The full angle-dependence of the four-loop cusp anomalous dimension in QED

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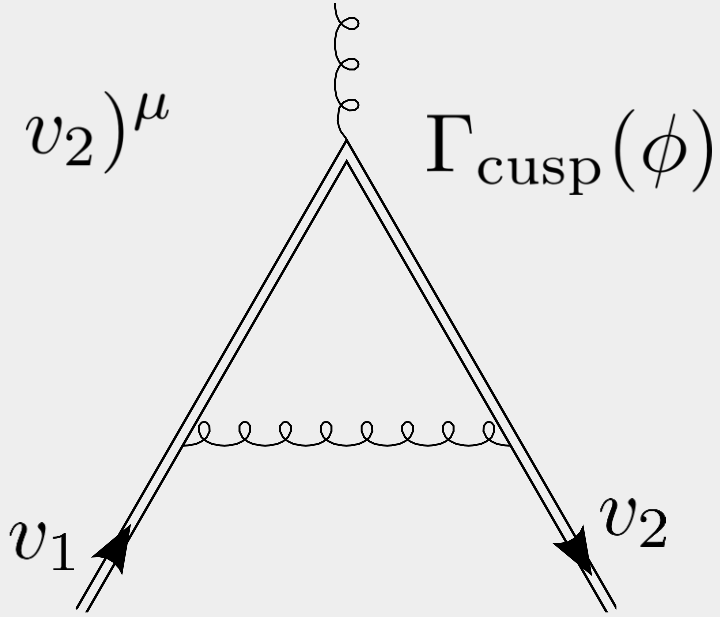
The angle-dependent cusp anomalous dimension

- Heavy quark scattering $q^\mu = m_Q(v_1 - v_2)^\mu$
- HQET: Renormalization group
- IR divergences of massive scattering
- Ingredient in resummation



[Korchemsky, Radyushkin, '86]

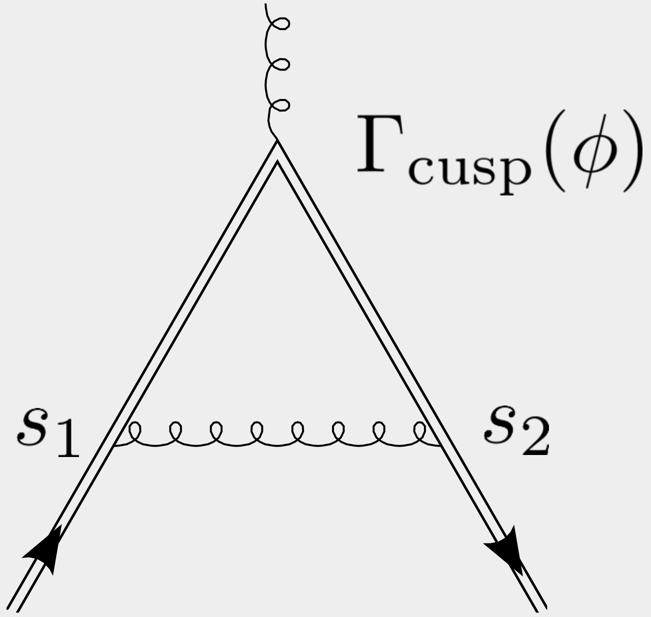
Wilson lines

- Momentum space: Eikonal propagators

$$\int \frac{\mathrm{d}^D k}{k^2 (k \cdot v_1)(k \cdot v_2)}$$

- Configuration space

$$\frac{i}{k \cdot v_1} = \int_0^\infty e^{i s_1\, k \cdot v_1} \mathrm{d}s_1$$



- $\Gamma_{\mathrm{cusp}}(\phi)$ in UV and IR divergences of cusped Wilson line

[Korchemsky, Radyushkin, '92]

$$W = \frac{1}{N} \mathrm{P} \exp \left(ig \int_C \mathrm{d}x^\mu A_\mu(x) \right)$$

