Diffeomorphism Covariance and the Quantum Schwarzschild Interior

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Summary

Goal:

To derive a family of dynamics for the Schwarzschild interior model by requesting the quantum framework to be covariant under residual diffeomorphisms.

Why?

- Background independence (diffeomorphism covariance) is a key feature of the formulation of General Relativity.
- Loop Quantum Gravity (LQG) is built based on this principle.
- Loop Quantum Cosmology (LQC) applies quantization techniques analogous to LQG to symmetry-reduced models, but does not require diffeomorphism covariance a priori.
- Engle & Vilensky (2018,2019) show for homogeneous isotropic LQC and Bianchi I models that a family of dynamics can be derived from residual diffeomorphism covariance [5, 6],
  - uniqueness can be achieved by requiring the Hamiltonian to have a minimal number of terms.
- Requiring the same for Schwarzschild interior model contributes to reduce ambiguities in its construction.

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  - uniqueness can be achieved by requiring the Hamiltonian to have a minimal number of terms.
- Requiring the same for Schwarzschild interior model contributes to reduce ambiguities in its construction.

Really fast-track Kantowski-Sachs

- Homogeneous model with spatial section of topology $S^2 \times \mathbb{R}$, equivalent to Schwarzschild, and geometry described by pairs (b, p_b) and (c, p_c), such that

$$\{b, p_b\} = G\gamma \quad \text{and} \quad \{c, p_c\} = 2G\gamma,$$

$$ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c| L_0^2} dx^2 + |p_c| d\Omega^2,$$

$$V = 4\pi |p_b| \sqrt{|p_c|}.$$

- Ashtekar-Barbero variables:

$$A_a^1 = -b \sin\theta \partial_a \phi, \qquad E_1^a = -\frac{p_b}{L_0}\phi^a$$

$$A_a^2 = b \partial_a \theta, \qquad E_2^a = \frac{p_b}{L_0} \sin\theta \theta^a$$

$$A_a^3 = \frac{c}{L_0}\partial_a x + \cos\theta \partial_a \phi, \qquad E_3^a = p_c \sin\theta x^a$$

- Hamiltonian Constraint (with choice of lapse $N = V^n$)

$$H_{c\ell}[N] = -\frac{V^{n+1}}{8\pi G \gamma^2} \operatorname{sgn} p_b \left[\frac{b^2 + \gamma^2}{p_c} + \frac{2bc}{p_b}\right].$$

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Covariance equation

- Residual diffeomorphisms: group of transformations preserving the form of (A, E).

$$\mathcal{L}_{\vec{v}} A^i_a(t) = \dot{A}^i_a = \frac{\partial A^i_a}{\partial b}\dot{b}(t) + \frac{\partial A^i_a}{\partial c}\dot{c}(t)$$
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Flow equations result in

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0$$

and the Hamiltonian transforms as

$$\dot{H}_{c\ell} = (n+1)H_{c\ell}.$$

- Transformations are non-canonical
  - We cannot directly apply the canonical recipe $\dot{F} = \{\Lambda, F\} \Rightarrow \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right]$.
  - Nevertheless, variables can be cast in a related form

$$\dot{F} = \frac{p_b}{\gamma G}\{b, F\} - \frac{c}{2\gamma G}\{p_c, F\}.$$

- Covariance equation for $\hat{H}$ (choosing the Weyl ordering for quantizing products):

$$(n+1)\hat{H} = \frac{1}{2i\gamma\ell_p^2}\left\{\hat{p}_b\left[\hat{b}, \hat{H}\right] + \left[\hat{b}, \hat{H}\right]\hat{p}_b\right\} - \frac{1}{4i\gamma\ell_p^2}\left\{\hat{c}\left[\hat{p}_c, \hat{H}\right] + \left[\hat{p}_c, \hat{H}\right]\hat{c}\right\}.$$

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2. Impose of preservation of the Bohr Hilbert space of LQC + hermiticity and parity symmetries of (b, p_b) and (c, p_c),
3. *Physical assumption*: quantization is resultant from the fact that holonomies can only be shrank to a minimum area Δ, which is dependent only on the absolute values of the momentum variables,
4. Define the *classical analogue (effective Hamiltonian)* H of the operator $\hat{H}$ as the preimage under quantization map.

$$H = |p_b|^{n+1} a_0 \operatorname{sgn}(p_b p_c) + |p_b|^{n+1} \sum_{k=1}^{M} \Bigg(a_k \operatorname{sgn}(p_b p_c) \cos(A_k(|p_c|)b) \cos\left(B_k(|p_c|)\frac{c}{|p_b|}\right)$$
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Classical asymptotic behavior

5. Expanding H for the limit of low curvatures ($b, c \to 0$), and matching terms of same order with H_{cl}, result in a system of equations to find a family of Hamiltonians, depending on the parameter M chosen

$$\mathcal{O}(1): \qquad -\frac{(4\pi)^n}{2G}|p_c|^{\frac{n-1}{2}} = a_0 + \sum_{k=1}^{M} a_k$$

$$\mathcal{O}(b): \qquad 0 = \sum_{k=1}^{M} c_k A_k$$

$$\mathcal{O}(c): \qquad 0 = \sum_{k=1}^{M} b_k B_k$$

$$\mathcal{O}(bc): \qquad -\frac{(4\pi)^n}{G\gamma^2}|p_c|^{\frac{n+1}{2}} = \sum_{k=1}^{M} d_k A_k B_k$$

$$\mathcal{O}(b^2): \qquad \frac{(4\pi)^n}{G\gamma^2}|p_c|^{\frac{n-1}{2}} = \sum_{k=1}^{M} a_k A_k^2$$

