PMC_∞
Infinite-Order Scale-Setting method using the Principle of Maximum Conformality and preserving the Intrinsic Conformality (iCF)

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Outline

- The Renormalization Scale Setting in QED/QCD
- State of the art about renormalization in QCD
- The PMC_∞ scale setting procedure
- PMC_∞ and the Event Shape Variables
- Comparison with CSS
- QED and Conformal Thrust
- Work in progress and future perspectives

Why The Scale Setting in QCD is a Key Issue?

- *To determine $\alpha s(Q^2)$ to the highest precision;*
- *To make precision tests of the QCD;*
- *To eliminate the renormalization scale ambiguity and the scheme dependence in the observables;*
- *To assess and reach the maximum sensitivity to NP.*

The Renormalization Scale Problem in QED

QED is not only perturbative :

- *No ambiguity in the renormalization scale in QED;*
- *The renormalization scale in QED is physical and set by the exchanged photon virtuality;*
- *An infinite series of Vacuum Polarization diagrams is resummed;*
- *The QED coupling is defined from physical observables (Gell Mann-Low scheme);*
- *No scheme dependence is left;*
- *Analyticity (space-like/time-like);*
- *Exact number of active leptons is set;*
- *Recover of a conformal-like series;*

Effective coupling

$$\alpha(Q) = \frac{\alpha_0}{1 - \Pi(Q)}$$

The VPF contains all β-terms

QED: a Theoretical Constraint for QCD

QCD $\longrightarrow$ Abelian Gauge Theory

In the limit : $N_c \longrightarrow 0$,

at fixed $\alpha = C_F \alpha_s$, $n_l = T n_F / C_F$

The scale setting procedure used in QCD must be consistent with the QED

Huet, S.J.Brodsky

The road to scale setting in QCD is paved with some misbeliefs

According to the Conventional practice :

- ***The renormalization scale is arbitrarily guessed or picked by judging its results «a posteriori»;***
- ***The renormalization scale is unique for each process;***
- ***The renormalization scale is a simple unphysical parameter;***
- ***The renormalization and factorization scales are equal;***

These assumptions are wrong for QED and than for QCD!

..Other methods such as PMS and FAC lead to incorrect and unphysical results violating important RG properties;

The Principle of Maximum Conformality is the principle underlying the BLM

S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983)

Observable in the initial parametrization

Stanley J. Brodsky, L.D.G.:
Phys. Rev. D 86, 085026 (2011)

$$\begin{aligned}\rho(Q^2) =& r_{0,0} + r_{1,0} a(Q) + [r_{2,0} + \beta_0 r_{2,1}] a(Q)^2 \\ &+ [r_{3,0} + \beta_1 r_{2,1} + 2\beta_0 r_{3,1} + \beta_0^2 r_{3,2}] a(Q)^3 \\ &+ [r_{4,0} + \beta_2 r_{2,1} + 2\beta_1 r_{3,1} + \frac{5}{2}\beta_1\beta_0 r_{3,2} + 3\beta_0 r_{4,1} \\ &+ 3\beta_0^2 r_{4,2} + \beta_0^3 r_{4,3}] a(Q)^4 + \mathcal{O}(a^5) \qquad (6)\end{aligned}$$

Mojaza, Matin and Brodsky, Stanley J. and Wu, Xing-Gang
Phys.Rev.Lett. 110 (2013) 192001

$r_{n,0}$ conformal coefficients

The β-terms are reabsorbed by RGE

$$\rho(Q^2) = r_{0,0} + r_{1,0} a(Q_1) + r_{2,0} a(Q_2)^2 + r_{3,0} a(Q_3)^3 + r_{4,0} a(Q_4)^4 + \mathcal{O}(a^5) \ ,$$

Conformal-like expansion

$$\ln \frac{Q_k^2}{Q^2} = \frac{R_{k,1} + \Delta_k^{(1)}(a) R_{k,2} + \Delta_k^{(2)}(a) R_{k,3}}{1 + \Delta_k^{(1)}(a) R_{k,1} + \left(\Delta_k^{(1)}(a)\right)^2 (R_{k,2} - R_{k,1}^2) + \Delta_k^{(2)}(a) R_{k,1}^2} \ .$$

PMC scales

Features of the PMC/PMC$_\infty$

- All terms associated with the beta-function are included into the running coupling;
- PMC agrees with the QED in the Abelian limit;
- PMC is consistent with the Conformal limit;
- No scale ambiguities;
- Results are scheme independent ;
- The PMC scale sets the correct number of active flavors;
- Transitivity Property is preserved;
- No renormalon n! growth in pQCD associated with the beta function;
- Resulting series is identical to conformal series! (CSR - Crewther Relation ;)
- PMC: One procedure from first principles for the whole SM and also for a theory of grand unification.