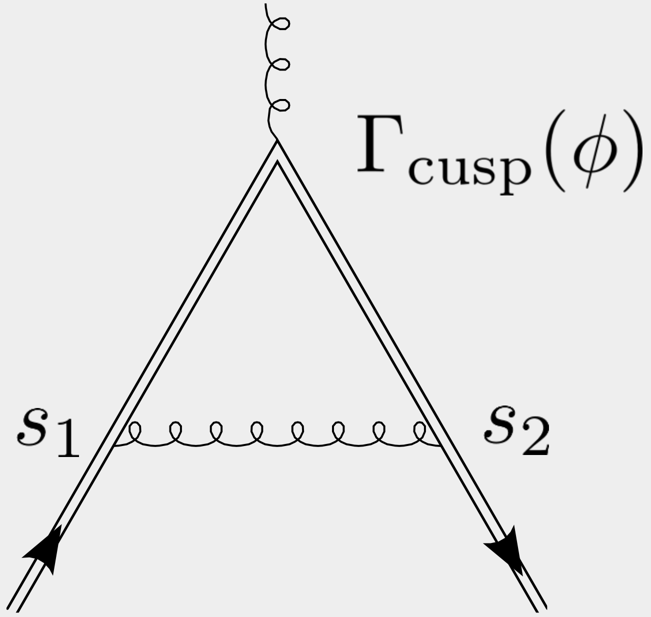
Wilson lines

- Momentum space: Eikonal propagators

$$\int \frac{\mathrm{d}^D k}{k^2(k\cdot v_1)(k\cdot v_2)}$$

- Configuration space

$$\frac{i}{k\cdot v_1+\delta} = \int_0^\infty e^{is_1\,(k\cdot v_1+\delta)}\mathrm{d}s_1$$



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Four-loop $\Gamma_{\text{cusp}}(\phi)$

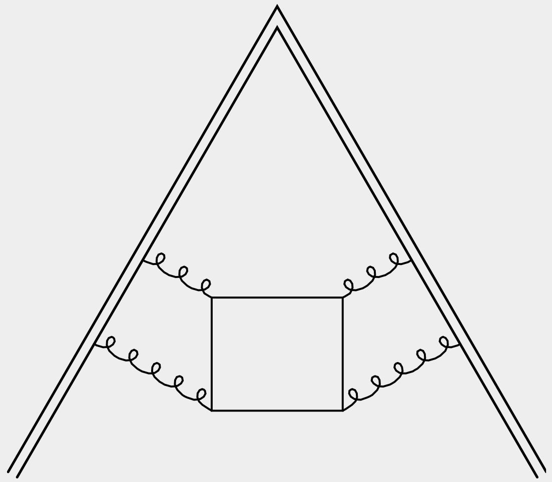
[Beneke, Braun, '95 / Grozin, Henn, Korchemsky, Marquard '16 / Henn, Lee, Smirnov, Smirnov, Steinhauser '16 / Henn, Smirnov, Smirnov, Steinhauser '16 / Grozin, Henn, Stahlhofen '17 / Davies, Vogt, Ruijl, Ueda, Vermaseren, '17 / Henn, Smirnov, Smirnov, Steinhauser, '18 / Grozin, '18 / Moch, Ruijl, Ueda, Vermaseren, Vogt '17 and '18 / Lee, Smirnov, Smirnov, Steinhauser '19 / Henn, Peraro, Stahlhofen, Wasser, '19]

- Many colour structures already computed

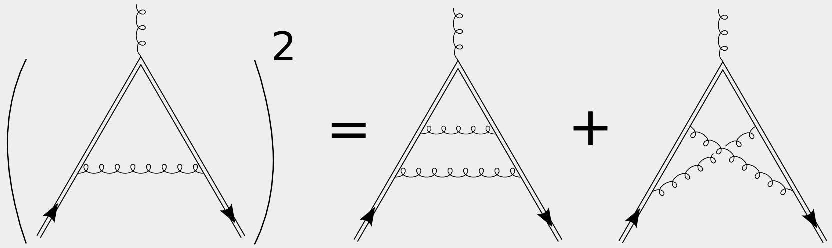
$(T_F n_f)^3 C_R, (T_F n_f)^2 C_R C_F, (T_F n_f)^2 C_R C_A$

