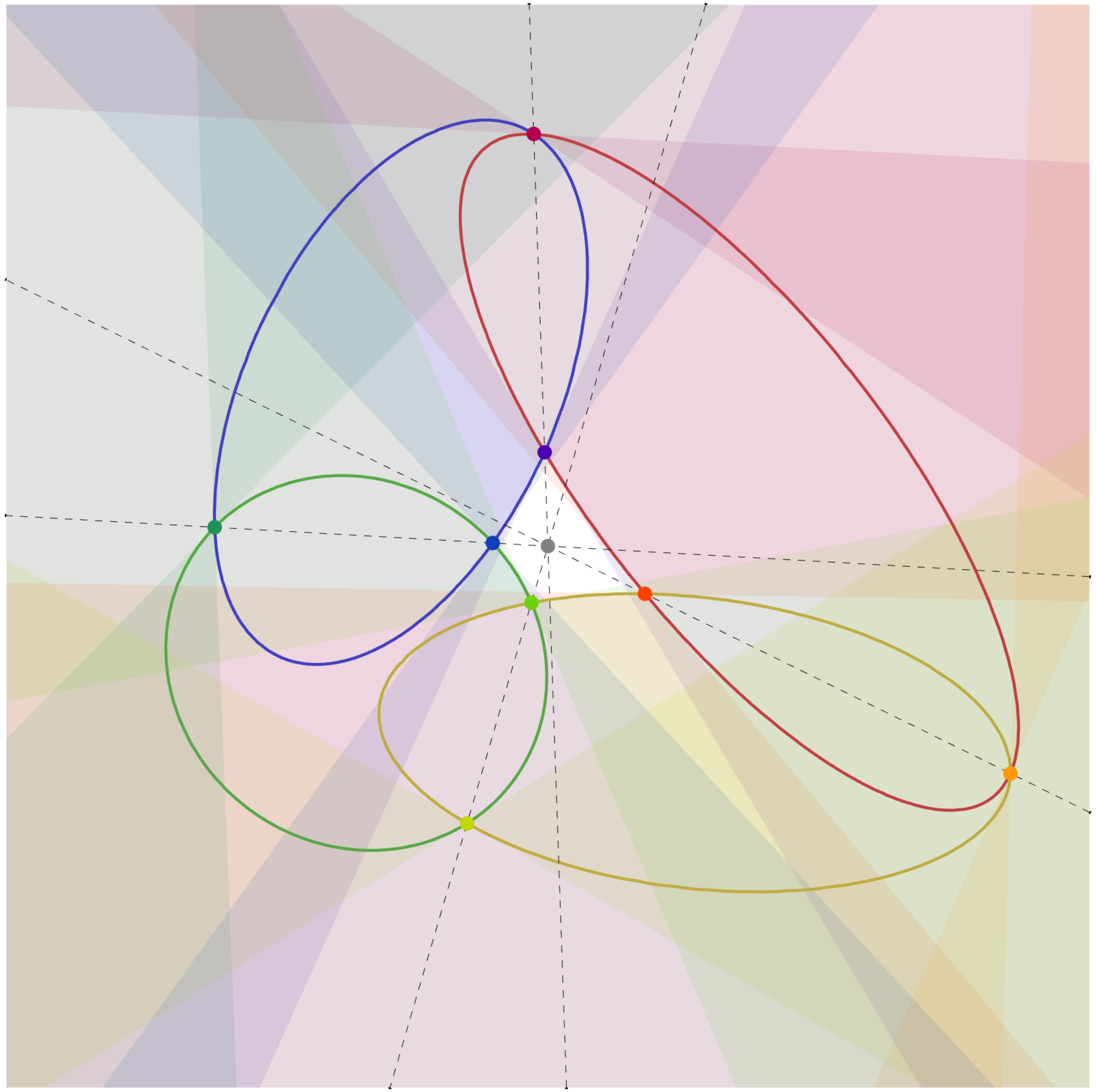
Numerical integration of loop integrals through local cancellation of threshold singularities

RADCOR-LoopFest 2021



Dario Kermanschah, 19.5.21

Perturbative QFT

Theoretical predictions for particle colliders

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Loop Integrals

Range of methods, usually increasingly difficult with number of loops, legs, scales

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renormalisation, KLN theorem

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UV: BPHZ

IR: [Anastasiou, Haindl, Sterman, Yang, Zeng: 2008.12293]

locally combining real & virtual: [Capatti, Hirschi, Pelloni, Ruijl: 2010.01068]

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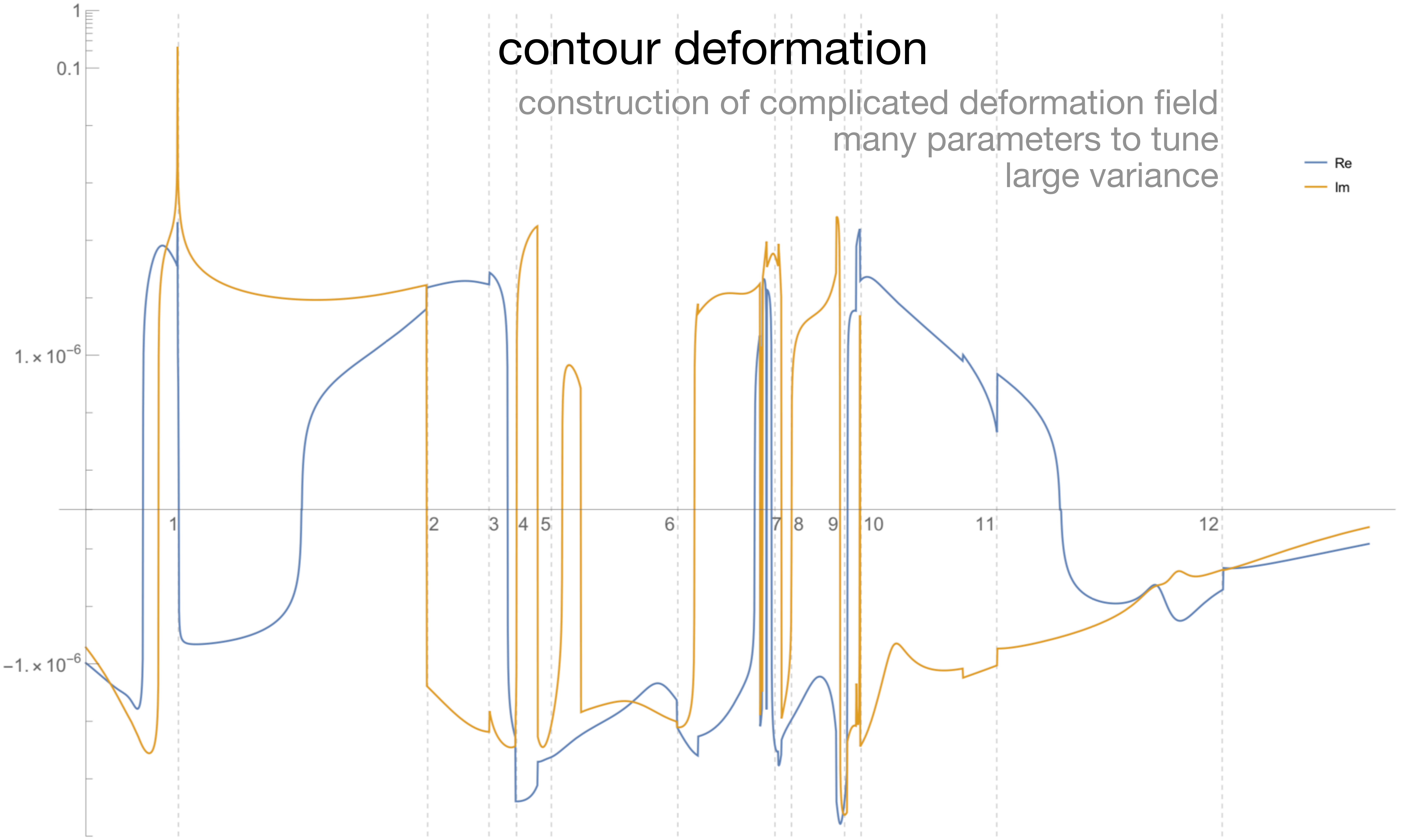

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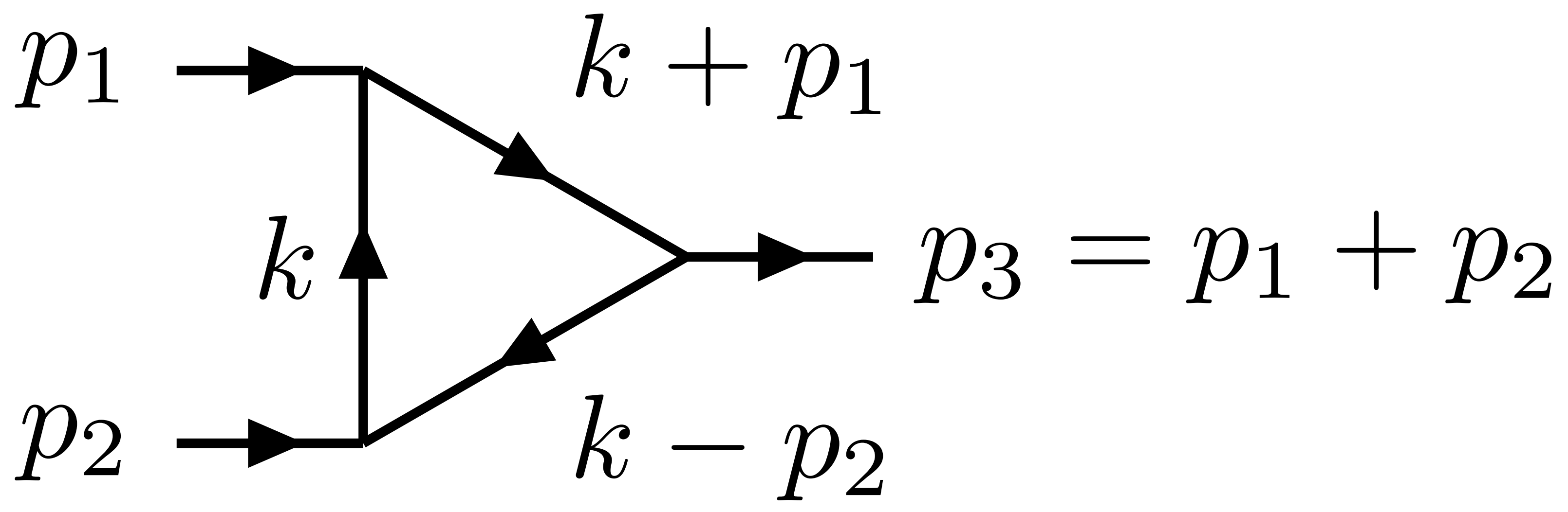
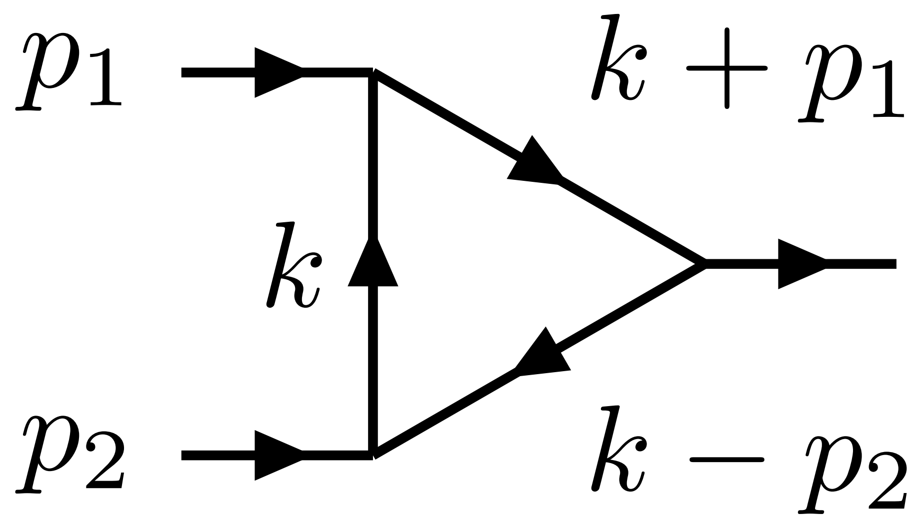
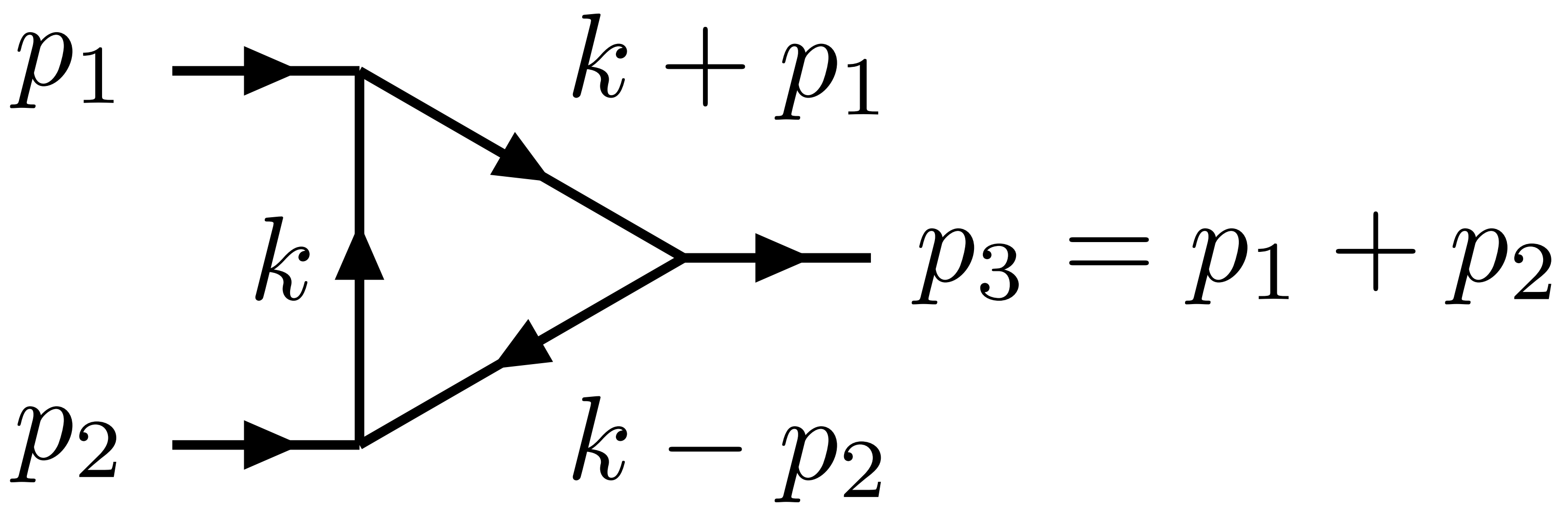

p_1
$k + p_1$
k
$p_3 = p_1 + p_2$
p_2
$k - p_2$



$$p_3 = p_1 + p_2 \quad = \lim_{\delta \to 0} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + \mathrm{i}\delta} \frac{1}{(k+p_1)^2 - m^2 + \mathrm{i}\delta} \frac{1}{(k-p_2)^2 - m^2 + \mathrm{i}\delta}$$



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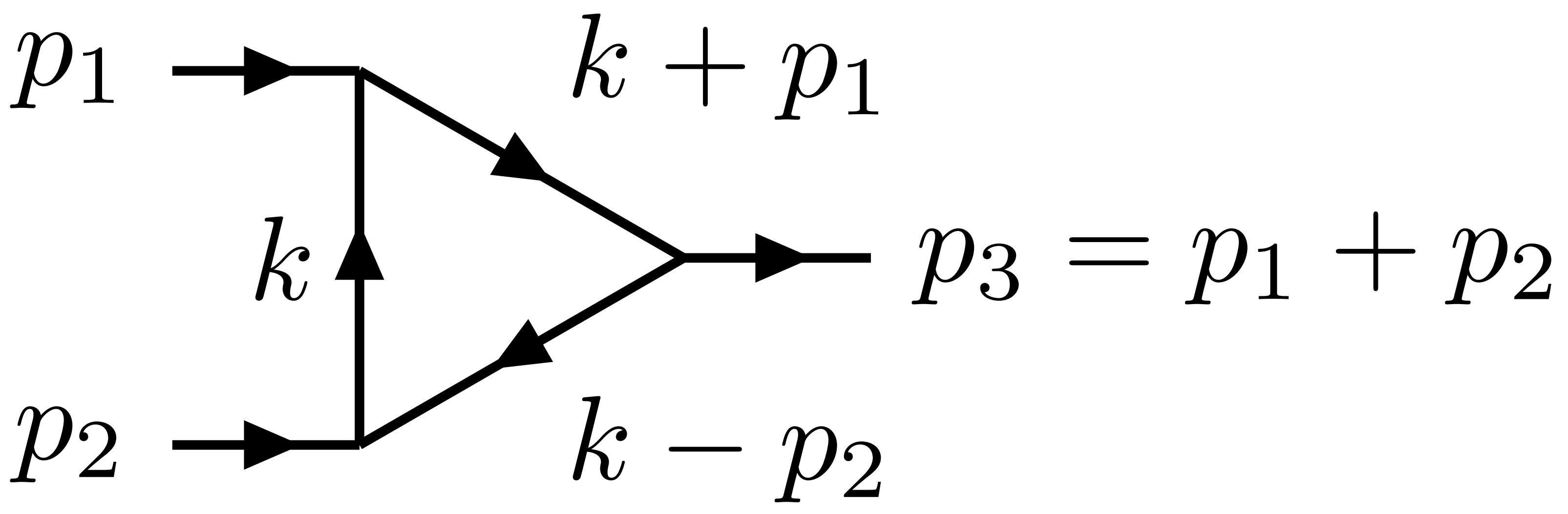

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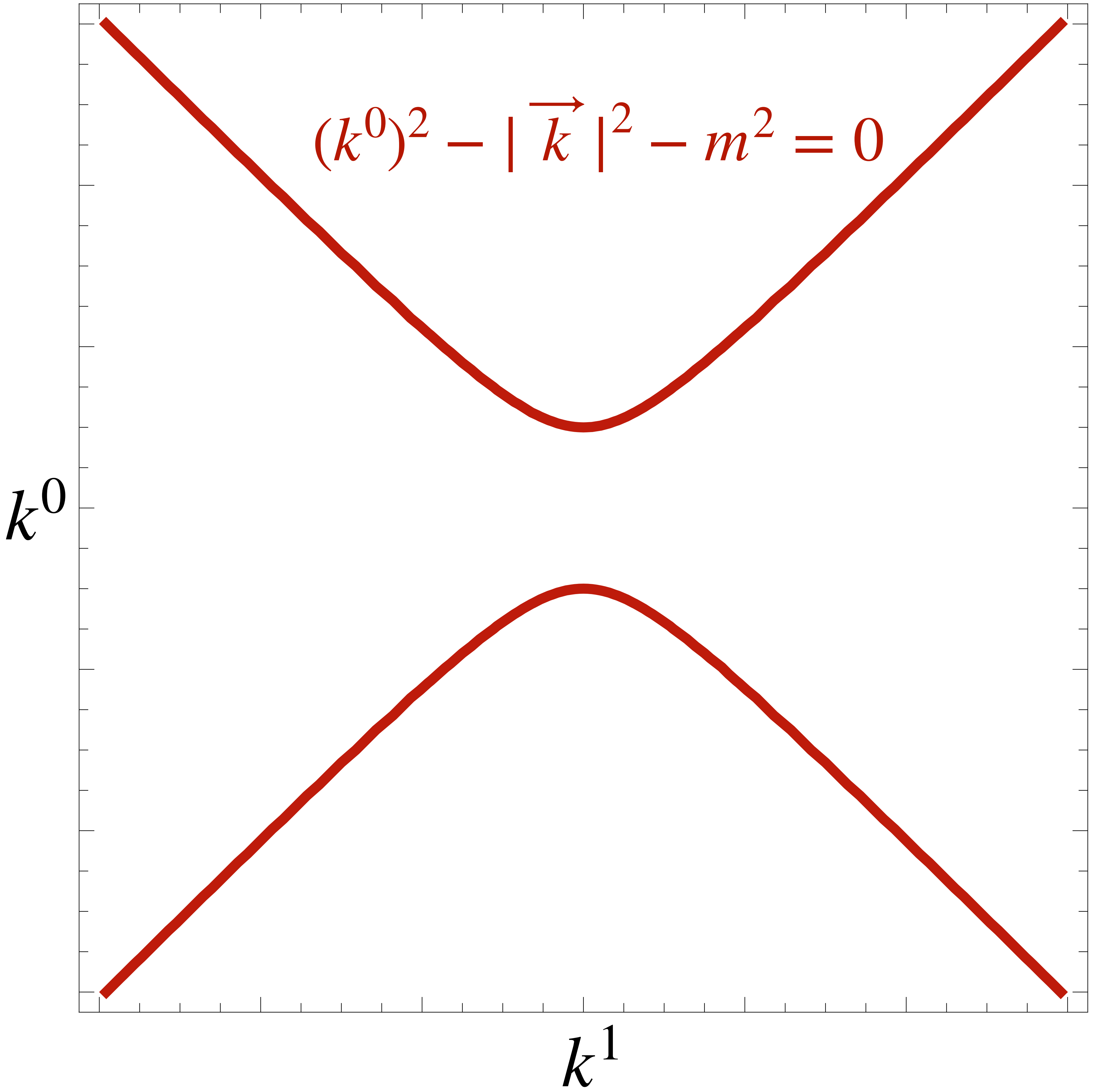

