Applications of the Perturbative Gradient Flow

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The Gradient Flow

- Origin in lattice QCD
- Introduce additional parameter $t \geq 0$, called *flow time* [Lüscher 2010]
- *Flowed fields* live in $D+1$ dimensions and fulfill differential *flow equations* like

$$\partial_t \Phi(t,x) = D_x \Phi(t,x) \quad \text{with} \quad \Phi(t,x)\big|_{t=0} = \phi(x)$$

- Flow equation similar to the heat equation (thermodynamics)

$$\partial_t u(t,\vec{x}) = \alpha \Delta u(t,\vec{x}) \quad \text{with} \quad \Delta = \sum_i \partial_{x_i}^2$$

- Fields at positive flow time smeared out with smearing radius $\sqrt{8t}$

$\Rightarrow$ Intuition: Regulates divergencies

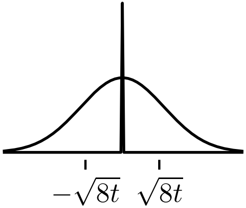

Figure: Sketch of smearing.

Gluon Flow Equation [Lüscher 2010]

$$\partial_t B_\mu^a = \mathcal{D}_\nu^{ab} G_{\nu\mu}^b \quad \text{with} \quad B_\mu^a(t,x)\Big|_{t=0} = A_\mu^a(x)$$

- Covariant derivative:

$$\mathcal{D}_\mu^{ab} = \delta^{ab}\partial_\mu - f^{abc} B_\mu^c$$

- Field-strength tensor:

$$G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + f^{abc} B_\mu^b B_\nu^c$$

- Similar equations for the quarks

Applications of the Gradient Flow

In lattice QCD:

- Inherent smearing for better continuum extrapolation [Lüscher 2010]
- New strategies for scale setting [Lüscher 2010; Borsányi et al. 2012; ...]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

Applications of the Gradient Flow

In lattice QCD:

- Inherent smearing for better continuum extrapolation [Lüscher 2010]
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- Composite operators do not require renormalization [Lüscher, Weisz 2011]

⇒ Can directly relate operators and observables in different schemes to each other, e.g. lattice and perturbative schemes:

- Extract parameters like α_s from lattice simulations [Fodor et al. 2012; Fritsch, Ramos 2013; ...; Harlander, Neumann 2016; ...; Artz, Harlander, **FL**, Neumann, Prausa 2019; ...]
- *Flowed operator product expansion* [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:
  - ★ Define the energy-momentum tensor of QCD on the lattice [Suzuki 2013; Makino, Suzuki 2014; Harlander, Kluth, **FL** 2018]
  - ★ Alternative determination of vacuum polarization functions on the lattice [Harlander, **FL**, Neumann 2020]
  - ★ ...

Lagrangian

- Construct it for the gradient flow as [Lüscher, Weisz 2011; Lüscher 2013]

$$\mathcal{L} = \mathcal{L}_{\mathsf{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

$$\mathcal{L}_{\mathsf{QCD}} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f (\not{D}^\mathrm{F} + m_f) \psi_f + \dots$$

- Construct flowed Lagrangian using Lagrange multiplier fields $L^a_\mu(t,x)$
- Flowed gluon Lagrangian:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr} \left[L^a_\mu T^a \left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b \right) \right]$$

- Similarly for the quarks

⇒ Flow equations automatically fulfilled

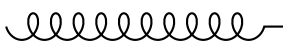
⇒ QCD Feynman rules + gradient-flow Feynman rules

Flowed Propagators and Flow Lines

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\, \mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

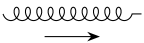
- Combined Feynman rule for the (flowed) gluon propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{B}_\nu^b(s,q)\right\rangle$:

p

s,ν,b  t,μ,a $= \delta^{ab}\dfrac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$

- No squared L_μ^a in $\mathcal{L}_B \Rightarrow$ no propagator
- Instead, mixed propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{L}_\nu^b(s,q)\right\rangle$ called *flow line*:

p

s,ν,b  t,μ,a $= \delta^{ab}\, \theta(t-s)\delta_{\mu\nu}\, e^{-(t-s)p^2}$

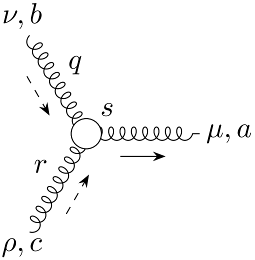
- Directed towards increasing flow time

Vertices

- Additional flow vertices from

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\,\mathrm{Tr}\left[L^a_\mu T^a\left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b\right)\right]$$

