(Extreme) isolated horizons

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Extereme isolated horizons

1. There was exciting progress in the isolated horizons (especially extreme case, $\kappa = 0$).
2. The case of extreme isolated horizons constitutes a hard part of the classification of stationary black holes.
3. Apparently, they can be created in physical processes [Kehle, Unger 22]
4. They have some applications in quantum gravity.

The goal of this talk is to present some recent developments in this topic

Isolated horizons [Ashtekar, Beetle, Lewandowski 01] [Ashtekar, Krishnan 03] [Lewandowski, Pawlowski 01;03]

1. Null hypersurface $\mathcal{H}$ fibered by null vector ℓ and degenerate metric (restriction of the spacetime metric)

$$\tilde{g}_{ij}\ell^i = 0, \quad \pi\colon \mathcal{H} \to \Sigma \ (\dim \Sigma = n, \ \text{compact}\ ) \tag{1}$$

2. Non-expanding condition

$$\mathcal{L}_\ell \tilde{g} = 0. \tag{2}$$

The metric is a pull-back $\tilde{g} = \pi^* g$ (g Riemannian on Σ).

3. Spacetime covariant derivative preserves tangent space to $\mathcal{H}$, restriction $\tilde{\nabla}_i$ (metric and torsion free). Isolated horizon condition (stronger than WIH in Abhays's talk)

$$\mathcal{L}_\ell \tilde{\nabla} = 0. \tag{3}$$

4. Introduce rotation form $\tilde{\nabla}_i \ell^j = \tilde{\omega}_i \ell^j$ and surface gravity $\kappa = \tilde{\omega}_i \ell^i = \mathrm{const.}$ Connection not uniquely determined by $\tilde{\omega}$ and $\tilde{g}$.

Isolated horizon data (abstractly): $(\mathcal{H}, \ell, \tilde{g}, \tilde{\nabla})$ satisfying (1), (2), (3).

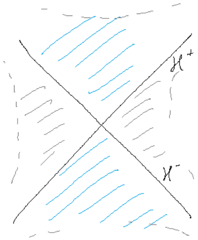
Isolated horizons (non-extreme case)

Non-extreme horizons $\kappa \neq 0$

1. Einstein equations determines the connection in terms of $\tilde{\omega}$ and $\tilde{g}$

$$\mathcal{L}_\ell \tilde{\omega} = 0,\ \mathcal{L}_\ell \tilde{g} = 0,\ \tilde{\omega}_i \ell^i = \kappa,\ \tilde{g}_{ij}\ell^i = 0. \tag{4}$$

2. In analytic case, there exists unique spacetime with this horizon as a Killing horizon (with bifurcated surface)



Black hole holograph [Racz, Wald] [Racz]

Extreme horizons $\kappa = 0$

1. Einstein equations imposes constraints on the data on Σ

$$0 \stackrel{!}{=} \nabla_{(\mu}\omega_{\nu)} + \omega_\mu\omega_\nu - \frac{1}{2}R_{\mu\nu} + \frac{1}{2}\lambda g_{\mu\nu} + \ldots, \qquad (\text{EIH}^n_\lambda)$$

   where $\tilde{\omega} = \pi^*\omega$, $\tilde{g} = \pi^* g$ (NHG data (g, ω))
2. Connection is left undetermined (higher order constraint)
3. For every data (g, ω) one can construct certain Kundt spacetime (Near Horizon Geometry NHG) [Lewandowski, Pawlowski, Jezierski] [Horowitz] [Reall]

$$ds^2 = F\rho^2 dv^2 + 2d\rho dv - 4\rho\omega_\mu dx^\mu dv + g_{\mu\nu}dx^\mu dx^\nu \qquad (5)$$

   where x are coordinates on Σ and $\mathcal{H} = \{\rho = 0\}$ (obtained by the Geroch type procedure applied to a spacetime with extreme horizon)

$$F(x) = \nabla^\mu\omega_\mu + 2\omega^\mu\omega_\nu - \lambda + \ldots. \qquad (6)$$

4. Non-uniqueness of embedding is a desireable feature (for NHG data).

Differences

EIH equation (closely related to Ricci solitons)

$$\nabla_{(\mu}\omega_{\nu)} + \omega_\mu\omega_\nu - \frac{1}{2}R_{\mu\nu} + \frac{1}{2}\lambda g_{\mu\nu} = 0, \qquad (\mathrm{EIH}^n_\lambda)$$

Extreme case:

1. All axisymmetric solutions on S^2 $(n = 2)$ found. They correspond to Kerr solutions [Lewandowski Pawlowski 01]
2. Generalized to higher dimensions with $U(1)^{n-1}$ symmetry (enhancing symmetry of NHG i.e. higher group of isometries than expected) [Kunduri, Lucietti 09] [Hollands, Ishibashi 10]

$$\nabla_\mu\Gamma - 2\omega_\mu\Gamma \text{ is a Killing vector for some } \Gamma > 0. \qquad (7)$$

Extreme versus non-extreme horizons

Geometry of extreme horizons is constraint intrinsically in contrast to the non-extreme case where it is constraint by global properties of the spacetime.