$(T_F n_f) C_R C_F^2, (T_F n_f) C_R C_F C_A$

Wilson line



- Casimir scaling $\Gamma_{\text{cusp}}(\phi) = C_R(\ldots) + \mathcal{O}(\alpha_s^4)$
- First non-planar corrections

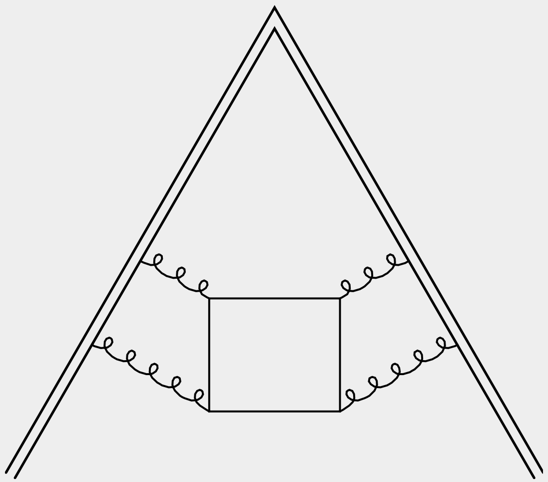


- Fermion box diagrams $n_f d_R d_F / N_R, (T_F n_f) C_R C_A^2$

Universality conjecture

- Use light-like cusp as effective coupling: [Grozin, Henn, Korchemsky, Marquard, '16]

$$a \sim \Gamma_{\text{cusp}}(\phi \to i\infty) \sim K \qquad \Gamma_{\text{cusp}}(\phi) = \sum_{k\geq 1} a^k \Omega^{(k)}(\phi)$$



the coefficients are matter independent

$$\Omega_{\text{QCD}}(\phi, a) = \Omega_{\text{YM}}(\phi, a) = \Omega_{\mathcal{N}=4}(\phi, a)$$

- Prediction for matter dependent terms

$$\Gamma_{\text{cusp}}(\phi) = \frac{\alpha_a}{\pi}\Omega^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\Omega^{(2)} + K^{(2)}\Omega^{(1)}\right) + \mathcal{O}(\alpha_s^3)$$

- Violated for some colour structures $n_f d_R d_F / N_R, (T_F n_f) C_R C_A^2$

[Brüser, Grozin, Henn, Stahlhofen, '19]

Universality conjecture

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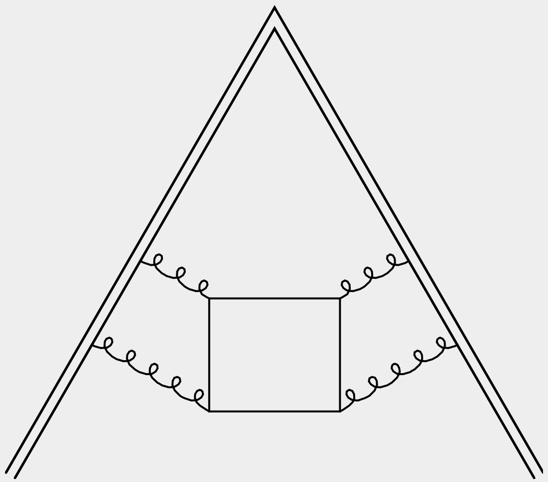