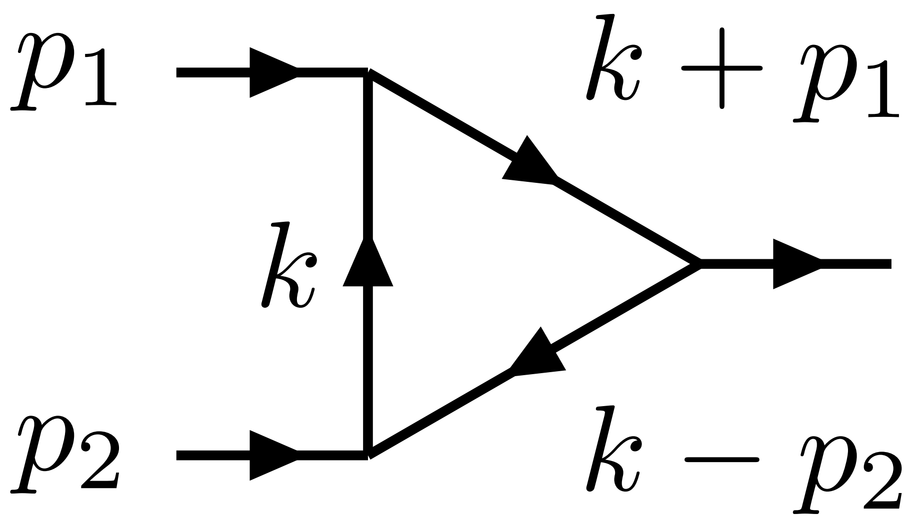

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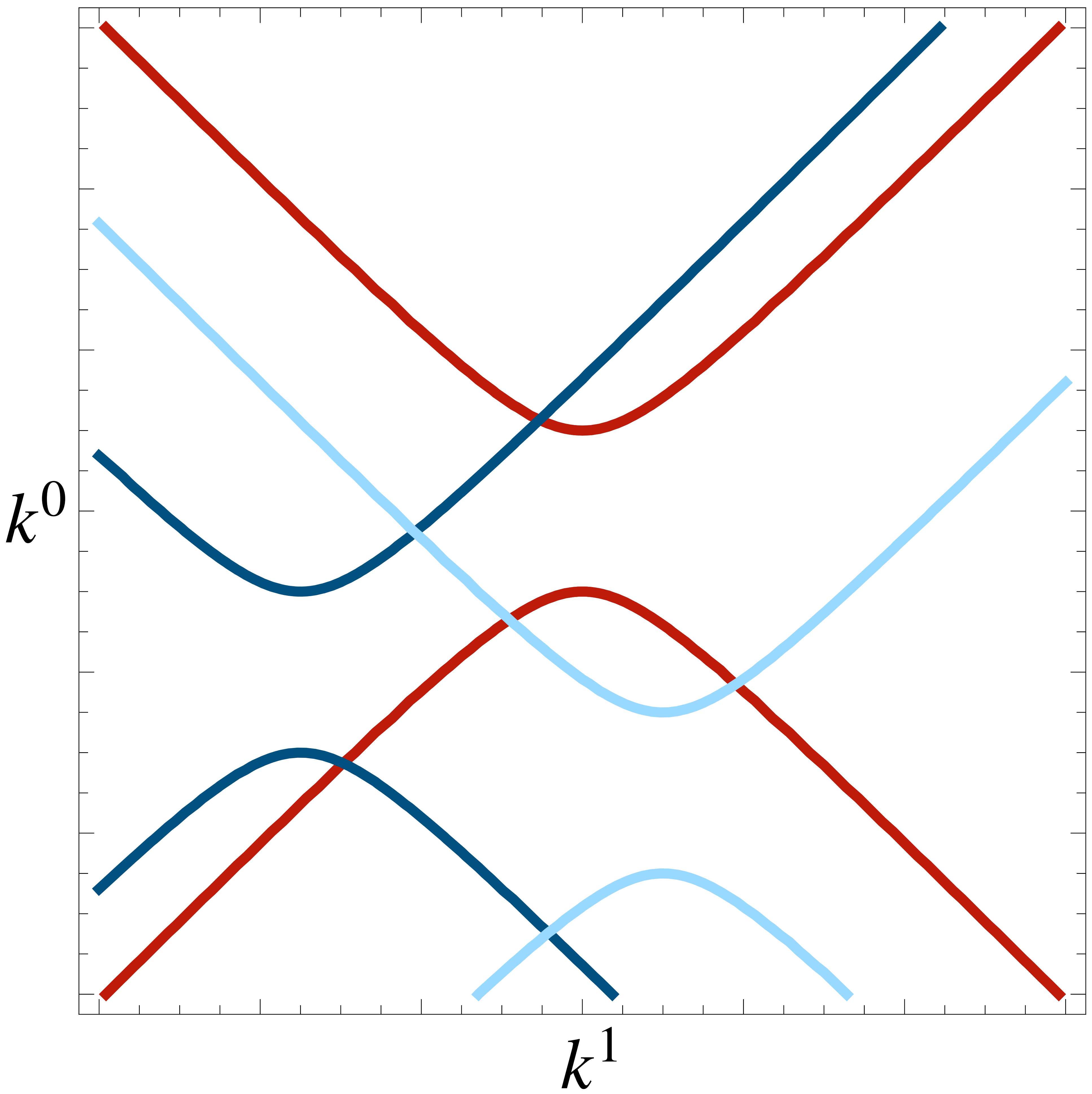

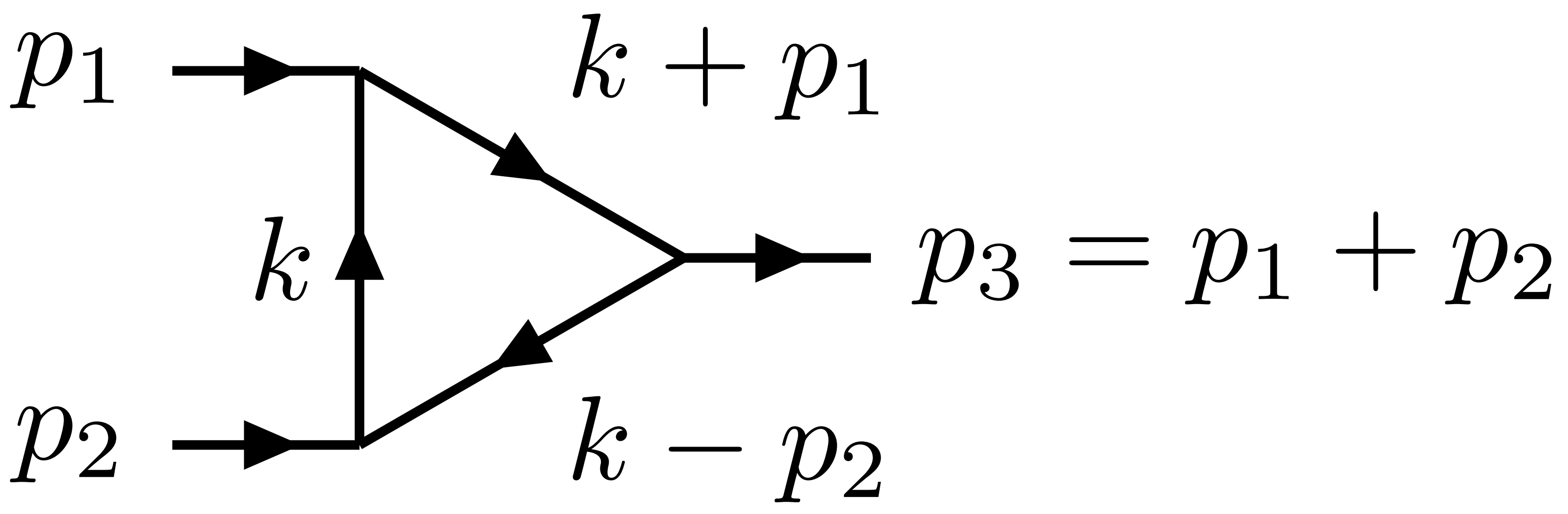

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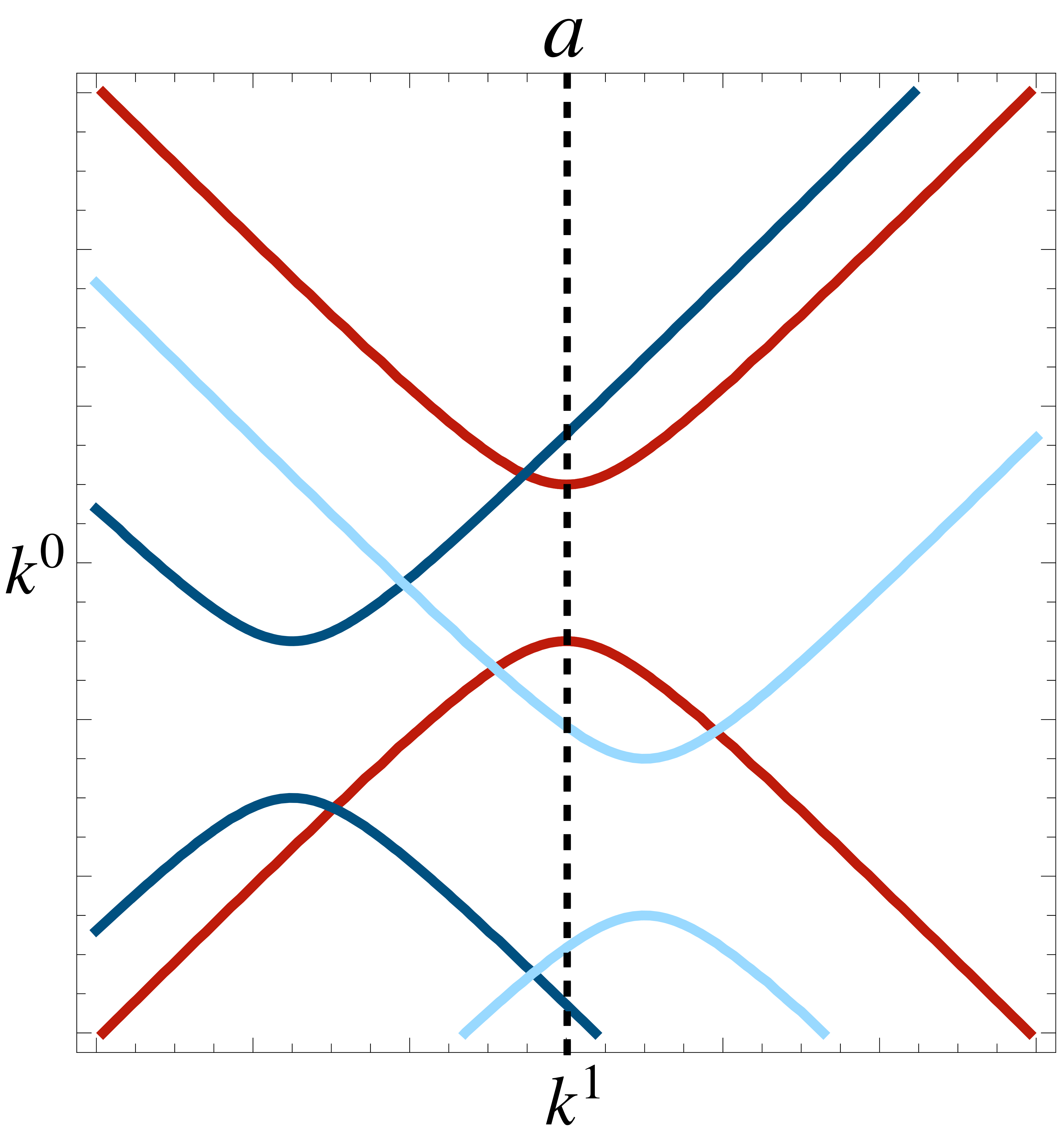

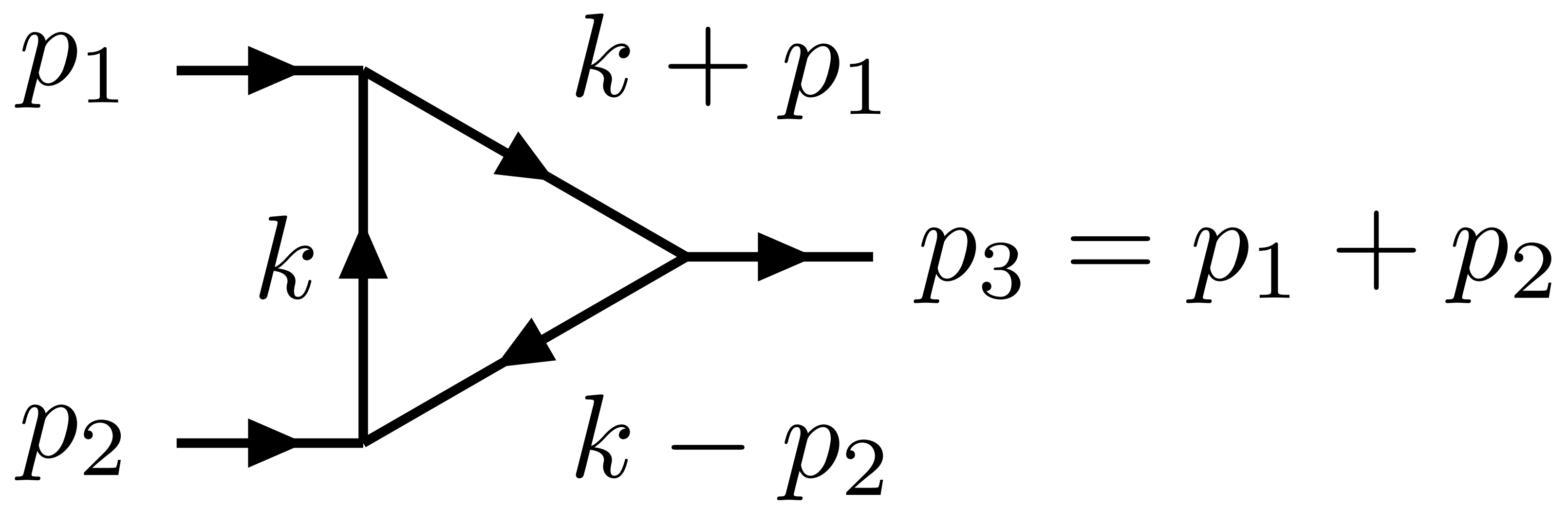

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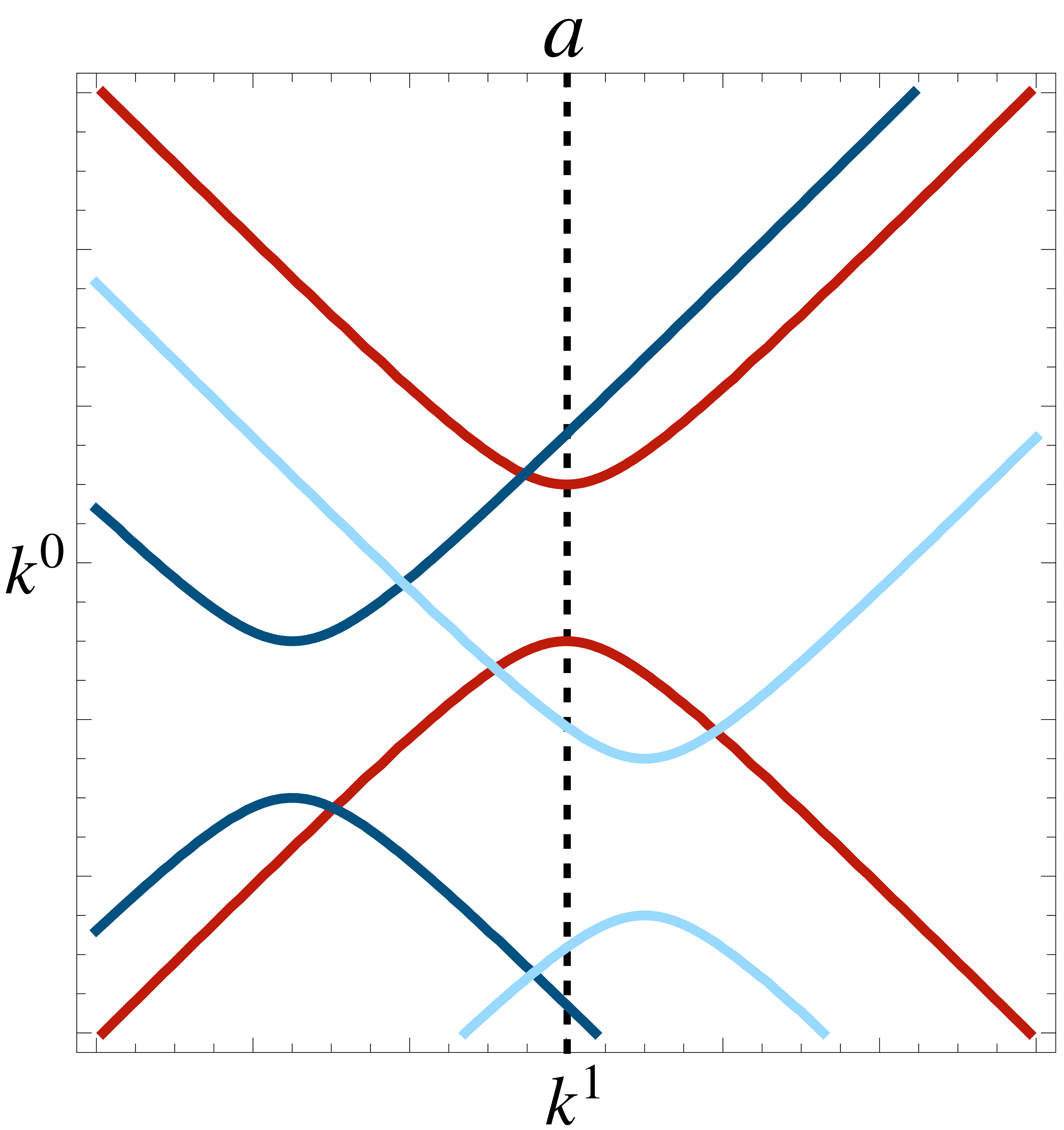