Structure of the talk (questions)

Problems for extreme (degenerate) isolated horizons:

1. **Rigidity problem** Does every non-static solution posses at least one symmetry

$$\mathcal{L}_K g = 0,\ \mathcal{L}_K \omega = 0$$

Does corresponding NHG spacetime have enhanced $SO(1,2)$ symmetry group?

2. **Staticity problem** The Killing vector ℓ has vanishing twist if $d\omega = 0$. The black hole is not rotating if $\omega = 0$

$$d\omega = 0 \overset{?}{\Longrightarrow} \omega = 0.$$

3. **Topology problem** Does Σ needs to be a sphere (especially $4d$ due to Hawking theorem)
4. **Killing embeddability** Classify possible spacetimes with isolated horizons as Killing horizons (classify isolated horizons with given NHG data)

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Killing vectors on compact manifolds

1. Killing laplacian $\Box K_\mu := \Delta K_\mu + R_\mu^\nu K_\nu$,

$$\nabla_{(\mu} K_{\nu)} = 0 \Longrightarrow \Box K_\mu = 0,\ \nabla_\mu K^\mu = 0 \qquad (8)$$

   On compact manofolds we have also $\Longleftrightarrow$.

2. Idea similar as in the case $\Delta\phi = 0 \Longleftrightarrow \phi = \mathrm{const}$

$$0 = \int \phi \Delta \phi = -\int \nabla_\mu \phi \nabla^\mu \phi \Longrightarrow \nabla_\mu \phi = 0 \qquad (9)$$

Killing vectors on compact manifolds

If $\nabla_\mu K^\mu = 0$ and there exists V and A_μ such that

$$\Box K_\mu = \partial_\mu V + A_\mu, \quad A_\mu K^\mu = 0 \qquad (10)$$

then $\nabla_{(\mu} K_{\nu)} = 0$.

Remarkable identity [Dunajski, Lucietti 23]

1. For $K_\mu = \nabla_\mu \Gamma - 2\omega_\mu \Gamma$ (Γ arbitrary function)

$$\Box K_\mu = -2\nabla_\nu K^\nu \omega_\mu + \nabla_\mu V + A_\mu \qquad \text{(RI)}$$

   where $V = \Delta\Gamma + 2\lambda\Gamma$ and $A_\mu = -2\Omega_{\mu\nu}K^\nu$ with $\Omega_{\mu\nu} = 2\nabla_{[\mu}\omega_{\nu]}$.
2. If we choose a nontrivial Γ such that $\nabla_\nu K^\nu = 0$ then K is a Killing vector field.

Missing piece existence of such nontrivial Γ:

1. Solution to the equation

$$0 = L\Gamma := -\nabla^\mu(\nabla_\mu \Gamma - 2\omega_\mu \Gamma)$$

   Operator L is elliptic and has discrete spectrum. Spectrum of $L^\dagger$ is complex conjugated to L

$$L^\dagger 1 = 0 \Longrightarrow 0 \in \operatorname{Spec} L^\dagger \Longrightarrow 0 \in \operatorname{Spec} L$$

Remarkable identity [Dunajski, Lucietti 23]

$$L\Gamma := -\nabla^\mu(\nabla_\mu\Gamma - 2\omega_\mu\Gamma)$$

Application of Krein-Rutnam theorem shows that

1. $\Gamma \neq 0$ (we can choose $\Gamma > 0$)
2. The 0 eigenvalue is multiplicity free and simple and every other eigenavlue has bigger real part

Interpretation: L is a generator of the Brownian motion with shift. Function Γ is the equilibrium probability distribution.

