Proving the dimension-shift conjecture

2011.13821 [hep-th] JHEP 04 (2021), 276

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RadCor/LoopFest 2021

Thursday 20th May

European Research Council
Established by the European Commission

Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Research funded by grant number 647356 (CutLoops)

The dimension-shift conjecture

- Made by Z. Bern , L. Dixon, D. C. Dunbar and D. Kosower (BDDK) hep-th/9611127 Phys.Lett.B 394 (1997), 105-115

- Simple relationship between GLUON amplitudes at ONE LOOP in different theories.

- Relates all-plus-helicity QCD amplitudes to MHV amplitudes in N=4 SYM

$$D = 4 - 2\epsilon$$

$$A_n^{\rm QCD}\left(1^+, 2^+, \ldots, n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2 \left[\frac{A_n^{\mathcal{N}=4}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle ij\rangle^4}\right]_{D \to D+4}$$

The dimension-shift conjecture

$$A_n^{\mathrm{QCD}}\left(1^+,2^+,\ldots,n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2\left[\frac{A_n^{\mathcal{N}=4}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)}{\langle ij\rangle^4}\right]_{\epsilon\to\epsilon-2}$$

- BDDK verified up to n=6
- Can we prove this after 25 years of advances in understanding?

Proven!

- THIS TALK:
  - How to prove
  - How to compute all-ε all-n forms in both theories

The dimension-shift conjecture

$$A_n^{\text{QCD}}\left(1^+, 2^+, \ldots, n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2 \left[\frac{A_n^{\mathcal{N}=4}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle ij\rangle^4}\right]_{\epsilon\to\epsilon-2}$$

- How far have we come with these particular theories and helicity configurations?

All-plus QCD

$$A_n^{\mathrm{QCD}}(1^+, 2^+, ..., n^+)$$

- Tree-Level

$$= 0$$

- One-loop

$$= \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \frac{\mathrm{tr}_-(i_1 i_2 i_3 i_4)}{\langle 12...n1 \rangle} + \mathcal{O}(\epsilon)$$

[Mahlon '93] [Bern, Chalmers, Dixon, Kosower '93]

- Two-loop
  - Planar: First n>4 result in QCD [Badger, Frellesvig, Zhang '13]
  - Planar: all-n for cut-constructible (n=7 full) [Dunbar, GRJ, Perkins '16] [Dunbar, Godwin, GRJ, Perkins '17]
  - Non-planar: first all-n partial amplitude at 2 loops [Dunbar, Godwin, Perkins, Strong '20]

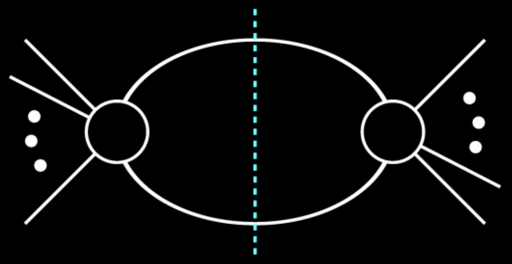
MHV in N=4

$$A_n^{\mathcal{N}=4}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

- Tree-Level

[Parke, Taylor '86] [Nair '88] [Bianchi, Elvang, Freedman '05]

$$A_n^{\text{tree}} = \frac{\langle ij \rangle^4}{\langle 12 \ldots n1 \rangle} = \frac{\delta^4}{\delta \eta_i^4} \frac{\delta^4}{\delta \eta_j^4} \left[\frac{\delta^{(8)}(|k\rangle \eta_{kA})}{\langle 12 \ldots n1 \rangle} \right]$$

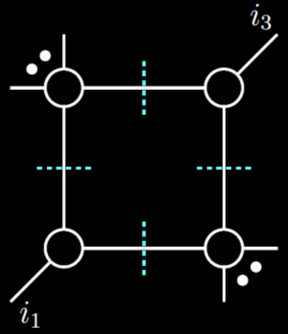


[Bern, Dixon, Dunbar, Kosower '94]