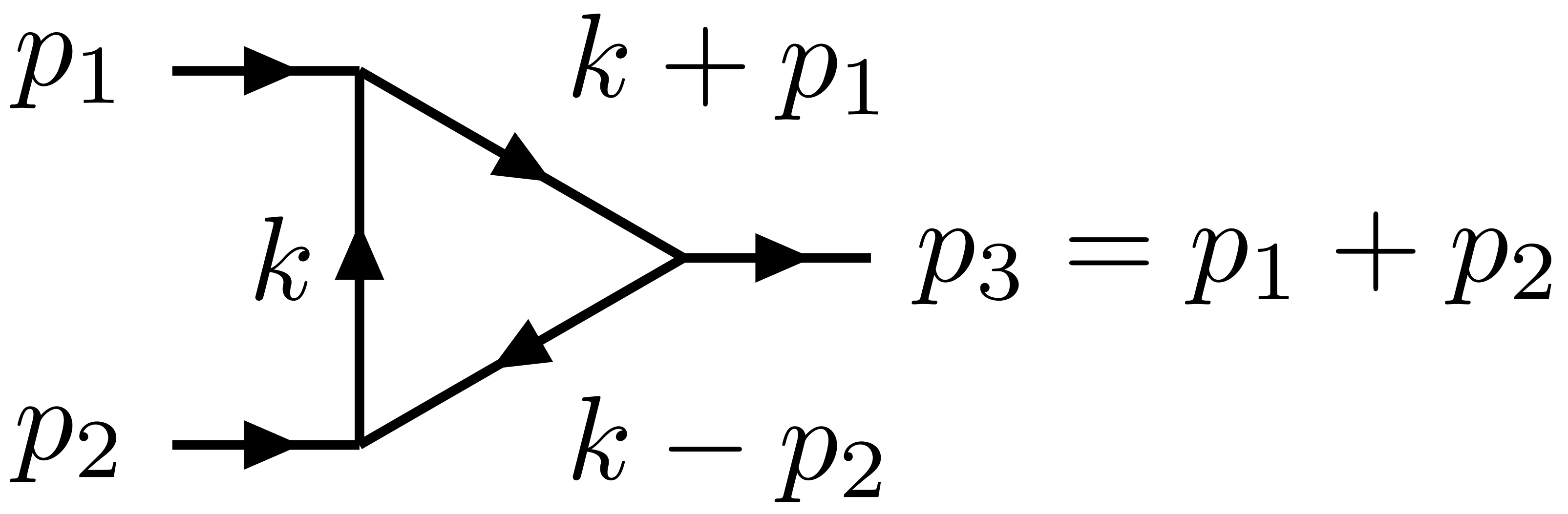

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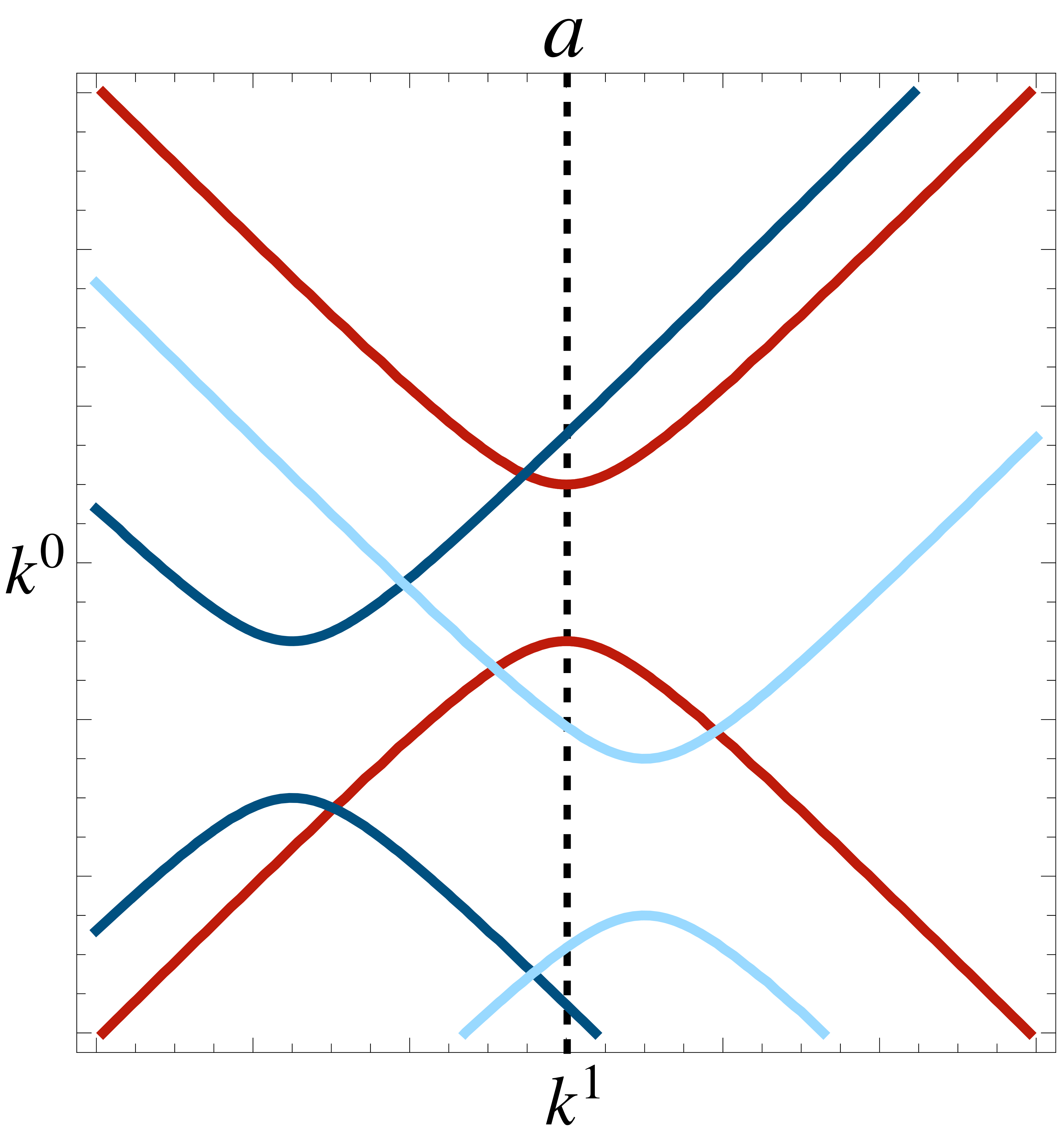

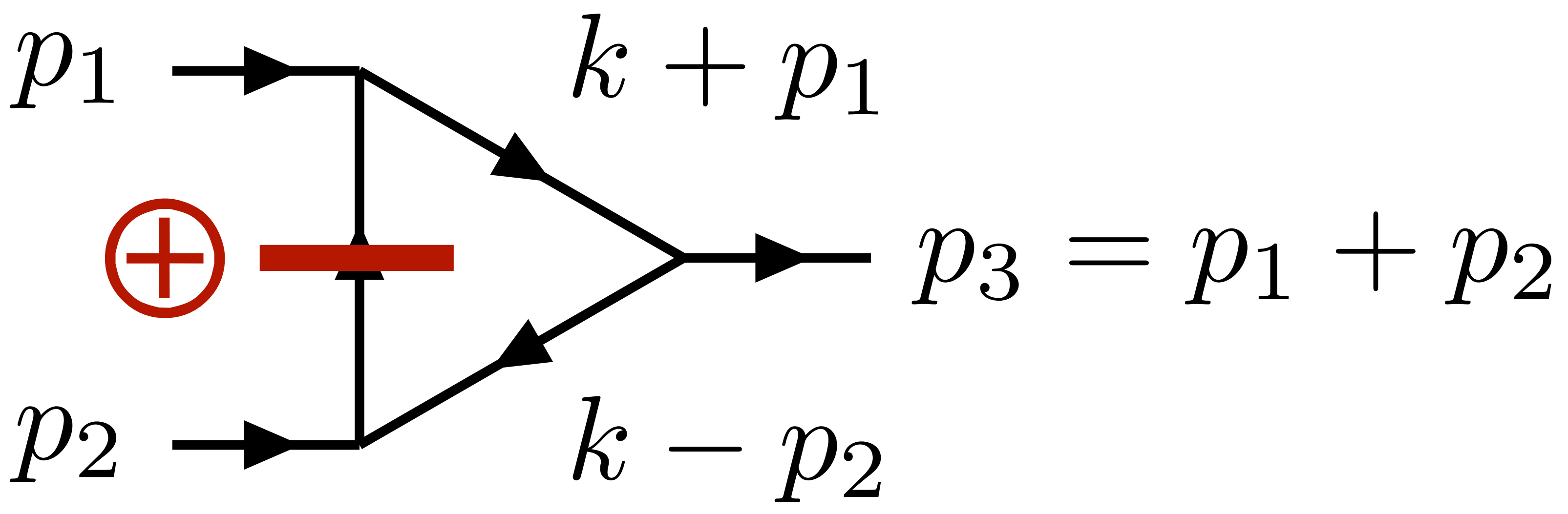

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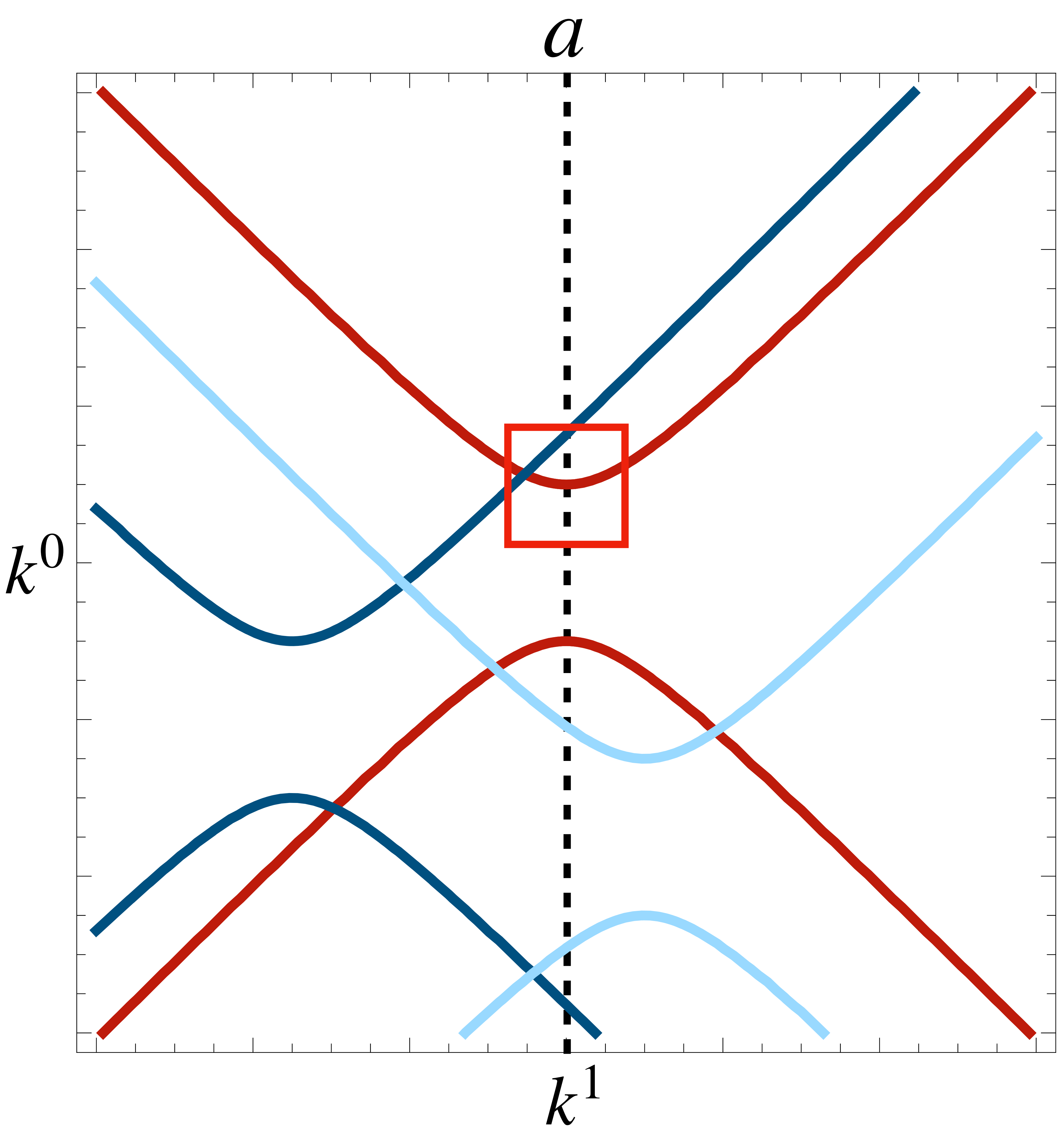

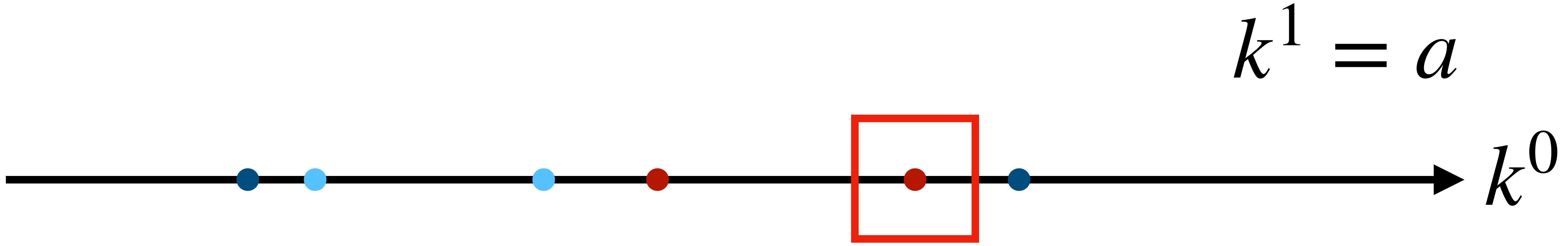

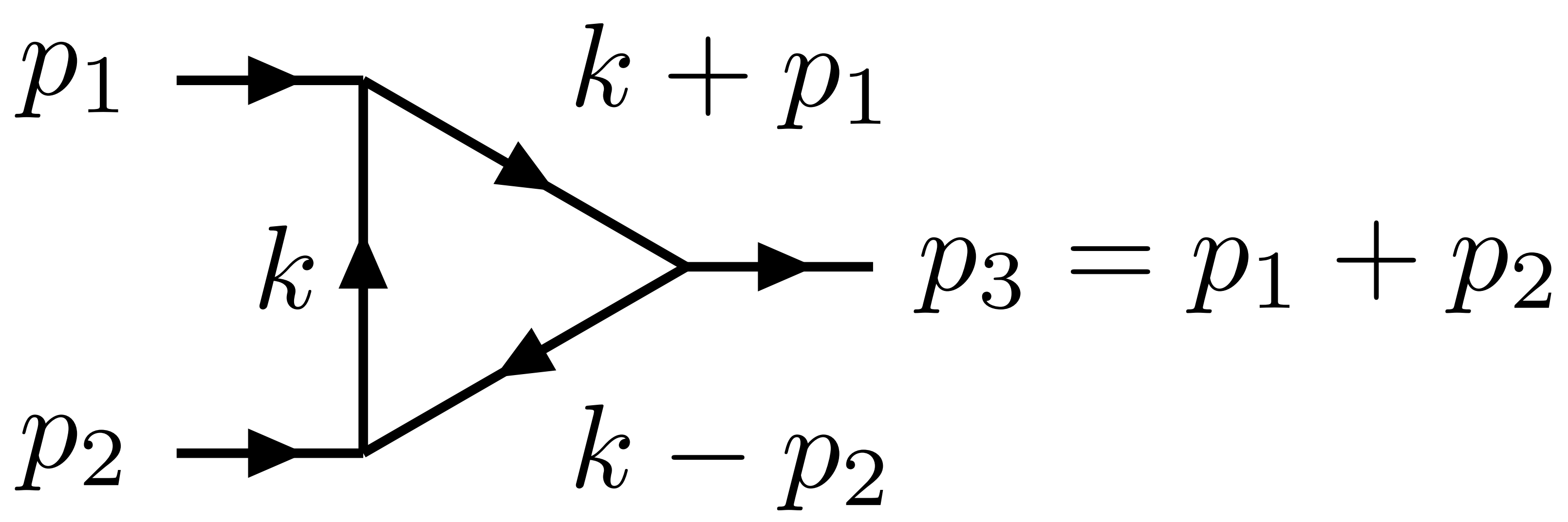

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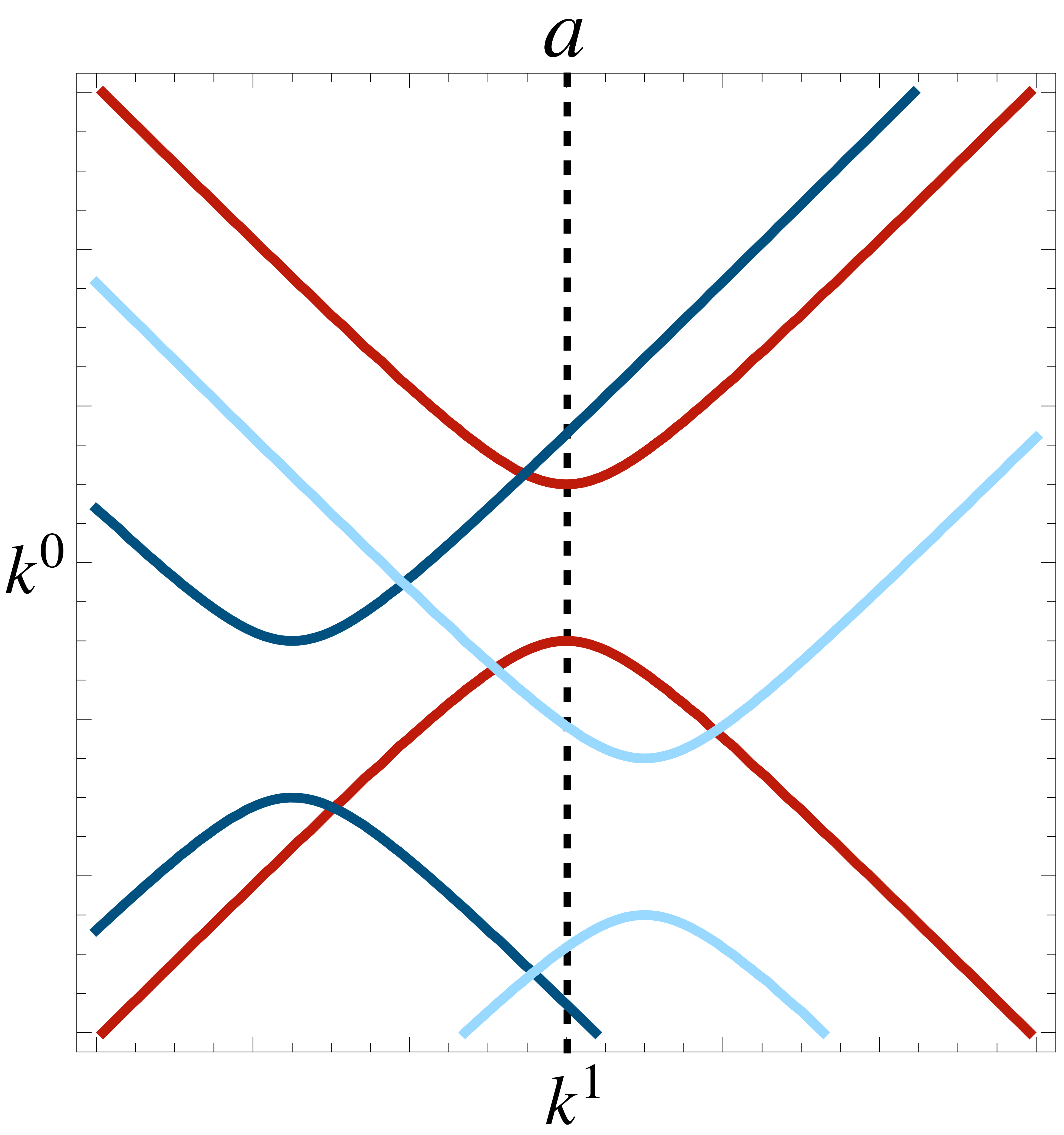