[Andersson, Mars, Simon]

Remarkable identity [Dunajski, Lucietti 23]

1. Remarkable identity simplifies for K satisfying Killing equation

$$0 = \Box K_\mu = \nabla_\mu V - 2\Omega_{\mu\nu} K^\nu$$

Equivalent to $K \lrcorner d\omega = -\frac{1}{2} dV$. Cartan formula provides

$$\dot{\omega} := \mathcal{L}_K \omega = dU, \quad U := K \lrcorner \omega - \frac{1}{2} V.$$

2. Lie derivative of NHG equation

$$0 = \nabla_{(\mu} \dot{\omega}_{\nu)} + 2\omega_{(\mu} \dot{\omega}_{\nu)}, \quad \Delta U + 2\omega_\mu \nabla^\mu U = 0$$

The only solution $U = \mathrm{const}$.

Axisymmetry theorem [Dunaski, Lucietti 23] [Colling et.al 24]

Every non-static solution to EIH posses a Killing K^μ satisfying $\mathcal{L}_K \omega = 0$.

Non-static because $0 = K^\mu = \nabla^\mu \Gamma - 2\Gamma \omega^\mu$ iff $\omega = d\frac{1}{2} \ln \Gamma$ is exact.

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Static solutions [Chrusciel, Reall, Tod 04] [Bahuad et al 23] [Wylie 23]

1. In one dimension, $g^1 = ds^2$, $s \in [0, L]$ and $\omega^1 = \phi ds$
$$0 = \dot{\phi} + \phi^2 + \frac{1}{2}\lambda$$
The only periodic solution $\phi = \pm\sqrt{-\lambda/2}$, $\lambda \leq 0$ (static).
2. Every solution to Einstein equation $R_{\mu\nu} = \lambda g_{\mu\nu}$ provides EIH^n_λ with $\omega = 0$ (static).
3. For $(g^1, \omega^1) \in \mathsf{EIH}^1_\lambda$ and $(g, 0) \in \mathsf{EIH}^n_\lambda$
$$(g^1 \oplus g, \omega^1 \oplus 0) \in \mathsf{EIH}^{n+1}_\lambda \tag{11}$$
The result is also static.
4. Topologically nontrivial construction locally as above.

For $\lambda = 0$ in these solutions $\omega = 0$.

Question

Are these all static solutions?

Static solutions [Chrusciel, Reall, Tod 04] [Bahuad et al 23] [Wylie 23]

1. Remarkable identity simplifies $\Omega_{\mu\nu} = 0$ thus $A_\mu = 0$

$$0 = \Box K_\mu = \nabla_\mu V \implies V = \Delta\Gamma + 2\lambda\Gamma = \text{const}\,. \tag{11}$$

2. An argument with the eigenspace of Laplacian for $\lambda \leq 0$ shows

$$\Gamma = \text{const} \implies K_\mu \propto \omega_\mu \tag{12}$$

3. $\nabla_\mu \omega_\nu = 0$ so locally we have a splitting. Global result follows from considerations of universal cover.

Staticity theorem [Chrusciel, Reall, Tod 04] [Bahuad et al 23] [Wylie 23]

If EIH^n_λ data is static then

1. for $\lambda = 0$, $\omega = 0$
2. for $\lambda < 0$, it is a type of solution described earlier,
3. for $n = 2$ and $\lambda \in \mathbb{R}$, $\omega = 0$. This holds also in the case with Maxwell field

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Topology 4d, $n = 2$ [Dobkowski-Rylko et.al 18,19] [Lewandowski, Kaminski 24-t.b.p]

Situation in 4d ($n = 2$) is special

1. Riemannian geometry is complex analysis plus scale

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} \tag{13}$$

2. Projection on anti-holomorphic covectors $P_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - i\epsilon_{\mu\nu})$

$$\pi_\mu = P_{\mu\nu}\omega^\nu, \quad D_\mu = P_{\mu\nu}\nabla^\nu \tag{14}$$

3. The EIH equation imposes $D_\mu \pi_\nu + \pi_\mu \pi_\nu = 0$ (it holds also in the presence of Maxwell field).

Surface Σ of genus ≥ 1 [to be published]

If f_μ satisfies

$$D_\mu f_\nu + f_\mu f_\nu = 0, \quad f_\mu = P_\mu{}^\nu f_\nu \tag{15}$$

then $f_\nu \equiv 0$.

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Topology theorem [to be published]

For surfaces of nonzero genus the only EM EIH solutions are surfaces with constant curvature, constant electromagnetic field and $\omega = 0$.

In higher dimensions weaker constraints [Kunduri, Lucietti 08] [Bahuad et al 23].