PMC∞ - Results for Event Shape Variables distributions at NNLO

Work in collaboration with S. J. Brodsky, S.Q.Wang , X.G. Wu and F. Sannino

- arXiv: 2104.12132 [hep]
- *Phys.Rev.D* 102 (2020) 1, 014015

Thrust and C-Par distribution at NNLO: process: e+e- →3jets

$$T = \frac{\max\limits_{\vec{n}} \sum\limits_i |\vec{p}_i \cdot \vec{n}|}{\sum\limits_i |\vec{p}_i|},$$

$$C = \frac{3}{2} \frac{\sum_{i,j} |\vec{p_i}||\vec{p_j}| \sin^2 \theta_{ij}}{\left(\sum_i |\vec{p_i}|\right)^2},$$

Sheng-Quan Wang, S.J. Brodsky, Xing-Gang Wu , Jian-Ming Shen, L.D.G., *Phys.Rev.D* 100 (2019) 9, 094010

Sheng-Quan Wang, S.J. Brodsky, Xing-Gang Wu , L.D.G., Jian-Ming Shen, *Phys.Rev.D* 102 (2020) 1, 014005

Sheng-Quan Wang, S.J. Brodsky, Xing-Gang Wu , L.D.G., Phys. Rev. D 99, no.11, 114020 (2019)

Distributions from EERAD and Event2 codes by:

A. Gehrmann-De Ridder, T. Gehrmann, E. W. N. Glover and G. Heinrich, Phys. Rev. Lett. **99**, 132002 (2007).
A. Gehrmann-De Ridder, T. Gehrmann, E. W. N. Glover and G. Heinrich, JHEP **0712**, 094 (2007).
S. Weinzierl, JHEP **0906**, 041 (2009).
S. Weinzierl, Phys. Rev. Lett. **101**, 162001 (2008).

Strong Coupling from RunDec program :

K. G. Chetyrkin, J. H. Kuhn and M. Steinhauser, Comput. Phys. Commun. **133**, 43 (2000).

PMC∞ preserves the iCF: *the intrinsic Conformality*

Observable: Single variable distribution at NNLO calculated at the initial scale μ_0

The iCF is an **RG invariant parametrization** with conformal coefficients and scales.

$$\frac{1}{\sigma_{tot}}\frac{O d\sigma(\mu_0)}{dO} = \frac{\alpha_s(\mu_0)}{2\pi}\frac{O d\bar{A}_O(\mu_0)}{dO} + \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 \frac{O d\bar{B}_O(\mu_0)}{dO} + \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 \frac{O d\bar{C}_O(\mu_0)}{dO} + \mathcal{O}(\alpha_s^4), \quad (3)$$

$$\begin{aligned} A_O(\mu_0) &= A_{Conf}, \\ B_O(\mu_0) &= B_{Conf} + \frac{1}{2}\beta_0 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right) A_{Conf}, \\ C_O(\mu_0) &= C_{Conf} + \beta_0 \ln\left(\frac{\mu_0^2}{\mu_{II}^2}\right) B_{Conf} + \\ &+ \frac{1}{4}\left[\beta_1 + \beta_0^2 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\right] \ln\left(\frac{\mu_0^2}{\mu_I^2}\right) A_{Conf} \end{aligned} \quad (4)$$

- No redefinition of the conformal terms at higher orders;
- No initial scale dependence left under a global change of scale;
- The scale dependence is explicit.
- The iCF is the most general RG invariant parametrization;
- Other parametrizations can be reduced to the iCF;

$$\begin{aligned} \sigma_I &= \left\{\left(\frac{\alpha_s(\mu_0)}{2\pi}\right) + \frac{1}{2}\beta_0 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 + \frac{1}{4}\left[\beta_1 + \beta_0^2 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\right] \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3\right\} A_{Conf} \\ \sigma_{II} &= \left\{\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 + \beta_0 \ln\left(\frac{\mu_0^2}{\mu_{II}^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3\right\} B_{Conf} \\ \sigma_{III} &= \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 C_{Conf} \end{aligned} \quad (6)$$

The conformal subsets are the fundamental blocks of the iCF

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\sigma_N = 0.$$

- Each subset is scale invariant.
- Any combination of conformal subsets is an invariant.
- I can define a scale for each subset preserving the scale invariance.
- If Aconf=0 the whole subset becomes null.

