Feynman integrals for binary systems of black holes

How to cope with multiple square roots?

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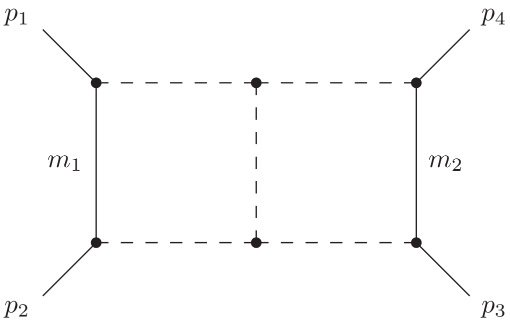
Gravitational waves

- The initial phase of the inspiral process of a binary system producing gravitational waves can be described by perturbation theory.
- Effective field theory methods provide a link between general relativity and particle physics.
- The **post-Newtonian expansion** is an expansion in the weak gravitational field limit and the small velocity limit.
- The **post-Minkowskian expansion** is an expansion in the weak gravitational field limit.

Buonanno, Damour, Goldberger, Rothstein, Bern, Cheung, Roiban, Shen, Solon, Zeng, Bini, Geralico, Levi, Porto, Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla, Laporta, Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove, Cristofoli, Kosower, Maybee, O'Connell, Blümlein, Maier, Marquard, Schäfer, Schneider, Parra-Martinez, Ruf, Kälin, Liu, Yang, ...

The H-graph

At the third post-Minkowskian order a two-loop double box graph contributes:



This is the most complicated graph entering the third post-Minkowskian order.

Kinematics

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2.$$

The external momenta are on-shell:

$$p_1^2 = p_2^2 = m_1^2, \qquad p_3^2 = p_4^2 = m_2^2.$$

- For a binary system the limit $|s| \ll t, m_1^2, m_2^2$ is relevant.
- A full relativistic calculation is helpfull.
- Distinguish two cases:

Equal mass case: $m_1 = m_2 = m.$
Unequal mass case: $m_1 \neq m_2.$

The H-graph: the essential numbers

- The equal mass case:
  - Two kinematic variables s/m^2, t/m^2.
  - 25 master integrals
  - 4 square roots
  - 17 dlog-forms

Bianchi, Leoni, '16

- The unequal mass case:
  - Three kinematic variables s/m_1^2, t/m_1^2, m_2^2/m_1^2.
  - 40 master integrals
  - 6 square roots
  - 29 dlog-forms

The method of differential equations

- Using integration-by-parts identities we derive the **differential equation**:

$$dI = AI.$$

Tkachov '81, Chetyrkin '81, Kotikov '91, Gehrmann, Remiddi '00

- The differential equation can be deformed to an **ε-form** (Henn '13):

$$A = \varepsilon \sum_k C_k \omega_k, \quad \omega_k = d \ln f_k.$$

The f_k's may contain square roots.

- The differential equation is solved in terms of **iterated integrals** (Chen '77). We are interested in the solution up to weight four.
- The iterated integrals are converted to standard functions (**multiple polylogarithms**).

The scalar double box integral

The scalar double box integral with **no dots** and **no irreducible scalar products** has up to weight four a rather simple expression in terms of multiple polylogarithms. In the equal mass case

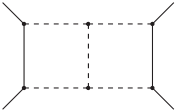
$$
\begin{aligned}
\frac{1}{\varepsilon^4 s^2 r_3} I_{1111111} &= 4\left[G(1,1,1;\bar{y}) + \zeta_2 G(1;\bar{y})\right]\varepsilon^3 \\
&+4\left\{2G(2,1,1,1;\bar{y}) + 2G(0,1,1,1;\bar{y}) - 2G(1,1,2,1;\bar{y}) - 2G(1,1,0,1;\bar{y})\right. \\
&+2\zeta_2\left[G(2,1;\bar{y}) + G(0,1;\bar{y}) - G(1,1;\bar{y})\right] - \zeta_3 G(1;\bar{y}) \\
&\left.+2\left[G(1,1,1;\bar{y}) + \zeta_2 G(1;\bar{y})\right]\left[G(1,\bar{x}) - 2\ln(\bar{x})\right]\right\}\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right), \\
s = -\frac{\bar{x}^2}{1-\bar{x}} m^2, &\quad t = -\frac{\bar{y}^2}{1-\bar{y}} m^2.
\end{aligned}
$$

and similar for the unequal mass case.

