

# Third order corrections to the semi-leptonic $b \rightarrow c$ and the muon decays

RADCOR and Loopfest 2021

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TTP KARLSRUHE

[based on: Fael, Schönwald, Steinhauser (arxiv:2011:13654)]



Institute for Theoretical Particle Physics

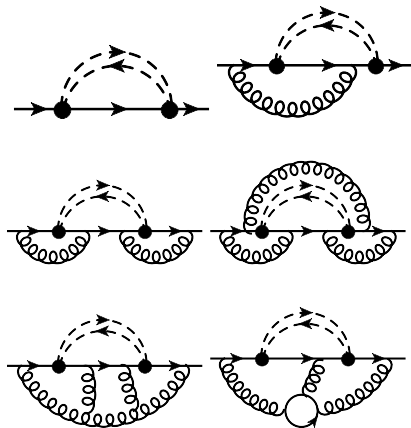


TRR 257 - Particle Physics Phenomenology  
after the Higgs Discovery

- 1 Motivation
- 2 Calculation
- 3 Results
- 4 Conclusions



- We calculate the inclusive decay rate to third order via the optical theorem, i.e. we consider the imaginary part of 5-loop forward scattering diagrams.
- We consider massless leptons, i.e. we have two dimensionful scales, the bottom mass  $m_b$  and the charm mass  $m_c$ .
- Analytical dependence on charm and bottom mass seems out of reach:  
 ⇒ consider expansion in mass difference



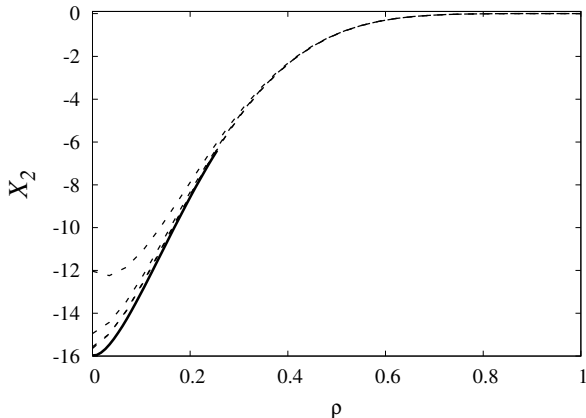
# The Heavy-Daughter Expansion

- Perform the expansion in the limit  $m_c \sim m_b$ :  $\delta = 1 - \rho = 1 - \frac{m_c}{m_b} \ll 1$
- Limit has been shown to converge well down to  $m_c/m_b \rightarrow 0$  at 2-loop order.

[Czarnecki, Dowling, Piclum (Phys. Rev. D 78 (2008))]

$$\Gamma(b \rightarrow c l \nu) = \Gamma_0 \left[ X_0 + C_F \sum_{n \geq 1} \left( \frac{\alpha_s}{\pi} \right)^n X_n \right]$$

with  $\Gamma_0 = G_F^2 m_b^2 |V_{cb}|^2 / 192 \pi^3$

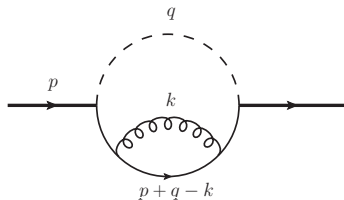


$$\delta = 1 - \frac{m_c}{m_b}$$

- We use the method of regions to perform the expansion. [Beneke, Smirnov (Nucl. Phys. B (1998))]
- Loop momenta can either scale hard  $k_i \sim m_b$  or ultra-soft  $k_i \sim \delta m_b$  (regions have been cross-checked with Asy). [Pak, Smirnov (Eur. Phys. J. C (2011))]
- The momentum of the electron-neutrino loop can be integrated trivially.
- The momentum of the lepton  $q$  has to scale ultra-soft, otherwise no imaginary part is generated. This reduces the number of regions to be considered.
- After the asymptotic expansion the integrals over the QCD loops ( $k_1, k_2, k_3$ ) have a definitive scaling in  $2p \cdot q + 2m_b^2 \delta$ :  
⇒ factorize out this dependence and perform the 1-loop tensor integral over  $q$  first.
- We are left with 3-loop integrals with integer powers in the end.