Ordered scale invariance: **scale invariance is preserved perturbatively independently from the process, the kinematics and the order.**

iCF underlies Scale Invariance

n limit splitted in $J_{/n}$ and $\bar{n}$ limit

$$\sigma_I = \left\{ \left(\frac{\alpha_s(\mu_0)}{2\pi}\right) + \frac{1}{2}\beta_0 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 + \frac{1}{4}\left[\beta_1 + \beta_0^2 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\right] \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 + \dots \right\} A_{Conf}$$

$$\sigma_{II} = \left\{ \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 + \beta_0 \ln\left(\frac{\mu_0^2}{\mu_{II}^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 + \dots \right\} B_{Conf}$$

$$\sigma_{III} = \left\{ \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 + \dots \right\} C_{Conf},$$

$$\vdots \qquad \ddots$$

$$\sigma_n = \left\{ \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n \right\} \mathcal{L}_{nConf}, \qquad (7)$$

$$\lim_{J_{/n}\to\infty} \sigma_I \to \left(\frac{\alpha_s(\mu_I)|_{n-2}}{2\pi}\right) A_{Conf}$$

$$\lim_{J_{/n}\to\infty} \sigma_{II} \to \left(\frac{\alpha_s(\mu_{II})|_{n-3}}{2\pi}\right)^2 B_{Conf}$$

$$\lim_{J_{/n}\to\infty} \sigma_{III} \to \left(\frac{\alpha_s(\mu_{III})|_{n-4}}{2\pi}\right)^3 C_{Conf}$$

$$\vdots \qquad \vdots$$

$$\lim_{J_{/n}\to\infty} \sigma_n \equiv \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n \mathcal{L}_{nConf} \qquad (9)$$

β terms

- **iCF is sufficient and necessary for scale invariance either for convergent or asymptotic series.**
- **Accuracy and Conformality can be controlled by varying the last scale in a physical range of values.**

$$\lim_{n\to\infty} \alpha_s(\mu_0)^n \sim a^n \text{ with } a < 1.$$

$$\lim_{\bar{n}\to\infty} \sigma_I \to \left(\frac{\alpha_s(\mu_I)}{2\pi}\right) A_{Conf}$$

$$\lim_{\bar{n}\to\infty} \sigma_{II} \to \left(\frac{\alpha_s(\mu_{II})}{2\pi}\right)^2 B_{Conf}$$

$$\lim_{\bar{n}\to\infty} \sigma_{III} \to \left(\frac{\alpha_s(\mu_{III})}{2\pi}\right)^3 C_{Conf}$$

$$\vdots \qquad \vdots$$

$$\lim_{\bar{n}\to\infty} \sigma_n \equiv \lim_{n\to\infty} \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n \mathcal{L}_{nConf} \to \text{Conformal Limit} \qquad (10)$$

Same conformal scales μ_N

New «*How to*» method

$$B_O(N_f) = C_F\left[C_A B_O^{N_c} + C_F B_O^{C_F} + T_F N_f B_O^{N_f}\right] \quad (12)$$

either for numerical or analytic calculations.

Find the roots of the β terms and vary the number of flavors.

$$B_{Conf} = B_O\left(N_f \equiv \frac{33}{2}\right),$$

$$B_{\beta_0} \equiv \log\frac{\mu_0^2}{\mu_I^2} = 2\frac{B_O - B_{Conf}}{\beta_0 A_{Conf}}$$

$$C_O(N_f) = \frac{C_F}{4}\left\{N_c^2 C_O^{N_c^2} + C_O^{N_c^0} + \frac{1}{N_c^2}C_O^{\frac{1}{N_c^2}} + N_f N_c \cdot C_O^{N_f N_c} + \frac{N_f}{N_c}C_O^{N_f/N_c} + N_f^2 C_O^{N_f^2}\right\}.$$

$$C_{Conf} = C_O\left(N_f \equiv \frac{33}{2}\right) - \frac{1}{4}\overline{\beta}_1 B_{\beta_0} A_{Conf}$$

β0 killing value

$$\overline{\beta}_1 \equiv \beta_1(N_f = 33/2) = -107.$$

$$C_{\beta_0} \equiv \log\left(\frac{\mu_0^2}{\mu_{II}^2}\right) = \frac{1}{\beta_0 B_{Conf}}\left(C_O - C_{Conf} - \frac{1}{4}\beta_0^2 B_{\beta_0}^2 A_{Conf} - \frac{1}{4}\beta_1 B_{\beta_0} A_{Conf}\right),$$

PMC∞ scales

$$\frac{1}{\sigma_{tot}}\frac{Od\sigma(\mu_I,\mu_{II},\mu_0)}{dO} = \left\{\overline{\sigma}_I + \overline{\sigma}_{II} + \overline{\sigma}_{III} + \mathcal{O}(\alpha_s^4)\right\}, \quad (20)$$

$$\alpha_I \equiv \alpha_s(\mu_I),\ \alpha_{II} \equiv \alpha_s(\tilde{\mu}_{II})$$