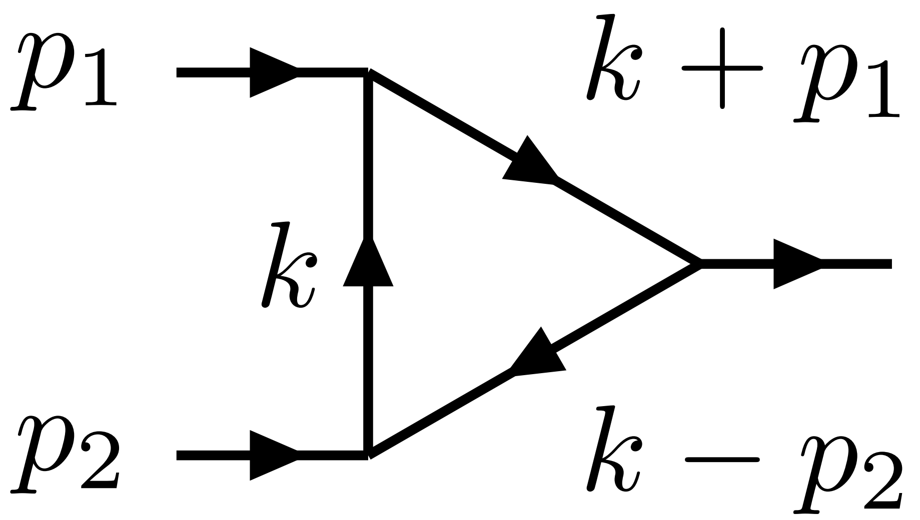

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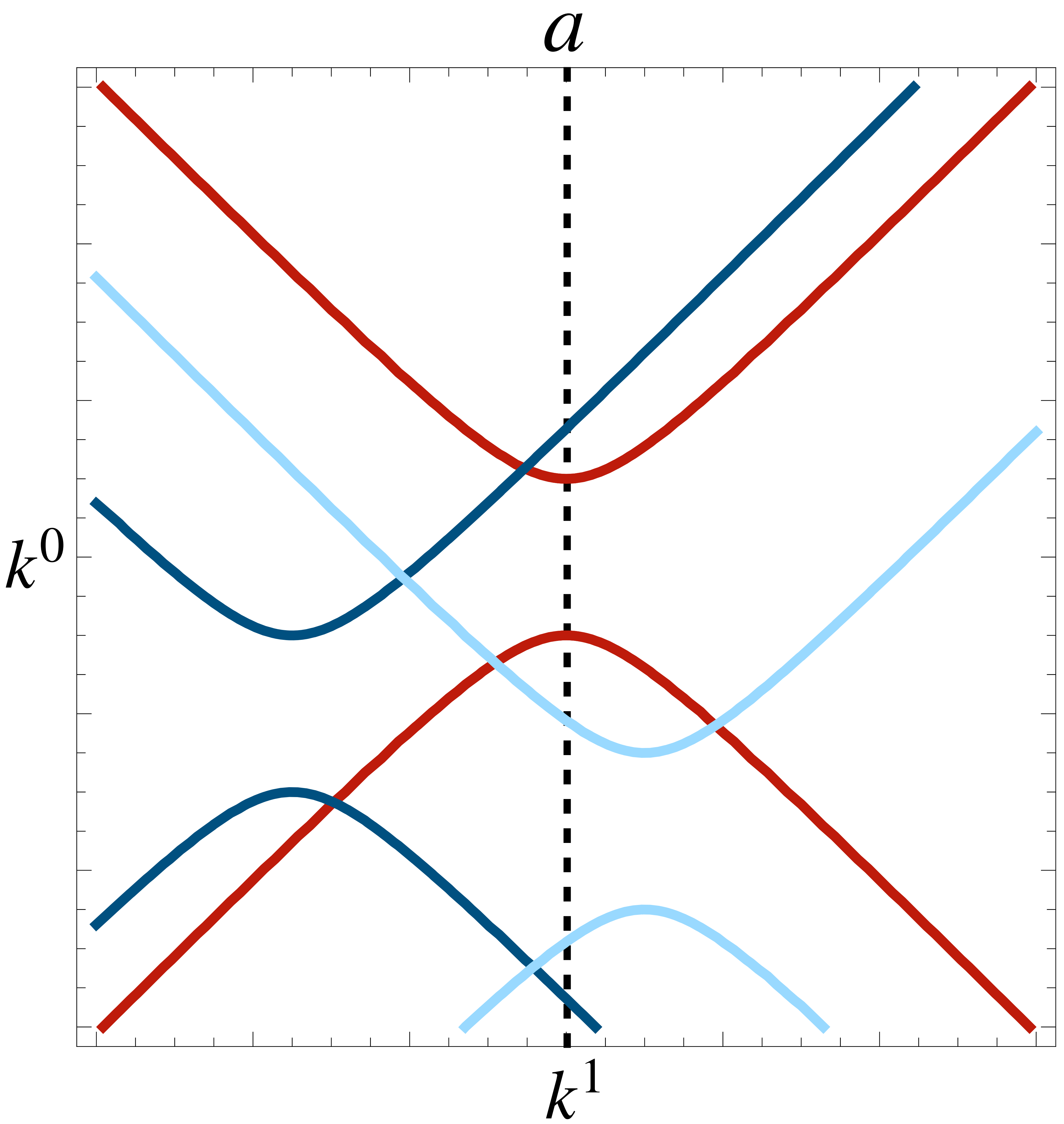
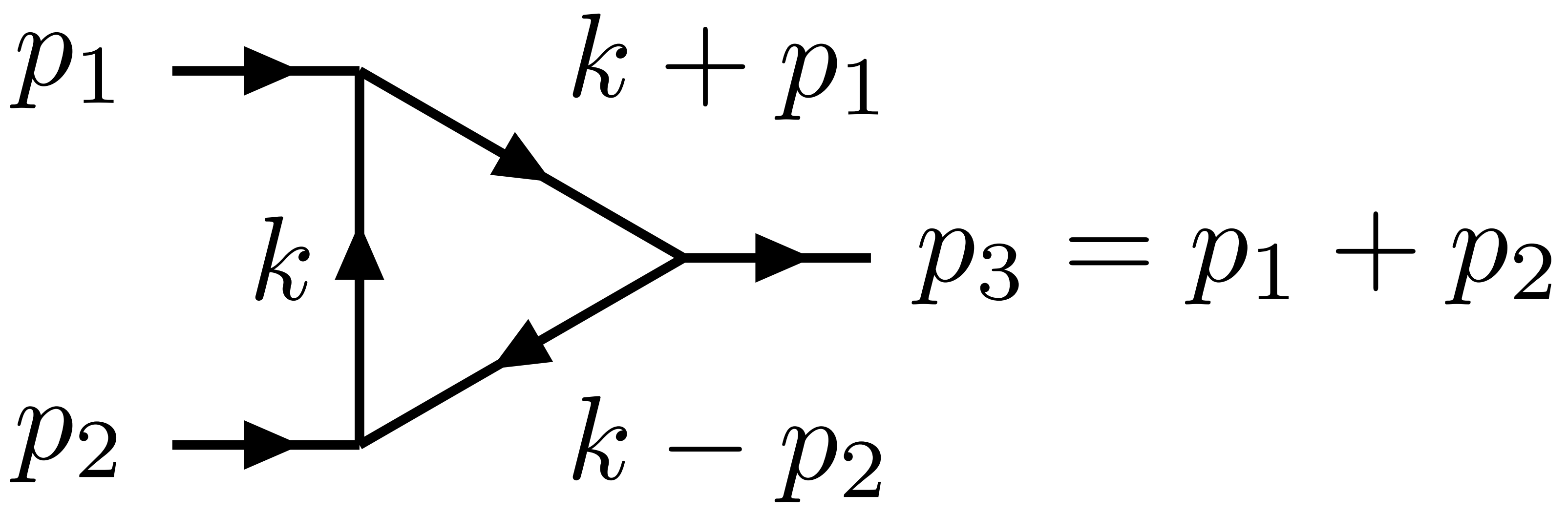


$$k^0 = \pm \sqrt{|\vec{k}|^2 + m^2 - \mathrm{i}\delta}$$

$$k^1 = a$$



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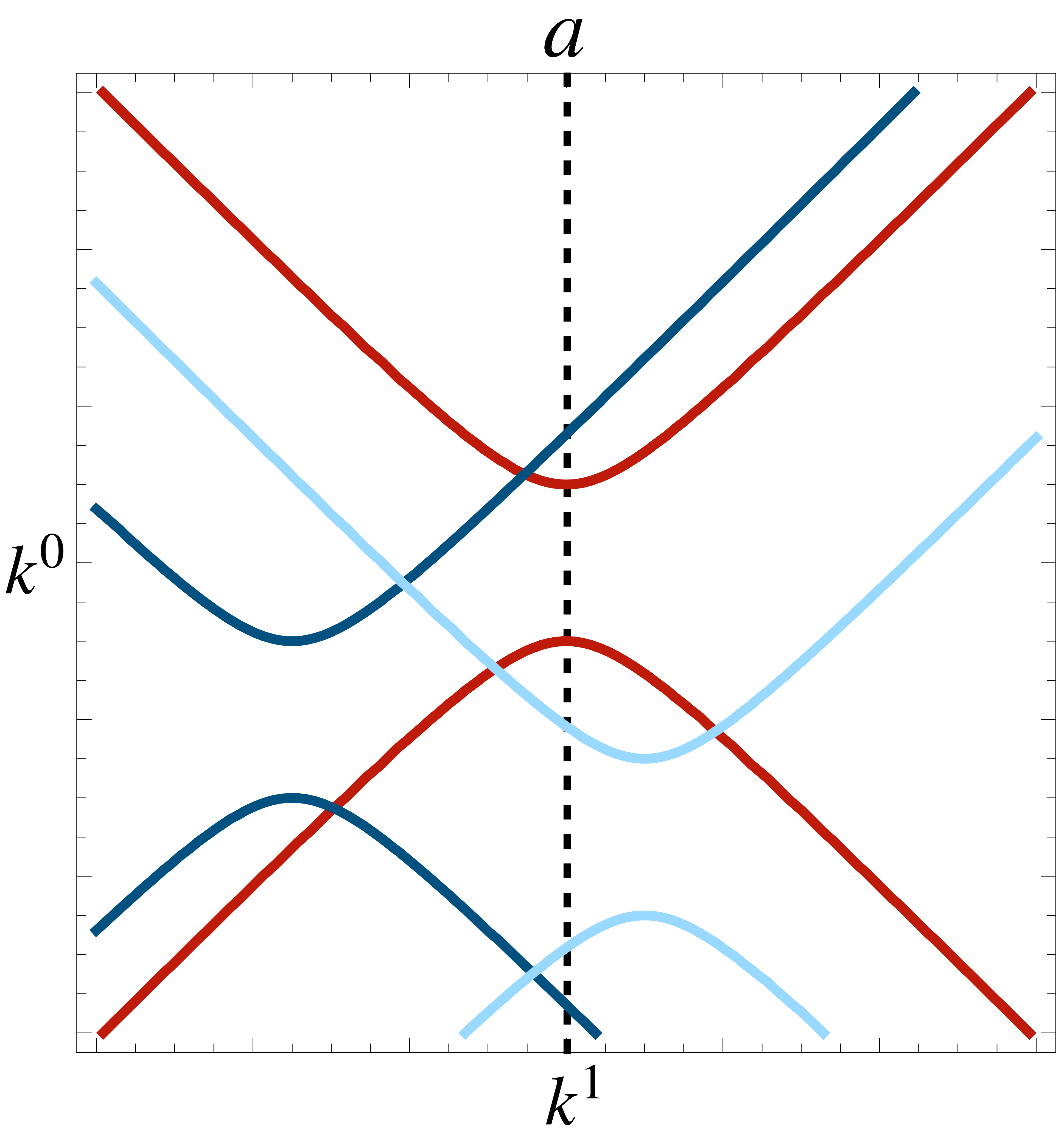

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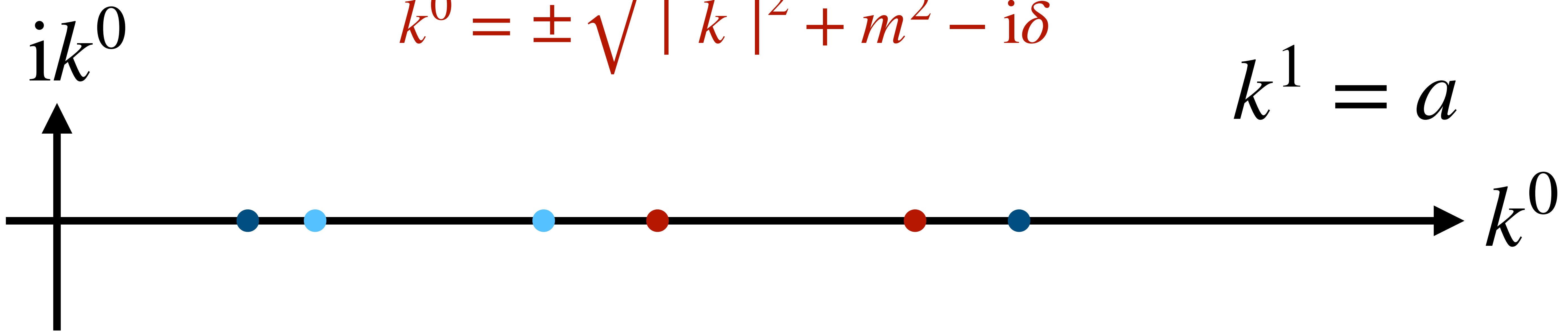

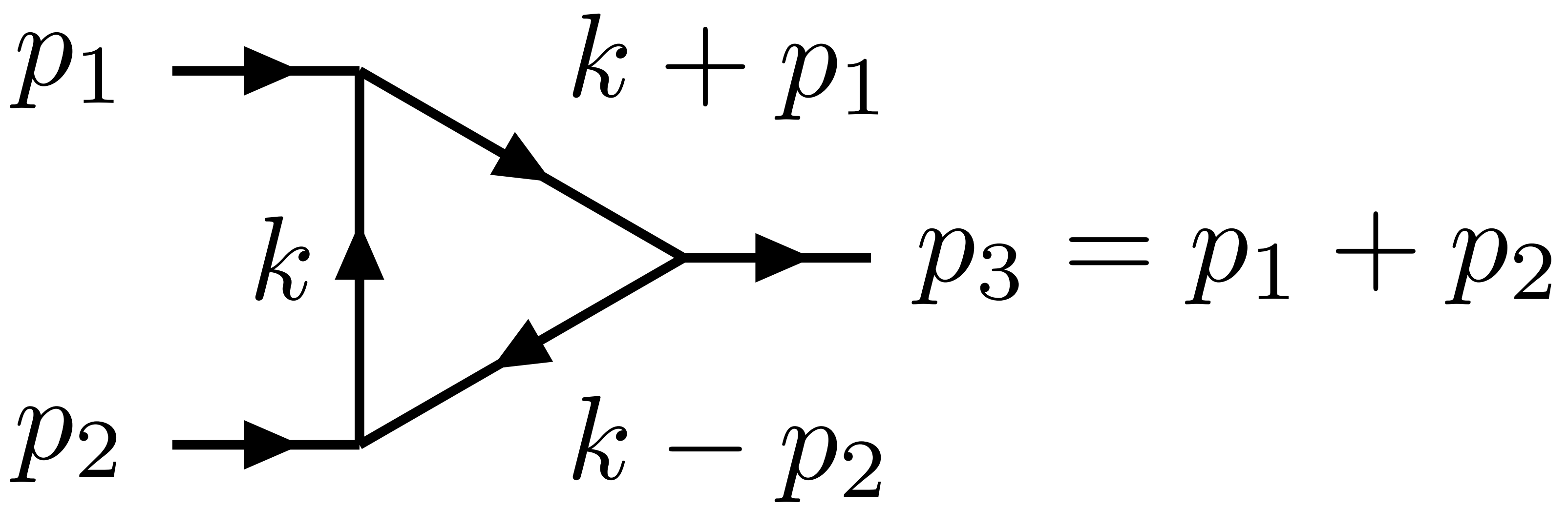

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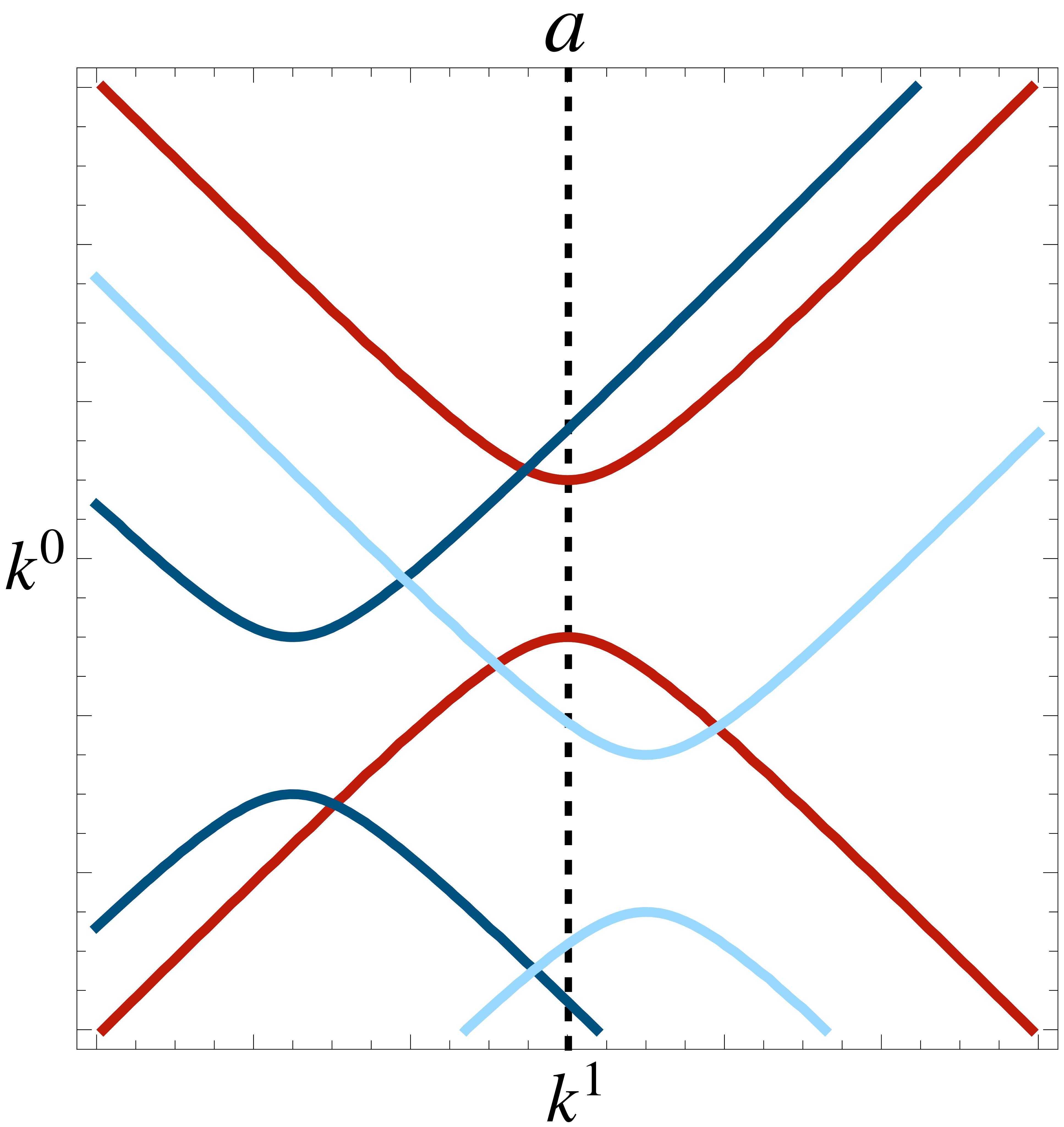