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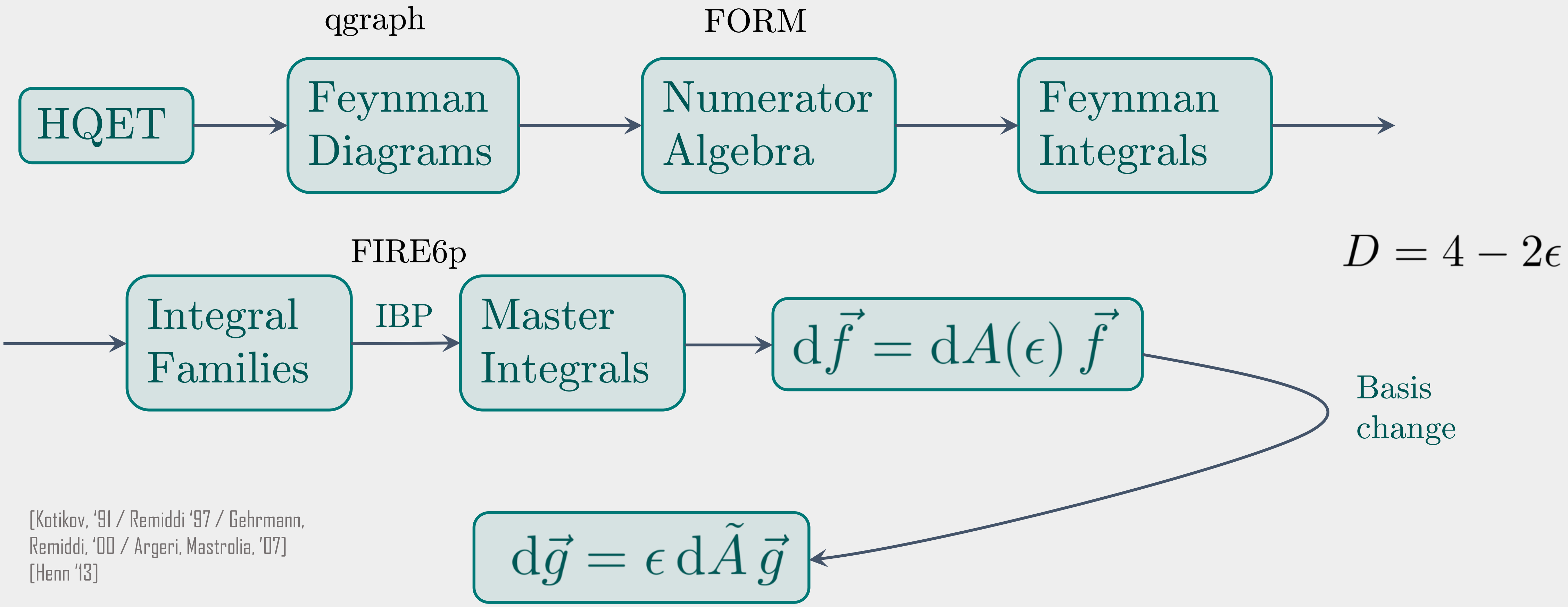
QED

[Brüser, Grozin, Henn, Stahlhofen, '19]

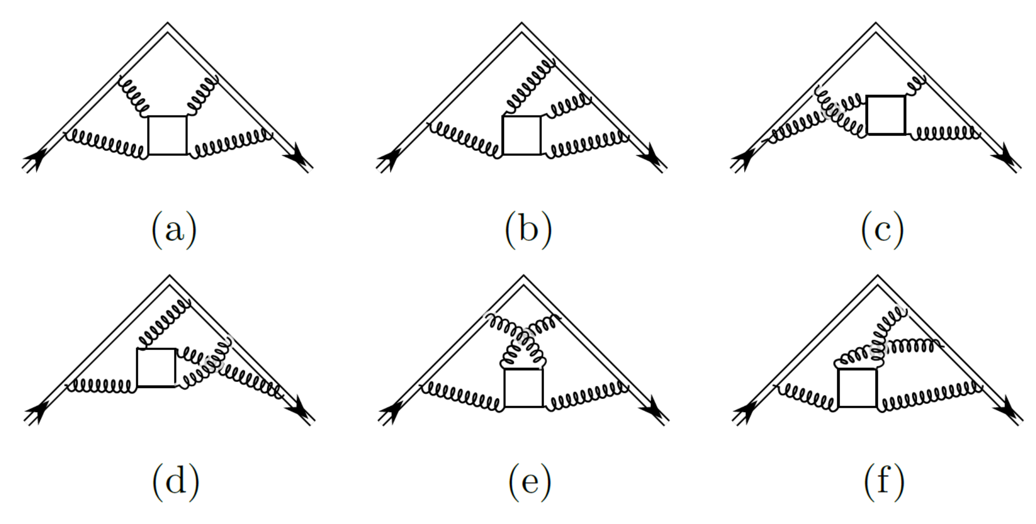
Workflow



[Kotikov, '91 / Remiddi '97 / Gehrmann, Remiddi, '00 / Argeri, Mastrolia, '07]
[Henn '13]

