Status of Birkhoff ’s theorem in polymerized semiclassical regime of Loop Quantum Gravity

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LC, Jerzy Lewandowski (2024) arXiv:2403.01910

Introduction

Collapse of a spherically symmetric cloud of dust

- Modified Einstein's equations
- Oppenheimer-Snyder model

EINSTEIN'S EQUATIONS

Classical theory

Collapse of spherically symmetric dust ball:

$$ds^2 = -N d\tau^2 + \frac{(E^\varphi)^2}{E^x}(dx + N^x\, d\tau)^2 + E^x d\Omega^2$$ (PG coordinates)

$$\{K_x(y_1), E^x(y_2)\} = 2\gamma\delta(y_1 - y_2)$$

$$\{K_\varphi(y_1), E^\varphi(y_2)\} = \gamma\delta(y_1 - y_2)$$

$G = c = 1$

$N = 1$ → Dust Gauge

$E^x = x^2$ → Areal Gauge

$$E^\varphi = \pm\frac{x}{\sqrt{1 + \varepsilon(\tau, x)}}$$

Polymerization

J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2020)
V. Husain, J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2022)

- Discretization: $x \longrightarrow x_j$

- Operators: $E^{\varphi} \Longrightarrow \hat{E}_j^{\varphi}$

$$K_{\varphi} \Longrightarrow \hat{U}_j = e^{i\,\bar{\mu}_j K_{\varphi}(x_j)}$$

- Polymerization: $\hat{U}_j = e^{i\,\bar{\mu}_j K_{\varphi}(x_j)}$ ⟶ $\frac{\hat{U}_j - \hat{U}_j^{\dagger}}{2\,i\,\bar{\mu}_j} = \frac{\sin(\bar{\mu}_j K_{\varphi}(x_j))}{\bar{\mu}_j}$

$$\bar{\mu}_j = \frac{\sqrt{\Delta}}{x_j}$$

Semiclassical theory

J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2020)
V. Husain, J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2022)

$$ds^2 = -d\tau^2 + \frac{(E^\varphi)^2}{x^2}(dx + N^x\, d\tau)^2 + x^2 d\Omega^2$$

$$\beta := \frac{\sqrt{\Delta}}{x} K_\varphi$$

$$\dot{E}^\varphi = -\frac{x^2}{\gamma\sqrt{\Delta}}\partial_x\left(\frac{E^\varphi}{x}\right)\sin\beta\cos\beta$$

$$\dot{K}^\varphi = \frac{\gamma x}{2(E^\varphi)^2} - \frac{\gamma}{2x} - \frac{\partial_x(x^3 sin^2\beta)}{2\gamma\Delta x}$$

Polymerized Einstein Field Equations (PEFE)

$$\rho = \frac{1}{8\pi x E^\varphi}\left[\frac{E^\varphi}{\gamma^2\Delta x}\partial_x(x^3 sin^2\beta) + \frac{x}{E^\varphi} + \frac{E^\varphi}{x} - 2\left(\frac{x^2}{E^\varphi}\right)\right]$$

$$N^x = -\frac{x}{\gamma\sqrt{\Delta}}\sin\beta\cos\beta$$

Interior $(\rho \neq 0, \quad \partial_x \rho = 0)$

M. Bojowald, J. D. Reyes, R. Tibrewala (2009)
K. Giesel, H. Liu, E. Rullit, P. Singh, S. A. Weigl (2023)

$$E^\varphi = \pm \frac{x}{\sqrt{1+\varepsilon(\tau,x)}}$$

PEFE
$$\begin{cases} \dot{\varepsilon} = \varepsilon' \sqrt{\frac{8\pi}{3}\rho x^2 + \varepsilon} \sqrt{1 - \frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{\varepsilon}{x^2}} \\ sin^2 \beta = \gamma^2 \Delta \left(\frac{8\pi}{3}\rho + \frac{\varepsilon}{x^2}\right) \end{cases}$$

$$\rho_c \coloneqq \frac{3}{8\pi\gamma^2\Delta}$$

$$x = \xi(T,R)$$
$$\tau = T$$
$$N^x = -\partial_T \xi$$
$$\varepsilon = E(R)$$

$$\longrightarrow \quad ds^2 = -dT^2 + \frac{(\partial_R \xi)^2}{1+E(R)} dR^2 + \xi^2 d\Omega^2$$

(LTB coordinates)

$$\left(\frac{\partial_T \xi}{\xi}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{E}{\xi^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{E}{\xi^2}\right)$$