- One-loop

$$= \frac{1}{4} A_n^{\text{tree}} \sum_{i_1, i_3 = 1}^{n} \operatorname{tr}(i_1 q_{i_1+1, i_3} i_3 q_{i_3+1, i_1}) I_4^{[i_1, i_1+1, i_3, i_3+1]} + \mathcal{O}(\epsilon)$$

[Britto, Cachazo, Feng '05]



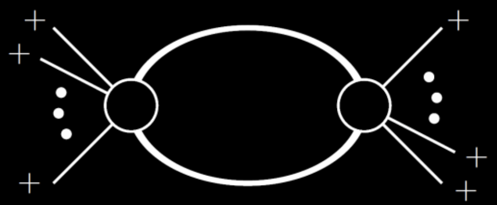
- Multi-loop
  - BDS ansatz to all loop order (exact for n<6) [Bern, Dixon, Smirnov '05]
  - Seven-loops n=6 [Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou '19]
  - Two loops n=9 [Golden, McLeod'21]

Proving the conjecture

$$A_n^{\mathrm{QCD}}\left(1^+,2^+,\ldots,n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2\left[\frac{A_n^{\mathcal{N}=4}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)}{\langle ij\rangle^4}\right]_{\epsilon\to\epsilon-2}$$

- Need to capture all-ε structure to all multiplicities
- To compute up to six points, BDDK use:
  - String-derived (worldline) formalism for N=4 amplitude
  - D-dimensional cuts for all-plus QCD

[Bern, Kosower '92]



$$A_n^{\mathrm{QCD}}\left(1^+,2^+,\ldots,n^+\right) = \;2A^{[0]}$$

Amplitude with scalar particle in the loop

Proving the conjecture

$$A_n^{\mathrm{QCD}}\left(1^+,2^+,\dots,n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2\left[\frac{A_n^{\mathcal{N}=4}(1^+,\dots,i^-,\dots,j^-,\dots,n^+)}{\langle ij\rangle^4}\right]_{\epsilon\to\epsilon-2}$$

- D-dimensional cuts nicer for both sides

$$\ell = l + \ell^{[-2\epsilon]} \qquad \Rightarrow \qquad \ell^2 = l^2 - \mu^2 = 0$$

$$\mu^2 \equiv -\left(\ell^{[-2\epsilon]}\right)^2$$

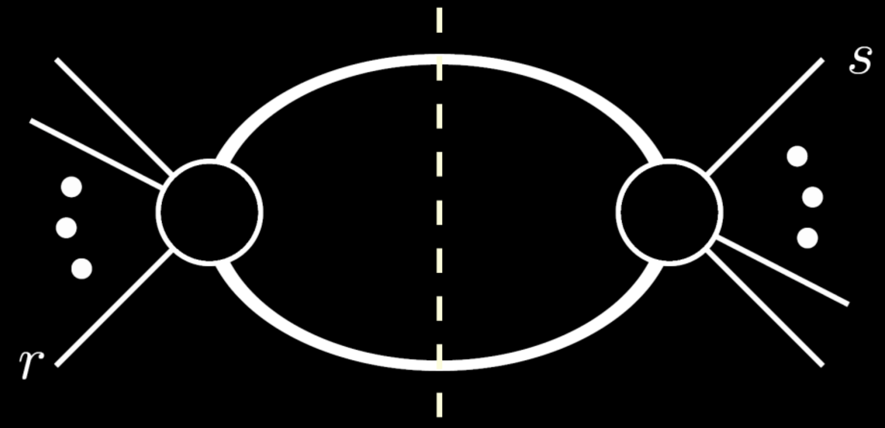
$$\int \frac{d^{4-2\epsilon}\ell}{\ell^2\cdots(\ell-q)^2} = -\int \frac{d\mu^2}{(-\mu^2)^{1+\epsilon}}\frac{d^4 l}{(l^2-\mu^2)\cdots((l-q)^2-\mu^2)}$$