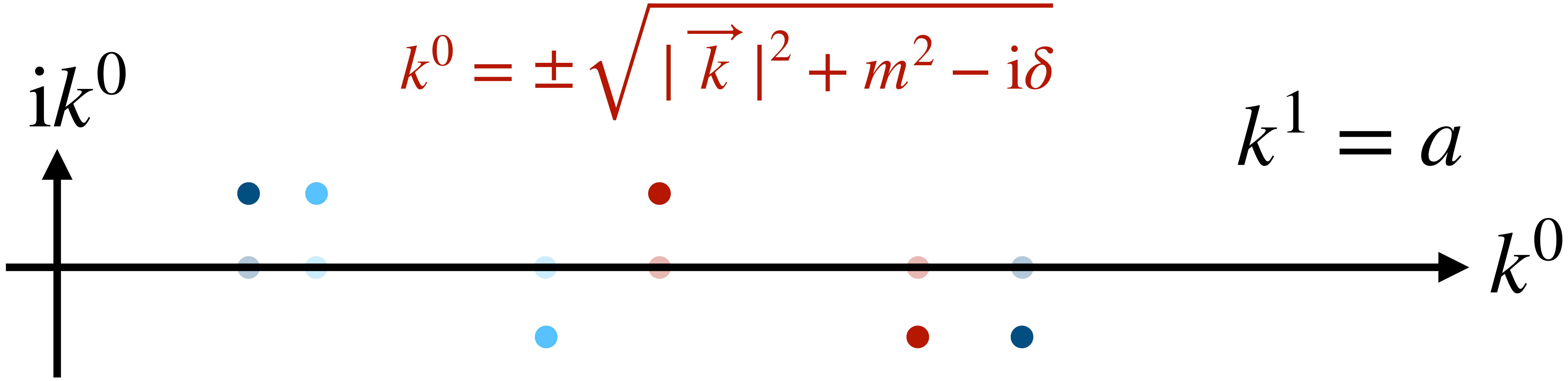

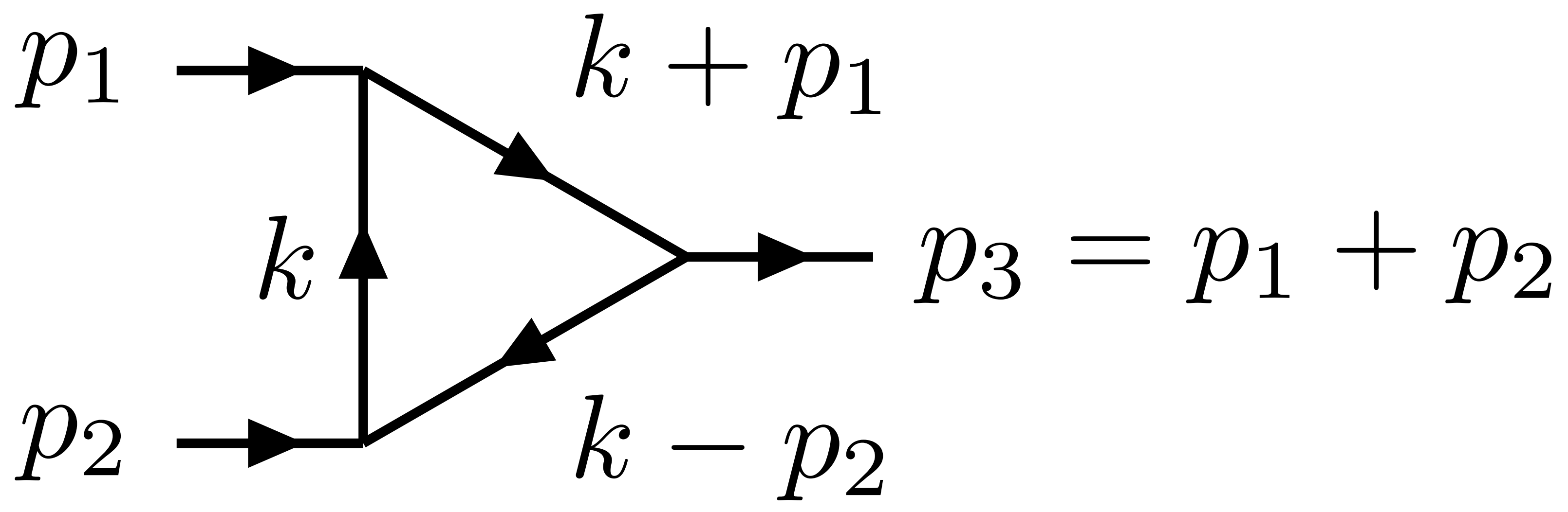

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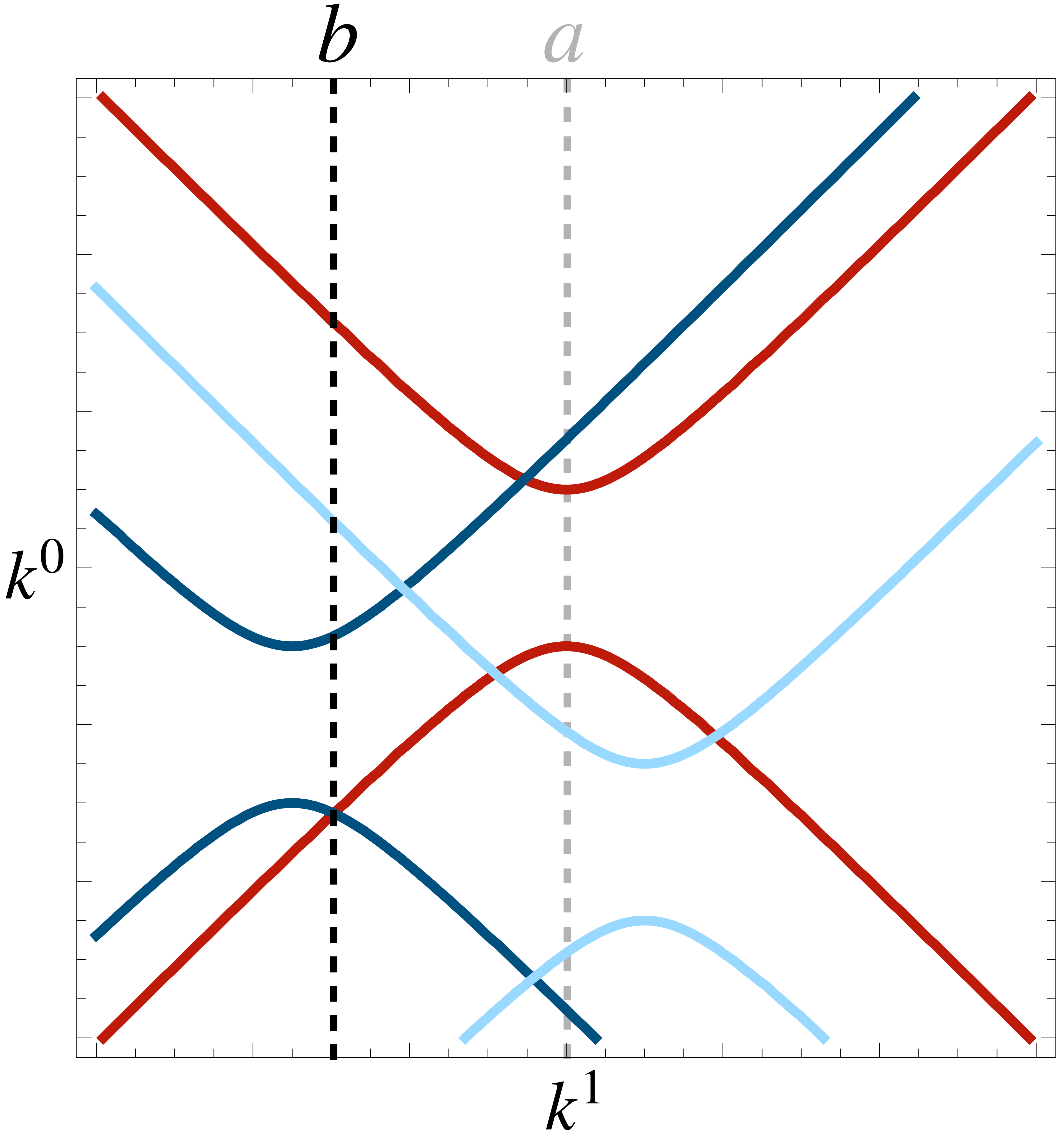

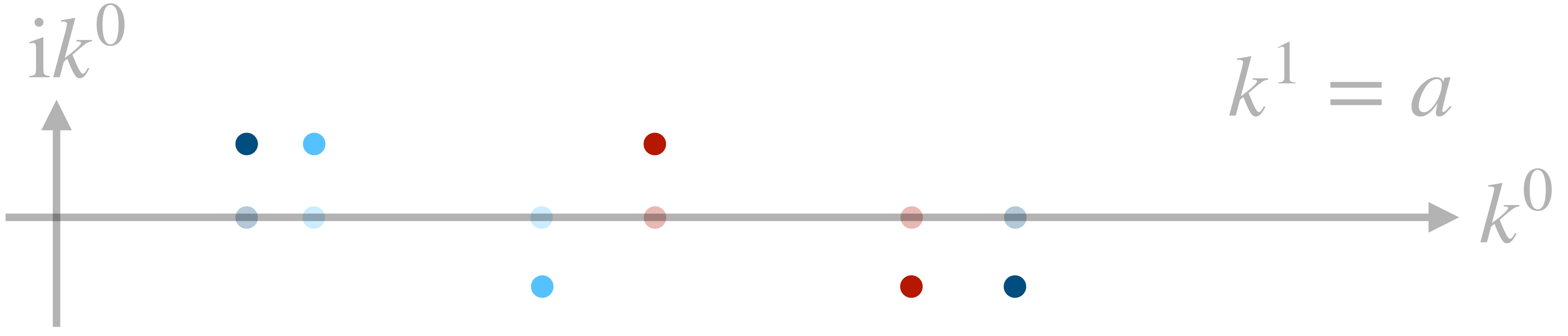

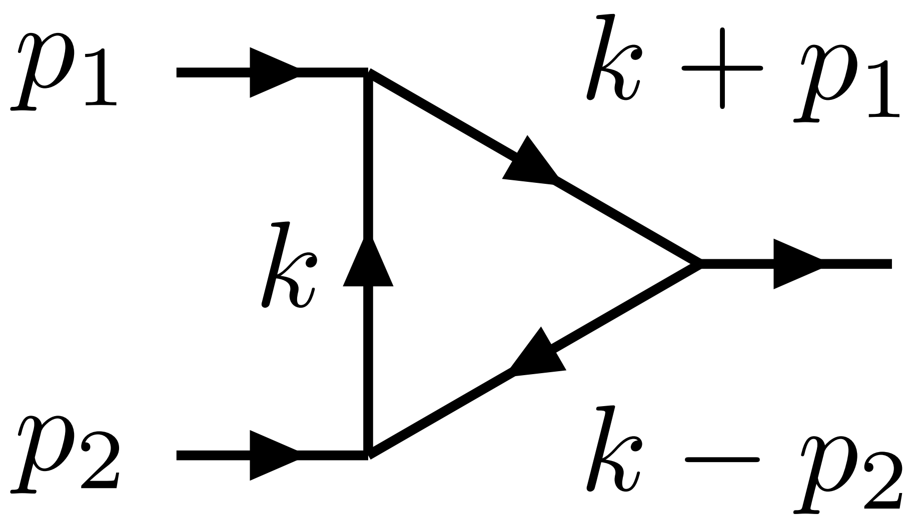

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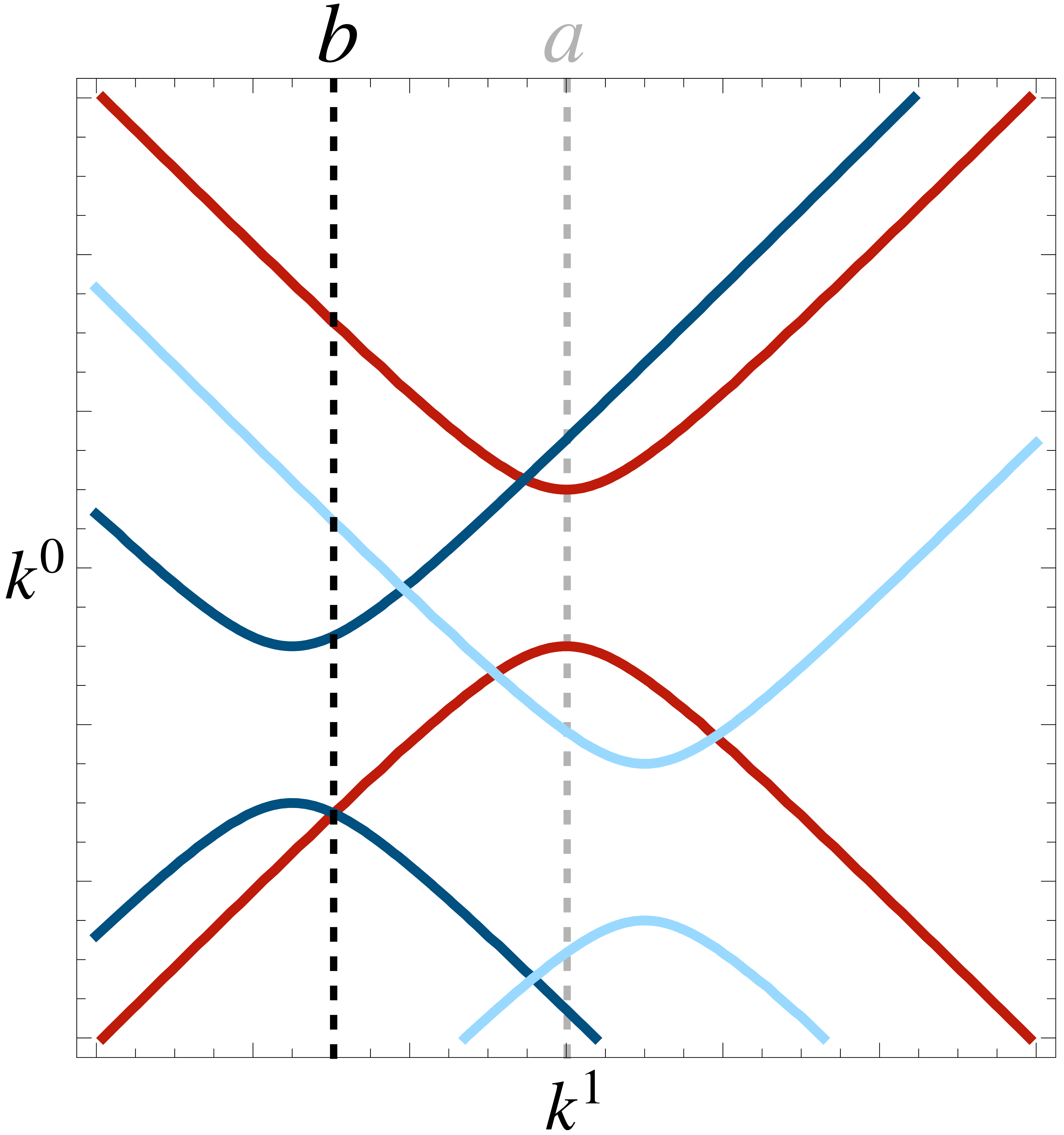

