Higher-order β-functions in the Standard Model and beyond

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May 20, 2021

Fermilab

What are β-functions?

What are β-functions

β-functions determine the energy dependence of coupling constants:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\alpha_i(\mu)}{\pi} = \beta_i\left(\{\alpha_j\}\,;\epsilon\right) \ .$$

β_i depends on all couplings $\{\alpha_j\}$ of the theory.

QCD: $\{\alpha_j\} = \left\{\frac{g_s^2}{4\pi}\right\}$

What are β-functions

β-functions determine the energy dependence of coupling constants:

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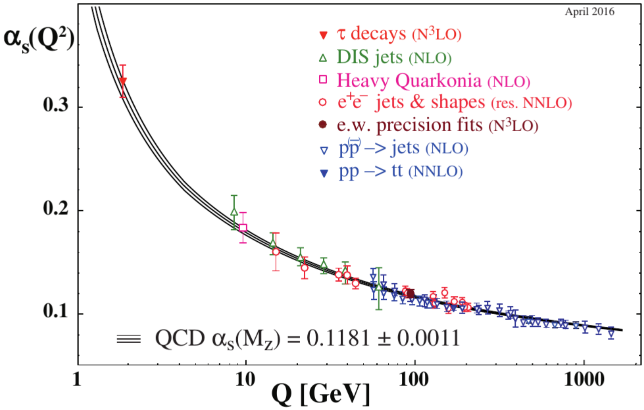
β_i depends on all couplings $\{\alpha_j\}$ of the theory.

SM: $\{\alpha_j\} = \left\{ \frac{\alpha_{\mathrm{QED}}}{\cos^2\theta_W}, \frac{\alpha_{\mathrm{QED}}}{\sin^2\theta_W}, \frac{g_s^2}{4\pi}, \frac{y_f^2}{4\pi}, \frac{\lambda}{4\pi} \right\}$

Asymptotic freedom

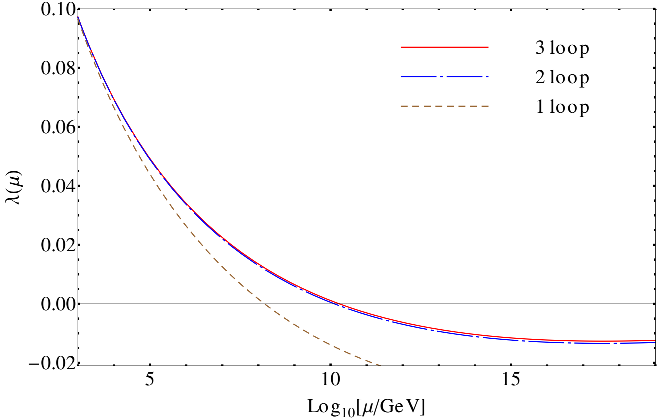


[Particle Data Group]

$\Rightarrow$ precision of α_s determinations made five-loop calculation necessary

Vacuum stability

Stability of the electroweak vacuum depends on sign of quartic coupling λ



[Zoller '13]

$\Rightarrow$ three-loop calculation necessary to gain confidence in results

State of the art for gauge theories

- 5-loop QCD β-function

  [Baikov, Chetyrkin, Kühn '16], [Herzog, Ruijl, Ueda, Vermaseren, Vogt '17], [Luthe, Maier, Marquard, Schroder '17]

State of the art for gauge theories

- 3-loop SM gauge coupling β-functions

  [Mihaila, Salomon, Steinhauser '12], [Bednyakov, Pikelner, Velizhanin '12]

- 3-loop SM Yukawa coupling β-functions

  [Chetyrkin, Zoller '12], [Bednyakov, Pikelner, Velizhanin '13,'14]

- 3-loop SM self-coupling β-functions

  [Chetyrkin, Zoller '12,'13], [Bednyakov, Pikelner, Velizhanin '13]

State of the art for gauge theories

- 3-loop 2HDM gauge and Yukawa coupling β-functions

  [FH, Mihaila, Steinhauser '19]
- 3-loop gauge coupling β-function for arbitrary gauge group

  [Poole, Thomsen '19]

Why don't we know the SM β-functions at 4 loops?

