Prospects for numerical calculation of massive multi-loop Feynman integrals using Mellin-Barnes representations

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Introduction

- The standard model is basically right and precision tests agree largely with it.
- If there is new physics, it must be hidden
- Precision = Discovery

Collider Physics at the Precision Frontier G. Heinrich, 2009.00516

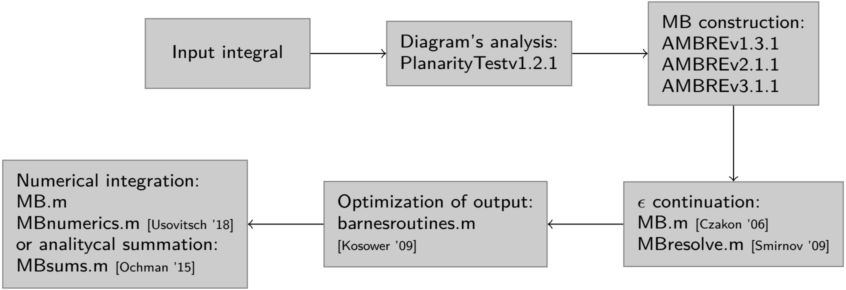
	analytic	numerical
pole cancellation	exact	with numerical uncertainty
control of integrable singularities	analytic continuation	less straightforward
fast evaluation	yes	depends
extension to more scales/loops	difficult	promising
automation	difficult	less difficult

"*New directions* in science are launched by new tools much more often than by new concepts."

F. Dyson

Computation of Feynman integrals with Mellin-Barnes (MB) method

Operational sequence of the MB-suite:



Construction of MB representations

- Feynman parametrization(textbook knowledge)

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j-1}\delta(1-\sum_{i=1}^{N} x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

- Mellin-Barnes master formula

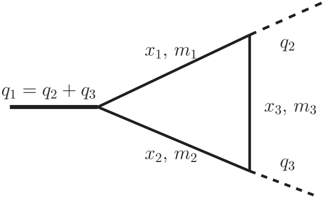
$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i}\; A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\;\Gamma(\lambda+z_1+\ldots+z_{n-1})$$

- integration over Feynman parameters

$$\int_0^1 \prod_{i=1}^{N} dx_i \; x_i^{n_i-1}\; \delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1+\ldots+n_N\right)}$$

- 1st and 2nd Barnes lemmas: analytical integration over some z_i if applicable

"One-loop" example:



$$U = x_1 + x_2 + x_3 \equiv 1$$
$$F_0 = -(q_2+q_3)^2 x_1 x_2 - q_2^2 x_1 x_3 - q_3^2 x_2 x_3$$
$$F = F_0 + U(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2)$$

$$G(X) \sim \int dz_1 dz_2 dz_3\, (-s x_1 x_2)^{z_1} (-q_2^2 x_1 x_3)^{z_2} (-q_3^2 x_2 x_3)^{z_3} \times \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)^{-z_1-z_2-z_3-N_\nu+d/2}$$

Beyond one-loop:

- $U(\vec{x}) \neq 1$
- complexity/dimensionality starts to depend on $U(\vec{x})$ structure
- nontrivial simplification of graph polynomials is needed

$x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$	3-dim representation
$(x_1 + x_2)(x_3 + x_4)$	2-dim representation
$(x_1 + x_2)(x_3 + x_4) \to$ $[x_1 \to v_1\xi_{11}, x_2 \to v_1\xi_{12}, \delta(1 - \xi_{11} - \xi_{12});$ $x_3 \to v_2\xi_{21}, \ldots] \to v_1v_2$	0-dim representation
$(x_1 + x_2)(x_3 + x_4)$ + BL	0-dim representation $^{*)}$

$*)\ (x_1 + x_2)^p \to \int dz_1 x_1^{z_1} x_2^{p-z_1} \Gamma(-z_1)\Gamma(-p + z_1)$
$\to \int dz_1 \Gamma(-z_1)\Gamma(-p + z_1)\Gamma(z_1 + 1)\Gamma(p - z_1 + 1)/\Gamma(p + 2)$