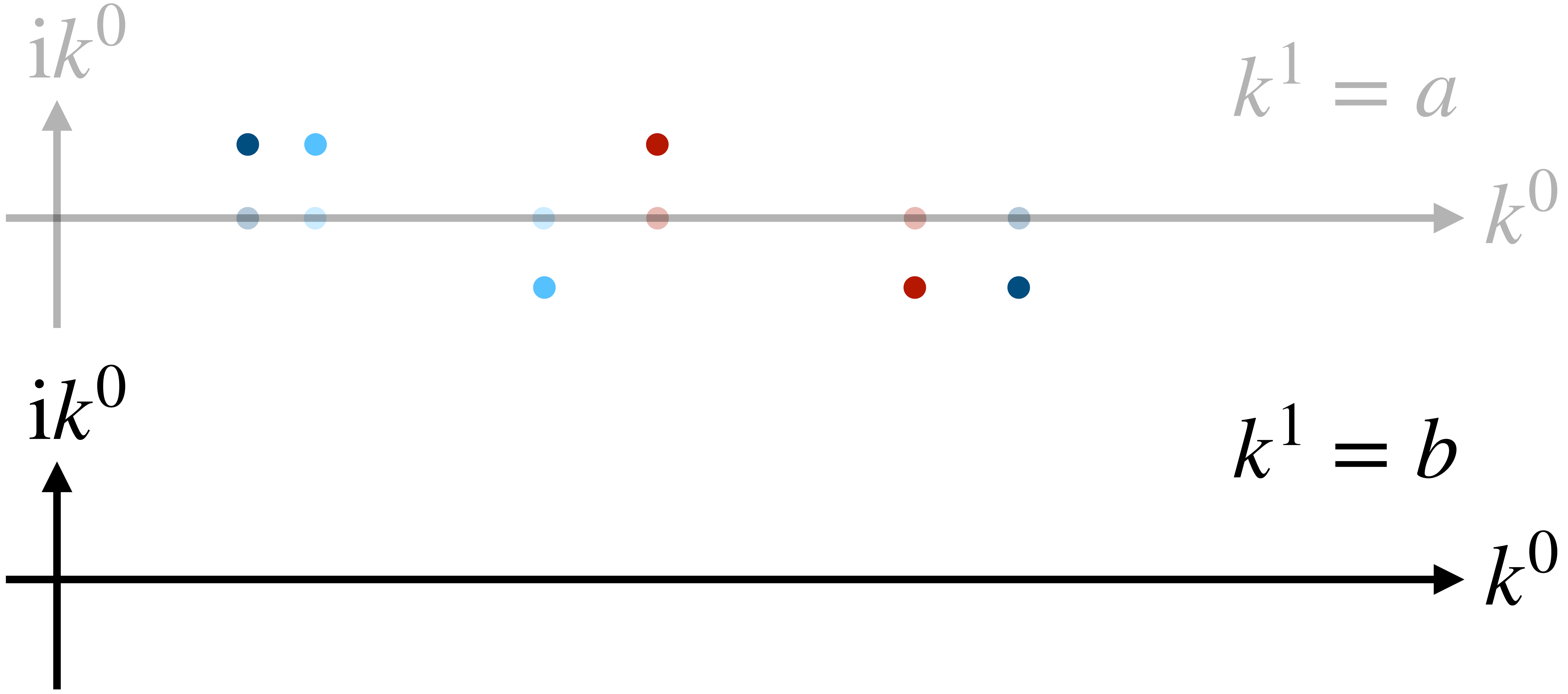

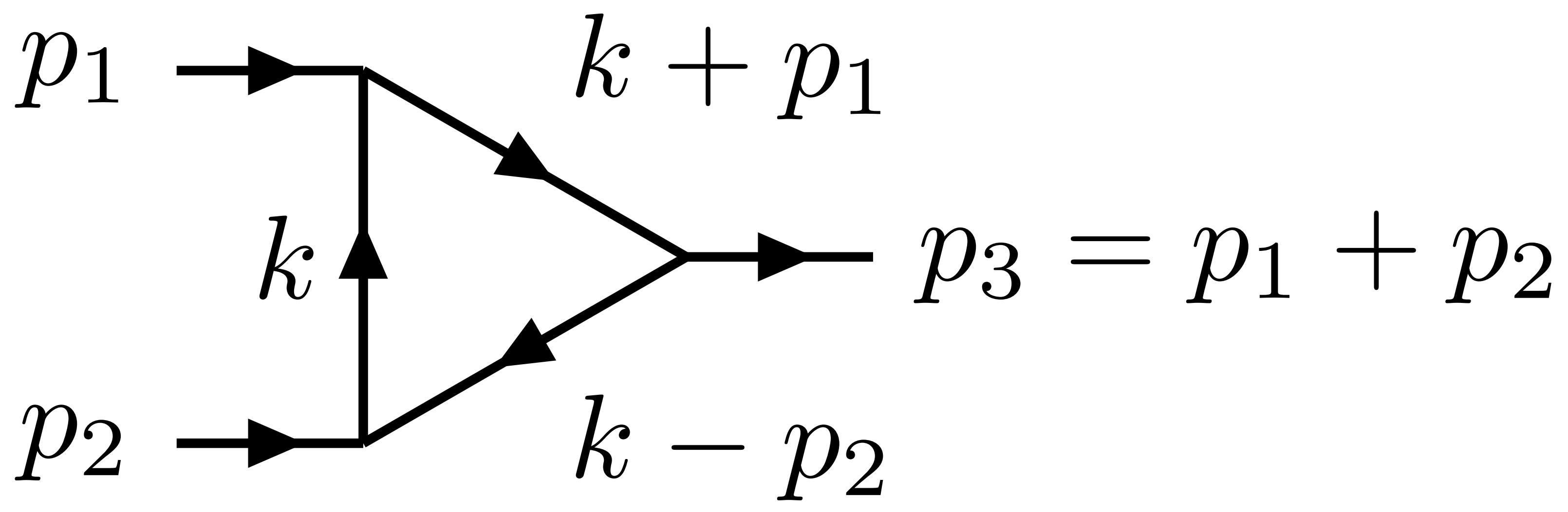

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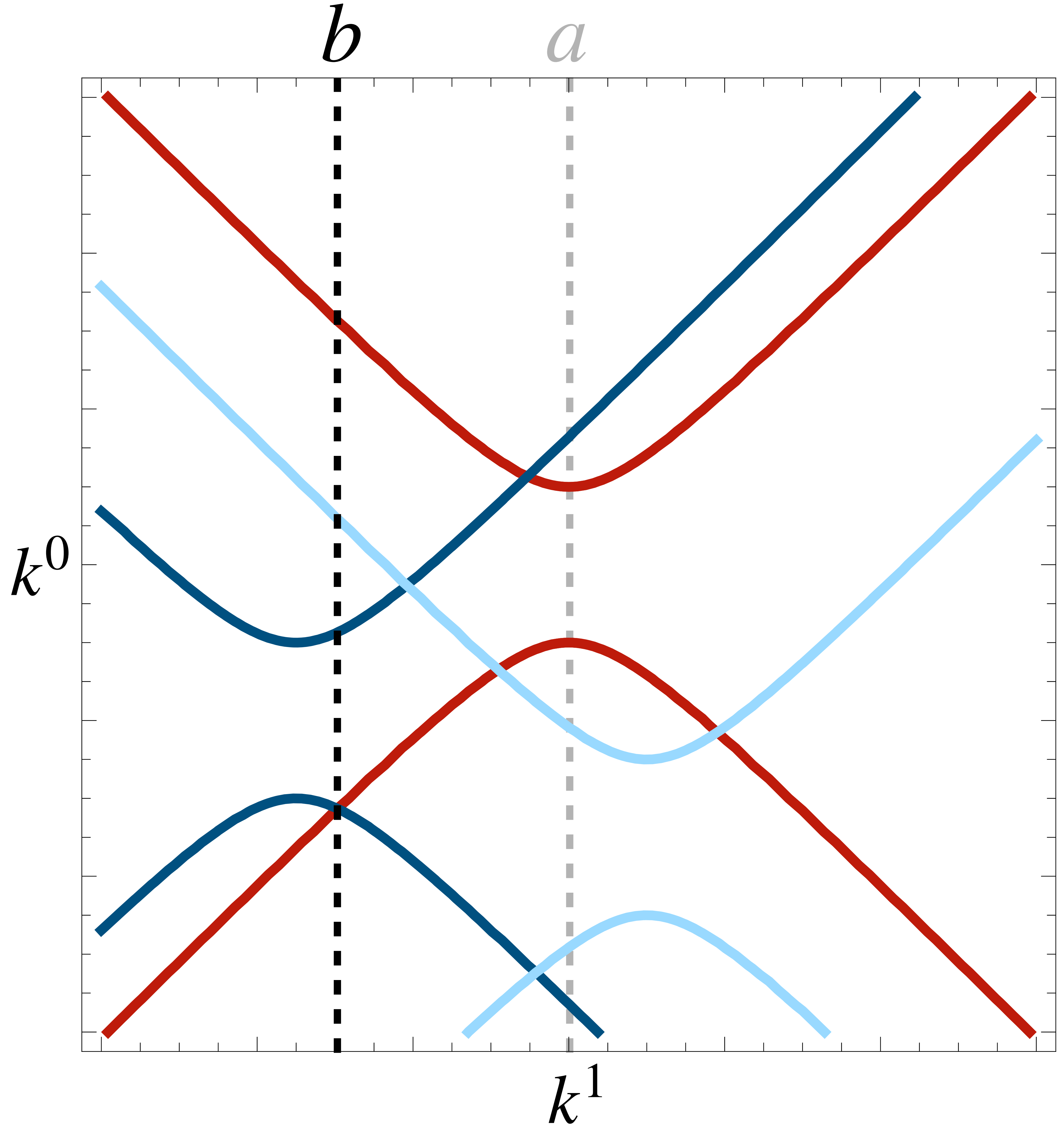
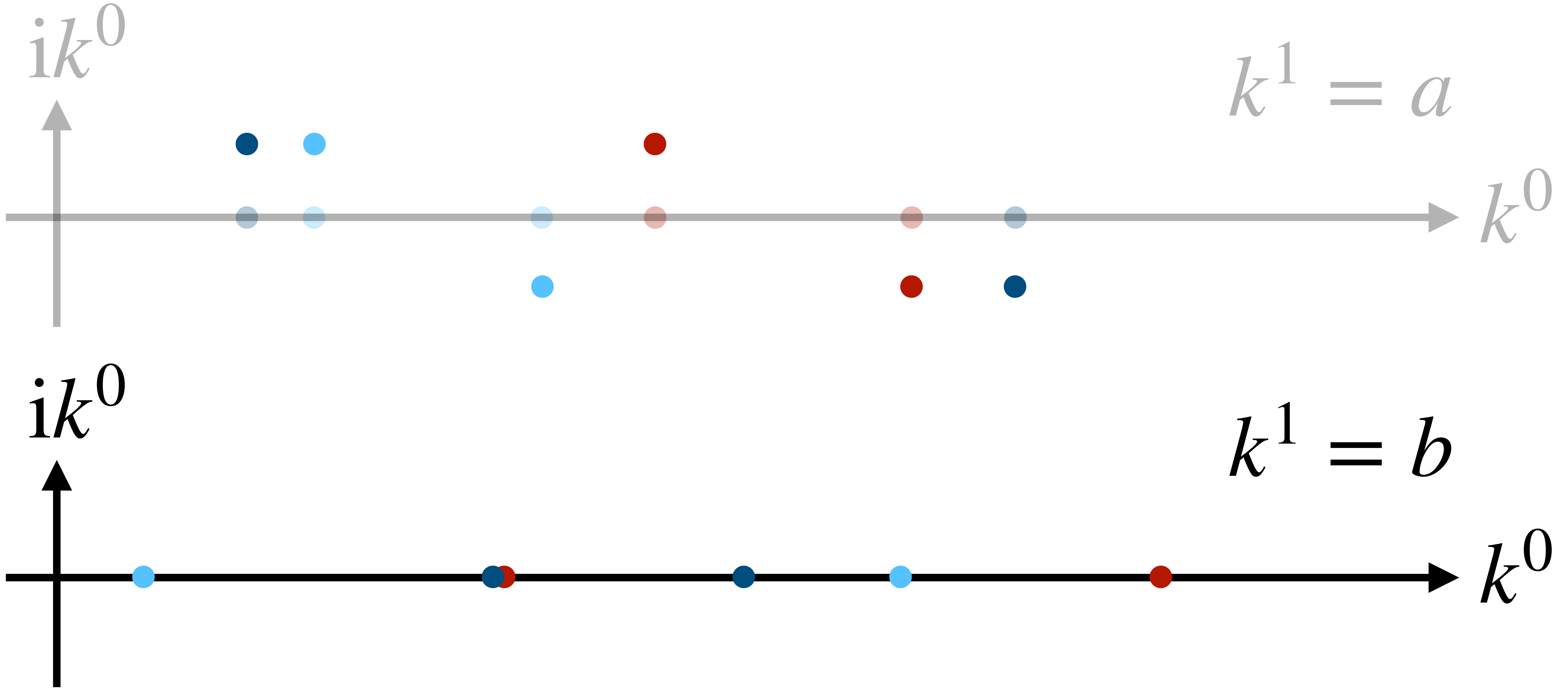
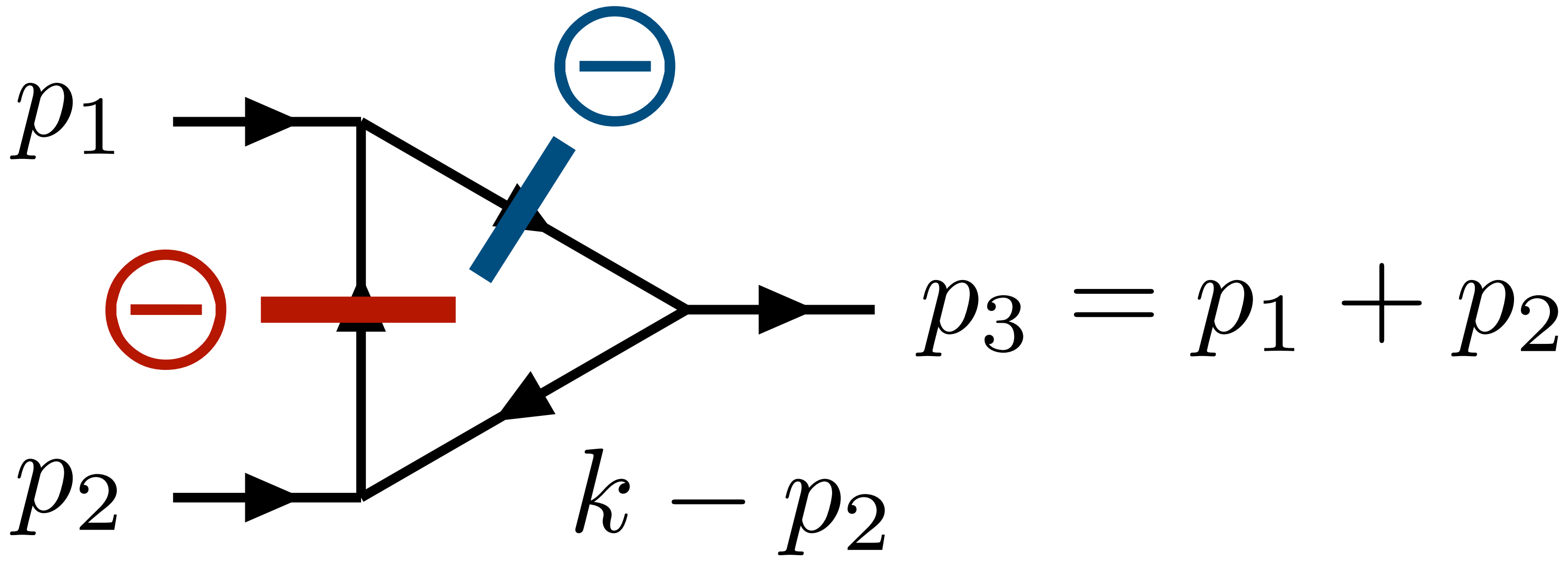
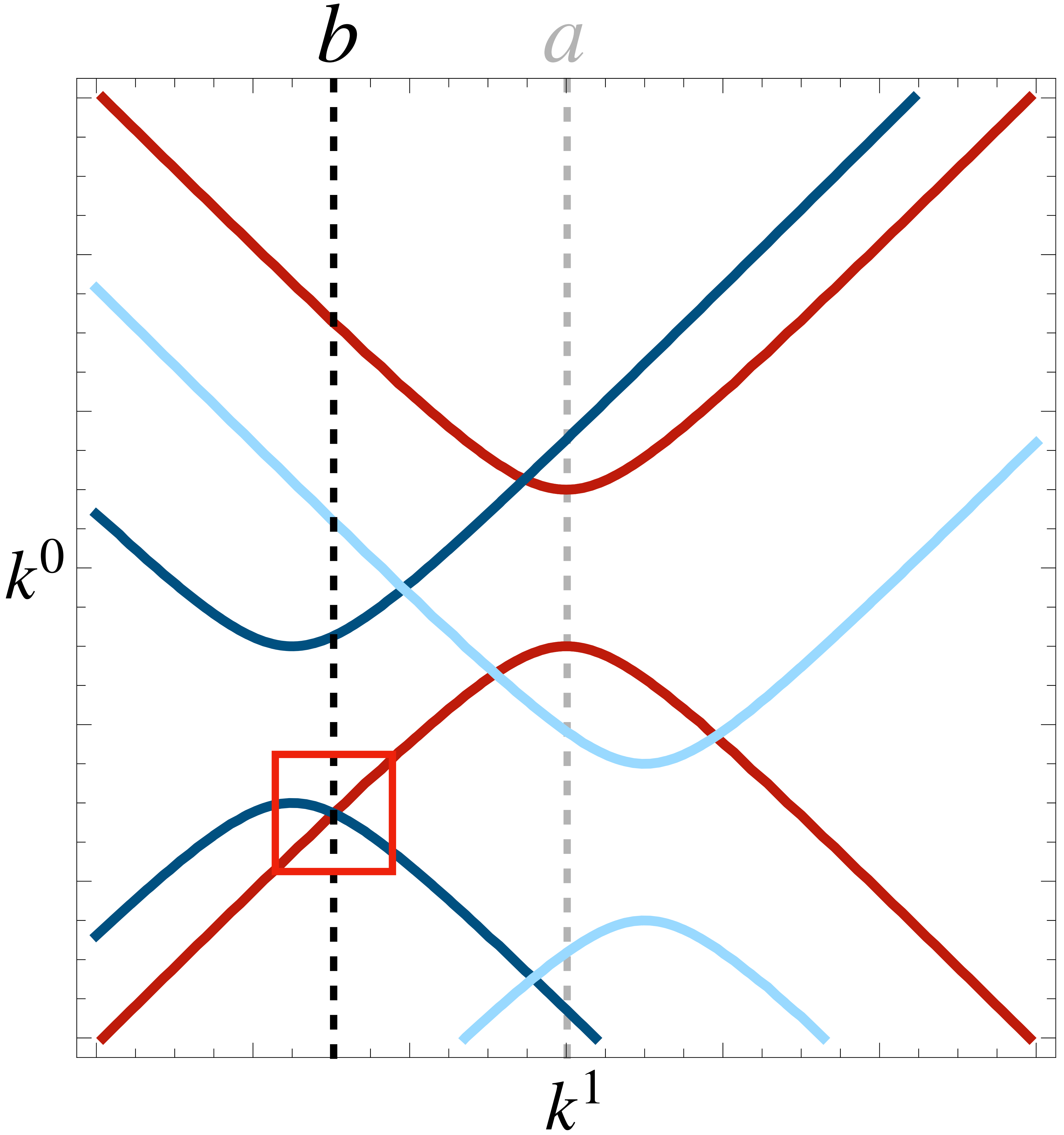
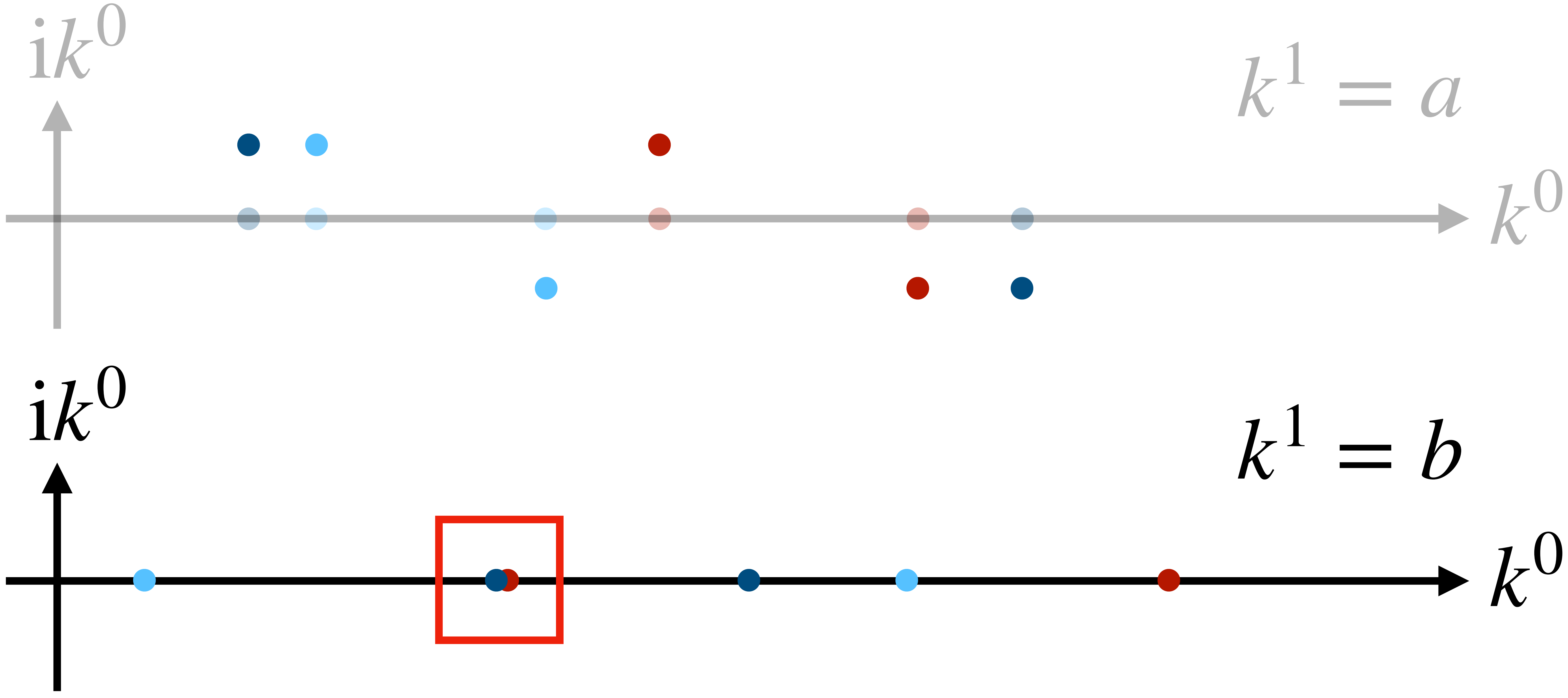


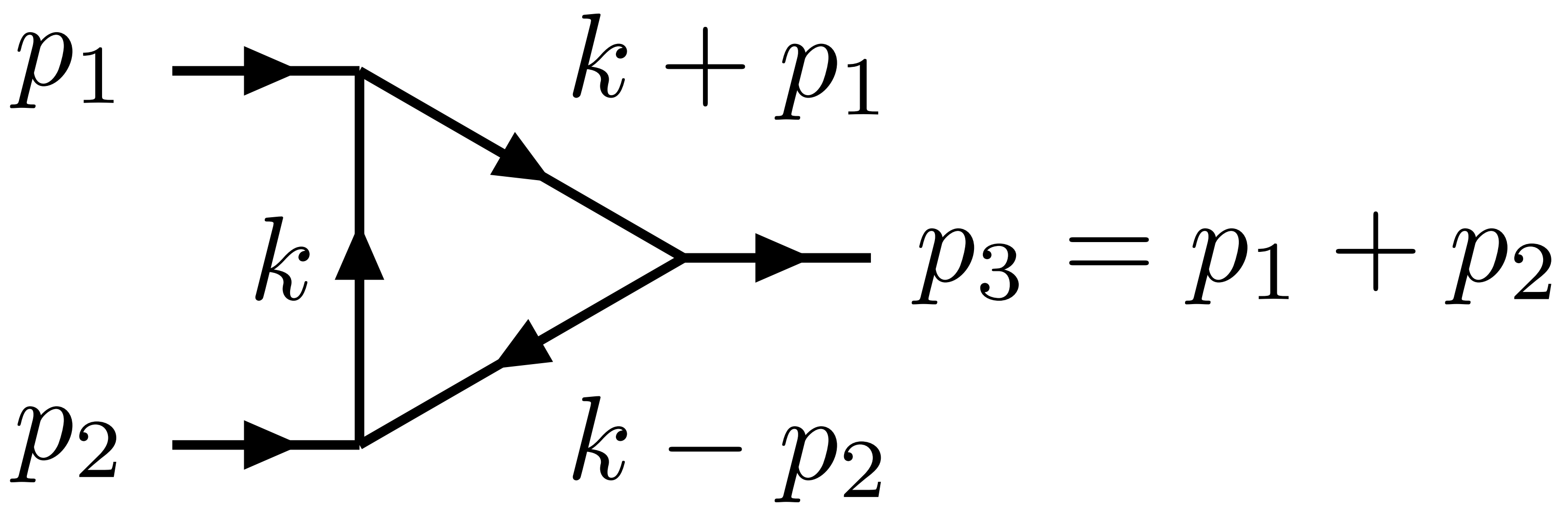




$$= \lim_{\delta \to 0} \int \frac{d^4 k}{(2\pi)^4} \boxed{\frac{1}{k^2 - m^2 + i\delta}} \boxed{\frac{1}{(k+p_1)^2 - m^2 + i\delta}} \frac{1}{(k-p_2)^2 - m^2 + i\delta}$$