Interior $(\rho \neq 0, \quad \partial_x \rho = 0)$

From $ds^2 = -dT^2 + \frac{(\partial_R \xi)^2}{1+E(R)} dR^2 + \xi^2 d\Omega^2$

$$x = \xi = a(T)\chi_k(R)$$
$$\varepsilon = -k\left(\frac{x}{a}\right)^2 = -k\chi_k^2 = E(R)$$

The Friedmann dust ball: $ds^2 = -dT^2 + a^2 dR^2 + a^2 \chi_k^2 d\Omega^2$ with $\chi_k(R) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}R)$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{k}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right)$$

Static Exterior $(\rho = 0)$

$$\dot{E}^{\varphi} = -\frac{x^2}{\gamma\sqrt{\Delta}}\partial_x\left(\frac{E^{\varphi}}{x}\right)\sin\beta\cos\beta = 0$$

$$E^{\varphi} = Ax = \pm\frac{x}{\sqrt{1+B}} \longrightarrow \dot{K}^{\varphi} = 0$$

$$sin^2\beta = \gamma^2\Delta\left(\frac{B}{x^2}+\frac{2M}{x^3}\right) \longrightarrow (N^x)^2 = \frac{2M}{x} - \frac{\alpha}{x^2}\left(\frac{M}{x}+\frac{B}{2}\right)^2 + B \qquad \alpha := 4\gamma^2\Delta$$

$$ds^2 = -d\tau^2 + A^2(dx + N^x\, d\tau)^2 + x^2 d\Omega^2$$

In Schwarzschild coordinates: $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r}+\frac{B}{2}\right)^2$$

Time dependent Exterior $(\rho = 0)$

$$\dot{E}^{\varphi} = -\frac{x^2}{\gamma\sqrt{\Delta}}\partial_x\left(\frac{E^{\varphi}}{x}\right)\sin\beta\cos\beta \neq 0 \quad \longrightarrow \quad E^{\varphi} = \pm\frac{x}{\sqrt{1+\varepsilon(\tau,x)}}$$

PEFE
$$\begin{cases} \dot{\varepsilon} = \varepsilon' \sqrt{\varepsilon + \frac{2M}{x}}\sqrt{1-\gamma^2\Delta\left(\frac{\varepsilon}{x^2}+\frac{2M}{x^3}\right)} \\ sin^2\beta = \gamma^2\Delta\left(\frac{\varepsilon}{x^2}+\frac{2M}{x^3}\right) \end{cases}$$

$\varepsilon = const$ or $\varepsilon = \varepsilon(\tau, x)$

If $\varepsilon \neq \varepsilon(\tau,x)$ then the only line element is given by $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r}+\frac{B}{2}\right)^2$

BIRKHOFF'S THEOREM !!

Oppenheimer-Snyder model

Oppenheimer-Snyder model

H. Ziaie, Y. Tavakoli (2020)
A. Parvizi, T. Pawlowski, Y. Tavakoli, J. Lewandowski (2022)
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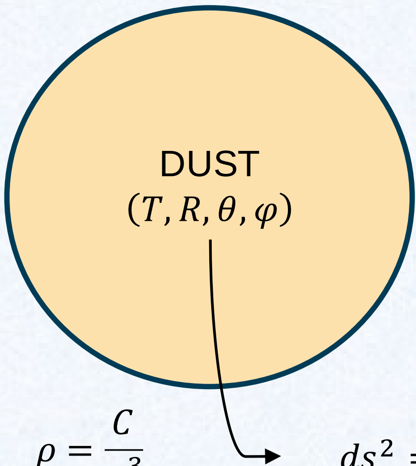
VACUUM
(t, r, θ, φ) ⟶

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} - \frac{k\chi_{k,0}^2}{2}\right)^2$$



$$\rho = \frac{C}{a^3}$$

$$ds^2 = -dT^2 + a^2 dR^2 + a^2 \chi_k^2 d\Omega^2 \quad \text{with} \quad \chi_k(R) = \frac{1}{\sqrt{k}} \sin\left(\sqrt{k}R\right)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{k}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right) + \left(\frac{8\pi}{3}\rho - \frac{k}{a^2}\right)\left(\frac{3}{2}\frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right)$$

Critical mass and horizons

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} - \frac{k\chi_{k,0}^2}{2}\right)^2$$

- If $M_- \leq M \leq M_+ \quad \Rightarrow \quad \nexists$ real solutions of $f(r) = 0$ (no horizons)

$$M_\pm^2 = \frac{\alpha}{216}\left[64 - 96\, k\chi_{k,0}^2 + 30\, k^2\chi_{k,0}^4 + k^3\chi_{k,0}^6 \pm \left(16 - 16\, k\chi_{k,0}^2 + k^2\chi_{k,0}^4\right)^{3/2}\right]$$