- Example:



$$= -\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu\right)$$

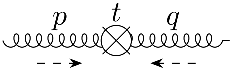
Observables

- Simple first observable: vacuum expectation value of the gluon action density

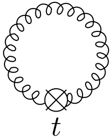
$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)$$

$$G^a_{\mu\nu}(t,x) = \partial_\mu B^a_\nu(t,x) - \partial_\nu B^a_\mu(t,x) + f^{abc} B^b_\mu(t,x) B^c_\nu(t,x)\,.$$

- Feynman rules like

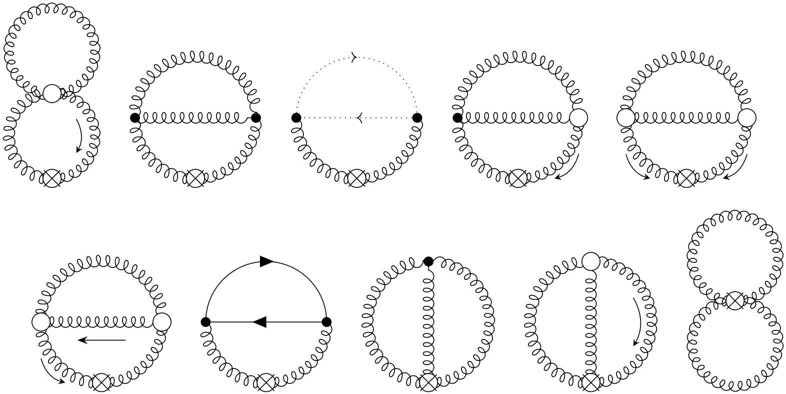
μ, a 

ν, b $= -g^2 \delta^{ab}(\delta_{\mu\nu} p \cdot q - p_\mu q_\nu)$

$\langle E(t) \rangle|_{\mathrm{LO}} =$ 

$= \frac{3\alpha_s}{4\pi t^2} \frac{N_A}{8}$ where $\alpha_s \equiv \frac{g^2}{4\pi}$

Feynman Diagrams for $\langle E(t) \rangle$ at Next-To-Leading Order (NLO)



Automatized Calculation

- qgraf [Nogueira 1991]
- q2e and exp [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]
- FORM [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013]
- Generate system of equations employing integration-by-parts-like relations [Tkachov 1981; Chetyrkin, Tkachov 1981]
- Kira [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] $\oplus$ FireFly [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020] $\Rightarrow$ talk by Johann Usovitsch
- Calculation of master integrals:
  - Direct integration with Mathematica
  - Expansion employing HyperInt [Panzer 2014]
  - Sector decomposition [Binoth, Heinrich 2000 + 2003] with FIESTA [Smirnov, Tentyukov 2008; Smirnov, Smirnov, Tentyukov 2009; Smirnov 2013] and in-house integration routines [Harlander, Neumann 2016]

$\langle E(t)\rangle$ through Next-To-Next-To-Leading Order (NNLO)

$$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2}\frac{N_A}{8}\left[1+\frac{\alpha_s}{4\pi}e_1+\left(\frac{\alpha_s}{4\pi}\right)^2 e_2+O(\alpha_s^3)\right]+O(m^2)$$

$$e_1 = e_{1,0}+\beta_0\,L(\mu^2 t)$$

$$e_2 = e_{2,0}+(2\beta_0\,e_{1,0}+\beta_1)\,L(\mu^2 t)+\beta_0^2\,L^2(\mu^2 t)$$

$$L(z) \equiv \ln(2z)+\gamma_E$$

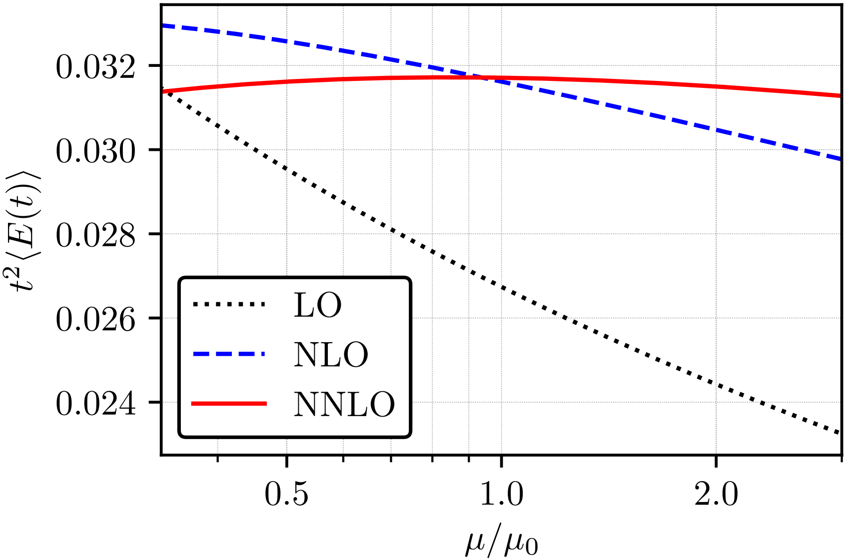
- $\langle E(t)\rangle$ finite after the QCD renormalization of α_s only!
- NLO [Lüscher 2010]:

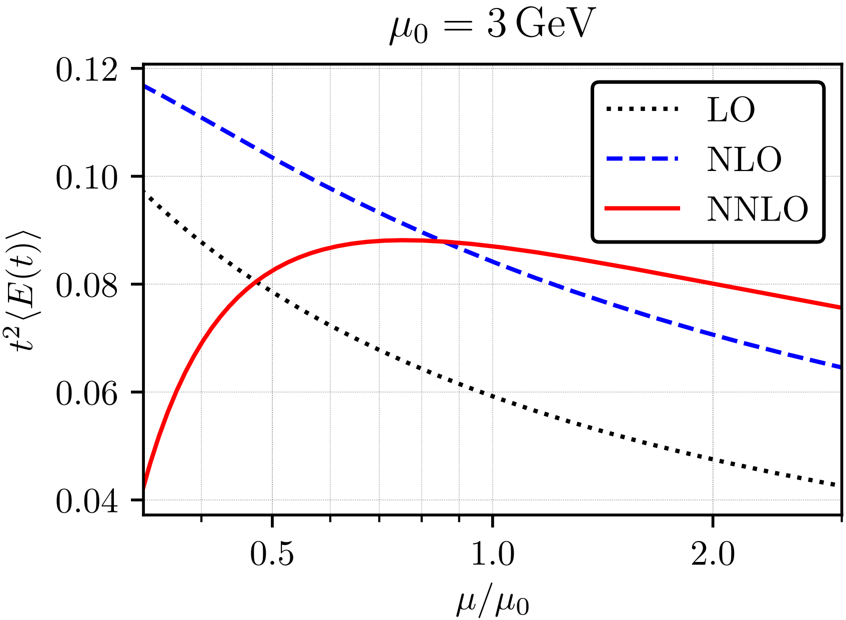
$$e_{1,0} = \left(\frac{52}{9}+\frac{22}{3}\ln 2-3\ln 3\right)C_A-\frac{8}{9}n_f T_R$$

- NNLO [Harlander, Neumann 2016; Artz, Harlander, **FL**, Neumann, Prausa 2019]:

$$e_{2,0} = 27.9786\,C_A^2-(31.5652\ldots)\,n_f T_R C_A+\left(16\zeta(3)-\frac{43}{3}\right)n_f T_R C_F+\left(\frac{8\pi^2}{27}-\frac{80}{81}\right)n_f^2 T_R^2$$

$\mu_0 = 130\,\mathrm{GeV}$




$\mu_0 = 3\,\mathrm{GeV}$
0.12
0.10
0.08
0.06
0.04
$t^2\langle E(t)\rangle$
0.5
1.0
2.0
μ/μ_0
LO
NLO
NNLO

Gradient Flow Coupling

$$\langle E(t)\rangle = \frac{3\alpha_{\rm s}}{4\pi t^2}\frac{N_{\rm A}}{8}\left[1+\frac{\alpha_{\rm s}}{4\pi}e_1+\left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 e_2+O(\alpha_{\rm s}^3)\right]+O(m^2)$$

- $\langle E(t)\rangle$ sensitive to $\alpha_{\rm s}$

$\Rightarrow$ Define *gradient flow coupling* [Lüscher 2010]:

$$\hat{\alpha}_\rho(\mu) \equiv \frac{32\pi\rho^2}{3N_{\rm A}\mu^4}\,\langle E(\rho/\mu^2)\rangle|_{m=0} \equiv \alpha_{\rm s}(\mu)\left[1+\sum_{n=1}^{\infty}\left(\frac{\alpha_{\rm s}(\mu)}{4\pi}\right)^n e_n(\rho)\right]$$

- Invert:

$$\alpha_{\rm s} = \hat{\alpha}_\rho\left[1-\frac{\hat{\alpha}_\rho}{4\pi}\,e_1(\rho)+\left(\frac{\hat{\alpha}_\rho}{4\pi}\right)^2(2e_1^2(\rho)-e_2(\rho))+\ldots\right]$$

$\Rightarrow$ Could extract $\alpha_{\rm s}$ from a lattice simulation of $\langle E(\rho/\mu^2)\rangle$

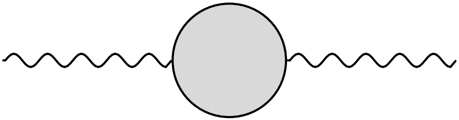
- Similar relation for m_f and a different observable

Current-Current Correlators

$$T(Q) \equiv \int \mathrm{d}^4x\, e^{\mathrm{i}Qx}\, \mathrm{T}j(x)j(0), \quad j = \bar{\psi}\Gamma\psi, \quad \Gamma = 1, \gamma_\mu, \gamma_5, \ldots$$

