Probing Quadratic Gravity with Binary Inspirals

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The Einstein-Hilbert Action

The Einstein-Hilbert Action gives us the Einstein Field Equations

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} + \mathcal{L}_M \right]$$

$$\text{ELE} \Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\text{Expanded} \Rightarrow \Box h_{\mu\nu} = 2\kappa \tau_{\mu\nu}$$

- $\kappa = 8\pi G = 8\pi / M_p^2$
- R is the only independent scalar which we can construct (up to second derivatives) of the metric
- The metric is split up as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to find the wave equation

Why Modify General Relativity?

General Relativity is the simplest theory coupling spacetime curvature to matter

Good reason to look at modified theories

- Interaction with quantum matter, should be a limit from any quantum theory of gravity

We can consider modified theories by adding terms to the Hilbert action, as long as they

- Are diffeomorphism invariant, scalar, etc
- Limit correctly to GR, and Newtonian gravity

What effect do these modifications have?

- Must look at strong gravity
- $\Rightarrow$ Binary Systems are an ideal testing ground

The Post-Newtonian Formalism

The Post-Newtonian (PN) formalism is an iterative expansion scheme in v/c, for arbitrarily precise solutions to the Einstein field equations

- Requires slow moving, weakly stressed sources (valid for inspiralling binary black holes up to $v/c = 0.5$)
- Naturally includes non-linearity and higher multipole characteristics
- Convention is to just track $1/c^n$, and call those terms "$\frac{n}{2}$PN order"

0PN order is called the "Newtonian" order, and GR only affects dynamics at higher orders

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \frac{1}{c} h^{(1)}_{\mu\nu} + \frac{1}{c^2} h^{(2)}_{\mu\nu} + \dots$$

Project Aim

To investigate gravitational waves from binary systems in the early inspiral phase, given by an effective field theory applicable only in the low energy/curvature regime

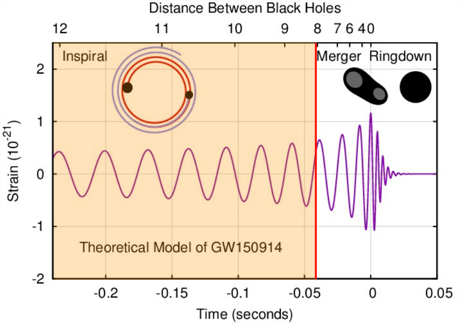

Figure 1: The orange section illustrates the early inspiral phase [1]

Modified Action I

We can modify GR by adding in all independent terms up to 4th derivatives of the metric [2]

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \beta R^2 + \gamma R^{\mu\nu}R_{\mu\nu}\right]$$

- These are unavoidable from one-loop renormalisation of matter with semi-classical gravity

Modified Action II

We can interpret the extra degrees of freedom from the quadratic terms as a massive spin-0 field ϕ, and a massive spin-2 field $\pi_{\alpha\beta}$. So essentially the quadratic terms can be rewritten as

$$\beta R^2 \to -\frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + m_\phi^2 \phi^2\right)$$
$$\gamma R^{\mu\nu} R_{\mu\nu} \to -\frac{1}{2}\left(\partial_\mu \pi^{\alpha\beta} \partial^\mu \pi_{\alpha\beta} + m_\pi^2 \pi^{\alpha\beta} \pi_{\alpha\beta}\right)$$

We introduce a source term that models a binary system of two point particles with masses m_a, and 4-velocities v_a^μ

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\left(\partial_\mu \pi^{\alpha\beta}\partial^\mu \pi_{\alpha\beta} + m_\pi^2 \pi^{\alpha\beta}\pi_{\alpha\beta}\right) - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + m_\phi^2\phi^2\right)\right] + \sum_{a=1}^{2} m_a \int dt\sqrt{-\tilde{g}_{\mu\nu}v_a^\mu v_a^\nu}$$

where the metric is conformally constructed as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \sqrt{2\kappa}\eta_{\mu\nu}\phi + \sqrt{4\kappa}\pi_{\mu\nu}$$

Linearised Equations Of Motion

Finding the linearised field equations for ϕ, and $\pi_{\mu\nu}$ gives Yukawa-like solutions, and the GR solution is what we expect

$$\tilde{h}_{\mu\nu}(x) \propto \frac{1}{\left|\vec{x} - \vec{y}_a(t_r)\right|}$$

$$\phi(x) = \sqrt{\frac{G}{4\pi}} \sum_{a=1}^{2} m_a \frac{e^{-m_\phi \left|\vec{x} - \vec{y}_a(t_r)\right|}}{\left|\vec{x} - \vec{y}_a(t_r)\right|}$$

$$\pi_{\mu\nu}(x) = -\sqrt{\frac{G}{2\pi}} \sum_{a=1}^{2} m_a \left(v_{\mu a} v_{\nu a} + \frac{1}{4}\eta_{\mu\nu}\right) \frac{e^{-m_\pi \left|\vec{x} - \vec{y}_a(t_r)\right|}}{\left|\vec{x} - \vec{y}_a(t_r)\right|}$$

The solutions look just like the regular GR potential, expect they have a exponential mass suppression. Note the signs, where the spin-0 field is attractive, while the spin-2 field is repulsive.