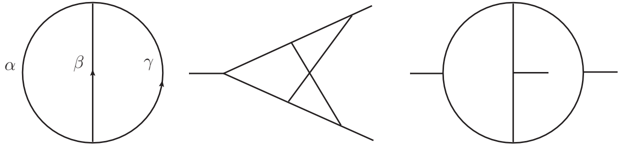
BL can be also applied without factorization, but this requires special transformation of z_i variables, see e.g., barnesroutines.m [D. Kosower, 2009]

$$\int_{-i\infty}^{i\infty} dz \Gamma(a + z)\Gamma(b + z)\Gamma(c - z)\Gamma(d - z) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

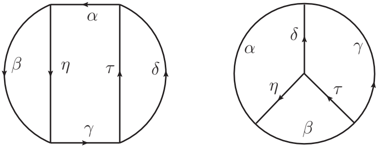
`AMBREv3.m`:

- topology based factorization - chain diagrams, Kinoshita '74

2-loop:



3-loop:



transformation/rescaling of Feynman parameters:

$$\{\vec{x}\}_i : \;\; x_k \to v_i \xi_{ik} \;\; \times \delta\left(1 - \sum_{k=1}^{\eta_i} \xi_{ik}\right),$$

where i denotes chain index and $k \in [1, \eta_i]$, with η_i - number of propagators in chain. δ-function keeps number of variables unchanged.

For any 2-loop diagram:

$$U_{\text{2-loop}} = v_1 v_2 + v_2 v_3 + v_1 v_3$$

For any "ladder" 3-loop diagram:

$$U_{\text{3-loop(I)}} = v_1 v_2 v_3 + v_1 v_2 v_4 + v_2 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_1 v_4 v_5 + v_3 v_4 v_5$$

For any "mercedes" 3-loop diagram:

$$U_{\text{3-loop(II)}} = v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_2 v_4 v_5 + v_3 v_4 v_5 \\ + v_1 v_2 v_6 + v_2 v_3 v_6 + v_1 v_4 v_6 + v_2 v_4 v_6 + v_3 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6$$

Cheng–Wu theorem: Delta function in the feyman parameters representation can be replaced by $\delta\left(\sum_{i\in\Omega} x_i - 1\right)$ where Ω is an arbitrary subset of the lines $1,\ldots,L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.

- 2-loop: $\delta(1-v_1-v_2)$, $U(\vec{v}) = v_3 + v_1 v_2$
  no additional MB integrations from U, similar to 1-loop cases
- 3-loop: $\delta(1-v_1-v_2-v_3)$
  - "ladder" - 2 additional MB integrations
  - "mercedes" - 4 additional MB integrations

To get minimal dimensionality:

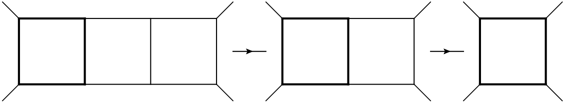
- 1-loop: $U(\vec{x}) \equiv 1$ whenever it's possible
- 2- and 3-loop: expression for F polynomial is not expanded

$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2$$

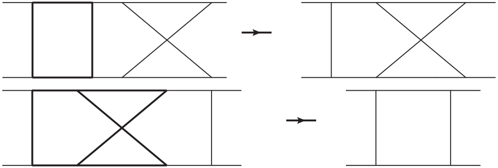
- Barnes lemmas with linear transformation of integration variables: $\vec{z}\,' = T\vec{z}$

AMBRE versions overview:

- iteratively to each subloop – loop-by-loop approach (LA): mostly for planar (AMBREv1.3.1 & AMBREv2.1.1)



- in one step to the complete U and F polynomials – global approach (GA): general (AMBREv3.1.1)
- combination of the above methods – Hybrid approach (HA) (next versions)