Important objects, since *vacuum polarization functions* $\Pi = \langle T \rangle$ contribute to many problems:

- Imaginary part related to decay rates (Z- and Higgs-bosons, hadronic R-ratio, ...) through optical theorem
- Directly contributes to anomalous magnetic moment, see talks by Massimo Passera and Syed Mehedi Hasan
  - Hadronic vacuum polarization (mainly relevant for muon) non-perturbative though
  - Traditionally obtained via dispersion relations from experimental data
  - Last year: First competitive lattice result disagrees with dispersive results [Borsanyi et al. 2020]



Operator Product Expansion

$$T(Q) \equiv \int \mathrm{d}^4x\, e^{\mathrm{i}Qx}\, \mathrm{T}j(x)j(0) \overset{Q^2\to\infty}{\sim} \sum_{k,n} C_n^{(k)}(Q)\mathcal{O}_n^{(k)}$$

- Operators up to dimension four:

$$\mathcal{O}_1^{(0)} \equiv \mathcal{O}^{(0)} = \mathbb{1}, \qquad \mathcal{O}_1^{(2)} \equiv \mathcal{O}^{(2)} = m_\mathrm{B}^2\,\mathbb{1},$$

$$\mathcal{O}_1^{(4)} \equiv \mathcal{O}_1 = \frac{1}{g_\mathrm{B}^2} F_{\mu\nu}^a F_{\mu\nu}^a, \qquad \mathcal{O}_2^{(4)} \equiv \mathcal{O}_2 = \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f \overleftrightarrow{\not{D}}^{\mathrm{F}} \psi_f,$$

$$\mathcal{O}_3^{(4)} \equiv \mathcal{O}_3 = m_\mathrm{B}^4\,\mathbb{1}$$

- Wilson coefficients $C_n^{(k)}(Q)$ known to high perturbative order for many currents and operators

Flowed Operator Product Expansion (Flowed OPE)

- Small flow-time expansion [Lüscher, Weisz 2011]:

$$\tilde{\mathcal{O}}_i(t,x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Invert to express (sums of) composite operators through flowed operators [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:

$$T = \sum_i C_i \mathcal{O}_i = \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv \sum_j \tilde{C}_j \tilde{\mathcal{O}}_j$$

$\Rightarrow$ *Flowed OPE*

- T defined in regular QCD expressed through better behaved flowed operators $\tilde{\mathcal{O}}_j$, which do not require renormalization
- Can relate T in lattice and perturbative schemes without scheme transformation

Flowed OPE for Current-Current Correlators

$$T(Q) \overset{Q^2\to\infty}{\sim} \tilde{C}^{(0)}(Q^2,t)\mathbb{1} + \tilde{C}^{(2)}(Q^2,t)m^2\mathbb{1} + \sum_n \tilde{C}_n(Q^2,t)\tilde{\mathcal{O}}_n(t)$$

with

$$\tilde{C}^{(0,2)}(Q^2,t) = C^{(0,2)}(Q^2) - \sum_n \tilde{C}_n(Q^2,t)\,\zeta_n^{(0,2)}(t),$$

$$\tilde{C}_n(Q^2,t) = \sum_k C_k(Q^2)\zeta_{kn}^{-1}(t)$$

Flowed Operators:

$$\tilde{\mathcal{O}}_1(t) = \frac{1}{\hat{\mu}^{2\epsilon}g^2}G^a_{\mu\nu}(t)G^a_{\mu\nu}(t),$$

$$\tilde{\mathcal{O}}_2(t) = \mathring{Z}_\chi \sum_{f=1}^{n_\mathrm{f}} \bar{\chi}_f(t)\overleftrightarrow{\not{D}}^{\mathrm{F}}(t)\chi_f(t),$$

$$\tilde{\mathcal{O}}_3(t) = m^4\mathbb{1}$$

Method of Projectors

[Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

- Define projectors

$$P_k[\mathcal{O}_i(x)] \equiv D_k\langle 0|\mathcal{O}_i(x)|k\rangle = \delta_{ik} + O(\alpha_{\rm s})$$

$$\Rightarrow \quad P_k[\tilde{\mathcal{O}}_i(t,x)] = \sum_j \zeta_{ij}(t) P_k[\mathcal{O}_j(x)]$$

- $\zeta_{ij}(t)$ only depend on t

$\Rightarrow$ Set all other scales to zero

$\Rightarrow$ No perturbative corrections to $P_k[\mathcal{O}_j(x)]$, because all loop integrals are scaleless:

$$\zeta_{ij}(t) = P_j[\tilde{\mathcal{O}}_i(t,x)]\Big|_{p=m=0}$$

$\Rightarrow$ Compute matrix elements of flowed operators and apply projectors

Calculation

- Example:

$$\zeta_n^{(0)}(t) = P^{(0)}[\tilde{\mathcal{O}}_n(t)] \equiv \langle \tilde{\mathcal{O}}_n(t) \rangle \Big|_{m_\mathrm{B}=0} \qquad \text{for} \qquad \mathcal{O}^{(0)} = \mathbb{1}$$