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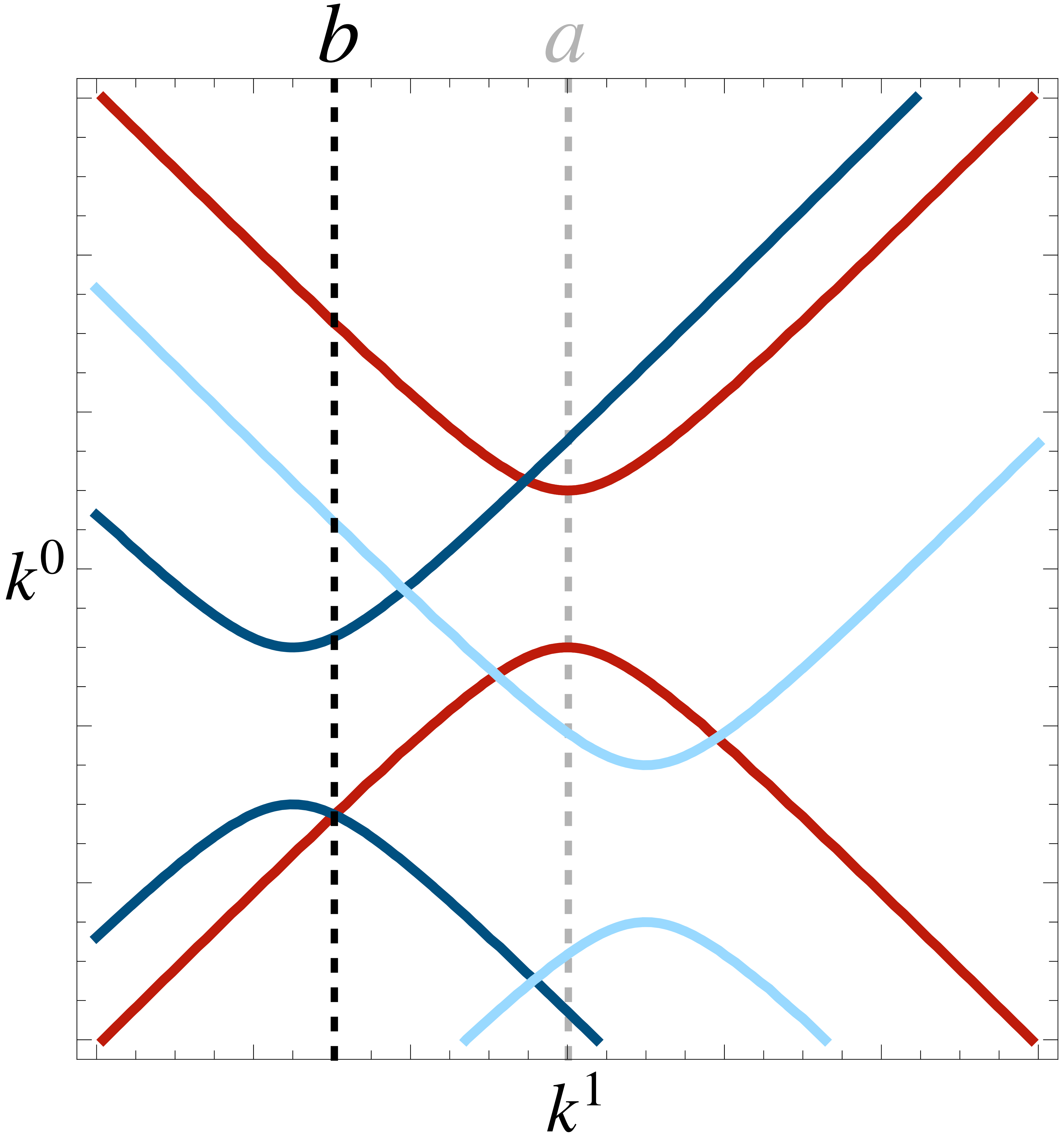

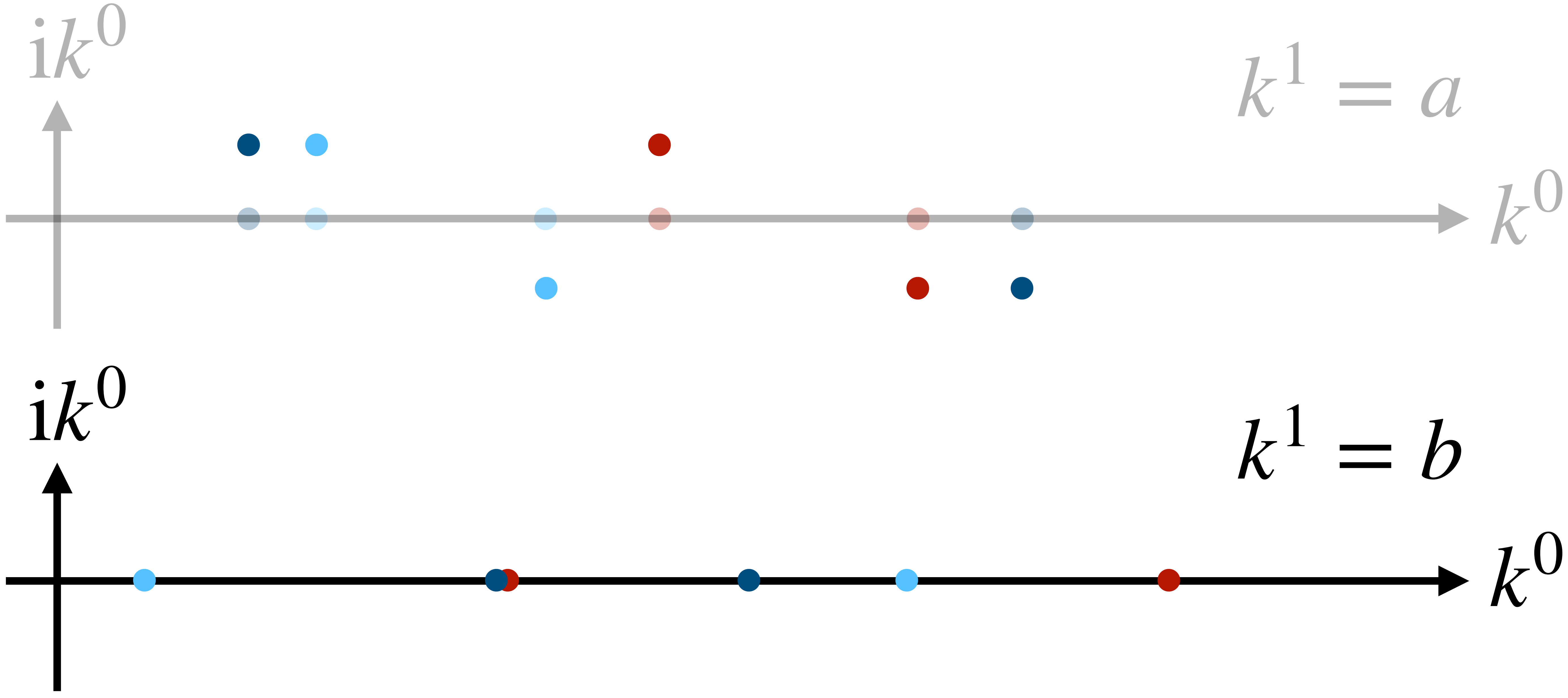

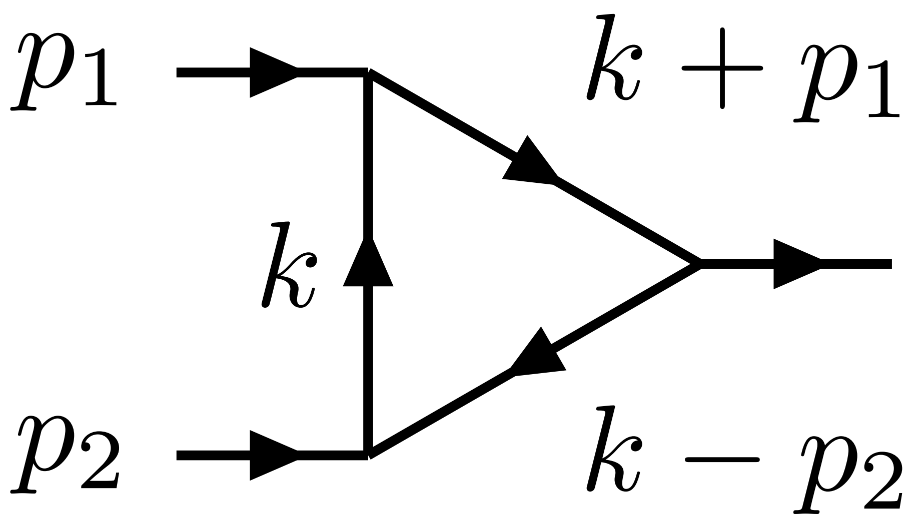

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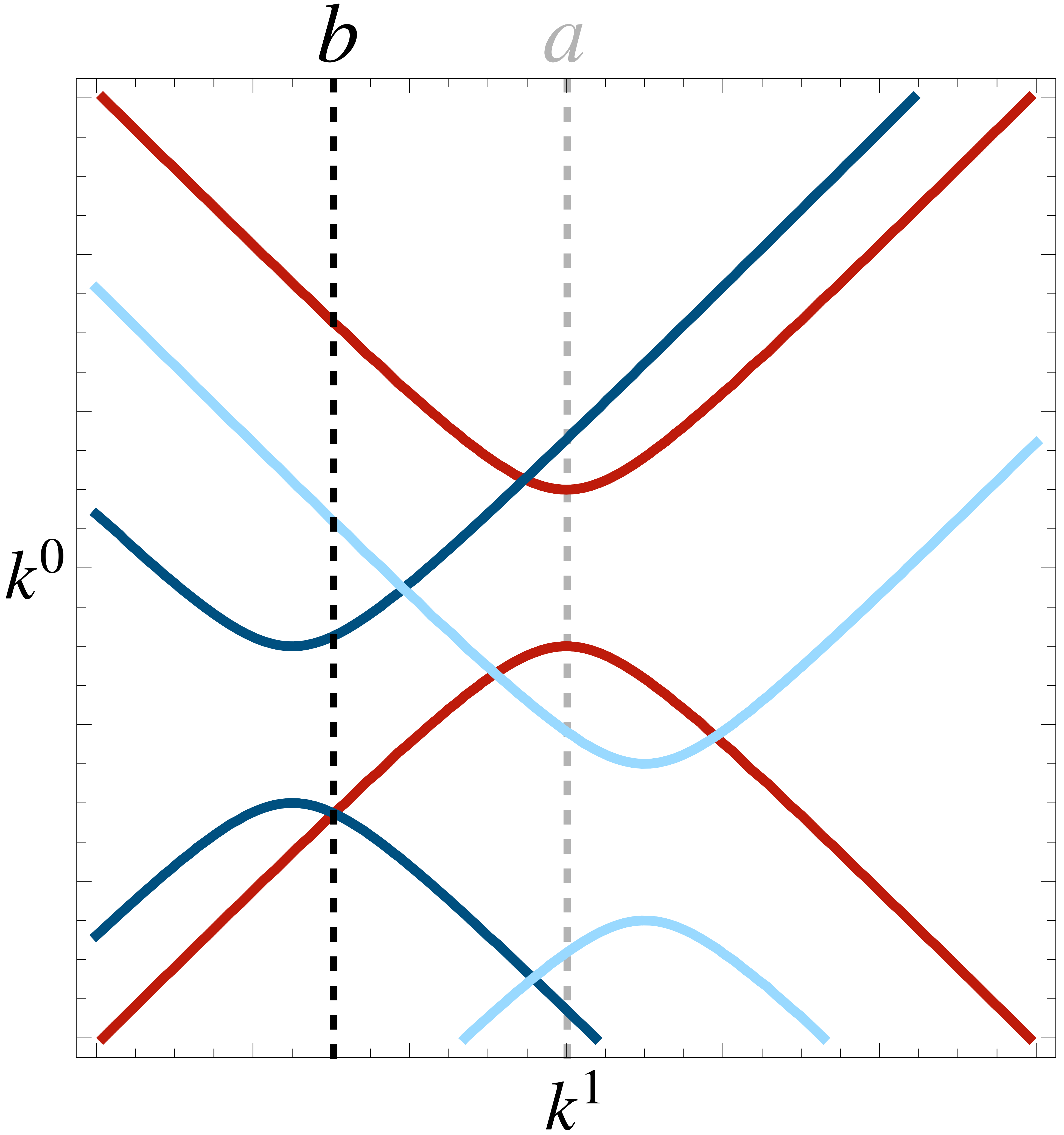