Examples, description, links to basic tools and literature:
http://prac.us.edu.pl/~gluza/ambre/

AMBREv3.1.1.m example:

```
invariants = {p1^2 -> 0, p2^2 -> 0, p1*p2 -> s/2};

d = 6 - 2 eps;(* by default d=4-2 eps *)

res = MBreprNP[{1}, {PR[k1, 0, n1] PR[k1 - k2, 0, n2] PR[k2, 0, n3]
PR[k1 - k2 + p1, m, n4] PR[k2 + p2, 0, n5] PR[k1 + p1 + p2, 0, n6]}, {k1, k2}]
Fauto::mode: F polynomial will be calculated in AUTO mode.
In order to use MANUAL mode execute Fauto[0].

Upoly = x[1] x[2]+x[1] x[3]+x[2] x[3]+x[1] x[4]+x[3] x[4]+x[1] x[5]+x[2] x[5]
+x[4] x[5]+x[2] x[6]+x[3] x[6]+x[4] x[6]+x[5] x[6]
Fpoly = m^2 Upoly x[4]-s x[1] x[4] x[5]-s x[1] x[2] x[6]-s x[1] x[3] x[6]
-s x[2] x[3] x[6]-s x[1] x[4] x[6]-s x[1] x[5] x[6]

{((-1)^(n1+n2+n3+n4+n5+n6) (m^2)^z1 (-s)^(6-2 eps-n1-n2-n3-n4-n5-n6-z1)
Gamma[3-eps-n3-n5] Gamma[3-eps-n2-n4-z1] Gamma[-z1] Gamma[3-eps-n1-n6-z2] Gamma[-z2]
Gamma[6-2 eps-n2-n3-n4-n5-n6-z1-z3] Gamma[6-2 eps-n1-n2-n3-n5-n6-z2-z3]
Gamma[6-2 eps-n1-n2-n3-n4-n6-z1-z2-z3] Gamma[-z3] Gamma[n2+z3] Gamma[n3+z3]
Gamma[n6+z2+z3] Gamma[-6+2 eps+n1+n2+n3+n4+n5+n6+z1+z2+z3])/(Gamma[n1] Gamma[n2]
Gamma[n3] Gamma[n4] Gamma[n5] Gamma[n6] Gamma[6-2 eps-n2-n3-n4-n5-z1]
Gamma[9-3 eps-n1-n2-n3-n4-n5-n6-z1] Gamma[6-2 eps-n1-n3-n5-n6-z2]
Gamma[6-2 eps-n1-n2-n4-n6-z1-z2])}
```

Numerical integration of MB integrals

In the most general form MB integral can be represented as follows:

$$I = \frac{1}{(2\pi i)^r} \int\limits_{-i\infty+z_{10}}^{+i\infty+z_{10}} \cdots \int\limits_{-i\infty+z_{r0}}^{+i\infty+z_{r0}} \prod_i^r dz_i \; f_S(Z) \frac{\prod\limits_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod\limits_{k=1}^{N_d} \Gamma(\Lambda_k)} f_\psi(Z)$$

$f_S(Z)$ depends on: Z – some subset of integration variables
S – kinematic parameters and masses

Λ_i : linear combinations of z_i, e.g., $\Lambda_i = \sum_l \alpha_{il} z_l + \gamma_i$

An example:

$$I^{0h0w}_{5,\epsilon^{-2}} = \frac{1}{2s} \frac{1}{2\pi i} \int\limits_{-i\infty-\frac{1}{2}}^{+i\infty-\frac{1}{2}} dz \left(\frac{M_Z^2}{-s}\right)^z \frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)}$$

Asymptotic behavior: $\Gamma(z)_{|z|\to\infty} = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\right]$

- core: ("smooth" function)

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \xrightarrow{|z|\to\infty} e^{z(\ln z - \ln(-z)) + \frac{1}{2}\ln z - \frac{5}{2}\ln(-z)}.$$