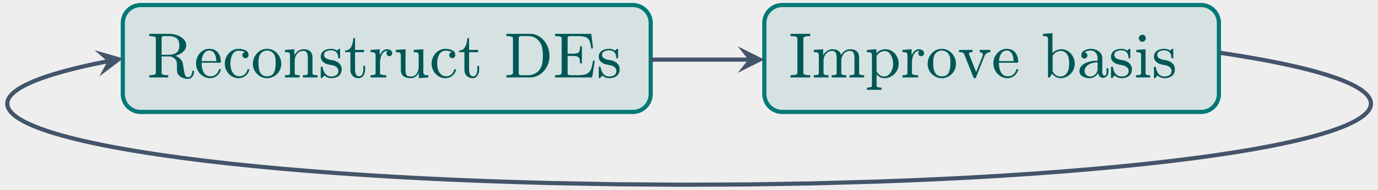
Integral families

- All integrals belong to one of six families:
  - $\sim$ 500 master integrals per family
  - $\sim$ 150 sectors per family
  - Largest sector has 17 masters



IBP reduction

- FIRE6p: “numeric” reduction over finite fields
  - Max Planck Supercomputer Cobra
  - ~ 60-100 compute nodes with 40 cores for 3-4 weeks
- Custom reconstruction algorithm for [Peraro, ’16 / Manteuffel, Schabinger, ’15 / Kotsireas, Mourrain, Pan, Cuyt, Lee, ’11]
  - Variables D, ϕ: 40 x 20 evaluations
  - Rational numbers: 5 evaluations
- Only DEs and Feynman diagrams reconstructed
  - Iterative procedure:



Canonical basis

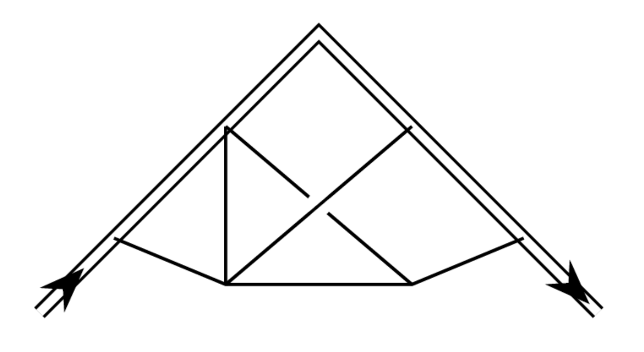
- Block structure: [R. Lee, '15]
  - Diagonal blocks
  - Off-diagonal blocks

$$\mathrm{d}\vec{g} = \epsilon\, \mathrm{d}\tilde{A}\, \vec{g}$$

- Standard algorithms: Epsilon, Fuchsia
- INITIAL:
  - E.g. 17 masters sector: 2 min to solve
  - One integral from canonical basis needed
  - Can often be found by analysing singularities of integrals
  - Test ~ $\mathcal{O}(1000)$ integrals

Beyond polylogarithms?

$$\mathrm{d}\vec{g} = \epsilon\,\mathrm{d}\tilde{A}\,\vec{g}$$



- One sector seems to have no canonical form
- Second-order DE at ϵ^0
- Also no form $\mathrm{d}\vec{g} = (\epsilon + \frac{1}{2})\,\mathrm{d}\tilde{A}\,\vec{g}$
- Integrals not relevant for pole and therefore not needed for $\Gamma^{(4)}_{\text{cusp}}$
- Matrices have singularities $x = e^{i\phi}$

$$\{x, 1+x, 1-x, 1+x^2, 1-x+x^2, \tfrac{1-\sqrt{-x}}{1+\sqrt{-x}}, \tfrac{1-\sqrt{-x}+x}{1+\sqrt{-x}+x}\}$$

