General covariance and dynamics with a Gauss law

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work in collaboration with Viqar Husain (arXiv: 2312.06079)

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Prelude: the Husain-Kuchar (HK) model

Given an $\mathfrak{su}(2)$-valued triad e_a^i and connection A_a^i ($a \in \{1, ..., 4\}$) on a 4d spacetime M, consider the generally covariant action

$$S = \frac{1}{2} \int_M d^4x \, \mathrm{Tr}(e \wedge e \wedge F)$$

where $F = dA + A \wedge A$.

Contrast with the 4d Palatini action: $\mathfrak{so}(3,1)$-valued *tetrads* replaced with $\mathfrak{su}(2)$-valued *triads*.

Canonical HK

Assuming $M = \mathbb{R} \times \Sigma$, the canonical decomposition of the action is straightforward:

$$S = \int dt \int_\Sigma d^3x (\tilde{E}^a_i \dot{A}^i_a - A^i_0 \tilde{G}_i - (e^i_0 E^a_i) \tilde{C}_a)$$

where $\tilde{E}^a_i = \det(e) E^a_i = \tilde{\epsilon}^{0abc} \epsilon_{ijk} e^j_b e^k_c$, and

$$\tilde{G}_i = -D_a \tilde{E}^a_i \approx 0 \quad \text{(Gauss)}$$
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No Hamiltonian constraint!

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- The theory is non-dynamical: the geometry of Σ does not evolve. But **not** topological: local degrees of freedom exist.
- There's an invertible spatial metric $g_{ab} = \delta_{ij} e^i_a e^j_b$, $a, b \in \{1, 2, 3\}$. Thus interesting three-geometries exist.

Enter HK2.0

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with only one constraint, a modified Gauss law with a source:

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No Hamiltonian and diffeomorphism constraints!

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In these theories, for any generator of diffeomorphisms v,

$$\mathcal{L}_v A = G\text{-transformations} + \text{equations-of-motion terms}$$

where G is the gauge group of the connection A.

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- For instance, canonical quantization via LQG methods yields a Hilbert space of spin network states with a finite number of charges ϕ sitting at the vertices.
- Would be interesting to look at the spinfoam and group field theory models of the theory (work currently underway).

Thank you!