Proving the conjecture

$$A_n^{\mathrm{QCD}}\left(1^+,2^+,\dots,n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2\left[\frac{A_n^{\mathcal{N}=4}(1^+,\dots,i^-,\dots,j^-,\dots,n^+)}{\langle ij\rangle^4}\right]_{\epsilon\to\epsilon-2}$$

$$I_m^{4-2\epsilon}[\mu^{2r}] = -\int\frac{d^4l\,d\mu^2}{(-\mu^2)^{1+\epsilon}}\frac{\left(\mu^2\right)^r}{(l^2-\mu^2)\cdots((l-q)^2-\mu^2)}$$

$$= -\epsilon(1-\epsilon)\cdots(r-1-\epsilon)I_m^{4+2r-2\epsilon}$$

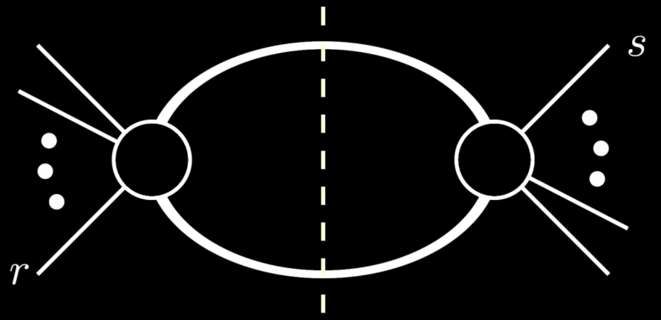


$$A_n^{\mathrm{QCD}}\Big|_{q_{rs}\text{ cut}}^{\mu^2\neq0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Bigg|_{q_{rs}\text{ cut}}^{\mu^2\neq0}$$

Proving the conjecture

$$A_n^{\mathrm{QCD}}\Big|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Big|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0}$$

- Need tree amplitudes with two equally-massive legs
  - Vector bosons, adjoint fermions, scalars
  - Expressions available



[Forde, Kosower '06]

[Rodrigo '06]

[Ferrario, Rodrigo, Talavera '06]

[Boels, Schwinn '11]

[Craig, Elvang, Kiermaier, Slatyer '11]

[Kiermaier '11]

Proving the conjecture

$$A_n^{\text{QCD}}\Big|_{q_{rs}\text{ cut}}^{\mu^2\neq 0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Big|_{q_{rs}\text{ cut}}^{\mu^2\neq 0}$$

- Simple relationships between all-n amplitudes in "MHV band"
  - All related to scalar amplitude
  - Coulomb-branch SUSY associates spin structure

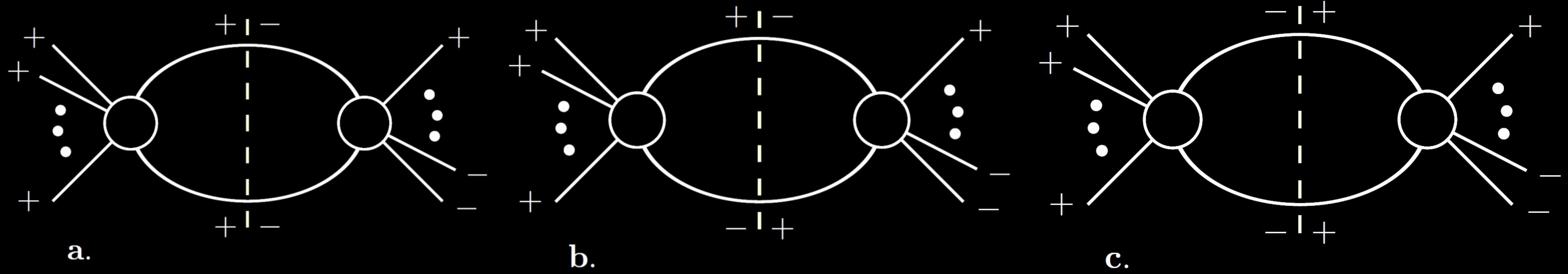
[Craig, Elvang, Kiermaier, Slatyer '11]
[Kiermaier '11]
[Elvang, Freedman, Kiermaier '11]

$$A_{\text{tree}}^{\text{MHV}-\text{band}} = \frac{[\lambda_n\lambda_1]^2\delta_{12}^{\chi}\delta_{34}^{\chi}}{m^2 q_{n2}^4} A_n^{\text{tree}}(\mathbf{1}^0, 2^+, 3^+, \ldots, (n-1)^+, \mathbf{n}^0)$$