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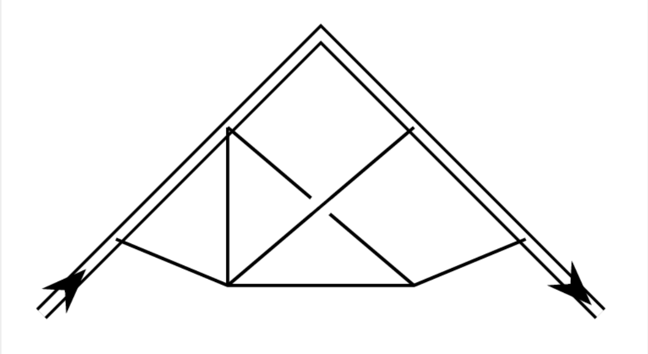


$$\left\{x, 1+x, 1-x, 1+x^2, 1-x+x^2, \frac{1-\sqrt{-x}}{1+\sqrt{-x}}, \frac{1-\sqrt{-x}+x}{1+\sqrt{-x}+x}\right\}$$

do not appear in $\Gamma^{(4)}_{\mathrm{cusp}}$

Results

- QED result:

one-loop function

$$A(x) = -\frac{1+x^2}{1-x^2}\log x - 1$$

$$\Gamma_{\text{cusp}}(x,\alpha) = \gamma(\alpha)A(x) + \left(\frac{\alpha}{\pi}\right)^4 n_f B(x) + \mathcal{O}(\alpha^5)$$

constant

main new result

- Conjectured result:

$$B_c(x) = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right) A(x) \approx -0.484\, A(x)$$