Conformal subsets

$$\eta = 3.51$$

$$\begin{aligned}
\overline{\sigma}_I &= A_{Conf}\frac{\alpha_I}{2\pi} \\
\overline{\sigma}_{II} &= (B_{Conf} + \eta A_{tot}A_{Conf})\left(\frac{\alpha_{II}}{2\pi}\right)^2 - \eta A_{tot}A_{Conf}\left(\frac{\alpha_0}{2\pi}\right)^2 \\
&\quad - A_{tot}A_{Conf}\frac{\alpha_0}{2\pi}\frac{\alpha_I}{2\pi} \\
\overline{\sigma}_{III} &= (C_{Conf} - A_{tot}B_{Conf} - (B_{tot} - A_{tot}^2)A_{Conf})\left(\frac{\alpha_0}{2\pi}\right)^3
\end{aligned} \quad (23)$$

$$\mu_I = \sqrt{s}\cdot e^{-\frac{1}{2}B_{\beta_0}}, \quad (1-T)<0.33$$

$$\tilde{\mu}_{II} = \begin{cases} \sqrt{s}\cdot e^{-\frac{1}{2}C_{\beta_0}\cdot\frac{B_{Conf}}{B_{Conf}+\eta\cdot A_{tot}A_{Conf}}}, & (1-T)<0.33 \\ \sqrt{s}\cdot e^{-\frac{1}{2}\left(\frac{C_1}{11B_1-\frac{2}{3}B_0}\right)}, & (1-T)>0.33 \end{cases}$$

$$\mu III = \mu 0 = \sqrt{s}$$

Red

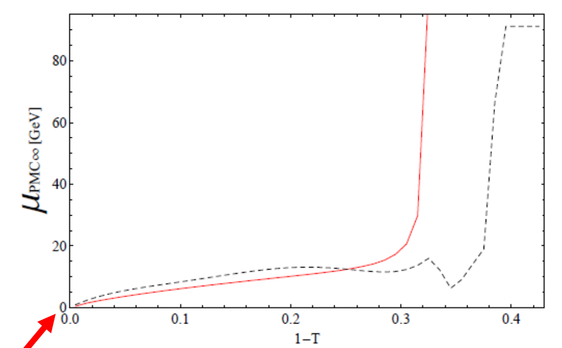
Dashed Black

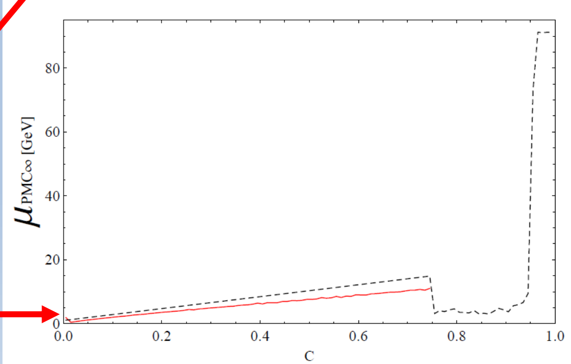
The PMC∞ scales are functions of the Event Shape Variable and of the physical scale of the process: $\sqrt{s}$

The PMC∞ determines the flow of the running coupling all over the spectrum of the observable reflecting the virtuality of quarks and gluons subprocesses

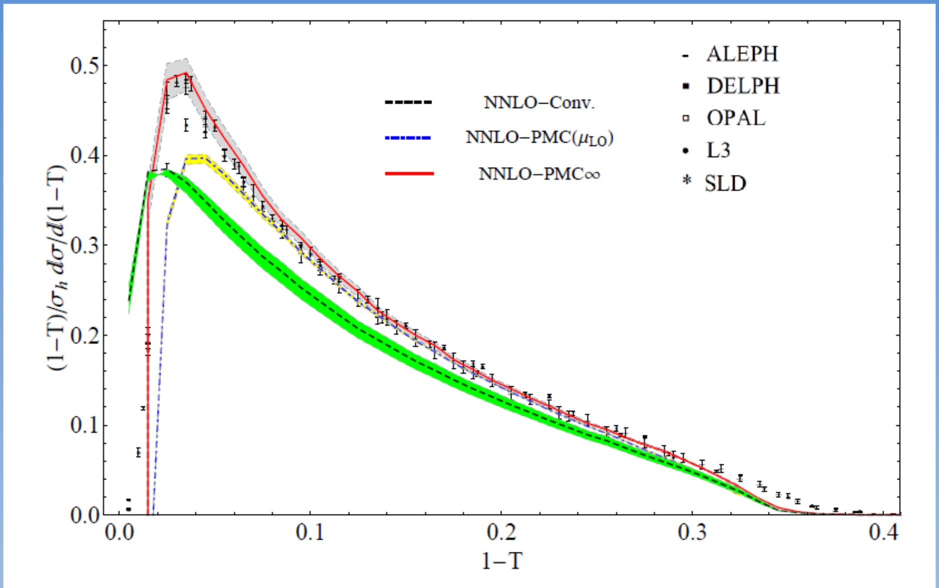
Kinematical constraints cancel the whole LO conformal subset: Aconf=0

