \otimes P^i + P^i \otimes J_i + S_i \otimes Q^i + Q^i \otimes S_i - \epsilon^{ijk} n_j \left(J_i \otimes Q^i + Q^i \otimes J_i\right)$$

A compatible $r-$martrix is given by

- $$r = \mathcal{P}^\alpha \otimes \mathcal{J}_\alpha = P^i \otimes J_i + Q^i \otimes S_i - \epsilon^{ijk} n_j\, Q^i \otimes J_i.$$

Amounts to equipping the Lie algebra $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \oplus \mathbb{R}^6$ with the structure of a classical double $D(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ viewed as $D(D(\mathfrak{sl}(2,\mathbb{R})))$ in the Lorentzian picture or $D(D(\mathfrak{su}(2)))$ in the Euclidean picture.

Classical r-matrix and Fock-Rosly Poisson structure

The the Fock and Rosly's Poisson structure is given in terms of a Poisson bivector

$$\{F, G\} = (dF \otimes dG)(B_{FR}) \quad \forall F, G \in \mathcal{C}^{\infty}(H)$$

where B_{FR} is given by the following:

Classical r-matrix and Fock-Rosly Poisson structure

Assign to each vertex $v \in V$ a classical r-matrix $r(v) = r^{\mathfrak{a}\mathfrak{b}}(v) T_{\mathfrak{a}} \otimes T_{\mathfrak{b}}$, such that their symmetric components are non-degenerate and agree for all vertices $v \in V$.
The Poisson bivector

$$B = \sum_{v \in V} r^{\mathfrak{a}\mathfrak{b}}_{(\mathfrak{a})}(v) \Big(\sum_{s(e)=v} T^{R_e}_{\mathfrak{a}} \wedge T^{R_e}_{\mathfrak{a}} + \sum_{t(e)=v} T^{L_e}_{\mathfrak{a}} \wedge T^{L_e}_{\mathfrak{b}} \Big)$$

$$+ \sum_{v \in V} r^{\mathfrak{a}\mathfrak{b}}(v) \Big(\sum_{v=t(e)=t(p)} T^{L_e}_{\mathfrak{a}} \wedge T^{Lp}_{\mathfrak{b}} + \sum_{v=t(e)=s(p)} T^{L_e}_{\mathfrak{a}} \wedge T^{R_p}_{\mathfrak{b}}$$

$$+ \sum_{v=s(e)=t(p)} T^{R_e}_{\mathfrak{a}} \wedge T^{Lp}_{\mathfrak{b}} \Big) + \sum_{v=s(e)=t(p)} T^{R_e}_{\mathfrak{a}} \wedge T^{R_p}_{\mathfrak{b}}$$

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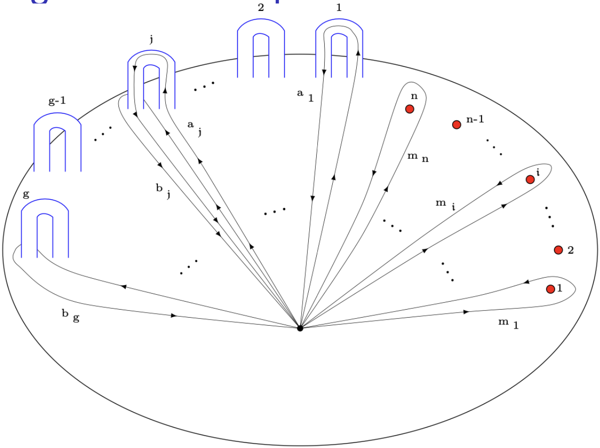
The algebra for $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$ on the space of holonomies

- For the Chern-Simons theory with compact and semisimple gauge groups, the Poisson structure on the space of holonomies was constructed by Alekseev, Grosse and Schomerus

- we follow aspects of the case of (non-compact and non-semisimple) semidirect product gauge groups of type $H = G \ltimes \mathfrak{g}^*$, where the Poisson structure was discussed and quantisation procedure developed and allow one to consider the algebra of function $C^\infty(G)$ Meusburger, Schroers

The algebra on the space of holonomies



. Generators of the fundamental group of a compact surface $\Sigma_{n,g}$ with n punctures

Subject to the relation

$$[b_g, a_g^{-1}] \cdot \ldots \cdot [b_1, a_1^{-1}] \cdot m_n \cdot \ldots \cdot m_1 = 1, \quad \text{with} \quad [b_i, a_i^{-1}] = b_i a_i^{-1} b_i^{-1} a_i$$

The algebra for on the space of holonomies

- Handle holonomies $A_j = \mathrm{Hol}(a_i), B_j = \mathrm{Hol}(b_j) \in SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$.
- Puncture holonomies $M_i = \mathrm{Hol}(m_i)$ are contained in fixed $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-conjugacy classes

  $$\mathcal{C}_{\mu_i s_i} = \{(v,\boldsymbol{x})\cdot(g_\mu, -\boldsymbol{s})\cdot(v,\boldsymbol{x})^{-1}\} | (v,\boldsymbol{x}) \in SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6,$$

  where μ_i label $SL(2,\mathbb{C})_{\mathbb{R}}$-congugacy classes and s_i are co-adjoint orbits of associated stabiliser Lie algebra.