Relating β-functions to counterterms for couplings

Bare and renormalized couplings are related by:

$$\alpha_i^0 = \mu^{2\epsilon} Z_{\alpha_i} \alpha_i$$

Taking the derivative w.r.t. μ and solving for β_i:

$$\beta_i = -\left[\epsilon \frac{\alpha_i}{\pi} + \frac{\alpha_i}{Z_{\alpha_i}} \sum_{j \neq i} \frac{\partial Z_{\alpha_i}}{\partial \alpha_j} \beta_j\right] \left(1 + \frac{\alpha_i}{Z_{\alpha_i}} \frac{\partial Z_{\alpha_i}}{\partial \alpha_i}\right)^{-1}$$

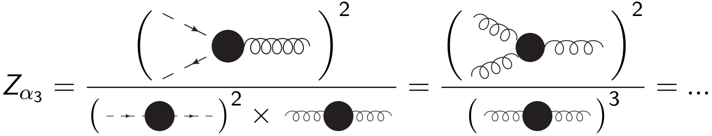
To obtain β_i at L loops, we need to know Z_{α_i} at L loops and all β_j at $L-1$ loops (at most)

Relating counterterms to Green's functions

Couplings renormalization constants computed via

$$Z_{\alpha_i} = \frac{Z_V^2}{\prod_\Phi Z_\Phi}$$

Slavnov-Taylor identities relate renormalization constants



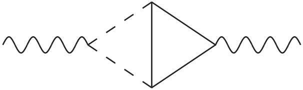
$\Rightarrow$ need to compute UV-poles of 2- and 3-point Green's functions

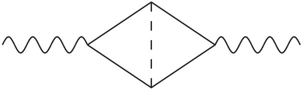
Poles do not depend on masses and momenta in $\overline{\mathrm{MS}}$ scheme

$\rightarrow$ neglect all particle masses

Relating counterterms to Green's functions

- $\mathcal{O}\left(10^5\right)$ diagrams per relevant Green's function at 4 loops
- 3 gauge parameters
- However, many diagrams share the same structure:





- Combine diagrams with same colour factors and topology into super-diagrams $\rightarrow \mathcal{O}\left(10^3\right)$

Treatment of γ_5

SM is a chiral theory, thus γ_5 appears

In four dimensions:

$$\{\gamma_5, \gamma_\mu\} = 0$$

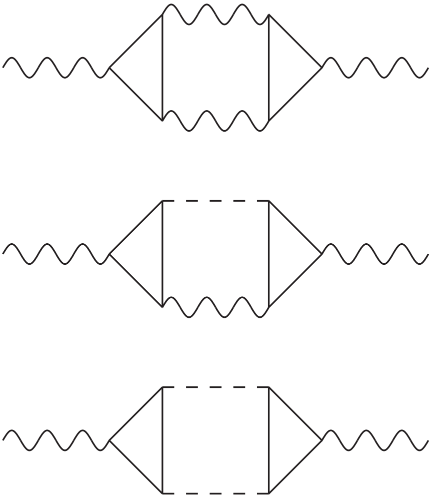
$$\gamma_5^2 = 1$$

$$\mathrm{tr}\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma\right) = -4i\epsilon_{\mu\nu\rho\sigma}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu'\nu'\rho'\sigma'} = g^{[\mu}_{[\mu'}\, g^{\ \nu}_{\ \nu'}\, g^{\ \rho}_{\ \rho'}\, g^{\ \sigma]}_{\ \sigma']}$$

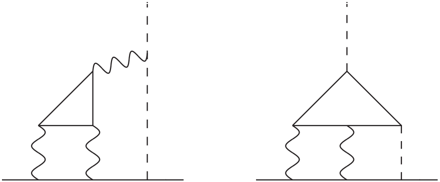
But what about D dimensions? (see also talks by Long Chen and Taushif Ahmed)

Treatment of γ_5



None of these diagrams contribute

Treatment of γ_5

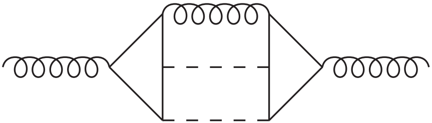


Only the second diagram contributes with an $\frac{1}{\epsilon}$ pole

$$\rightarrow \mathrm{tr}\left(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma\right) = -4 i \epsilon_{\mu\nu\rho\sigma} + \mathcal{O}(\epsilon)$$