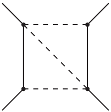
Bianchi, Leoni, '16; Kreer, S.W., '21

The system of master integrals

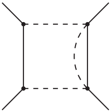
We are interested in **all** master integrals up to **weight four**. In particular:



(the remaining master integrals)
unequal: 4 masters
equal: 3 masters



unequal: 5 masters
equal: 4 masters



unequal: 3 masters
equal: 3 masters

The roots

$$
\begin{aligned}
r_1 &= \sqrt{-s\left(4m_1^2 - s\right)}, \\
r_2 &= \sqrt{-s\left(4m_2^2 - s\right)}, \\
r_3 &= \sqrt{\left[\left(m_1 - m_2\right)^2 - t\right]\left[\left(m_1 + m_2\right)^2 - t\right]}, \\
r_4 &= \sqrt{-s\left[4m_1^2 m_2^4 - s\left(m_1^2 - t\right)^2\right]}, \\
r_5 &= \sqrt{-s\left[4m_1^4 m_2^2 - s\left(m_2^2 - t\right)^2\right]}, \\
r_6 &= \sqrt{\left[\left(m_1 - m_2\right)^2 - s - t\right]\left[\left(m_1 + m_2\right)^2 - s - t\right]}.
\end{aligned}
$$

Why bother?

- Multiple square roots appear not only in the Feynman integrals associated to the H-graph, but also in Feynman integrals associated to other processes, for example Bhabha scattering, Drell-Yan, etc.

  Henn, Smirnov '13; Bonciani et al. '16

- As we study more two-loop integrals with several kinematic variables, multiple square roots become more frequent.

- We want to learn how to handle them.

  (Alternative approach based on an ansatz: Heller, von Manteuffel, Schabinger '19, Heller '21)

Rationalisation: A simple example

Consider a dlog-form with a square root:

$$\omega = d\ln\left(\frac{2m_1^2 - s - r_1}{2m_1^2 - s + r_1}\right) = \frac{2s}{r_1}\left(\frac{ds}{s} - \frac{dm_1^2}{m_1^2}\right)$$

The transformation

$$-\frac{s}{m_1^2} = \frac{(1-x)^2}{x}$$

rationalises the square root:

$$\omega = 2d\ln(x) = 2\frac{dx}{x}$$

Rationalisation

Assume that the differential equation is in **ε-dlog-form**, where the arguments of the logarithms contain square roots:

- There are algorithms to rationalise square roots.

  Besier, van Straten, S.W. '18; Besier, Wasser, S.W. '19

- If we can simultaneously rationalise all square roots, all integrals can be expressed in terms of multiple polylogarithms

- It can be hard to prove, that the set of square roots cannot be rationalised simulataneously.

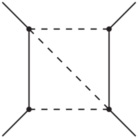
  Besier, Festi, Harrison, Naskrecki '19

- Even if one can prove that a set of square roots cannot be rationalised simulataneously, this does not imply that the Feynman integrals cannot be expressed in terms of multiple polylogarithms.

  Heller, von Manteuffel, Schabinger '19

Eliminating one square root

- The last square root r_6 appears up to weight 4 only in one master integrals.
- This master integral can be computed in the Feynman parameter representation and evaluates to multiple polylogarithms.
- This holds in the equal mass case and in the unequal mass case.



Brown ’08, Panzer ’14, Heller, von Manteuffel, Schabinger ’19

The equal mass case

- The remaining 24 master integrals involve up to weight 4 only three square roots.
- These square roots can be rationalised simultaneously.
- All master integrals evaluate up to weight 4 to multiple polylogarithms.

Kreer, S.W. '21

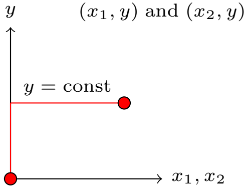
The unequal mass case

- Each master integral is a linear combination of iterated integrals.
- Each dlog-form contains either 0, 1 or 2 roots.
- Each iterated integral of the master integrals contains up to weight 4 no more than 3 distinct roots.
- In the remaining 39 master integrals up to weight 4 each occuring triple of distinct roots can be rationalised simultaneously.
  In other words: we **can rationalise simultaneously any occuring triple** (r_i, r_j, r_k) from $\{r_1, r_2, r_3, r_4, r_5\}$.

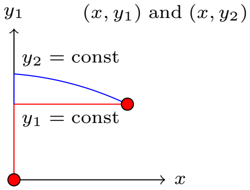
Using different rationalisations

We would like to use different rationalisations for different iterated integrals. Are we allowed to do so?

- Different rationalisations may correspond to different parametrisations of the same integration path.



- Different rationalisations may correspond to different integration paths.