Proving the conjecture

$$A_n^{\mathrm{QCD}}\Big|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Big|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0}$$

- MHV has three types of cut in a given axial gauge



Proving the conjecture

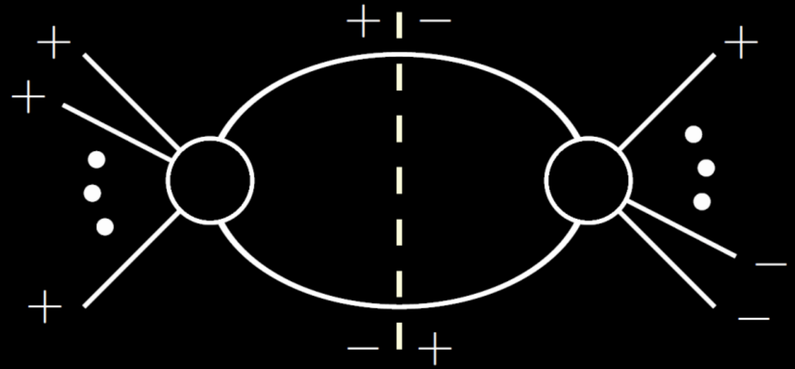
$$A_n^{\text{QCD}}\Big|_{q_{rs}\ \text{cut}}^{\mu^2\neq 0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Big|_{q_{rs}\ \text{cut}}^{\mu^2\neq 0}$$

- MHV has three types of cut in a given axial gauge $\quad p_\chi \cdot q_{rs} = 0$

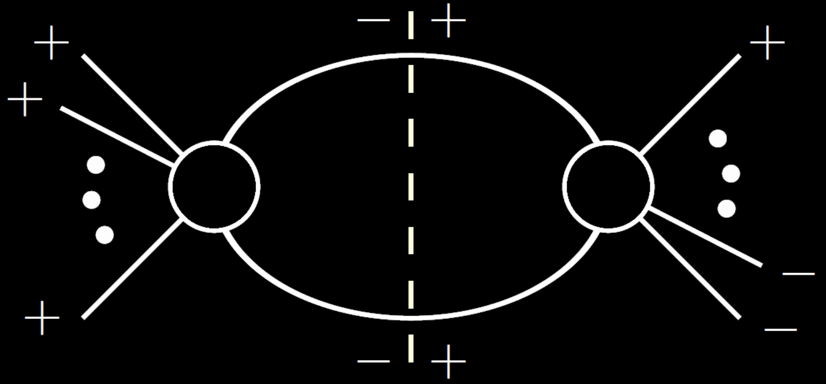
$$A^{\mathcal{N}=4} = A^{[1]} + 4A^{[\frac{1}{2}]} + 6A^{[0]}$$

$$A^{[1]}\Big|_{\mathbf{b}.\text{cut}} = 2A^{[0]}\Big|_{\mathbf{b}.\text{cut}}$$

$$A^{[\frac{1}{2}]}\Big|_{\mathbf{b}.\text{cut}} = -2A^{[0]}\Big|_{\mathbf{b}.\text{cut}}$$



b.



c.