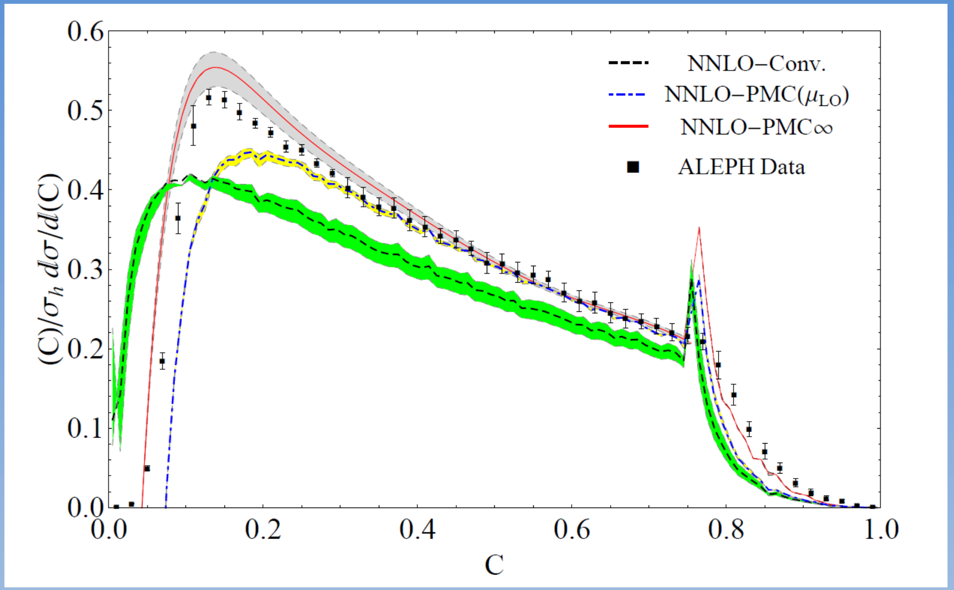
Correct physical behavior in the nonperturbative region (unlike other methods e.g. PMS and FAC)





Comparison with Conv. Scale Sett.





$\bar{\delta}[\%]$	Conv.	PMC(μ_{LO})	PMC$_\infty$
$0.10 < (1-T) < 0.33$	6.03	1.41	1.31
$0.21 < (1-T) < 0.33$	6.97	2.19	0.98
$0.33 < (1-T) < 0.42$	8.46		2.61
$0.00 < (1-T) < 0.33$	5.34	1.33	1.77
$0.00 < (1-T) < 0.42$	6.00	-	1.95

$\bar{\delta}[\%]$	Conv.	PMC(μ_{LO})	PMC$_\infty$
$0.00 < (C) < 0.75$	4.77	0.85	2.43
$0.75 < (C) < 1.00$	11.51	3.68	2.42
$0.00 < (C) < 1.00$	6.47	1.55	2.43

Errors: 85% depends on not-yet calculated orders.

PMC∞ improves the precision of the pQCD predictions.

We can use the standard criteria to evaluate the accuracy and the conformality at NNLO

The last unknown scale fixed to the last known leads to stable results.