$\ln z - \ln(-z) = i\pi\,\mathrm{sign}(\Im m\, z)$

$z = z_0 + i\,t, \;\; t \in (-\infty, \infty), \; |z| \to \infty \Leftrightarrow t \to \pm\infty$

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \longrightarrow \mathbf{e}^{-\pi|\mathbf{t}|} \frac{1}{|t|^2} \;\; \text{(nice suppression?)}$$

- kinematics: (oscillations)
  in Minkowskian case $s \to s + i\delta\, (s > 0)$ $\quad \to \frac{1}{p^2 - m^2 + i\delta}$

$$\left(\frac{M_Z^2}{-s}\right)^z = e^{z\ln(-\frac{M_Z^2}{s} + i\,\delta)} \longrightarrow e^{i\,t\ln\frac{M_Z^3}{s}} \mathbf{e}^{-\pi\mathbf{t}}, s > 0$$

$e^{-\pi|t|}$ and $e^{-\pi t}$ cancel each other when $t \to -\infty$ and oscillations are **NOT** damped any more by an exponential factor

Minimal Dimensionality vs Asymptotic Behavior

Minimal Dimensionality:

$$G(X) \sim \frac{U(x)^{N_\nu - d(L+1)/2}}{\left(F_0(x) + U(x)\sum_i m_i^2 x_i\right)^{N_\nu - dL/2}} \sim \prod_i \left(m_i^2 x_i\right)^{z_i} \frac{U(x)^{N_\nu - d(L+1)/2 + \sum_i z_i}}{F_0(x)^{N_\nu - dL/2 + \sum_i z_i}}$$

- number of additional integrations (beyond massless case) = number of massive propagators i (+)
- we lost all information about the threshold behaviour (-)

Full expansion in $F(x,s)$ (two-loop case with one kinematical invariant):

$$F(x,s) = -s\, x_i x_j x_k + \left(\sum m_i^2 - s\right) x_i x_j x_k + m_i^2\, x_i x_j x_k$$

- thresholds are clearly separated, below thresholds integrals are "euclidean", expansion is also required for cases with massive external lines (e.g., QED) (+)
- integrals have higher dimensionality (-)
- for a wide class of diagrams the "euclidean" behavior (NO cancelation of exponential damping factor) and can be straightforwardly integrated (++)
- requires new factorization approach (individually for different F polynomials) (?)

Numerical integration approaches

- integration over contours parallel to imaginary axis
  - requires combination of different types of transformation to finite integration region $(-\infty, +\infty) \to [0, 1]$

$$t_i \to \ln\left(\frac{x_i}{1 - x_i}\right), \quad t_i \to \tan\left(\pi(x_i - \frac{1}{2})\right)$$

  - slow numerical convergence
  - can be improved by new integration methods/libraries (QMC(pySecDec), CUBA, Vegas II, ...)
- contours deformation (restoring of the exponential damping factor)
  - steepest descent method - $z_i = z_{i0} + f_i(t_1, ..., t_n) + it_i$ (Gluza, Jeliński, Kosower '17) only one-dimensional cases
  - rotation of integration contours - $z_i = z_{i0} + (i + \theta)t_i$ (Freitas '10)
    works well for certain integrals, but is not general
    core of MB integral becomes non-smooth

Contour shifts (MBnumerics)

Related and auxiliary Software

MBnumerics

Project: I. Dubovyk, T. Riemann, J. Usovitsch (jusovitsch,googlemail.com)
Software: Johann Usovitsch
Publications: https://doi.org/10.18452/19484 , https://doi.org/10.1016/j.cpc.2006.07.002, https://doi.org/10.1016/j.
To be cited by users in publications, for details see README_copyright in the downloaded tarball.
Features: MBnumerics is a software for evaluation of MB integrals in the Minkowski kinematics
Download: http://us.edu.pl/~gluza/ambre/packages/mbnumerics.tgz