Proving the conjecture

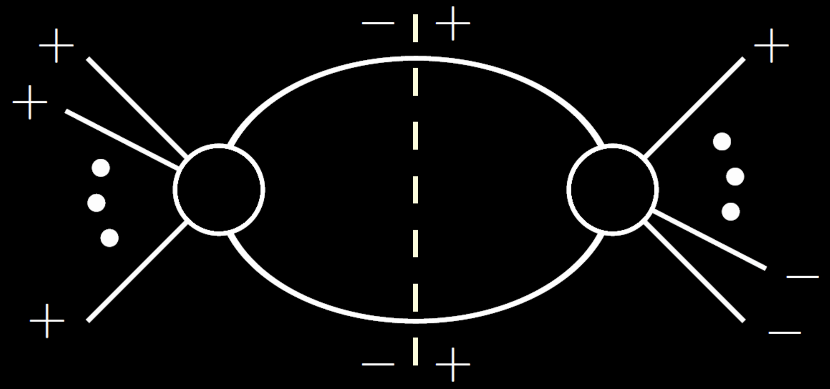
$$A_n^{\text{QCD}}\Big|_{q_{rs}\ \text{cut}}^{\mu^2\neq 0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Big|_{q_{rs}\ \text{cut}}^{\mu^2\neq 0}$$

- MHV x MHV cut similar to massless (4D) cuts

$$p_\chi \cdot q_{rs} = 0$$

$$\int d^4\eta_{l_r} d^4\eta_{l_s} A_L^{\text{MHV tree}}(\text{-}\boldsymbol{l}_r^1, r, ..., (s-1), \boldsymbol{l}_s^1) \times A_R^{\text{MHV tree}}(\text{-}\boldsymbol{l}_r^1, r, ..., s-1, \boldsymbol{l}_s^1)$$

$$= \frac{\delta^{(8)}\left(\langle\lambda_i|\eta_{iA}\right)}{\mu^4} A_L(\text{-}\boldsymbol{l}_r^0, r^+, ..., (s-1)^+, \boldsymbol{l}_s^0) A_R(\text{-}\boldsymbol{l}_s^1, s^+, ..., (r-1)^+, \boldsymbol{l}_r^1)$$



c.

apply functional derivatives….

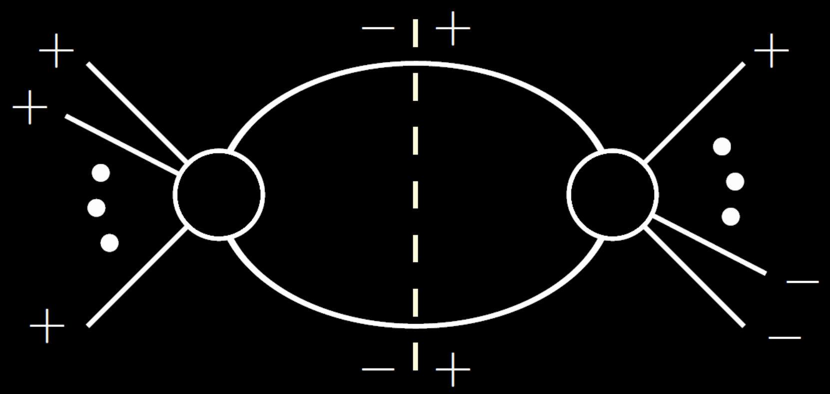
Proving the conjecture

$$A_n^{\mathrm{QCD}}\Big|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0} = A_n^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Big|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0}$$

- MHV x MHV cut similar to massless (4D) cuts

$$p_\chi \cdot q_{rs} = 0$$

$$= \frac{\langle ij\rangle^4}{2\mu^4} A_n^{\mathrm{QCD}}\Big|_{q_{rs}\ \mathrm{cut}} \qquad \square$$