γ_5 and the four-loop gauge coupling beta functions

This fails however at four loops:



In [Bednyakov, Pikelner '15], [Zoller '15] a non-cyclic trace was used $\rightarrow$ Result depends on reading point

SM gauge coupling β-functions at 4 loops

Weyl consistency conditions

In the framework of the local renormalization group Osborn's equation [Osborn '89, '91] can be derived:

$$\partial_I \tilde{A} = T_{IJ} B^J$$

This equation gives rise to the so-called Weyl consistency conditions, relations between coefficients of tensor structures of the β functions.

Weyl consistency conditions

Most general Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\sum_u F^{A_u}_{u,\mu\nu} F^{A_u\mu\nu}_u + \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a + i\psi^\dagger_i\bar{\sigma}^\mu(D_\mu\psi)_i$$
$$-\frac{1}{2}\left(Y_{aij}\psi_i\psi_j + \mathrm{h.c.}\right)\phi_a - \frac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d.$$

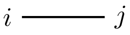
Covariant derivatives are defined by

$$D_\mu\phi_a = \partial_\mu\phi_a - i\sum_u g_u\, V^{A_u}_{u,\mu}(T^{A_u}_{\phi,u})_{ab}\phi_b$$
$$D_\mu\psi_i = \partial_\mu\psi_i - i\sum_u g_u\, V^{A_u}_{u,\mu}(T^{A_u}_{\psi,u})_{ij}\psi_j$$

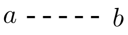
Weyl consistency conditions

Graph-tensor identification rules [Poole, Thomsen '19]:

$= G^2_{AB}$



$= \delta_{ij}$



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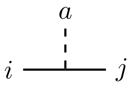
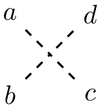
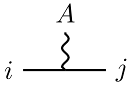


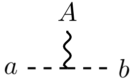
$= y_{aij}$



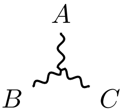
$= \lambda_{abcd}$



$= (T^A)_{ij}$



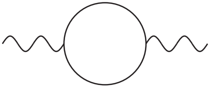
$= (T^A_\phi)_{ab}$



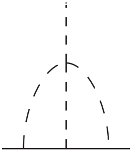
$= G^{-2}_{AD} f^{DBC}$

Weyl consistency conditions

Examples:



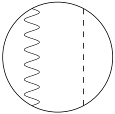
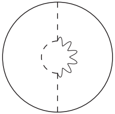
$$= \mathrm{Tr}\left(T^A T^B\right)$$



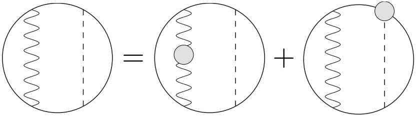
$$= y_b \tilde{y}_a y_c \lambda_{bcda}$$

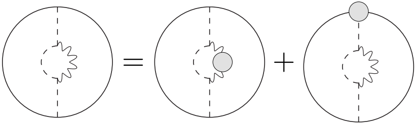
Weyl consistency conditions

$\tilde{A}$ can be decomposed into coefficients a_i and tensor structures, e.g. at three loops [Poole, Thomsen '19]:

$\tilde{A} \supset a_{10}^{(3l)}$  $+ a_{11}^{(3l)}$ 

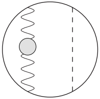
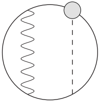
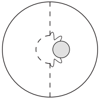
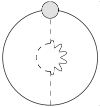
Derivative acts on couplings, corresponding to gauge lines, fermion-scalar vertices and quartic scalar vertices:

∂_I 

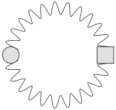
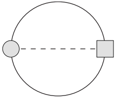
∂_I 

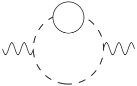
Weyl consistency conditions

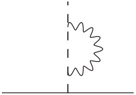
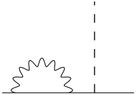
Overall, we get:

$\partial_I \tilde{A} \supset a_{10}^{(3l)} \Big($  $+$  $\Big) + a_{11}^{(3l)} \Big($  $+$  $\Big)$

In a similar way T_{IJ} and B^J can be decomposed:

$T_{IJ} \supset t_1^{(1l)}$  $+ t_4^{(2l)}$ 

$B^J \supset g_6^{(2l)}$  $+ g_7^{(2l)}$ 

$+ n_1^{(1l)}$  $+ n_2^{(1l)}$ 

Weyl consistency conditions

Identifying tensor structures, we obtain 4 equations:

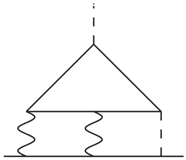
$$a_{10}^{(3l)} = t_1^{(1l)} g_6^{(2l)} , \qquad a_{11}^{(3l)} = t_1^{(1l)} g_7^{(2l)}$$

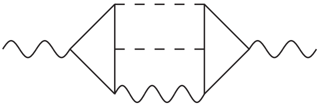
$$2a_{10}^{(3l)} = t_4^{(2l)} n_1^{(1l)} , \qquad 2a_{11}^{(3l)} = t_4^{(2l)} n_2^{(1l)}$$

Which can be solved for

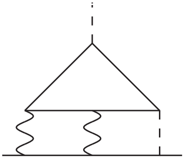
$$g_7^{(2l)} n_2^{(1l)} = g_6^{(2l)} n_1^{(1l)}$$

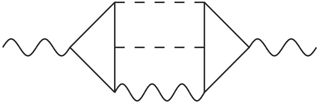
Taking a closer look at the γ_5 contributions



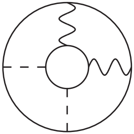


Taking a closer look at the γ_5 contributions



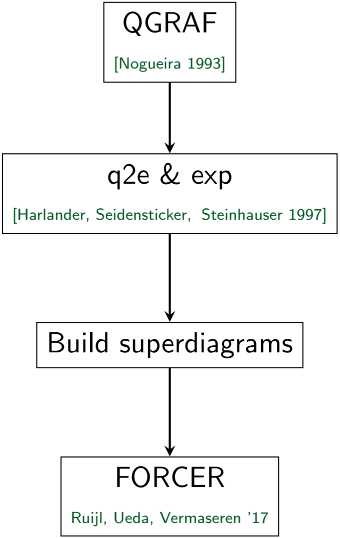


Let's connect the external lines in each of them...



All problematic contributions can be expressed through unproblematic ones [Poole, Thomsen '19]

Calculational setup



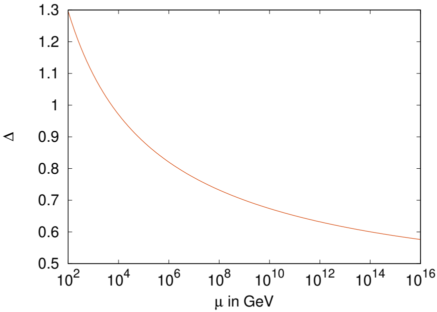
Gauge coupling β-functions in the SM at four loops

Combining the various ingredients we obtained the 4-loop gauge coupling β-functions in the SM [Davies, FH, Poole, Steinhauser, Thomsen '19]:

$$\begin{aligned}
\beta_1 = {} & \frac{\alpha_1^2}{(4\pi)^2}\left(\frac{82}{5}\right) + \frac{\alpha_1^2}{(4\pi)^3}\left(\frac{398\alpha_1}{25} + \frac{54\alpha_2}{5} + \frac{176\alpha_3}{5} - \frac{34\alpha_4}{5}\right) + \frac{\alpha_1^2}{(4\pi)^4}\Big(-\frac{388613\alpha_1^2}{6000} + \frac{123\alpha_1\alpha_2}{40} - \frac{548\alpha_1\alpha_3}{75} \\
& + \frac{789\alpha_2^2}{16} - \frac{12\alpha_2\alpha_3}{5} + \frac{1188\alpha_3^2}{5} - \frac{2827\alpha_1\alpha_4}{200} - \frac{471\alpha_2\alpha_4}{8} - \frac{116\alpha_3\alpha_4}{5} + \frac{189\alpha_4^2}{4} + \frac{54\alpha_1\alpha_7}{25} + \frac{18\alpha_2\alpha_7}{5} - \frac{36\alpha_7^2}{5}\Big) \\
& + \frac{\alpha_1^2}{(4\pi)^5}\Big[-\alpha_1^3\left(\frac{143035709}{1080000} + \frac{1638851\zeta_3}{5625}\right) - \alpha_1^2\alpha_2\left(\frac{3819731}{24000} - \frac{16529\zeta_3}{125}\right) - \alpha_1^2\alpha_3\left(\frac{3629273}{6750} - \frac{720304\zeta_3}{1125}\right) \\
& + \alpha_1\alpha_2^2\left(\frac{572059}{14400} - \frac{6751\zeta_3}{75}\right) - \frac{69\alpha_1\alpha_2\alpha_3}{25} + \alpha_1\alpha_3^2\left(\frac{333556}{675} - \frac{274624\zeta_3}{225}\right) - \alpha_2^3\left(\frac{117923}{2880} + \frac{3109\zeta_3}{5}\right) \\
& - \alpha_2^2\alpha_3\left(\frac{41971}{90} - \frac{7472\zeta_3}{15}\right) - \alpha_2\alpha_3^2\left(\frac{1748}{3} - \frac{2944\zeta_3}{5}\right) + \alpha_3^3\left(\frac{6116}{15} - \frac{18560\zeta_3}{9}\right) + \alpha_1^2\alpha_4\left(\frac{8978897}{72000} + \frac{2598\zeta_3}{125}\right) \\
& - \alpha_1\alpha_2\alpha_4\left(\frac{42841}{800} + \frac{1122\zeta_3}{25}\right) - \alpha_1\alpha_3\alpha_4\left(\frac{2012}{75} - \frac{408\zeta_3}{25}\right) - \alpha_2^2\alpha_4\left(\frac{439841}{960} - \frac{616\zeta_3}{5}\right) + \alpha_2\alpha_3\alpha_4\left(\frac{1468}{5} - \frac{1896\zeta_3}{5}\right) \\
& - \alpha_3^2\alpha_4\left(\frac{11462}{45} - \frac{3184\zeta_3}{5}\right) + \alpha_1\alpha_4^2\left(\frac{29059}{160} - \frac{357\zeta_3}{25}\right) + \alpha_2\alpha_4^2\left(\frac{71463}{160} + \frac{639\zeta_3}{5}\right) + \alpha_3\alpha_4^2\left(\frac{1429}{5} - 240\zeta_3\right) \\
& - \alpha_4^3\left(\frac{13653}{40} + \frac{102\zeta_3}{5}\right) + \frac{3627\alpha_1^2\alpha_7}{500} + \frac{1917\alpha_1\alpha_2\alpha_7}{50} + \frac{889\alpha_2^2\alpha_7}{20} - \frac{1926\alpha_1\alpha_4\alpha_7}{25} - \frac{162\alpha_2\alpha_4\alpha_7}{5} - \frac{474\alpha_4^2\alpha_7}{5} \\
& - \frac{1269\alpha_1\alpha_7^2}{25} - \frac{981\alpha_2\alpha_7^2}{5} + \frac{1188\alpha_4\alpha_7^2}{5} + \frac{624\alpha_7^3}{5}\Big]
\end{aligned}$$

For the three gauge couplings, the 4-loop corrections amount to 8%, 5% and 127% of the 3-loop corrections.

127% ???



$$\Delta = \frac{|\alpha_3^{(4l)} - \alpha_3^{(3l)}|}{|\alpha_3^{(3l)} - \alpha_3^{(2l)}|}$$

Electroweak corrections cancel pure α_3 terms at 3-loops.