# Asymptotic Expansion – Example

Look at the 1-loop integral (we already integrated out the electron-neutrino loop):



$$\sim \int \frac{dq dk}{[q^2]^\alpha [(p+q)^2 - m_c^2]^2 [k^2] [(p+q+k)^2 - m_c^2]}$$

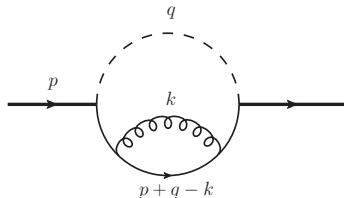
- case 1:  $q$  has to be ultra-soft,  $k$  is hard;

$$\rightarrow \int \frac{dq}{[q^2]^\alpha [2p \cdot q + 2m_b^2 \delta]^2} \times \int \frac{dk}{[k^2] [(k+p)^2 - m_b^2]}$$

- We see an explicit factorization.

# Asymptotic Expansion – Example

Look at the 1-loop integral (we already integrated out the electron-neutrino loop):



$$\sim \int \frac{dq dk}{[q^2]^\alpha [(p+q)^2 - m_c^2]^2 [k^2] [(p+q+k)^2 - m_c^2]}$$

- case 2:  $q$  and  $k$  are ultra-soft;

$$\begin{aligned} &\rightarrow \int \frac{dq dk}{[q^2]^\alpha [2p \cdot q + 2m_b^2 \delta]^2 [k^2] [2p \cdot k + \underbrace{2p \cdot q + 2m_b^2 \delta}_{\text{fixed combination}}]} \\ &= \int \frac{dq}{[q^2]^\alpha [2p \cdot q + 2m_b^2 \delta]^\beta} \times \int \frac{dk}{[k^2] [2p \cdot k + 1]} \end{aligned}$$

- Integrals can be factorized through definite power counting in the asymptotic expansion.



- We can always perform the integrations over the electron-neutrino loop and lepton momenta analytically via 1-loop tensor reduction. The remaining loop integration have the following scalings:

	scaling	n. regions
$\mathcal{O}(\alpha_s)$	h, u	2
$\mathcal{O}(\alpha_s^2)$	hh, hu, uu	4
$\mathcal{O}(\alpha_s^3)$	hhh, huu, hhu, uuu	8

- In case a single region with either hard or ultra-soft scaling remains we can also integrate it out analytically.
- The remaining two- or three-loop integrals have integer powers of the propagators and can be reduced to master integrals via IBP reduction.
- Since we expand up to  $\mathcal{O}(\delta^{12})$  we have to reduce about 25M three-loop integrals with positive and negative indices up to 12. We used FIRE together with LiteRed for this task.

[\[Smirnov,Chuharev \(2020\), Lee \(2013\)\]](#)

Different kinds of master integrals appear in hard or ultra-soft regions:

- hard regions: up to three-loop on-shell master integrals.  
[Melnikov, van Ritbergen (Nucl. Phys. B (2000) ; Lee, Smirnov (JHEP (2011))]
- soft regions: three-loop ultra-soft master integrals with eikonal propagators
  - up to 2-loop integrals expressible in terms of  $\Gamma$ -functions
  - 3-loop integrals computed for the  $m^{\text{OS}} - m^{\text{kin}}$  relation at  $O(\alpha_s^3)$  [see talk by Matteo Fael].  
[Fael, KS, Steinhauser (2020); hep-ph/2011.11655]

Renormalization:

- For the renormalization of the decay width the wave function and mass renormalization constants with two massive quarks need to be known in the expansion  $m_c \sim m_b$  up to  $O(\alpha_s^3)$ .
- Previously they were only known in the expansion  $m_c \ll m_b$  and numerically for larger values of  $m_c$ . [Bekavaz, Grozin, Seidel, Steinhauser (JHEP (2007))]
- We computed them analytically and expanded around the equal mass limit to obtain the needed quantities. [see talk by Matthias Steinhauser]

$$\Gamma(b \rightarrow c l \nu) = \Gamma_0 \left[ X_0 + C_F \sum_{n \geq 1} \left( \frac{\alpha_S}{\pi} \right)^n X_n \right], \quad X_n = \sum_{j=5}^{\infty} \delta^j X_{n,j}$$

$$C_F X_3 = \delta^5 \left[ \frac{533858}{1215} - \frac{20992 a_4}{81} + \frac{8744 \pi^2 \zeta_3}{135} - \frac{6176 \zeta_5}{27} - \frac{16376 \zeta_3}{135} - \frac{2624 l_2^4}{243} + \frac{5344 \pi^2 l_2^2}{1215} \right. \\ \left. + \frac{179552 \pi^2 l_2}{405} - \frac{39776 \pi^4}{6075} - \frac{1216402 \pi^2}{3645} \right] + \mathcal{O}(\delta^6),$$

with  $l_2 = \ln(2)$ ,  $a_4 = \text{Li}_4(1/2)$  and  $\zeta_i = \sum_{j=1}^{\infty} 1/j^i$ .