The Conservation Equation

In order to calculate the waveform we can invoke the conservation equation

$$\tilde{\nabla}_{\mu}\tilde{T}^{\mu\nu} = 0$$

Then the relative acceleration to Newtonian order is

$$\vec{a} = -\frac{GM}{r^2}\hat{n}\left(1 + 2e^{-m_{\phi}r}(m_{\phi}r + 1) - 3e^{-m_{\pi}r}(m_{\pi}r + 1)\right)$$

where $r = |\vec{y}_1 - \vec{y}_2|$, and $\hat{n} = (\vec{y}_1 - \vec{y}_2)/r$. The spin-0 field is indeed attractive, while the spin-2 field is repulsive.

Energy Balance Equation

Using the acceleration we find the energy

$$E = -\frac{Gm_1m_2}{r}\left(\frac{1}{2} + 2e^{-m_\phi r} - 3e^{-m_\pi r}\right)$$

We see that the spin-0 field takes away energy, while the spin-2 field gives energy to the black hole.

Since energy lost from the black hole is what we observe, we assume the balance equation

$$\frac{dE}{dt} = -\mathcal{F}$$

The far-field flux will be highly suppressed for the massive fields.

The Stationary Phase Approximation

We can assume a GW signal with amplitude $A(t)$, and phase $\Phi(t)$ takes the form [5]

$$h(t) = 2A(t)\cos\Phi(t)$$

In the Stationary Phase Approximation (SPA), the frequency domain signal is given by the following

$$\tilde{h}(f) = \frac{\sqrt{2\pi}A(t_f)}{\sqrt{\ddot{\Phi}(t_f)}}e^{i\psi(f)}$$

$$\psi(f) = 2\pi f t_f - \pi/4 - \Phi(t_f)$$

where the parameter t_f is given by the time when $d\Phi(t)/dt = 2\pi f$.

The Inspiral Waveform

Using the energy balance equation the leading corrections to the GR waveform phase of the binary inspiral is given as

$$\Phi = -\frac{x^{-5/2}}{16\nu}\left[1 + e^{-m_\phi \frac{GM}{x}}\left(\frac{5}{2} - \frac{5}{3} m_\phi \frac{GM}{x}\right) - e^{-m_\pi \frac{GM}{x}}\left(\frac{15}{4} - \frac{5}{2} m_\pi \frac{GM}{x}\right)\right]$$

where $x \equiv (GM\Omega)^{\frac{2}{3}}$ is a frequency-related parameter, and $\nu = \frac{m_1 m_2}{M^2}$ is the symmetric mass ratio.

Inside the bracket, the terms with x^0 are associated with the Newtonian (quadrupole) order, whereas the x^{-1} are associated with the dipole order.

The Fourier Coefficients

Following the SPA prescription we obtain the relation for $\Phi(t_f)$

$$\Phi(t_f) \propto \frac{3}{128\nu} \sum_{i=-4}^{0} \varphi_i (GM\pi f)^{(i-5)/3}$$

In GR the -1PN order coefficient is zero, however in our modified case it is not zero

$$-1\text{PN} \quad \varphi_{-2} = \frac{451928}{27} m_\phi GM e^{-m_\phi \delta} - \frac{225964}{9} m_\pi GM e^{-m_\pi \delta}$$

where $\delta = 4GM\left(8\pi GMf\right)^{-2/3}$

Parameters of Quadratic Gravity

The absolute deviation of the -1PN phase has been constrained from gravitational wave data to be $|\delta\varphi_{-1PN}| < 10^{-2}$

$$\left|\frac{451928}{27} m_\phi GM e^{-m_\phi \delta} - \frac{225964}{9} m_\pi GM e^{-m_\pi \delta}\right| \lesssim 10^{-2}$$

Taking the typical values $f = 75\text{Hz}$, and $M = 30M_\odot$, the masses should satisfy the inequalities separately

$$m_{\phi,\pi} \gtrsim 7.1 \cdot 10^{-12}\text{eV}$$

Rewritten as limits on the original dimensionless parameters of quadratic gravity

$$\begin{aligned} 0 \leq \quad &\gamma \quad \lesssim 1.5 \cdot 10^{78} \\ -\frac{\gamma}{4} \leq \quad &\beta \quad \lesssim -1.1 \cdot 10^{77} \end{aligned}$$

Summary

We were able to recast the quadratic gravity degrees of freedom as a massive spin-0 and spin-2 field alongside the usual massless spin-2 graviton, and derived linear, lowest order field equations

To Newtonian order, they respectively act as attractive, and repulsive Yukawa potentials

Found the -1PN correction to the GW phase of an inspiralling binary system in quadratic gravity

Placed constraints on quadratic gravity from GW observations from the LIGO, and Virgo Collaborations

References

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[2] K. S. Stelle, Phys. Rev. D **16**, 953 (1977). doi:10.1103/PhysRevD.16.953

[3] L. Blanchet, T. Damour and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995) Erratum: [Phys. Rev. D **54**, 1860 (1996)] doi:10.1103/PhysRevD.51.5360, 10.1103/PhysRevD.54.1860 [gr-qc/9501029].