- If $M \geq M_+ \quad \cup \quad M \leq M_- \quad \Rightarrow \quad \exists$ 2 real solutions to $f(r) = 0$

$$r_- = \left(\frac{\alpha M}{2}\right)^{1/3} + \frac{1 - 2\, k\chi_{k,0}^2}{6M}\left(\frac{\alpha M}{2}\right)^{2/3} + \frac{\left(1 - k\chi_{k,0}^2\right)^2}{24\, M}\alpha + O\left(\alpha^{4/3}\right)$$

$$r_+ = 2M - \frac{\left(1 - k\chi_{k,0}^2\right)^2}{8\, M}\alpha + O\left(\alpha^{4/3}\right)$$

k=0

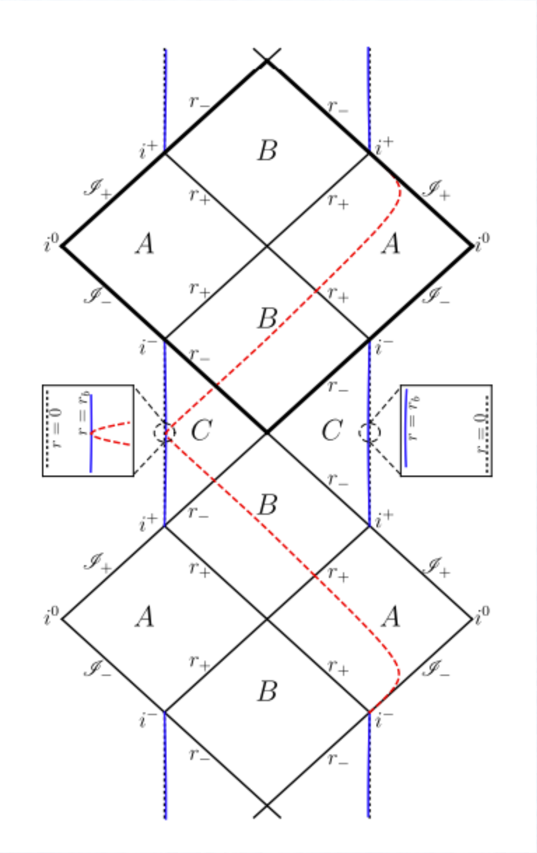
H. Ziaie, Y. Tavakoli (2020)
A. Parvizi, T. Pawlowski, Y. Tavakoli, J. Lewandowski (2022)
J. Lewandowski, Y. Ma, J. Yang, C. Zhang (2023)

- $f(r) = 1 - \frac{2M}{r} + \alpha \frac{M^2}{r^4}$
- $M_- = 0, \quad M_+ = \frac{4}{3\sqrt{3}}\sqrt{\alpha}$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho\right)\left(1 - \frac{\rho}{\rho_c}\right)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) + 4\pi\frac{\rho^2}{\rho_c}$$

Exact solution to the PEFE with $B = 0$



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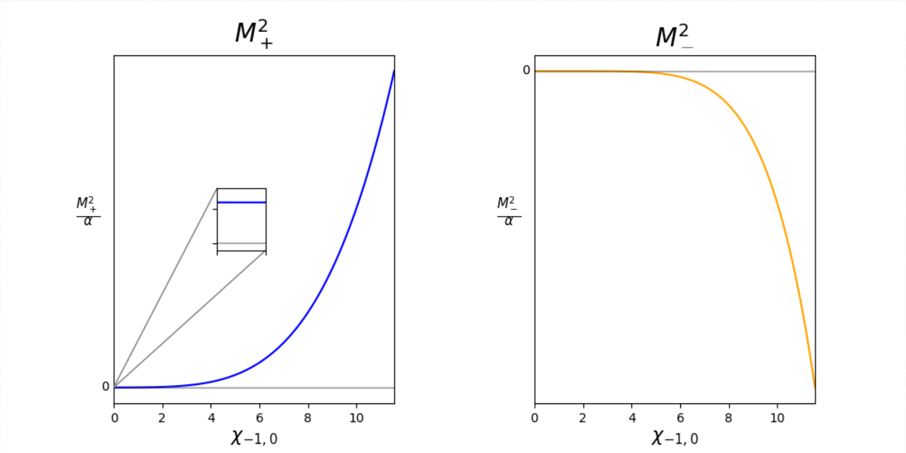
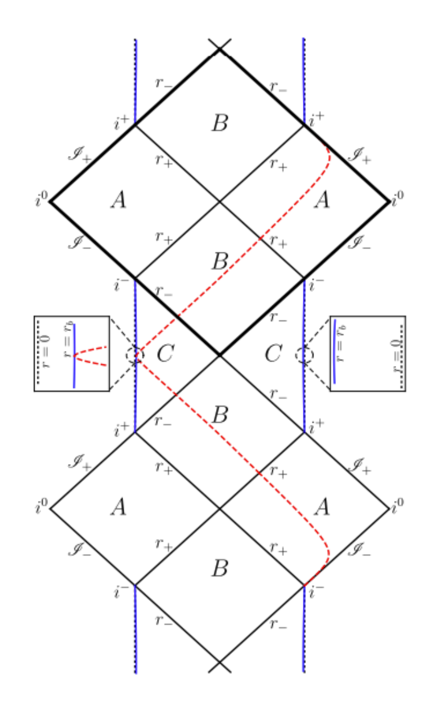
C. Zhang