- We have calculated the expansion up to  $\delta^{12}$  (for general color factors).
- A subset of color factors has been independently been computed up to  $\delta^9$ . [Czaron, Czarnecki, Dowling (2021)]

- We see a good convergence at the physical point of  $\rho = m_c/m_b \approx 0.28$ .

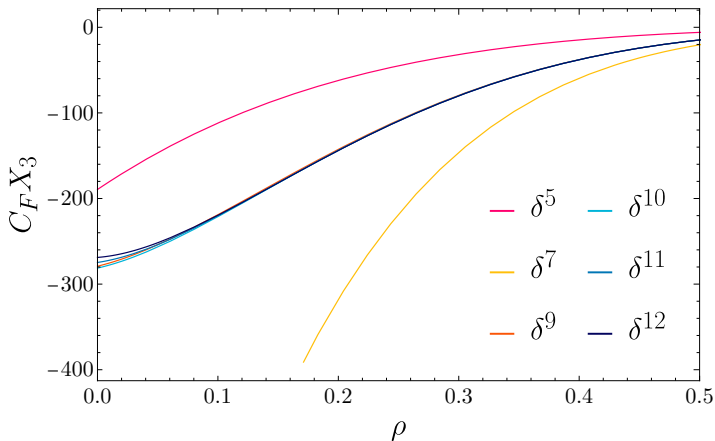
- We find:

$$X_3(\rho = 0.28) = -68.4 \pm 0.3$$

- We use the difference of the last two expansion terms to estimate the uncertainty.

- For  $\rho \rightarrow 0$  we can extract values for  $b \rightarrow ul\nu$ :

$$X_3^u = -202 \pm 20$$



# Convergence – Muon Decays

- Specifying the color factor to QED and setting  $\rho = m_e/m_\mu \approx 0$  we get the 3-loop contributions to the muon decay.

- We find:

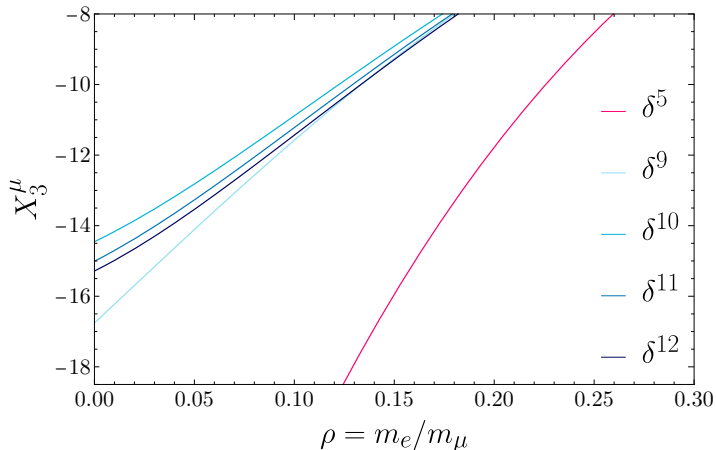
$$X_3^\mu = -15.3 \pm 2.3$$

- This leads to the shift:

$$\Delta\tau_\mu = (-9 \pm 1) \cdot 10^{-8} \mu\text{s}$$

- The current experimental value reads:

$$\tau_\mu = (2.1969811 \pm 0.0000022) \mu\text{s}$$



- The total decay rate of quarks expressed in terms of on-shell masses converges poorly:

$$\Gamma_{sl} \sim 1 - 1.72 \frac{\alpha_s(m_b)}{\pi} - 13.1 \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 - 163 \left( \frac{\alpha_s(m_b)}{\pi} \right)^3$$