- Can actually reuse results:

$$\zeta_1^{(0)}(t) = \langle \tilde{\mathcal{O}}_1(t) \rangle \Big|_{m=0} = \frac{1}{\hat{\mu}^{2\epsilon} 4\pi\alpha_\mathrm{s}} \langle G^a_{\mu\nu}(t) G^a_{\mu\nu}(t) \rangle \Big|_{m=0} = \frac{1}{\hat{\mu}^{2\epsilon} \pi\alpha_\mathrm{s}} \langle E(t) \rangle |_{m=0}$$

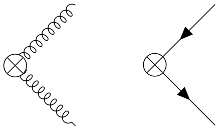
- For some other projectors we just need higher order bare mass terms, e.g.

$$\zeta_n^{(2)}(t) \equiv Z_m^2 \frac{1}{2!} \frac{\partial^2}{\partial m_\mathrm{B}^2} \langle \tilde{\mathcal{O}}_n(t) \rangle \Big|_{m_\mathrm{B}=0} \qquad \text{for} \qquad \mathcal{O}^{(2)} = m_\mathrm{B}^2 \, \mathbb{1}$$

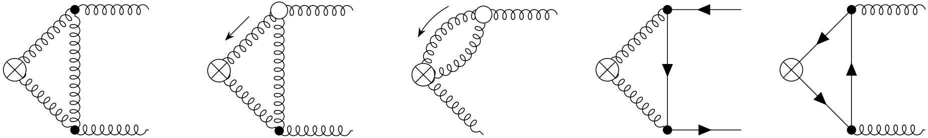
- Similarly, we can reuse results for $\tilde{\mathcal{O}}_2(t, x)$ as well

Example Diagrams for the Mixing of $\mathcal{O}^{(4)}_{1/2}(x)$ and $\tilde{\mathcal{O}}_{1/2}(t,x)$

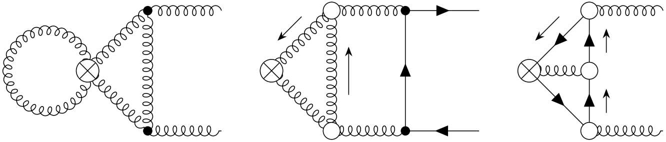
- LO:



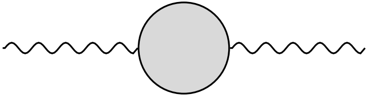
- NLO:



- NNLO:



Hadronic Vacuum Polarization and the Flowed OPE



Alternative determination of hadronic vacuum polarization, e.g. for anomalous magnetic moment:

$$\Pi_{\rm VV}(Q) \overset{Q^2\to\infty}{\simeq} \tilde{C}^{(0)}(Q^2,t) + \tilde{C}^{(2)}(Q^2,t)m^2 + \sum_n \tilde{C}_n(Q^2,t)\langle\tilde{\mathcal{O}}_n(t)\rangle,$$

$$\tilde{C}^{(0,2)}(Q^2,t) = C^{(0,2)}(Q^2) - \sum_n \tilde{C}_n(Q^2,t)\,\zeta_n^{(0,2)}(t),$$

$$\tilde{C}_n(Q^2,t) = \sum_k C_k(Q^2)\zeta^{-1}_{kn}(t)$$

- $C^{(0,2)}(Q^2)$ and $C_k(Q^2)$ available in [Chetyrkin, Gorishny, Spiridonov 1985; Chetyrkin, Harlander, Kühn, Steinhauser 1997]
- $\zeta_n^{(0,2)}(t)$ and $\zeta^{-1}_{kn}(t)$ computed in [Harlander, **FL**, Neumann 2020]
- $\langle\tilde{\mathcal{O}}_n(t)\rangle$ have to be computed on the lattice

Conclusion

- Gradient flow useful in lattice QCD (scale setting, smearing, non-renormalization)
- Cross-fertilization between lattice and perturbative QCD
- Can extract α_s and m_f from lattice simulations
- Flowed operator product expansion can be applied manifold:
  - Alternative determination of hadronic vacuum polarization for anomalous magnetic moment
  - Define energy momentum tensor on the lattice
    - Already used in several studies of thermodynamics of QCD
  - Ongoing project: Apply to operators of electroweak Hamiltonian to compute hadronic contributions to flavor physics without scheme transformation