Path (in-) dependence

- A single iterated integral $I_\gamma(\omega_1, \dots, \omega_r)$ is in general **path dependent**.
- The linear combination of iterated integrals in the ε^j-term of the i-th master integral $I_i^{(j)}$ is path independent.
  - this is ensured by the integrability condition of the differential equation $dA - A \wedge A = 0$.
  - we may use different integration paths for $I_{i_1}^{(j_1)}$ and $I_{i_2}^{(j_2)}$
- We would like to split

$$I_i^{(j)} = I_{i,a}^{(j)} + I_{i,b}^{(j)}$$

and use different integration paths for $I_{i,a}^{(j)}$ and $I_{i,b}^{(j)}$.

Only allowed if $I_{i,a}^{(j)}$ and $I_{i,b}^{(j)}$ are path independent.

Path independence

Bar notation for the tensor algebra (all ω_j's closed):

$$
\begin{aligned}
\left[\omega_1 | \omega_2 | \dots | \omega_r \right] &= \omega_1 \otimes \omega_2 \otimes \dots \otimes \omega_r, \\
d \left[\omega_1 | \omega_2 | \dots | \omega_r \right] &= \sum_{j=1}^{r-1} \left[\omega_1 | \dots | \omega_{j-1} | \omega_j \wedge \omega_{j+1} | \omega_{j+2} | \dots | \omega_r \right].
\end{aligned}
$$

Associate to a linear combination of iterated integrals

$$
I = \sum_{j=1}^{r} \sum_{i_1,\dots,i_j} c_{i_1 \dots i_j} I_\gamma \left(\omega_{i_1}, \dots, \omega_{i_j} \right) \quad \Rightarrow \quad B = \sum_{j=1}^{r} \sum_{i_1,\dots,i_j} c_{i_1 \dots i_j} \left[\omega_{i_1} | \dots | \omega_{i_j} \right]
$$

The linear combination I is **path independent if and only if $dB = 0$**.

Chen '77

Path independence

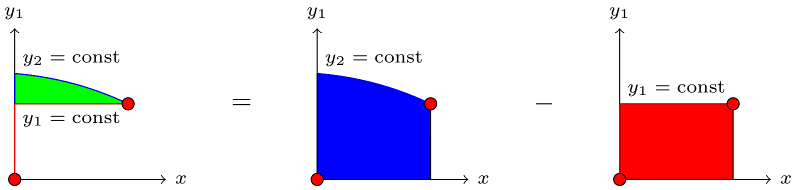
- Observation: $\omega_i \wedge \omega_j$ may involve **less square roots** than $\omega_i \otimes \omega_j$.
- Although $I^{(j)}_{i,a}$ and $I^{(j)}_{i,b}$ may be path dependent, we may find $I^{(j)}_{\mathrm{subtr}}$ **compatible with two rationalisations** such that

$$\left(I^{(j)}_{i,a} - I^{(j)}_{\mathrm{subtr}} \right) \quad \text{and} \quad \left(I^{(j)}_{i,b} + I^{(j)}_{\mathrm{subtr}} \right)$$

  are path independent.
- We may then evaluate $(I^{(j)}_{i,a} - I^{(j)}_{\mathrm{subtr}})$ with a rationalisation corresponding to an integration path γ_a and $(I^{(j)}_{i,b} + I^{(j)}_{\mathrm{subtr}})$ with a rationalisation corresponding to an integration path γ_b.

Path independence

Subtraction terms obtained from Stokes' theorem by integrating $\omega_i \wedge \omega_j$:



Another pitfall: Trailing zeros

May occur already for different parametrisations of the same integration path:

Consider the transformation $x = 2x'$ and the dlog-form

$$\omega_0 = d\ln(x) = d\ln(2x') = d\left[\ln(2) + \ln(x')\right] = d\ln(x')$$

We have

$$\int\limits_0^{x_f} \omega_0 = \ln(x_f) \neq \ln(x_f) - ln(2) = \ln(x_f') = \int\limits_0^{x_f'} \omega_0$$

Trailing zeros

The left-hand side is a divergent integral:

$$\int_0^{x_f} \frac{dx}{x} = \ln(x_f)$$

What we actually mean:

- We introduce a lower cutoff as a regulator λ
- We employ a "renormalisation scheme" and remove all $\ln(\lambda)$-terms.
- A transformation $x = 2x'$ induces a change of the "renormalisation scheme".

Solution: **Isolate** all **trailing log's** in x' and **substitute**

$$\ln(x') \rightarrow \ln(x') + \ln(2)$$

Conclusions

- The H-graph is relevant to gravitational waves
- The differential equation for the master integrals involves six square roots
- Techniques for the case, where not all square roots can be rationalised simultaneously
- Equal mass case: Up to weight 4 the result can be expressed in terms of multiple polylogarithms.
- Unequal mass case: Up to weight 4 result contains 3499 iterated integrals, 3493 can be expressed in terms of multiple polylogarithms, we are working on the remaining 6 ...