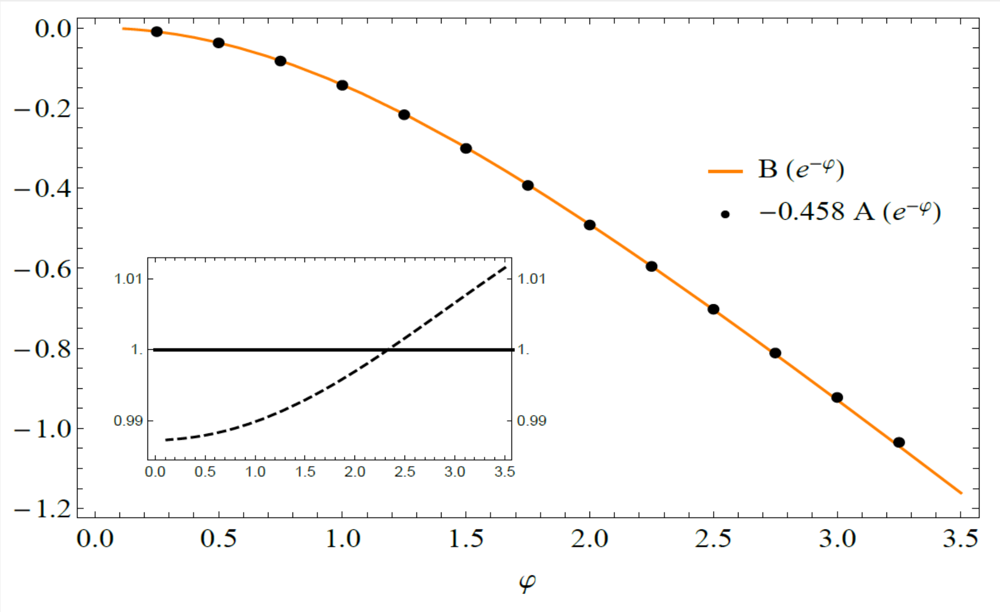
polylogarithms

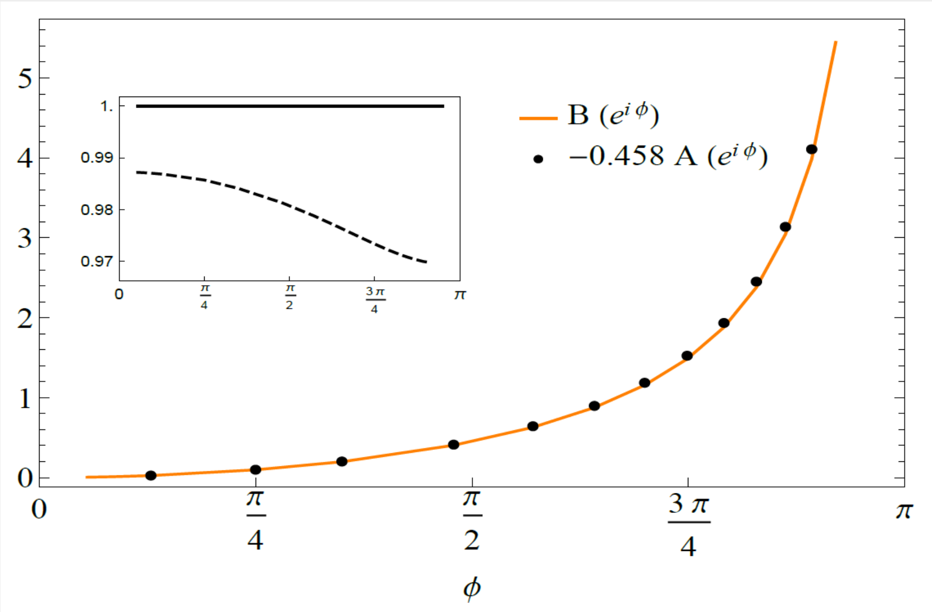
$$B(x) = \frac{1+x^2}{1-x^2}B_1 + \frac{x}{1-x^2}B_2 + \frac{1-x^2}{x}B_3 + B_4$$

Results

- Very well approximated by rescaled one-loop function

$B_c(x) \approx -0.484\, A(x)$





$\varphi = -i\phi$ Minkowskian angle

$x = e^{i\phi} = e^{-\varphi}$

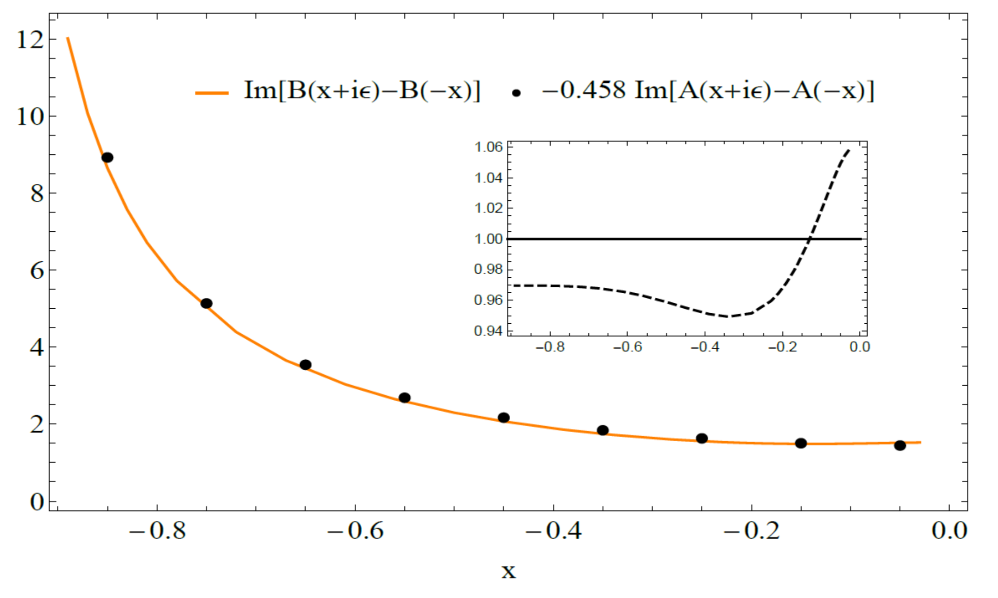
$$B(x) = \frac{1+x^2}{1-x^2}B_1 + \frac{x}{1-x^2}B_2 + \frac{1-x^2}{x}B_3 + B_4$$