$$\delta = \left|\frac{\sigma(2M))-\sigma(M/2)}{2\sigma(M)}\right|$$

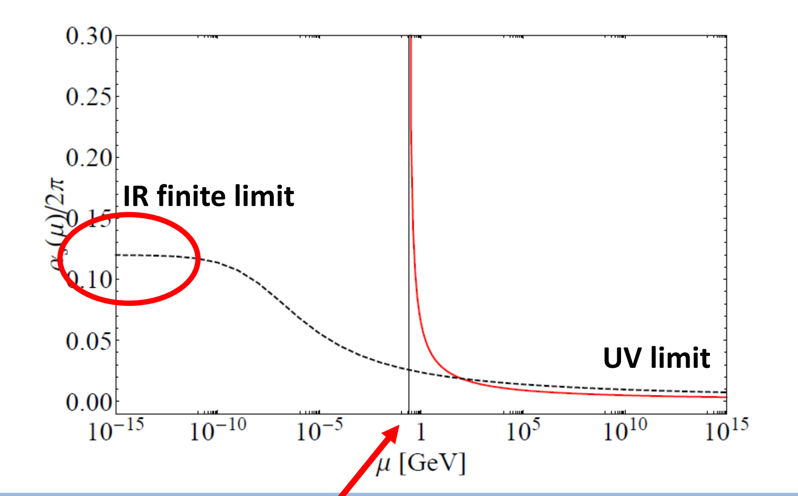
M = $\sqrt{s}$ = Z0 mass

The error due to the PMC∞ is 1.5% of the whole error $\approx 0.029 - 0.036\ \%$

Thrust in the QCD conformal window

- Banks-Zaks: UV+IR fixed points

L.D.G. , F. Sannino, S.Q. Wang, X.G. Wu, arXiv: 2104.12132 [hep]



$$\Lambda = \mu_0 \left(1 + \frac{|x^*|}{x_0}\right)^{\frac{1}{2B|x^*|}} e^{-\frac{1}{2Bx_0}}$$

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha_s}{2\pi}\right) = -\frac{1}{2}\beta_0 \left(\frac{\alpha_s}{2\pi}\right)^2 - \frac{1}{4}\beta_1 \left(\frac{\alpha_s}{2\pi}\right)^3 + O\left(\alpha_s^4\right)$$

2-loop solution: Lambert function

$$\frac{dx}{dt} = -Bx^2(1+Cx)$$

$$We^W = z$$

with:

$$W = \left(-\frac{1}{Cx} - 1\right)$$

$$z = e^{-\frac{1}{Cx_0}-1}\left(-\frac{1}{Cx_0} - 1\right)\left(\frac{\mu^2}{\mu_0^2}\right)^{-\frac{B}{C}}.$$

The general solution for the coupling is:

$$x = -\frac{1}{C}\frac{1}{1+W}.$$

Raising the number of flavors Nf , we can compare the two methods CSS and PMC∞ all over the entire energy range from 0 up to ∞.

QCD Conformal Window: $\frac{34N_c^3}{13N_c^2-3} < N_f < \bar{N}_f$

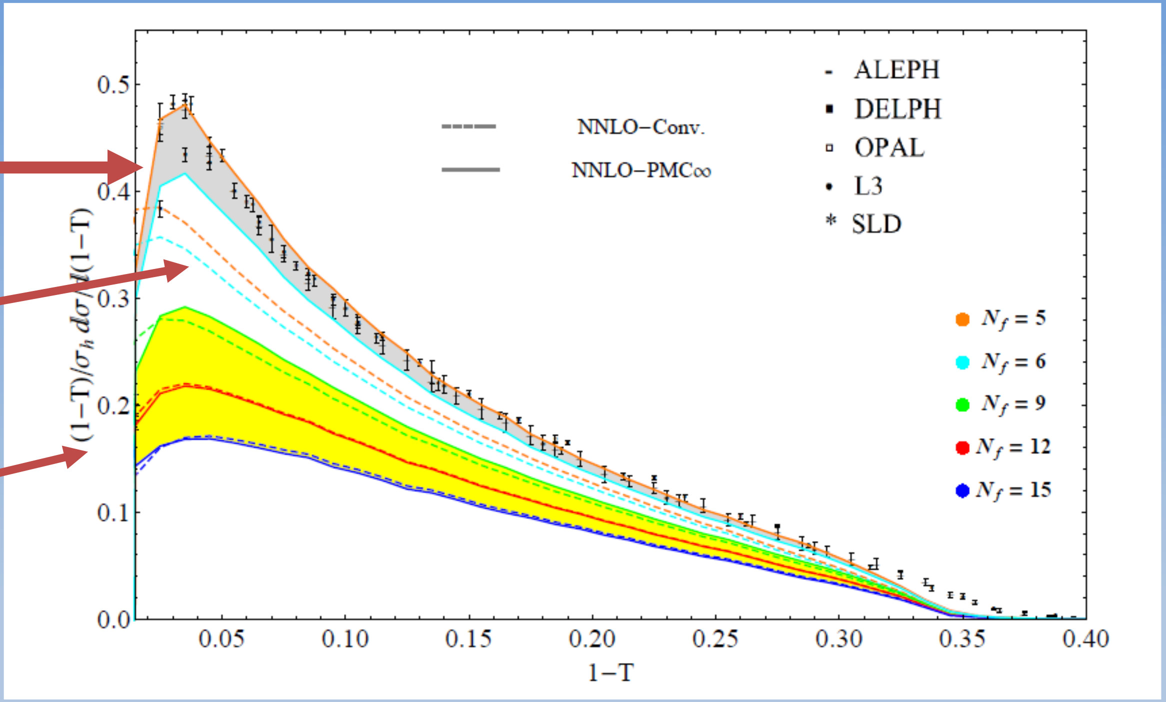
$$\bar{N}_f = x^{*-1}(x_0) \simeq 15.219 \pm 0.012,$$