k= -1

- $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} + \frac{\chi^2_{-1,0}}{2}\right)^2$ Exact solution to the PEFE with $B = \chi^2_{-1,0}$





$$\left(\frac{\dot a}{a}\right)^2 = \left(\frac{8\pi}{3}\rho + \frac{1}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

$$\left(\frac{\ddot a}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right) + \left(\frac{8\pi}{3}\rho + \frac{1}{a^2}\right)\left(\frac{3}{2}\frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

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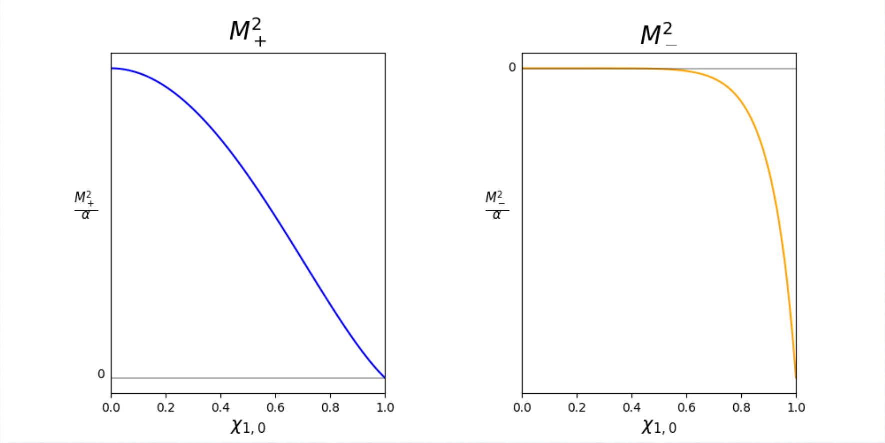
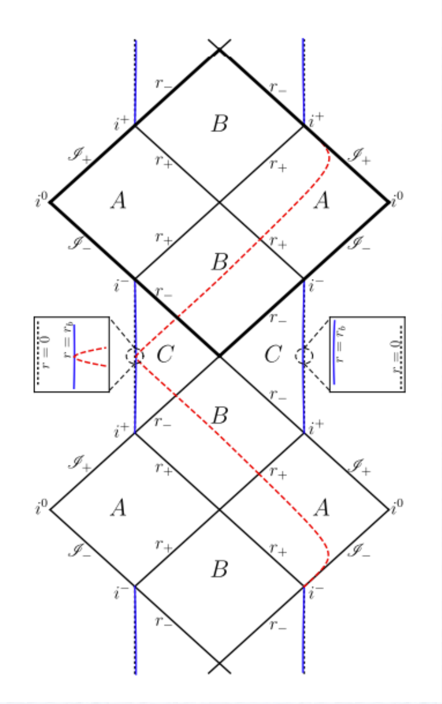
k=1

- $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} - \frac{\chi_{1,0}^2}{2}\right)^2$ Exact solution to the PEFE with $B = -\chi_{1,0}^2 < 0$





$$\left(\frac{\dot a}{a}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{1}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

$$\left(\frac{\ddot a}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right) + \left(\frac{8\pi}{3}\rho - \frac{1}{a^2}\right)\left(\frac{3}{2}\frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

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Conclusion

- Two independent methods lead to the very same metric.
- Static exterior solutions to the Einsten's equation are Schwarzschild-like but depend on two parameters (M and B).
- There may exist other non-static solutions.
  If this possibility is ruled out $\Rightarrow$ Birkhoff's theorem.