- gives high accuracy results up to certain dimensionality of MB integrals
- can produce huge cascade of lower-dimensional integrals

http://prac.us.edu.pl/∼gluza/ambre

Needs:
1. MB.m
2. CUBA [Hahn '05, '16]
3. CERNlib

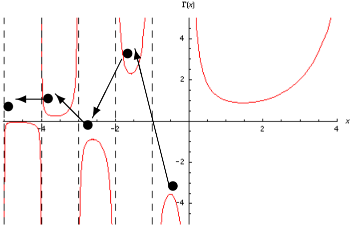
Two basic observations for shifting z follows

$$\int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + n + Im[z_k], \ldots) \qquad I_{orig}$$

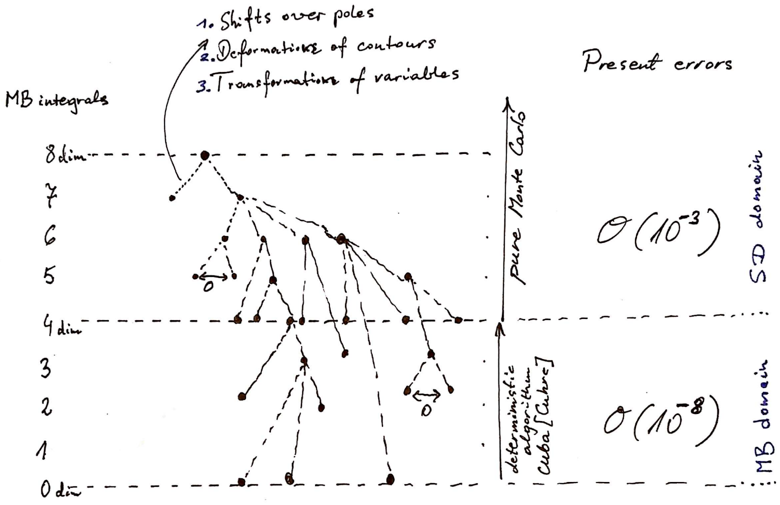
$$= \quad Residue[\int dz_1 \ldots \cancel{dz_k} \ldots I]_{Re[z_k]+n} \qquad I_{Res}$$

$$+ \quad \int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + (n+1) + Im[z_k], \ldots) \qquad I_{new}$$

1. Residues **lower** dimensionality of original MB integrals.
2. Integral after passing a pole (proper shifts) **can be made smaller**.

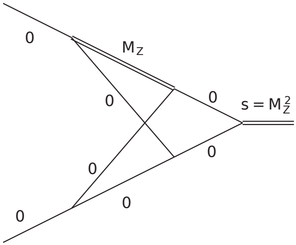


Top-bottom approach to evaluation of multidimensional MB integrals



2-loop

Bosonic corrections: $\sin^2\theta^{\rm b}_{\rm eff}$ [Dubovyk, Freitas, Gluza, Riemann, Usovitsch '16]

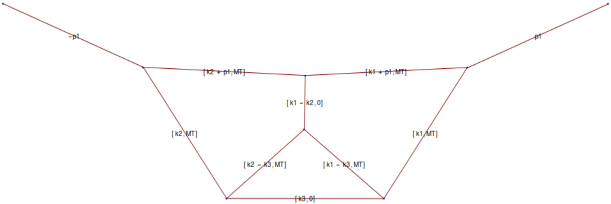


Minkowskian results (constant part, $s = (p_1 + p_2)^2 = M_Z^2 = 1$):