Beyond the SM

Beta functions at order 4-3-2

- Statements concerning γ_5 hold for any gauge theory
- BSM-landscape is vast $\rightarrow$ dedicated computation for each model unfeasible
- Ansatz by [Poole, Thomsen '19] covers general theory at order 4-3-2

Beta functions at order 4-3-2

- Directly computing each coefficient using real fields possible, but cumbersome
- Results available in the literature fix 487/510 coefficients
- Adding SM + ν_R and the type-I 2HDM gives 4 more constraints
- Remaining coefficients need a model with a scalar charged under multiple non-abelian gauge groups

Ambiguities

There is one problem with Yukawa matrices starting from 3 loops:

$$Z_f = 1 - K_\epsilon \left(\sqrt{Z_f}^\dagger \Sigma(Q^2) \sqrt{Z_f} \right)$$

Square root: $Z_f = \sqrt{\hat{Z}_f}^\dagger U^\dagger U \sqrt{\hat{Z}_f}$
$\to$ can only determine $\sqrt{Z_f}$ up to unitary rotation

Ambiguities

Issue with anomalous dimension (similar for β-function):

$$\gamma_f = \sqrt{Z_f}^{-1} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \sqrt{Z_f}$$
$$= U^\dagger \sqrt{\hat{Z}_f}^{-1} \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \sqrt{\hat{Z}_f} \right) U + U^\dagger \mu \frac{\mathrm{d}}{\mathrm{d}\mu} U$$

Choice $U = 1$ leads to poles in γ_f (and β) starting from 3 loops [Bednyakov, Pikelner, Velizhanin '14], [FH, Mihaila, Steinhauser '17]

Do the ambiguities lead to issues?

RG-finiteness

The divergent part of *any* set of MS/$\overline{\text{MS}}$ RG functions (β_I, γ) satisfy

$$\gamma^{(n)} \in \mathfrak{g}_F \quad \text{and} \quad \beta_I^{(n)} = -\left(\gamma^{(n)} g\right)_I, \qquad n \geq 1.$$

This property of the RG functions is referred to as RG-finiteness.

[FH, Thomsen '21]

The B-function

It is possible to define an improved β-function [Fortin, Grinstein, Stergiou '12]:

$$B_I = \beta_I - (\hat{\upsilon}\, g)_I$$

B is invariant under transformations of the fields with G_F
$(\hat{\upsilon}\, g)_I$ can be computed directly [Fortin, Grinstein, Stergiou '12] and coincides with $\left(\gamma^{(n)}\, g\right)_I$ [FH, Thomsen 21]

What else can we learn?

In a next step, we plan to derive the relations at orders 5-4-3.

- Allows to determine 3-loop scalar β-function, many coefficients already known [Steudtner '21]
- Investigate γ_5 at this order (3-loop scalar β-function is safe)
- Does the number of relations grow faster than the number of coefficients?
- Are there non-trivial relations for pure gauge theories at high orders?
- Can one combine Weyl consistency conditions with relations regarding the transcendality structure [Baikov, Chetyrkin '19]?
- ...

Conclusion

- In non-chiral theories 5-loop computations are feasible
- Weyl consistency conditions allow to circumvent an explicit treatment of γ_5 in certain cases
- Computed SM gauge coupling β-functions at 4 loops
- Results for a general gauge-Yukawa theory at order 4-3-2 coming soon [Davies, FH, Thomsen TBP]

Backup

Do the ambiguities lead to issues?

Ambiguous terms are unitary $\rightarrow$ do not enter quantities invariant under flavour rotations like:

$$\mathrm{Tr}\left(Y_u^\dagger Y_u\right), \mathrm{Tr}\left(Y_d^\dagger Y_d\right), \ldots$$

$\rightarrow$ not an issue when running observables

Do the ambiguities lead to issues?

The key observation [FH, Thomsen '21]:

- Poles are elements of the Lie-Algebra $\mathfrak{g}_F$ of the flavour group of SM (2HDM) ($G_F = \mathrm{U}(3)^5 \times U(1(2))$)

Do the ambiguities lead to issues?

The key observation [FH, Thomsen '21]:

$$\left(\frac{\partial}{\partial \ln \mu} + \left(\epsilon\beta_I^{(-1)} + \beta_I\right)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\left(\beta_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

$$= 0 \text{ iff } \beta_I^{(n)} = -(\gamma^{(n)} g)_I \text{ with } \gamma^{(n)} \in \mathfrak{g}_F$$