- Also the  $\overline{\text{MS}}$  scheme usually behaves poorly, since the scale has to be chosen rather low.
- Different threshold masses like the **PS** [Beneke (1998)] , **1S** [Hoang, Ligeti, Manohar (1998)] or **kinetic mass** [Bigi, Shifman, Uraltsev, Vainshtein (1996)] have been proposed to improve the convergence.
- We see a much better behavior in the convergence for the schemes used for the global fits of inclusive quantities.
- E.g. for the kinetic mass:

$$m_b^{\text{kin}}, m_c^{\text{kin}} : \quad \Gamma(b \rightarrow c l \nu) / \Gamma_0 = 0.633 (1 - 0.066 - 0.018 - 0.007) \approx 0.575$$

$$m_b^{\text{kin}}, \overline{m}_c(3 \text{ GeV}) : \quad \Gamma(b \rightarrow c l \nu) / \Gamma_0 = 0.700 (1 - 0.116 - 0.035 - 0.010) \approx 0.587$$

$$m_b^{\text{kin}}, \overline{m}_c(2 \text{ GeV}) : \quad \Gamma(b \rightarrow c l \nu) / \Gamma_0 = 0.648 (1 - 0.087 - 0.018 - 0.0003) \approx 0.580$$

## BLM and non-BLM part

$$\Gamma(b \rightarrow cl\nu) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} X_0 \left[ 1 + C_F \sum_{n \geq 1} \left( \frac{\alpha_s}{\pi} \right)^n Y_n \right]$$

$$Y_2 = \beta_0 Y_2^{\beta_0} + Y_2^{\text{rem}}$$

$$Y_3 = \beta_0^2 Y_3^{\beta_0} + Y_3^{\text{rem}}$$

	$Y_1$	$Y_2^{\text{rem}}$	$\beta_0 Y_2^{\beta_0}$	$Y_3^{\text{rem}}$	$\beta_0^2 Y_3^{\beta_0}$
$m_b^{\text{OS}}, m_c^{\text{OS}}$	-1.72	3.08	-16.17	48.8	-212.1
$m_b^{\text{kin}}, m_c^{\text{kin}}$	-0.94	0.33	-4.08	-5.4	-15.4
$m_b^{\text{kin}}, \bar{m}_c(3 \text{ GeV})$	-1.67	-3.39	-3.85	-97.7	69.1
$m_b^{\text{kin}}, \bar{m}_c(2 \text{ GeV})$	-1.25	-1.21	-2.43	-68.8	67.9
$\bar{m}_b(\bar{m}_b), \bar{m}_c(3 \text{ GeV})$	3.07	-21.81	35.17	-56.7	119.4
$m_b^{\text{PS}}, \bar{m}_c(2 \text{ GeV})$	-0.47	-6.10	-2.31	-93.1	-7.19
$m_b^{1\text{S}}, \bar{m}_c(2 \text{ GeV})$	-3.59	-0.98	-19.39	-39.83	-80.22
$m_b^{1\text{S}}, m_c$ via HQET	-1.38	0.73	-7.05	5.04	-38.09

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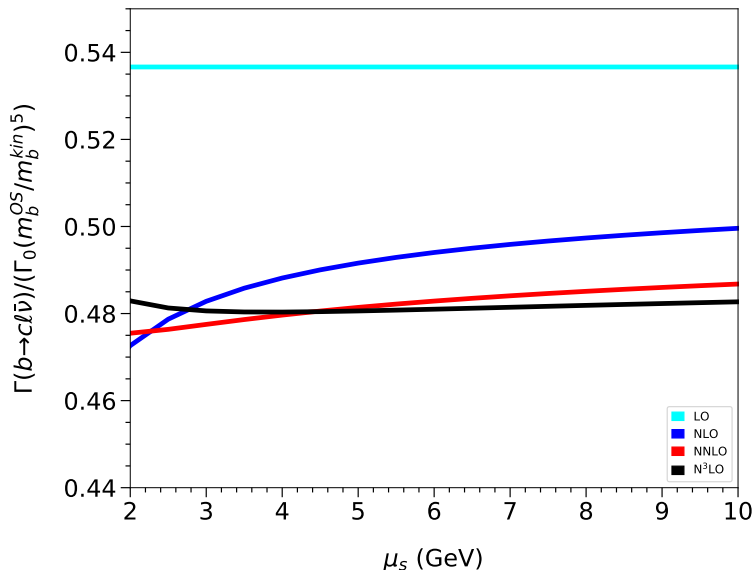
$$Y_3 = \beta_0^2 Y_3^{\beta_0} + Y_3^{\text{rem}}$$

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$m_b^{\text{1S}}, m_c$ via HQET	-1.38	0.73	-7.05	5.04	-38.09



# Different Renormalization Schemes – kinetic mass

- $m_b$  is expressed in the kinetic scheme.
- $m_c$  is expressed in the  $\overline{\text{MS}}$  scheme at 2 GeV.