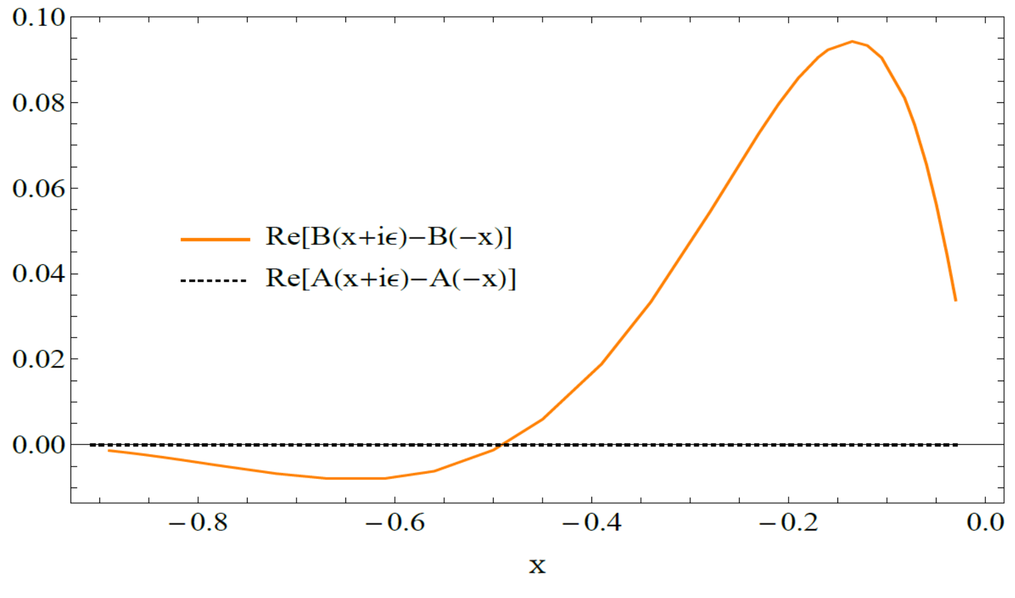
$$A(x) = -\frac{1+x^2}{1-x^2}\log x - 1$$

Results

- Deviation in analytic continuation

$$B_c(x) \approx -0.484\, A(x)$$





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$$A(x) = -\frac{1+x^2}{1-x^2}\log x - 1$$

Checks and limits

- Small angle (low-energy) limit $\phi \to 0, \quad x \to 1$
  - agreement with known terms up to $\mathcal{O}(\phi^6)$ [Grozin, Henn, Stahlhofen, '17]

- Light-like (high-energy) limit $\phi \to i\infty, \quad x \to 0$ [Lee, Smirnov, Smirnov, Steinhauser, '19 / Henn, Peraro, Stahlhofen, Wasser, '19]

$$B(x) = \log x \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3} \right) \qquad B_c(x) = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3} \right) A(x)$$

- Anti-parallel lines (threshold) limit $\phi \to \pi, \quad x \to -1$ [Lee, Smirnov, Smirnov, Steinhauser, '16]
  - conformal transformation ⇒ quark-antiquark potential

$$B(x) = \frac{\pi}{\phi - \pi} \left(\frac{79\pi^2}{72} - \frac{23\pi^4}{48} + \frac{5\pi^6}{192} + \frac{l_2\pi^2}{2} + \frac{l_2\pi^4}{12} - \frac{l_2^2\pi^4}{4} - \frac{61\pi^2\zeta_3}{24} + \frac{21\pi^2\zeta_3 l_2}{4} \right) + \mathcal{O}(\phi - \pi)$$

$$l_2 = \log(2)$$

Summary and outlook

- $\Gamma^{(4)}_{\rm cusp}$ in QED $\qquad n_f d_R d_F / N_R$
- High computational power needed for IBP reduction
- Size of DEs show automation of canonical form
- Result surprisingly close to rescaled one-loop function

- Missing terms: $(T_F n_f) C_R C_A^2, \qquad d_R d_A / N_R$