Backup

Quark Flow Equation [Lüscher 2013]

$$\partial_t \chi = \Delta \chi \quad \text{with} \quad \chi(t,x)|_{t=0} = \psi(x)\,,$$
$$\partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta} \quad \text{with} \quad \overline{\chi}(t,x)|_{t=0} = \overline{\psi}(x)$$

$$\Delta = (\partial_\mu + B_\mu^a T^a)(\partial_\mu + B_\mu^b T^b)\,, \qquad \overleftarrow{\Delta} = (\overleftarrow{\partial}_\mu - B_\mu^a T^a)(\overleftarrow{\partial}_\mu - B_\mu^b T^b)$$

- $\chi(t,x)$ and $\overline{\chi}(t,x)$: flowed quark and anti-quark fields
- $\psi(x)$ and $\overline{\psi}(x)$: fundamental quark and anti-quark fields

Solving the Flow Equations

- Split the flow equation into a linear part and a remainder [Lüscher 2010]

$$\partial_t B_\mu^a = \partial_\nu \partial_\nu B_\mu^a + R_\mu^a \quad \text{with} \quad B_\mu^a(t,x)\Big|_{t=0} = A_\mu^a(x)$$

- Solved by

$$B_\mu^a(t,x) = \int_y K_{\mu\nu}(t,x-y) A_\nu^a(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s,x-y) R_\nu^a(s,y)$$

with the integration kernel

$$K_{\mu\nu}(t,x) = \int_p e^{\mathrm{i}p\cdot x} \delta_{\mu\nu} e^{-tp^2} \equiv \int_p e^{\mathrm{i}p\cdot x} \widetilde{K}_{\mu\nu}(t,p)$$

Propagators

- Gluon flow equation solved by

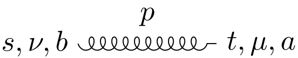
$$B^a_\mu(t,x) = \int_y K_{\mu\nu}(t,x-y)A^a_\nu(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s,x-y)R^a_\nu(s,y)$$

$$K_{\mu\nu}(t,x) = \int_p e^{\mathrm{i}p\cdot x}\delta_{\mu\nu}e^{-tp^2} \equiv \int_p e^{\mathrm{i}p\cdot x}\widetilde{K}_{\mu\nu}(t,p)$$

- Flowed gluon propagator contains the fundamental gluon propagator:

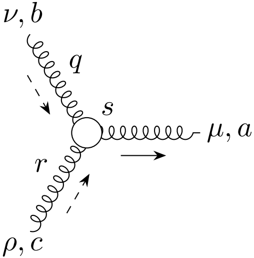
$$\left\langle \widetilde{B}^a_\mu(t,p)\widetilde{B}^b_\nu(s,q)\right\rangle\Big|_{\mathrm{LO}} = \widetilde{K}_{\mu\rho}(t,p)\widetilde{K}_{\nu\sigma}(s,q)\left\langle \widetilde{A}^a_\rho(p)\widetilde{A}^b_\sigma(q)\right\rangle$$

⇒ Combined Feynman rule:

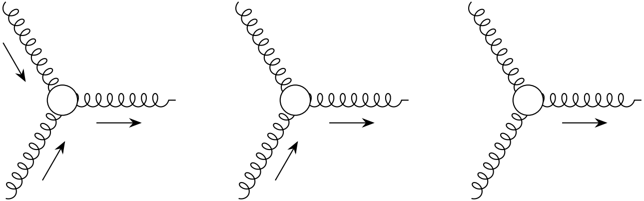


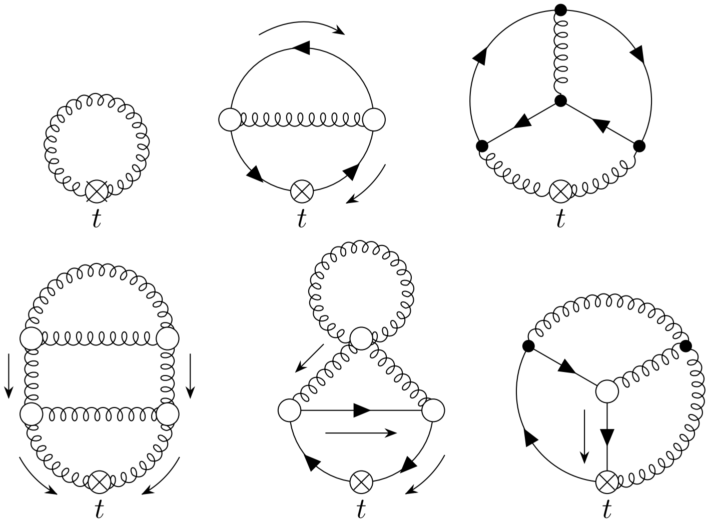
$$= \delta^{ab}\frac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$$