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Definition Of Terms

The mass terms are

$$m_\phi^2 = \frac{1}{3\kappa(4\beta + \gamma)}$$

$$m_\pi^2 = \frac{1}{2\kappa\gamma}$$

The Θ term is defined as

$$\Theta \equiv \frac{\nu}{5GM}(t_c - t)$$

The symmetric mass ratio terms are

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\nu = \frac{\mu}{M}$$

Quadratic Gravity As An Effective Field Theory

We cutoff our Lagrangian at quadratic order to avoid non renormalisability at the 2-loop level

Stelle [2] noted the negative norm states of the massive spin-2 field

- We must interpret this as an effective field theory

Quick calculation to show realm of validity

$$M_p^2 R > \alpha R^{\text{quad}} \Rightarrow M_p^2 p^2 > \alpha p^4 (\text{In momentum space})$$
$$\Rightarrow M_p^2/r^2 > \alpha/r^4$$
$$m_{\phi,\pi} \approx M_p^2/\alpha \Rightarrow m_{\phi,\pi} r > 1$$

We can then see that far-field plane waves $e^{-i(\omega t - \vec{k}\vec{x})}$ are suppressed

$$v^2 \approx GM/r < 1 < m_{\phi,\pi} r \Rightarrow m_{\phi,\pi} > \Omega^2 \approx \omega^2$$
$$\Rightarrow k^2 = \omega^2 - m_{\phi,\pi}^2 < 0$$

Recasting The Lagrangian

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \beta R^2 + \gamma R^{\mu\nu}R_{\mu\nu}\right]$$

Setting $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$, and $\alpha = \beta + \frac{\gamma}{4}$

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \alpha R^2 + \gamma S^{\mu\nu}S_{\mu\nu}\right]$$

Using Lagrange multipliers, and the following conformal transformation

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \Omega^2 = (1 + \sqrt{2\kappa}\phi)$$

$$S = \int d^4x\sqrt{-g}\left[\frac{\widetilde{R}}{2\kappa} + \pi^{\mu\nu}\widetilde{S}_{\mu\nu} - \frac{1}{4\gamma}\pi^{\mu\nu}\pi_{\mu\nu} - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + m_\phi^2\phi^2\right)\right]$$

Separating $\pi^{\mu\nu}$ from $\widetilde{h}^{\mu\nu}$ we obtain the final result

Details On The Waveform Calculations

We start from the balance equation

$$\frac{dE}{dt} = -\mathcal{F}$$

Using the chain rule our energy balance equation becomes

$$\frac{dE}{dx}\frac{dx}{d\varphi}\frac{d\varphi}{d\Theta}\frac{d\Theta}{dt} = -\mathcal{F}$$

We can calculate the third term by

$$\frac{d\varphi}{dt} = \Omega \quad \Rightarrow \quad \frac{d\varphi}{d\Theta} = -\frac{5}{\nu}x^{3/2}$$

The usual GR flux in terms of x

$$\mathcal{F} = \frac{32}{5G}\nu^2 x^5$$

Details On The Fourier Calculations

To get φ as a function of time t from x, we rewrite x as a function of time

$$\begin{aligned} x(t) &= f(\Theta(t)) \\ \Rightarrow \varphi(t) &= g(\Theta(t)) \end{aligned}$$

To find t_f we take derivatives

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= 2\pi f \\ \Rightarrow \frac{d\varphi(t)}{dt} &= \pi f \\ \Rightarrow t_f &= \dots \end{aligned}$$

Insert everything back into $\psi(f)$ to obtain the Fourier coefficients

Details On The Fourier Calculations

[6] Z. Cao, P. Galaviz and L. F. Li, Phys. Rev. D **87**, no. 10, 104029 (2013) doi:10.1103/PhysRevD.87.104029 [arXiv:1608.07816 [gr-qc]].

Cao *el al.* had a constraint of $|\beta| \propto 10^{80}$. Our results being 3 orders of magnitude stronger.

The method they used was to consider an $f(R)$ gravity theory which then can be simplified down to a coupled Einstein-Klein-Gordon equation. The numerical analysis was done to obtain the bounds on the coefficient.