Thrust in the Conformal Window: Conv.S.S. and PMC_∞

PMC_∞ : the number of active flavors is in agreement with the SM: Gray : 5<nf<6

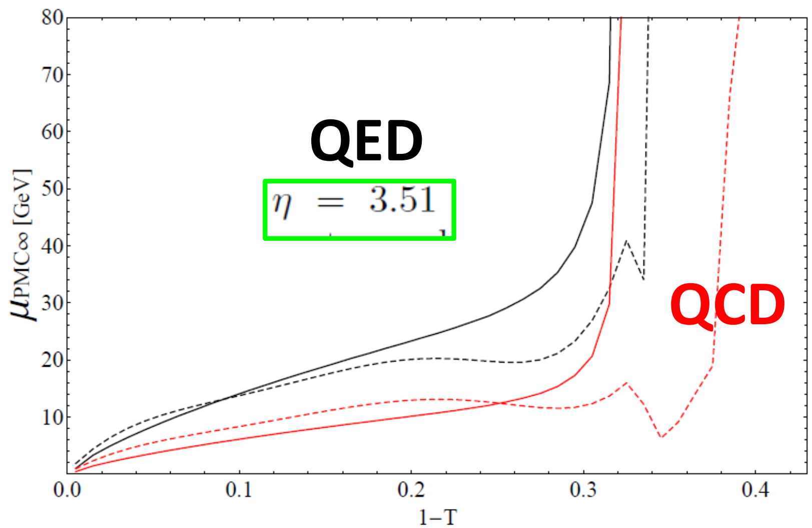
CSS deviates from the conformal distribution

Yellow: Conformal thrust : 9<nf<15



The PMC_∞ is the natural extension of the conformal thrust out of the QCD conformal window.

QED Thrust 3-Jet at NNLO, Limit Nc ⟶ 0 is consistent with PMC$_\infty$



Color factor rescaling for QED:
NA=1, CF=1, TR=1,CA=0,NA=0,NF=Nl

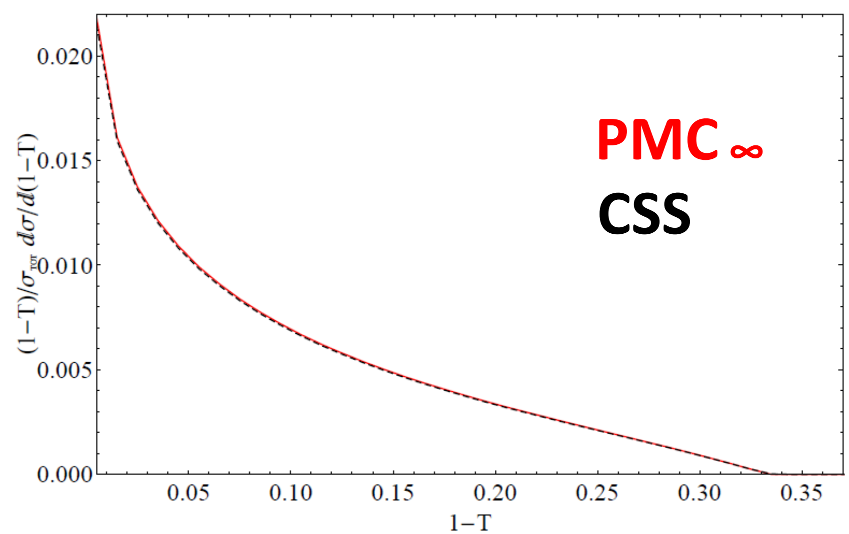
$$\beta_n/C_F^{n+1} \text{ and } \alpha_s \cdot C_F$$

$$\beta_0 = -\frac{4}{3}N_l \text{ and } \beta_1 = -4N_l$$

QED/QCD PMC$_\infty$ scales differ by the scheme $\overline{\text{MS}}$ factor reabsorption 5/3