Vertices II



$$= -\mathrm{i} g f^{abc} \int_0^\infty \mathrm{d}s \, \left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu} q_\rho - 2\delta_{\mu\rho} r_\nu\right)$$




t
t
t
t
t
t

General Integral Form (massless)

$$I(D,t,\{t_f^{\text{up}}\},\{T_i\},\{a_i\}) = \left(\prod_{f=1}^{F}\int_0^{t_f^{\text{up}}} \mathrm{d}t_f\right)\int_{k_1,\dots,k_L} \frac{\exp[-(T_1 q_1^2 + \cdots + T_N q_N^2)]}{q_1^{2a_1}\cdots q_N^{2a_N}}$$

- k_i are loop momenta and $\int_{k_i} \equiv \int \frac{\mathrm{d}^D k_i}{(2\pi)^D}$
- q_i are linear combinations of k_j (and external momenta if present)
- a_i are propagator powers
- t_f are flow-time integration variables with boundaries t_f^{up}
- T_i are nonnegative linear combinations of t and t_f, e.g. $T_1 = t + t_1 - 2t_3$

Integration-by-Parts Relations

[Tkachov 1981, Chetyrkin, Tkachov 1981]

$$\int_{k_1,\ldots k_L} \frac{\partial}{\partial k_i^\mu} \left(\tilde{q}_j^\mu \frac{1}{P_1^{a_1} \ldots P_N^{a_N}} \right) = 0$$

- Can easily be adopted to gradient-flow integrals
- Similar new relations:

$$\int_0^{t_f^{\text{up}}} \mathrm{d}t_f \partial_{t_f} F(t_f, \ldots) = F(t_f^{\text{up}}, \ldots) - F(0, \ldots)$$

$\Rightarrow$ Linear relations between the integrals

$\Rightarrow$ Build a system of equations and solve it with `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] $\oplus$ `FireFly` [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020]

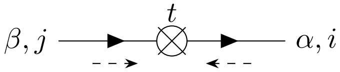
$\Rightarrow$ All integrals are expressed through master integrals

Quark Condensate

- Another observable is the quark condensate

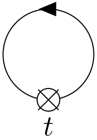
$$S(t,x) \equiv \sum_{f=1}^{n_\mathrm{f}} \bar{\chi}_f(t,x)\chi_f(t,x)$$

- Feynman rule:



$$= \delta_{ij}\delta_{\alpha\beta}$$

$$\langle S(t)\rangle|_{\mathsf{LO}} =$$



$$= -\frac{N_\mathrm{C}}{8\pi^2 t}\sum_{f=1}^{n_\mathrm{f}} m_f + O(m^2)$$

- Sensitive to the quark mass m_f

⇒ Could be used to extract m_f from lattice data

Renormalization and Ringed Fermions

- Flowed quarks require a field renormalization Z_χ:

$$S(t,x) = Z_\chi \sum_{f=1}^{n_\mathrm{f}} \bar{\chi}_{f,\mathrm{bare}}(t,x)\chi_{f,\mathrm{bare}}(t,x)$$

$\Rightarrow$ Renormalization-scheme dependent

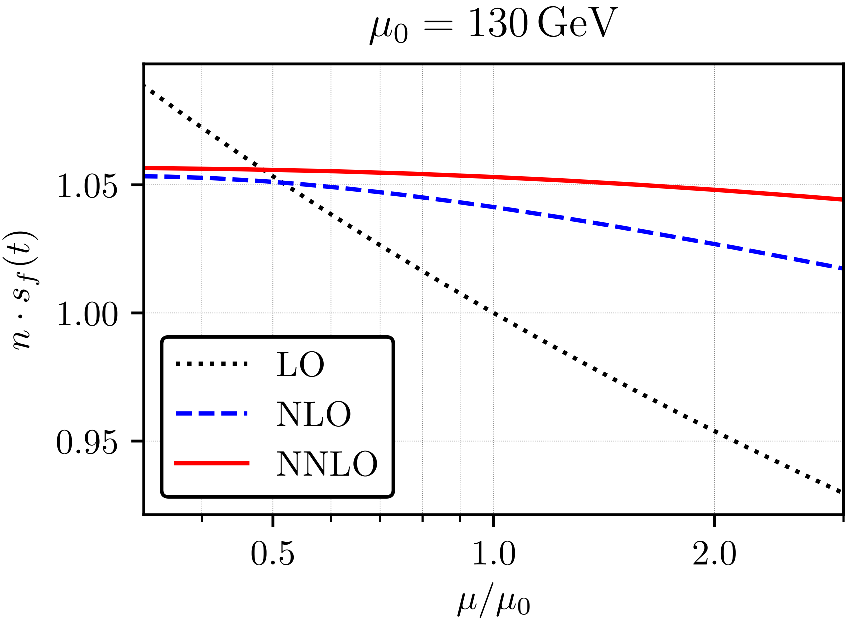
- Introduce "ringed" fermion fields [Makino, Suzuki 2014]

$$\chi_f(t,x) \to \mathring{\chi}_f(t,x) \equiv \left(\frac{\mathring{Z}_\chi}{Z_\chi}\right)^{1/2} \chi_f(t,x)$$