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Embeddings [Li, Lucietti 15] [Katona Lucietti 24], [Katona 24]

1. Consider metric in null gaussian coordinates (v, ρ, x),

$$ds^2 = \tilde{F}\rho^2 dv^2 + 2d\rho dv - 4\rho\tilde{\omega}_\mu dx^\mu dv + \tilde{g}_{\mu\nu}dx^\mu dx^\nu$$

   where $\ell = \partial_v$ (Killing vector), $\mathcal{H} = \{\rho = 0\}$ and null gaussian condition ∂_ρ tangent to null affinine geodesic.

2. There is a gauge freedom in the choice of coordinates (Diff_Σ and arbitrary shifts in v). The second can be fixed (to constant shift) by

$$\partial_\rho\sqrt{\tilde{g}}|_{\rho=0} = \mathrm{const}\,.$$

3. We can write Einstein equations in term of the Taylor expansion

$$\tilde{g}_{\mu\nu} = g^{[0]}_{\mu\nu}(x) + \rho g^{[1]}_{\mu\nu}(x) + \ldots, \quad g^{[0]}_{\mu\nu} = g_{\mu\nu} \tag{15}$$

   and similar expansions for $\tilde{F}$ and $\tilde{\omega}$.

We would like to solve Eintein's equations recursively in ρ.

Embeddings [Li, Lucietti 15] [Katona Lucietti 24], [Katona 24]

1. The Einstein equations give recurence relations

   $$A_n(g^{[n]}) = l.o.t, \quad A_n \text{ differential operator depending on } g \text{ and } \omega$$

   Once $g^{[n]}$ is know $F^{[n]}$ and $\omega_\mu^{[n]}$ are determined algebraically.
2. Tensor $g_{\mu\nu}^{[1]}$ contains informations equivalent to choice of connection. It is constraint by $A_1(g^{[1]}) = 0$. [Kolanowski, Lewandowski 20]
3. With this choice of gauge, A_n are elliptic operators of the second order.
4. Elliptic operators on a compact manifold have finite kernels and cokernels thus equations $A_n(g^{[n]}) = B_n$ have finite ambiguity and finite obstruction.
5. In every step of expansion, only finite number of ambiguities appear and the finite number of constraints.

Embeddings [Li, Lucietti 15] [Katona Lucietti 24], [Katona 24]

1. **The case of spherically symmetric EIH with $\lambda > 0$ (Schwarzschild-dS NHG data)**
    1. A_n have vanishing kernels and cokernels for $n \geq 2$
    2. A_1 has one dimensional kernel.

    Schwarzschild-de Sitter is the unique stationary spacetime with Schwarzschild-dS isolated horizon and moreover such horizons are unique horizons for corresponding NHG data (except NHG spacetime).

    [Katona Lucietti 24], [Katona 24]
2. **The case of Kerr EIH with** $\lambda = 0$ every A_n has a nontrivial kernel.

    [Horowitz et al 23]

Summary

1. Extreme isolated horizons are now better understood
   1. in vacuum for $\lambda \leq 0$
   2. for $\lambda > 0$ static solutions still mysterious
2. The case of horizons in presence of EM field is largely open.

Outlook:

1. Natural generalization when ℓ does not define fibration, but winds around some compact manifold (cosmological Cauchy horizons in Taub-NUT)
2. In such case there is no space of rays (at least globally).
3. Axisymmetric solutions classified for $n = 2$.

[Dobkowski-Rylko, Lewandowski, Ossowski]

Reminder: FAU^2 conference 25-27.06 Erlangen, Germany

Non-trivial bundle structure [Dobkowski-Rylko, Lewandowski, Ossowski]

1. General axisymmetric metric on S^2

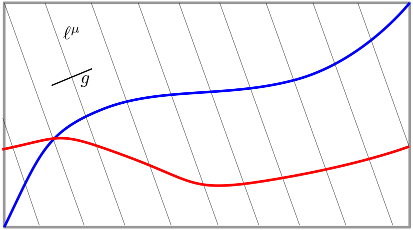
$$g = P(x)^{-1}dx^2 + P(x)d\phi^2,\ \ \phi \in [0, L],\ \ x \in [-1, 1] \tag{15}$$

2. Conditions for P in final points for metric to be smooth

$$P(\pm 1) = 0, \quad P'(\pm 1) = \pm\frac{4\pi}{L}. \tag{16}$$

3. EIH equation reduces to ODE for P and Γ (determining ω). General solution does not satisfy $P'(1) = -P'(-1)$ (Γ always smooth).

1. If ℓ does not define fibration the space of rays not defined



2. Metric defined separately on two discs with gluing by Hopf bundle map (as on picture). Proper choice of ℓ allows $P'(1) \neq -P'(-1)$.
3. All smooth axisymmetric solutions classified.