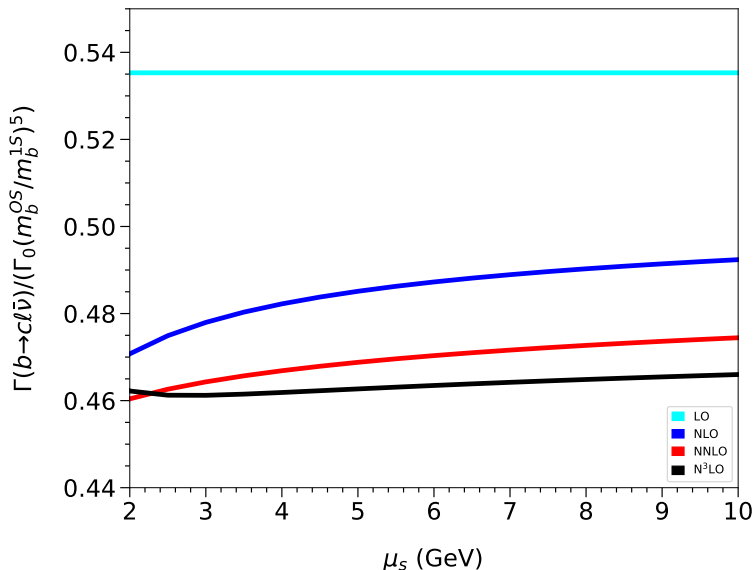


# Different Renormalization Schemes – 1S mass

■  $m_b$  is expressed in the 1S scheme.

■  $m_c$  is expressed through meson masses via the HQET relation:

$$m_b^{\text{OS}} - m_c^{\text{OS}} = \bar{m}_B - \bar{m}_D + \dots$$

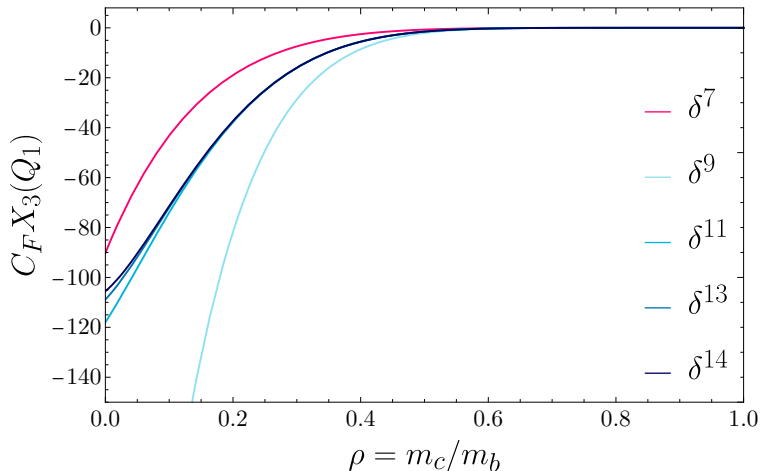


# Moments of Differential Distributions

- The method can be used to calculate inclusive moments of differential distributions.
- For example we can calculate- $q^2$  moments:

$$Q_n = \frac{1}{\Gamma_0} \int dq^2 \left( \frac{q^2}{m_b^2} \right)^n \frac{d\Gamma}{dq^2}$$

Preliminary



## Conclusions

- We have computed the  $\alpha_s^3$  corrections to the width of  $b \rightarrow c\ell\nu$ .
- We performed an expansion in the limit  $1 - m_c/m_b \ll 1$  and demonstrated its good convergence.
- The result is one of the few third order corrections involving two mass scales.
- The results are also relevant for  $b \rightarrow u\ell\nu$  and the muon decay.

## Outlook

- The method of calculation can be applied for the calculation of moments of the differential distributions.

# Backup



- Denominator structures:

- solid lines:  $1/(2k_i \cdot p + 1)$ , with  $p^2 = 1$
- double lines:  $1/(2k_i \cdot p)$ , with  $p^2 = 1$
- dotted lines:  $1/k_i^2$

- Analytically calculated via direct integration, symbolic summation and differential equations using the packages Sigma [Schneider (2007-)] and HarmonicSums [Ablinger et al (2011-)], verified with PSLQ.

- Only  $\zeta$  values appear in the results.

- Already needed for the calculation of the relation between the OS and kinetic mass (although to lower order in  $\epsilon$ ).