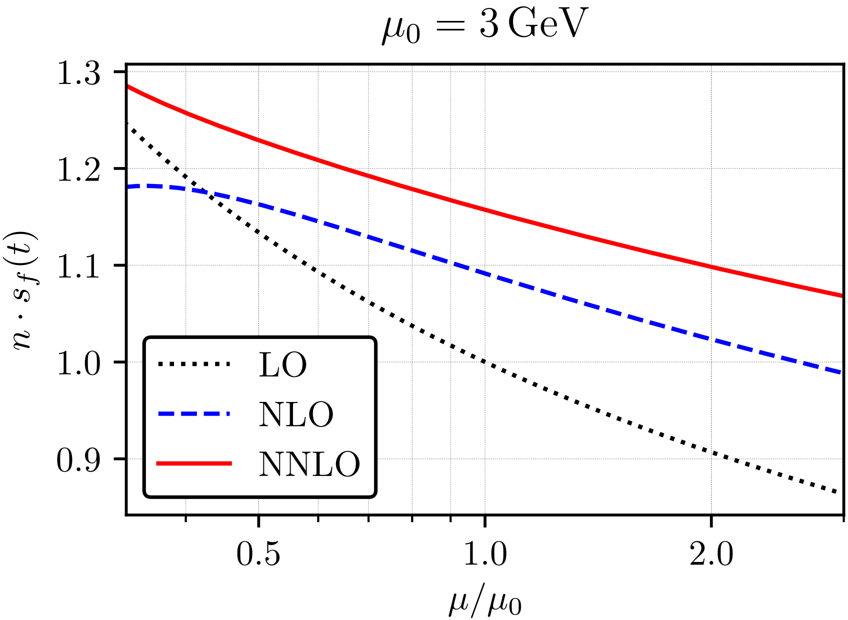
$$\mathring{Z}_\chi(t,\mu) = -\frac{2N_\mathrm{C} n_\mathrm{f}}{(4\pi t)^2}\left\langle R(t)\right\rangle^{-1}, \quad R(t,x) = \sum_{f=1}^{n_\mathrm{f}} \bar{\chi}_f(t,x) \overleftrightarrow{\not{D}}^\mathrm{F} \chi_f(t,x)$$

$\Rightarrow$ $\mathring{S}(t,x) = \frac{\mathring{Z}_\chi}{Z_\chi} S(t,x) \propto \frac{S(t,x)}{\langle R(t,x)\rangle}$ scheme independent

- However, have to calculate $\langle R(t,x)\rangle$ as well



$\mu_0 = 130\,\mathrm{GeV}$
1.05
1.00
0.95
$n \cdot s_f(t)$
0.5
1.0
2.0
μ/μ_0
LO
NLO
NNLO


$\mu_0 = 3\,\mathrm{GeV}$
1.3
1.2
1.1
1.0
0.9
$n \cdot s_f(t)$
LO
NLO
NNLO
0.5
1.0
2.0
μ/μ_0

Calculation II

- Due to $\mathcal{O}_3/\tilde{\mathcal{O}}_3 \propto m^4 \mathbb{1}$, the mixing matrix of the dimension-four operators schematically reads

$$\zeta = \begin{pmatrix} \zeta_{2\times 2} & \vec{\zeta}_3 \\ \vec{0}^T & 1 \end{pmatrix}$$

- $\vec{\zeta}_3$ are again just higher order bare mass terms:

$$\zeta^{\mathrm{B}}_{n3}(t) = P_3^{(4)}[\tilde{\mathcal{O}}_n(t)] \equiv \frac{1}{4!} \frac{\partial^4}{\partial m_{\mathrm{B}}^4} \langle \tilde{\mathcal{O}}_n(t) \rangle \Big|_{m_{\mathrm{B}}=0}$$

- $\mathcal{O}_1$ and $\mathcal{O}_2$ already appeared in our calculation for the EMT:

$$\delta_{\mu\nu} \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix} = \delta_{\mu\nu} \begin{pmatrix} F^a_{\rho\sigma} F^a_{\rho\sigma} \\ \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f(x) \overleftrightarrow{\not{D}}^{\mathrm{F}} \psi_f(x) \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{O}}_{2,\mu\nu} \\ \hat{\mathcal{O}}_{4,\mu\nu} \end{pmatrix}$$

$\Rightarrow$ $\zeta_{2\times 2}$ can be extracted from our results for the EMT [Harlander, Kluth, **FL** 2018]

- Due to the sophisticated renormalization of $\vec{\zeta}_3$ we need higher order terms in ϵ though