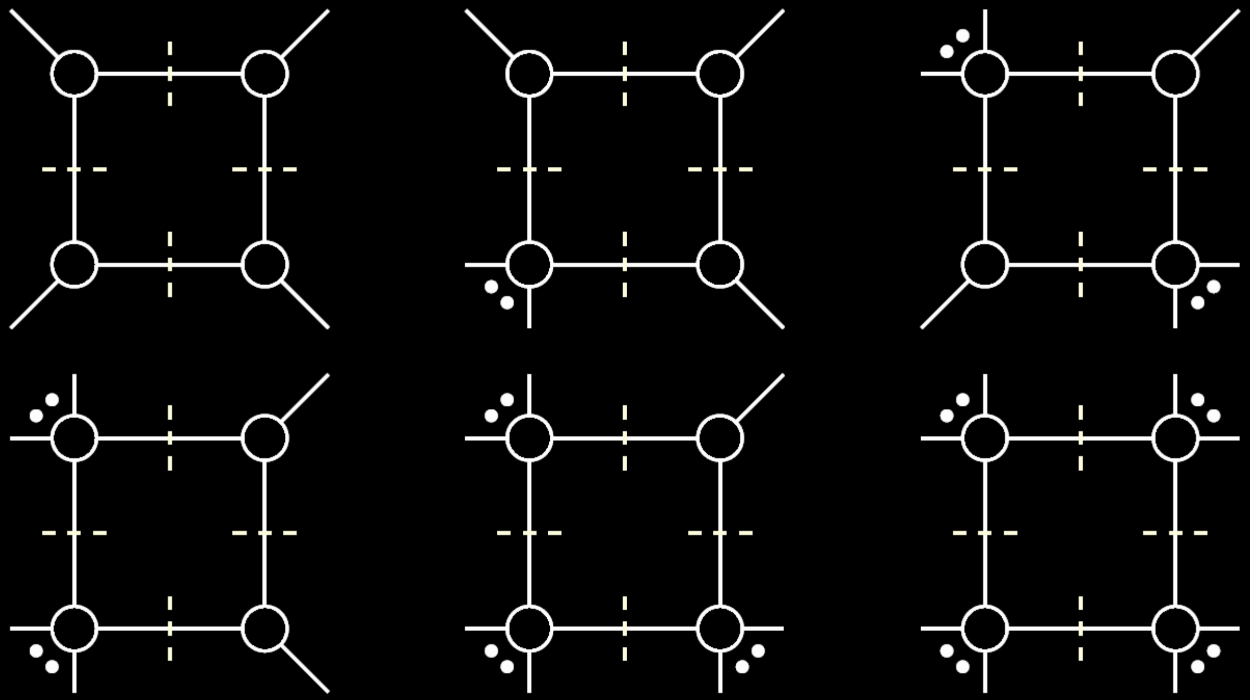


c.

Closed forms at all-multiplicity

- Boxes:
  - Can simply use 4D cuts on N=4 side
  - All-plus side: Two-mass easy from UV truncation

$$\frac{1}{\mu^4}\left[2\times\frac{1}{2}\sum_{\alpha_\pm} A^{\text{tree}}\times A^{\text{tree}}\times A^{\text{tree}}\times A^{\text{tree}}\Big|_{\mathcal{O}(\mu^6)}\right]$$