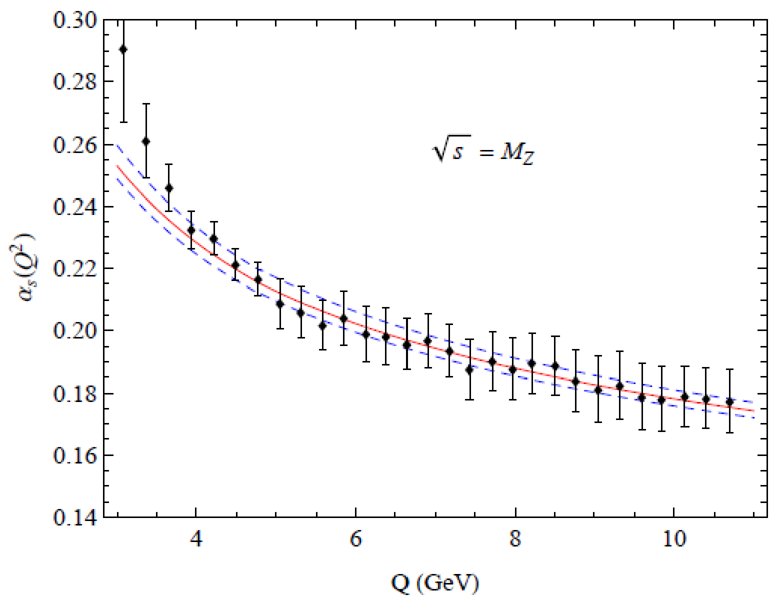
$$\alpha(Q^2) = \frac{\alpha}{\left(1 - \Re e \Pi^{\overline{\text{MS}}}(Q^2)\right)},$$

Analytic with leptons+quarks+W



The QED thrust

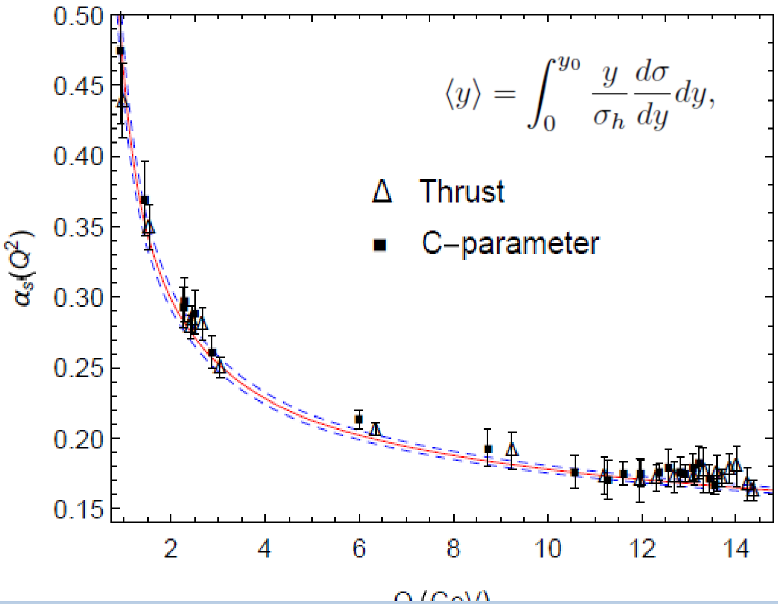
Novel method for the precise determination of $\alpha_s(Q)$



Sheng-Quan Wang, S.J. Brodsky,
Xing-Gang Wu , Jian-Ming Shen, L.D.G.,
Phys.Rev.D 100 (2019) 9, 094010

Asymptotic behavior of $\alpha_s(Q)$
determined from only one experiment

Mean values



T and C-par for e+e- → 3-Jet at a single $\sqrt{s}$

1-T

$$\alpha_s(M_Z^2) = 0.1185 \pm 0.0011(\text{exp.}) \pm 0.0005(\text{the.}) = 0.1185 \pm 0.0012,$$

C-par

$$\alpha_s(M_Z^2) = 0.1193^{+0.0009}_{-0.0010}(\text{exp.})^{+0.0019}_{-0.0016}(\text{the.}) = 0.1193^{+0.0021}_{-0.0019},$$

$$\alpha_s(M_Z^2) = 0.1181 \pm 0.0011$$

Summary

- The PMC_{∞} is based on the PMC and it preserves the iCF;
- The iCF underlies an *ordered* scale invariance;
- We have shown «how to» easily apply PMC_{∞} ;
- Event shape variables results for T and C-par are in very good agreement with data in a wide range of values;
- Thrust in the Conformal Window and in the Nc=>0 limit shows consistency with the PMC_{∞} ;
- The PMC_{∞} eliminates the scale ambiguity and improves the precision of the QCD predictions at any order;
- Measurement of αs agrees with the world average and with the asymptotic behavior;

.. in the short term

- Applications of the PMC_{∞} to other fundamental and interesting processes are in progress.

Thank you!