The space $\tilde{\mathcal{A}}_{g,n}$ of graph connections or holonomies is given by

$$\begin{aligned}\tilde{\mathcal{A}}_{g,n} \; = \; & \{(M_1, ..., M_n, A_1, B_1, ..., A_g, B_g) \\ & \in \mathcal{C}_{\mu_1 s_1} \times ... \times \mathcal{C}_{\mu_n s_n} \times (SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6)^{2n} | \\ & [A_g, B_g^{-1}] \cdot ... \cdot [A_1, B_1^{-1}] \cdot M_n \cdot ... \cdot M_1 = 1\}.\end{aligned}$$

The moduli space $\mathcal{M}_{g,n}$ of flat $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$-connections on a surface $\Sigma_{g,n}$ is then

$$\mathcal{M}_{g,n} = \tilde{\mathcal{A}}_{g,n} / \sim$$

where $\sim$ denotes silmultaneous conjugation.

Fock-Rosly algebra for the gauge group $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$

The algebra $\tilde{\mathcal{F}}$ for gauge group $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$ on a genus g surface $\Sigma_{g,n}$ with n punctures is the commutative Poisson algebra

$$\tilde{\mathcal{F}} = S\left(\bigoplus_{\mathfrak{k}=1}^{n+2g} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\right) \otimes \mathcal{C}^{\infty}(SL(2,\mathbb{C})_{\mathbb{R}})$$

where $S\left(\bigoplus_{\mathfrak{k}=1}^{n+2g} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\right)$ symmetric envelope of the real Lie algebra $\bigoplus_{\mathfrak{k}=1}^{n+2g} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, i.e. the polynomials with real coefficients on the vector space $\bigoplus_{\mathfrak{k}=1}^{n+2g} \mathbb{R}^6$.

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Quantum algebra and representations

The quantum algebra is

$$\hat{\mathcal{F}} = U\left(\bigoplus_{k=1}^{n+2g} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\right) \hat{\otimes}\, \mathcal{C}^{\infty}\left(SL(2,\mathbb{C})_{\mathbb{R}}^{n+2g}, \mathbb{C}\right)$$

with multiplication defined by

$$(\xi \otimes F) \cdot (\eta \otimes K) = \xi \cdot_{U\eta} \otimes FK + i\hbar\eta \otimes F\{\xi \otimes 1, 1 \otimes K\},$$

where $\xi, \eta \in \bigoplus_{k=1}^{n+2g} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, $F, K \in \mathcal{C}^{\infty}\left(SL(2,\mathbb{C})_{\mathbb{R}}^{n+2g}, \mathbb{C}\right)$ and $\cdot_U$ denotes the multiplication in $U\left(\bigoplus_{k=1}^{n+2g} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\right)$. The irreps are labelled by n $SL(2,\mathbb{C})_{\mathbb{R}}$-conjugacy classes $\mathcal{C}_{\mu_i} = \{gg_{\mu_i}g^{-1} | g \in (SL(2,\mathbb{C})_{\mathbb{R}}\}$, $i = 1, ..., n$, and the irreducible unitary Hilbert space representation $\Pi_{s_i} : N_{\mu_i} \to V_{s_i}$ of stabilisers $N_{\mu_i} = \{g \in SL(2,\mathbb{C})_{\mathbb{R}} | gg_{\mu_i}g^{-1} = g_{\mu_i}\}$ of chosen elements $g_{\mu_1}, ..., g_{\mu_n}$ in the conjugacy classes.

Symmetries and the quantum double $D(SL(2,\mathbb{C})_{\mathbb{R}})$

- $D(SL(2,\mathbb{C})_{\mathbb{R}})$ provides a transformation group algebra associated to the puncture and handle algebras
- one obtains a representation of the quantum double on the representation space of the quantum Fock-Rosly algebra for $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$.

Concluding Remarks

- In the context of CS theory, the implementation of reality conditions amount to extending the internal gauge group to higher dimensional CS analogue. In this case, from 6d complex $SL(2,\mathbb{C})$ to 12d real $SL(2,\mathbb{C})_{\mathbb{R}} \ltimes \mathbb{R}^6$.
- The quantum double $D(SL(2,\mathbb{C})_{\mathbb{R}})$ viewed as the double of a double $D(SU(2) \bowtie AN(2))$ provides a feature for quantum symmetries of the quantum theory for the model.
- Interesting to explore this model with cosmological constant.

THANK YOU!!!