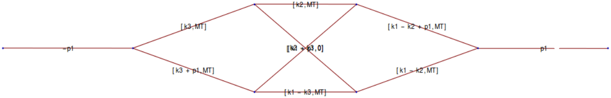
Analytical:	$-0.778599608979684 - 4.123512593396311 \cdot i$
MBnumerics$_{[3\text{-dim}]}$:	$-0.778599608324769 - 4.123512600516016 \cdot i$
MB$_{[4\text{-dim}]}$:	$-0.778524251263640 - 4.123498264231095 \cdot i$
SecDec:	big error [**2016**], $-0.77 - \mathrm{i} \cdot 4.1$ [**2017**], $-0.778 - \mathrm{i} \cdot 4.123$ [**2019**]
pySecDec+rescaling:	$-0.778598 - i \cdot 4.123512$ [**2020**]

3-loops

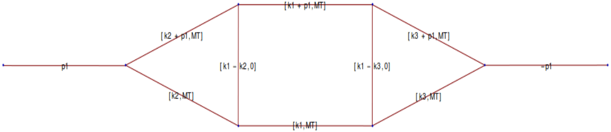
3-loop integrals for Z, W self-energy at $O(\alpha^2 \alpha_s)$
MB-method $\rightarrow$ single scale integrals with one or two equal masses



- MERC



- NP



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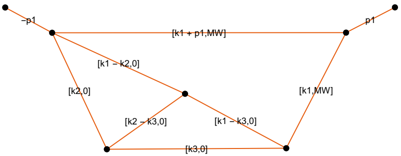

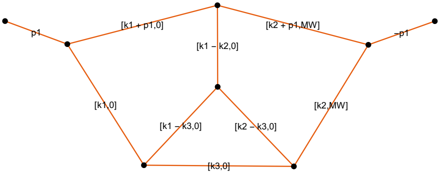

1-dim

$$-18.779406962 - 6.390785027i$$

6-dim

$$-22.5213 + 4.74442i \pm (0.001 + 0.001i)$$

$$I = -\frac{1}{(-s)^{1+3\epsilon}} \int_{-i\infty}^{+i\infty} \prod_{i=1}^{4} dz_i \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-\epsilon - z_1)\Gamma(-z_1)\Gamma(1+2\epsilon+z_1)}{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon-z_1)}$$
$$\times \frac{\Gamma(-2\epsilon - z_{12})\Gamma(1-\epsilon+z_2)\Gamma(1+z_{12})\Gamma(1+\epsilon+z_{12})\Gamma(1+3\epsilon+z_3)\Gamma(1-\epsilon-z_4)}{\Gamma(1-z_2)\Gamma(2+\epsilon+z_{12})}$$
$$\times \frac{\Gamma(-\epsilon-z_2)\Gamma(-z_2)\Gamma(1+z_3-z_4)\Gamma(-z_4)\Gamma(-z_3+z_4)\Gamma(-3\epsilon-z_3+z_4)}{\Gamma(1-4\epsilon-z_3)\Gamma(2+2\epsilon+z_3-z_4)}.$$

$$I = \frac{3}{s} \int_{-i\infty-\frac{17}{28}}^{+i\infty-\frac{17}{28}} dz_3 \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-1-z_3)\Gamma(-z_3)(\Gamma(1-z_3)\Gamma(-z_3) - \Gamma(-2z_3))}{\Gamma(1+z_3)\Gamma(-2z_3)}$$
$$\times \Gamma(1+z_3)\psi^{(2)}(1).$$

Conclusions

- Dimensionality of MB representations strongly depends on topology, number of legs and loops, internal and external masses.
- For certain classes of Feynman integrals, MB method gives very compact and well integrable representations, but in general, the method is not universal.
- Expansion in dimensional regularization parameter ϵ doesn't change the general structure of integrals (no integrable singularities).
- Linear transformations of integration variables in MB representations are a key ingredient for the simplification of MB integrals (Barnes lemmas and more).
- Proper handling of (pseudo)thresholds allows obtaining "always Euclidean" representation but at cost of higher-dimensional representation (an alternative to MBnumerics).
- Two loop applications with restrictions mentioned above are at a very satisfactory level.
- Three loop applications look promising but still need improvements, especially for non-planar cases.

Thank you!