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Minimality matching literature

6. Minimality: Requiring Hamiltonian to have a minimum number of shifts ($M = 2$) results in

$$H = -\frac{V^{n+1}\,\mathrm{sgn}(b)}{8\pi G\gamma^2 p_c}\left(\gamma^2 + 2p_c\,\mathrm{sgn}\,p_b\,\frac{\sin(A_1 b)}{A_1}\,\frac{\sin\left(B_1\frac{c}{|p_b|}\right)}{B_1} + \frac{4\sin^2\left(\frac{A_2}{2}b\right)}{A_2^2}\right),$$

where A_1, A_2, B_1 are arbitrary functions of $|p_c|$ only.

→ Given the presence of the ratio $\frac{c}{|p_b|}$, formulations based on μ_0 prescription are not covariant under residual diffeomorphisms (as expected).

→ Choosing the $\bar{\mu}$ prescription $\left(A_1 = \sqrt{\frac{\Delta}{|p_c|}}, B_1 = \sqrt{|p_c|\Delta}, A_2 = 2A_1\right) \Rightarrow H$ matches Chiou (2008) [4] for $n = 1$ (harmonic time gauge), and Joe & Singh (2015) [7] for $n = 0$ (proper time).

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Comments on AOS Model

- Ashtekar, Olmedo & Singh (2018): The most well-developed and physically viable model proposed so far in the literature ([1, 2] + Ashtekar's lecture on Summer School)

  Conditions:

  1. Regularizing loops enclose a physical area equal the area gap Δ *at the transition surface that replaces the classical singularity*
  2. Parameters δ_b and δ_c of regularization are Dirac observables (constant on dynamical trajectories),

  Consequences:

  - ✓ Expansion and shear diverge at the horizon just as in classical GR,
  - ✓ Transition surface always occurs in a regime where quantum gravity effects are expected to be relevant (Kretschmann scalar $\sim$ Planck scale),
  - ✓ Radii and Komar masses of trapped and anti-trapped surfaces nearly symmetric.
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- ⋆ *Attention point*: polymerized version of the classical lapse $N_{cl} = \frac{\gamma}{b}\,\mathrm{sgn}\,(p_c)\sqrt{|p_c|}$
  - ✓ Simplifies the solution by decoupling (b, p_b) and (c, p_c) components;

$$H_{c\ell}[N] = -\frac{1}{2G\gamma}\left(p_b\left(b+\frac{\gamma^2}{b}\right)+2cp_c\right) = H_b[N_{c\ell}] + H_c[N_{c\ell}].$$

Polymerized version replaces $\frac{1}{b}$ by $f(b) = \frac{\delta_b}{\sin(\delta_b b)}$, keeping the decouple.

- × δ_b depends only on (b, p_b) and δ_c only on (c, p_c), thus H is not covariant under residual diffeomorphisms;
  - ✓ however, key physical predictions calculated so far are invariant under residual diffeomorphisms → good motivation to seek an effective Hamiltonian that is exactly covariant under residual diffeomorphisms, even if it is more mathematically complex.
- × $f(b)$ needs an infinite number of terms to be quantized in $\mathcal{H}_{Bohr}$:
  1. Casting $f(b)$ into countable linear combination of shift operators requires its decomposition into a Fourier series;
  2. $f(b)$ is not absolutely integrable → Fourier series exists only in the distributional sense, with infinite number of non-zero terms;
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Comments on AOS Model

- ⋆ *Attention point*: polymerized version of the classical lapse $N_{cl} = \frac{\gamma}{b}\,\mathrm{sgn}\,(p_c)\sqrt{|p_c|}$
  - ✓ Simplifies the solution by decoupling (b, p_b) and (c, p_c) components;

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  Polymerized version replaces $\frac{1}{b}$ by $f(b) = \frac{\delta_b}{\sin(\delta_b b)}$, keeping the decouple.

- × δ_b depends only on (b, p_b) and δ_c only on (c, p_c), thus H is not covariant under residual diffeomorphisms;
  - ✓ however, key physical predictions calculated so far are invariant under residual diffeomorphisms → good motivation to seek an effective Hamiltonian that is exactly covariant under residual diffeomorphisms, even if it is more mathematically complex.

- × $f(b)$ needs an infinite number of terms to be quantized in $\mathcal{H}_{Bohr}$:
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Conclusion

1. Requiring Diffeomorphism Covariance:
   - ✓ allows to derive the quantum dynamics of the framework;
   - ✓ mitigates ambiguities related to regularizing prescription;
   - ✓ matches literature proposals when minimality is required;
2. AOS model:
   - lapse decoupling the components lead to attention points;
   - given the great results so far, there is good motivation to seek an effective Hamiltonian that is exactly covariant under residual diffeomorphisms, even if it is more mathematically complex.

Thank You!

Questions, suggestions, job offers??

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