$\sim \mu^4$

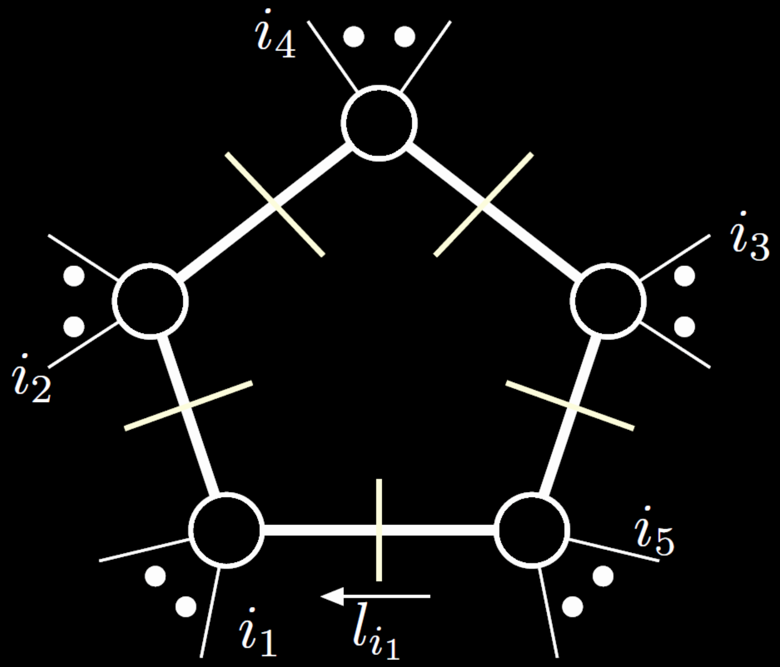
$\sim \mathcal{O}(\mu^6)$

$$=\frac{1}{2}\frac{\text{tr}(i_1 q_{i_1 i_3} i_3 q_{i_3+1,i_1-1})}{\langle 12 \ldots n1\rangle}$$

Closed forms at all-multiplicity

- Pentagons:
  - Generalised D-dimensional cuts
  - Solution very simple, general for all-massive case

[GRJ '20]



$$l_{i_1}^{\mu} = -\frac{\mathrm{tr}_5\left(q_{i_1 i_2} q_{i_2 i_3} q_{i_3 i_4} q_{i_4 i_5} q_{i_5 i_1} \gamma_\mu\right)}{2\,\mathrm{tr}_5\left(q_{i_1 i_2} q_{i_2 i_3} q_{i_3 i_4} q_{i_4 i_5}\right)}$$

$$A_n^{\mathrm{QCD}} = \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \frac{\mathrm{tr}_-(i_1 i_2 i_3 i_4)}{\langle 12 \ldots n1 \rangle} + \mathcal{O}(\epsilon)$$

$$\mathrm{tr}_- = \frac{1}{2}\left(\mathrm{tr} - \mathrm{tr}_5\right)$$

NON TRIVIAL CHECK
Verified up to n=17

Summary

- Dimension shift conjecture proven

- All-epsilon, all-n forms computed

- Very different cut constructions for each
  - Two-mass-easy boxes: UV vs helicity

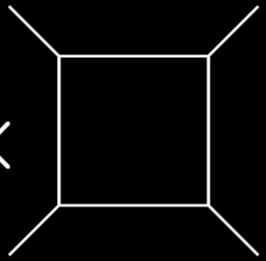
- (Purity of MHV in N=4 not all-epsilon (work in progress))

MHV in N=4

$$A_n^{\mathcal{N}=4}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

- Uniformly Transcendental amplitude (UT)

$$A_4^{\text{MHV}} = A_4^{\text{tree}} \times st \times$$



$$= 2A_4^{\text{tree}} \times \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\log(\text{-}s) + \log(\text{-}t)\right) + \ldots\right]$$

- Stronger property: Purity
  - Integral x leading singularity
  - MPLs have no kinematic coefficients

[Cachazo ‘08]

[Arkani-Hamed, Bourjaily, Cachazo, Trnka ‘12]

[Henn ‘13]

MHV in N=4

$$A_n^{\mathcal{N}=4}(1^+, ..., i^-, ..., j^-, ..., n^+)$$

- Uniformly Transcendental amplitude (UT)
  - All-epsilon was computed up to n=6 by BDDK

$$A_6^{\text{MHV;1-loop}} = \frac{1}{4} A^{\text{tree}} \Big[- \sum_{1<j_1<j_2\leq 6} \text{tr}\big((j_1+1) q_{j_1+1,j_2+1} (j_2+1) q_{j_2+1,j_1+1}\big) I_4^{4-2\epsilon,(j_1,j_2)} - 2\epsilon \left(\sum_{j=1}^{6} \text{tr}_5(j+1, j+2, j+3, j+4) I_5^{6-2\epsilon,(j)} + \text{tr}(123456) I_6^{6-2\epsilon} \right) \Big]$$

Higher-in-epsilon structure

NOT PURE n>5

✓ $\sqrt{\Delta_5}$

✗ $\text{tr}_5(123456)$