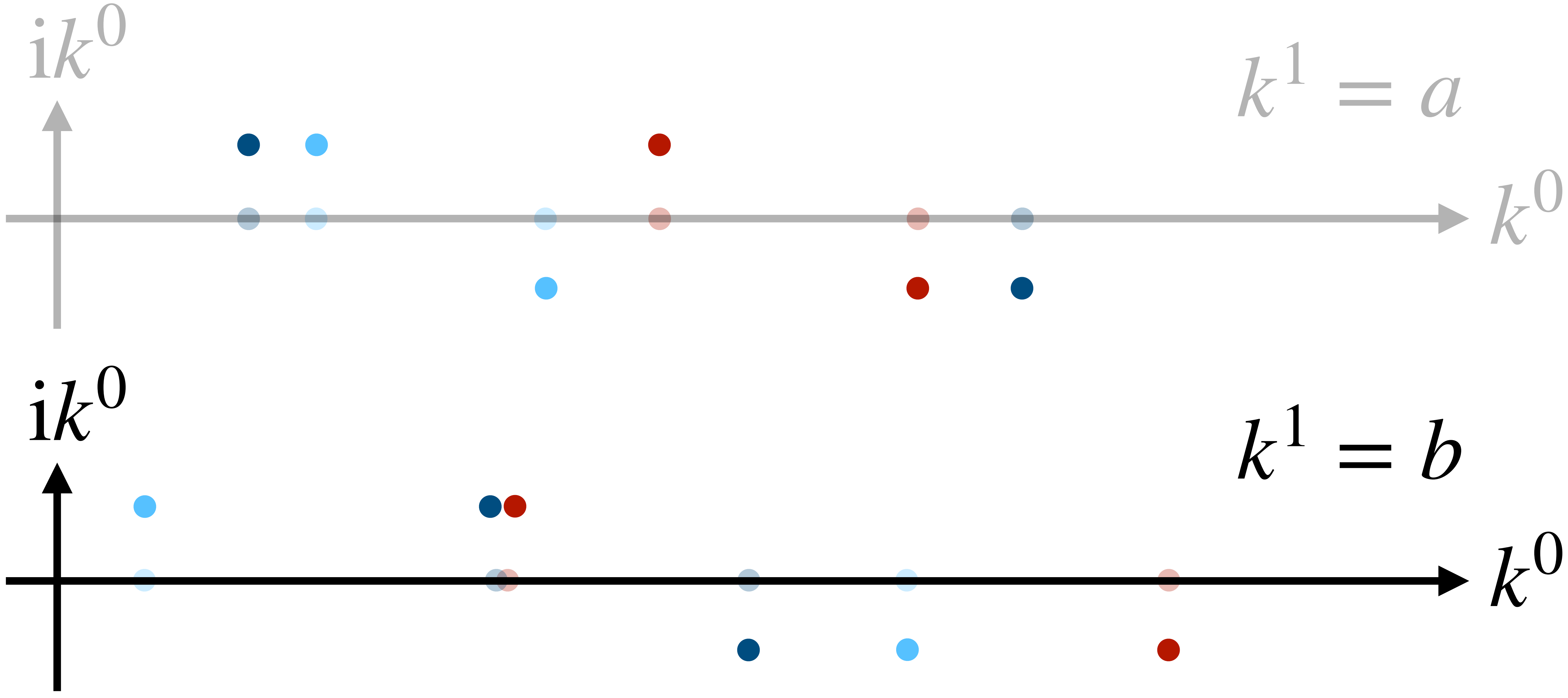

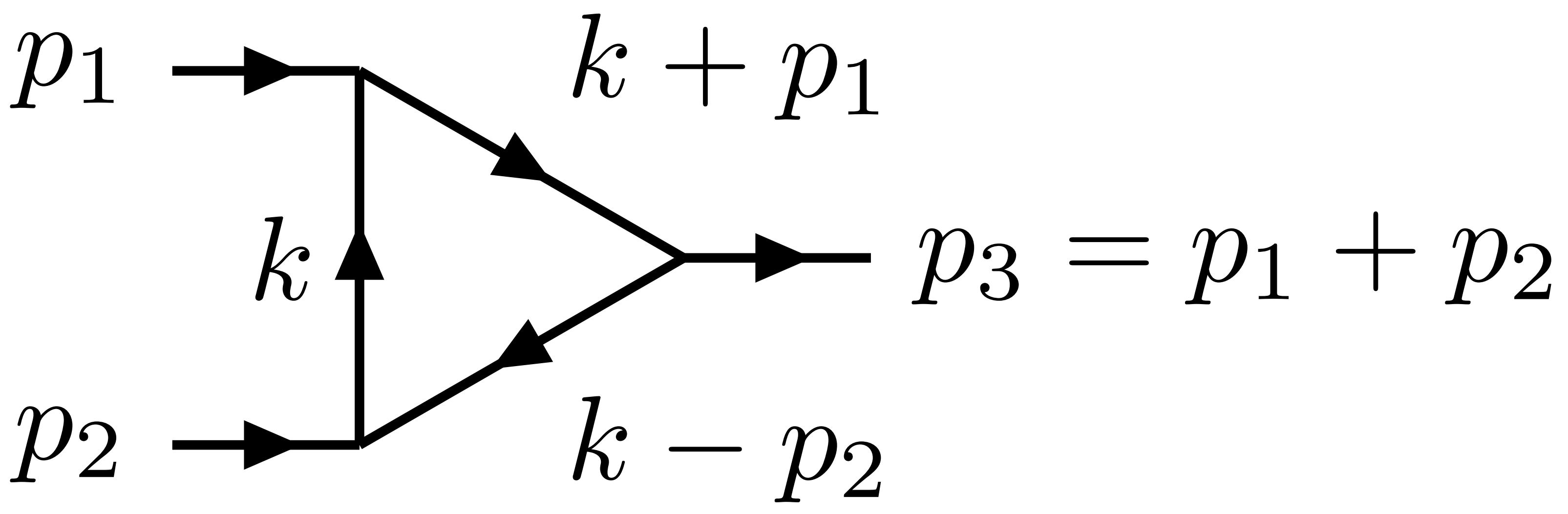

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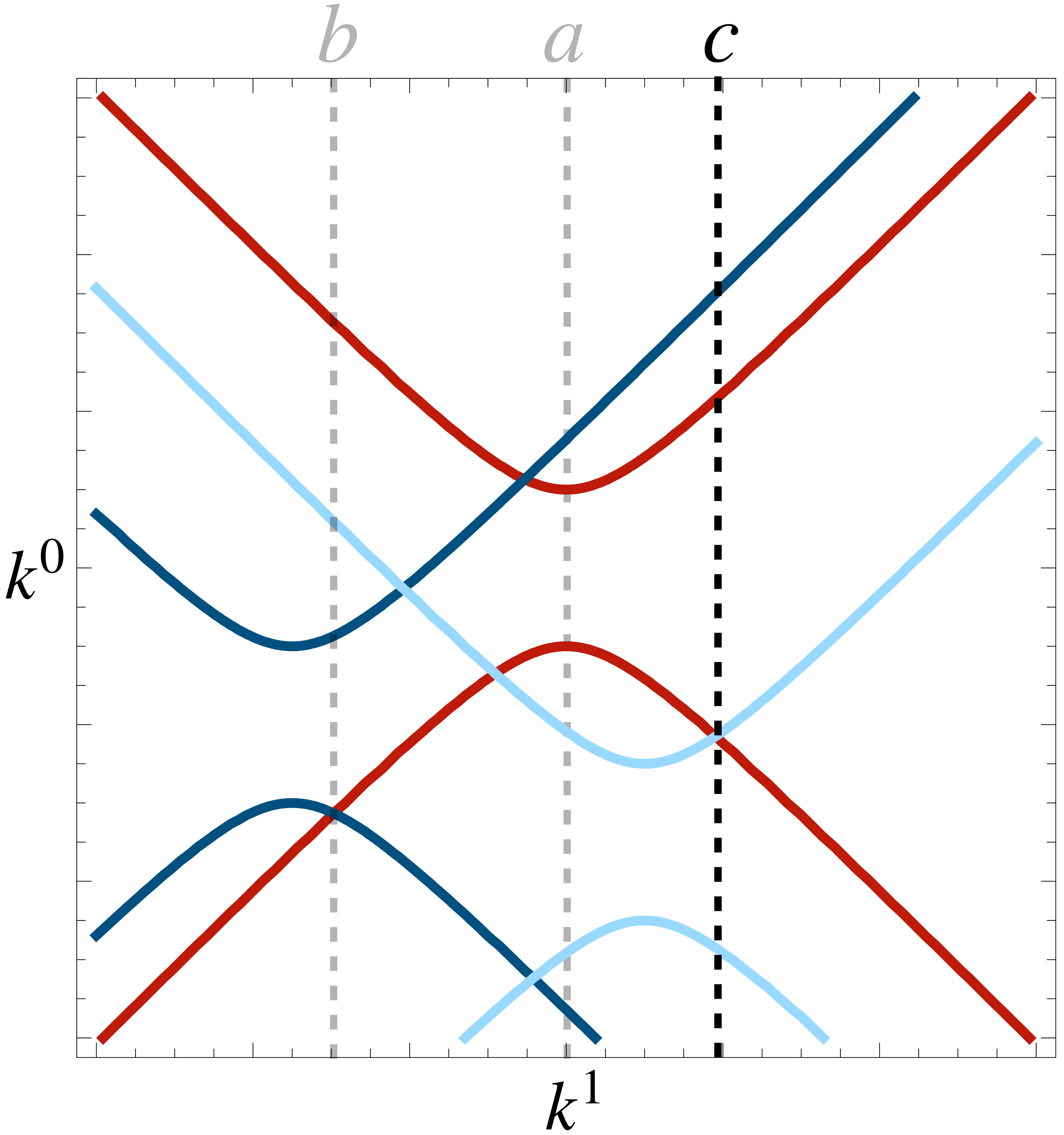

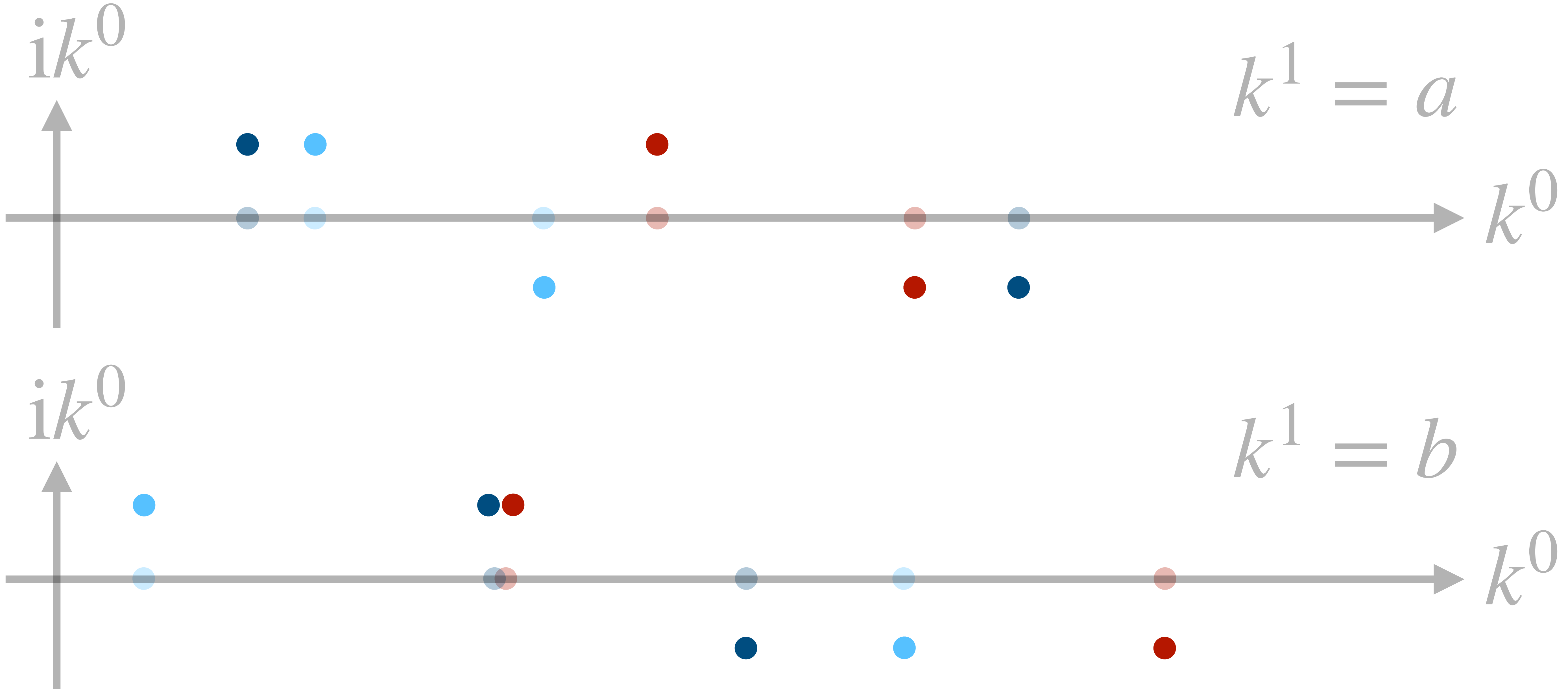

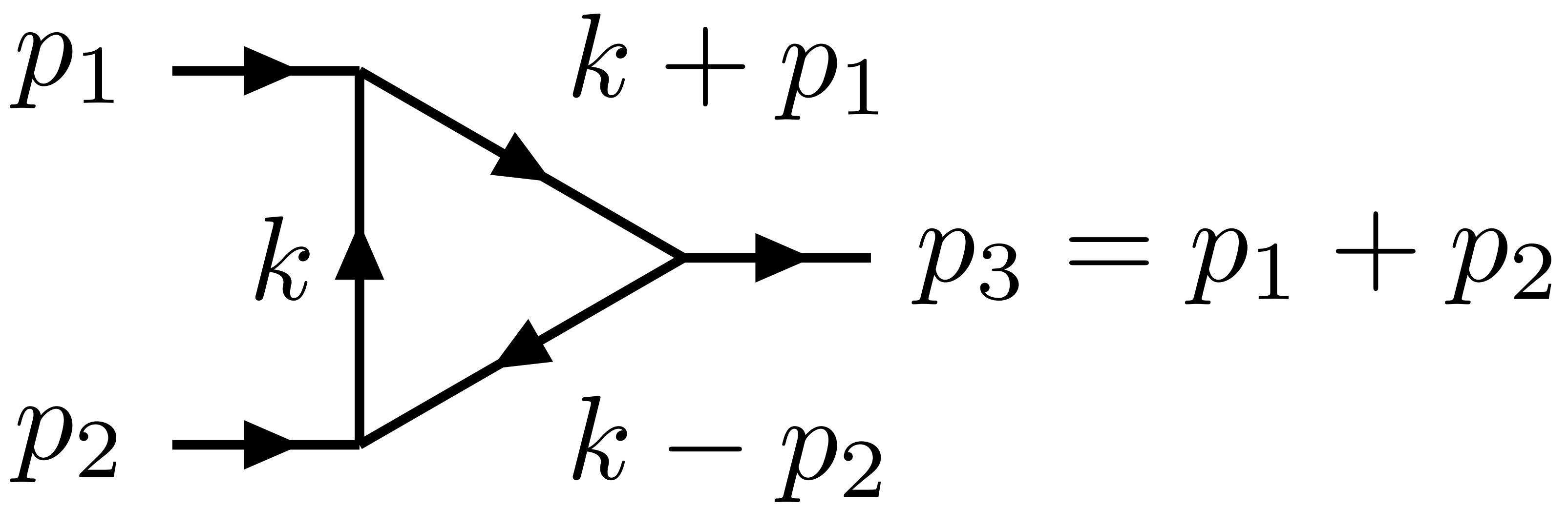

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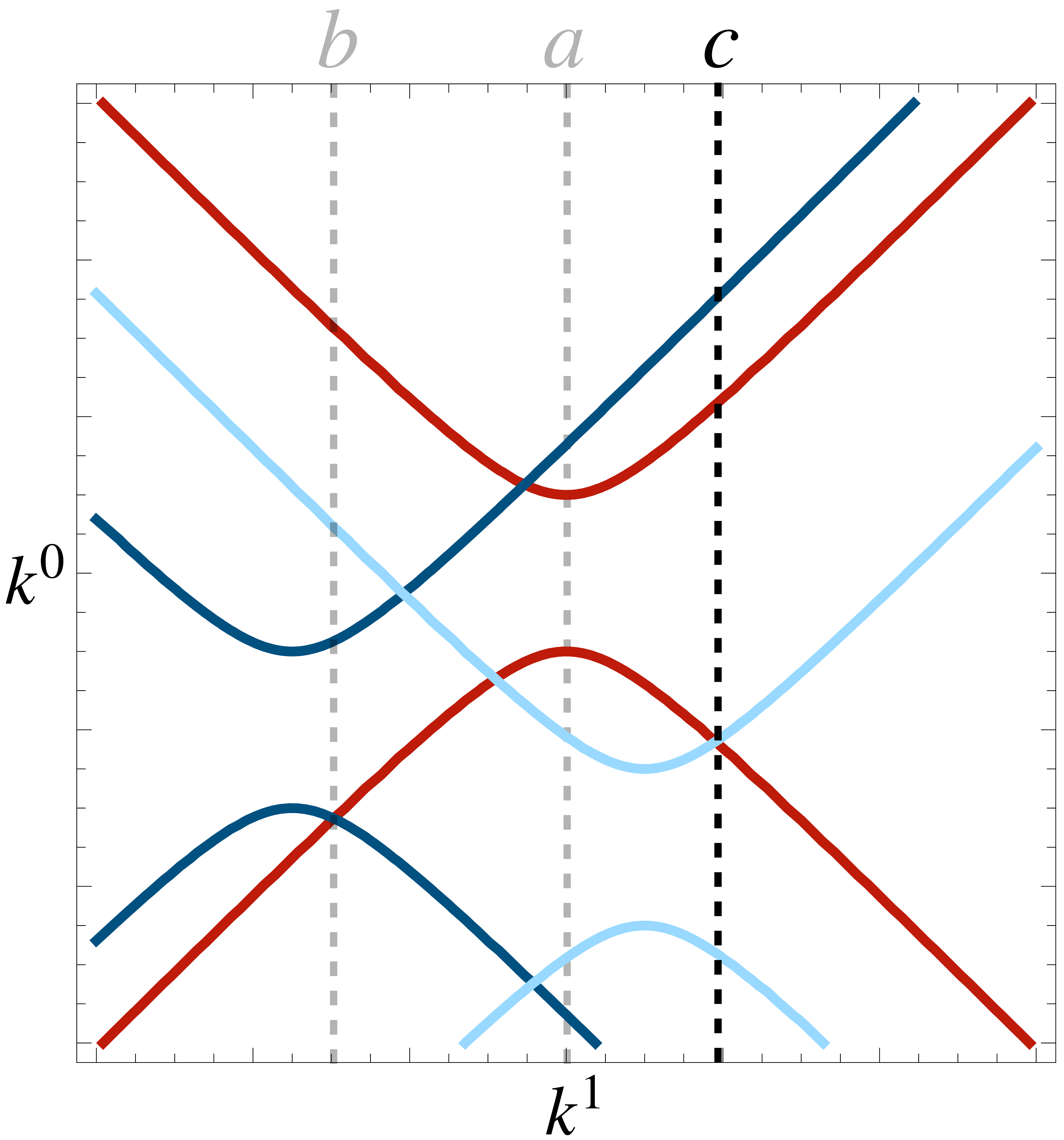

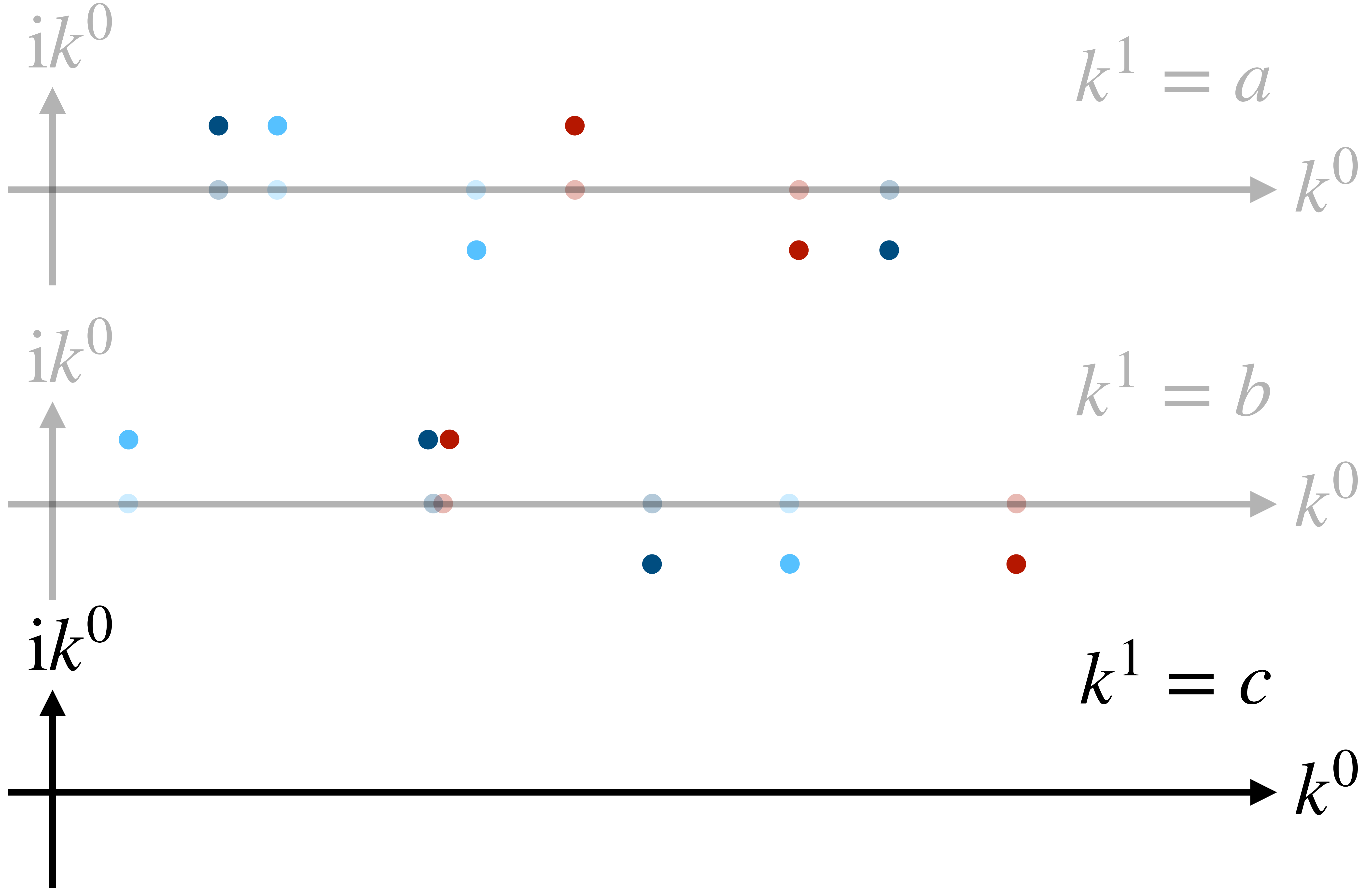

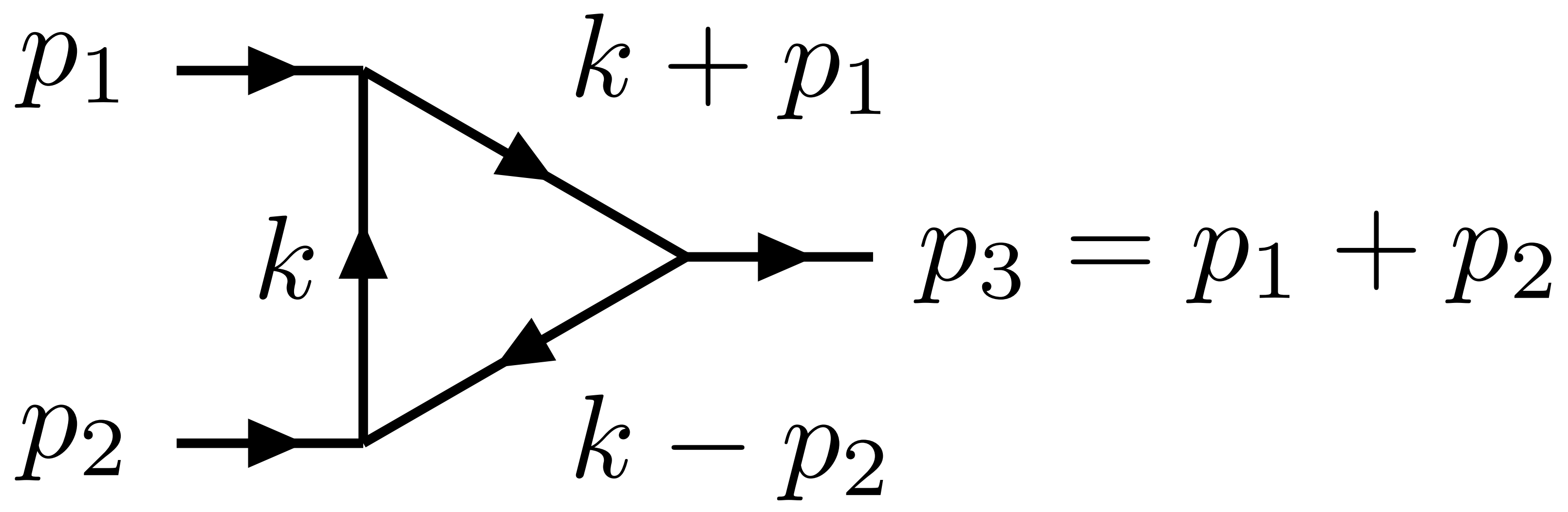

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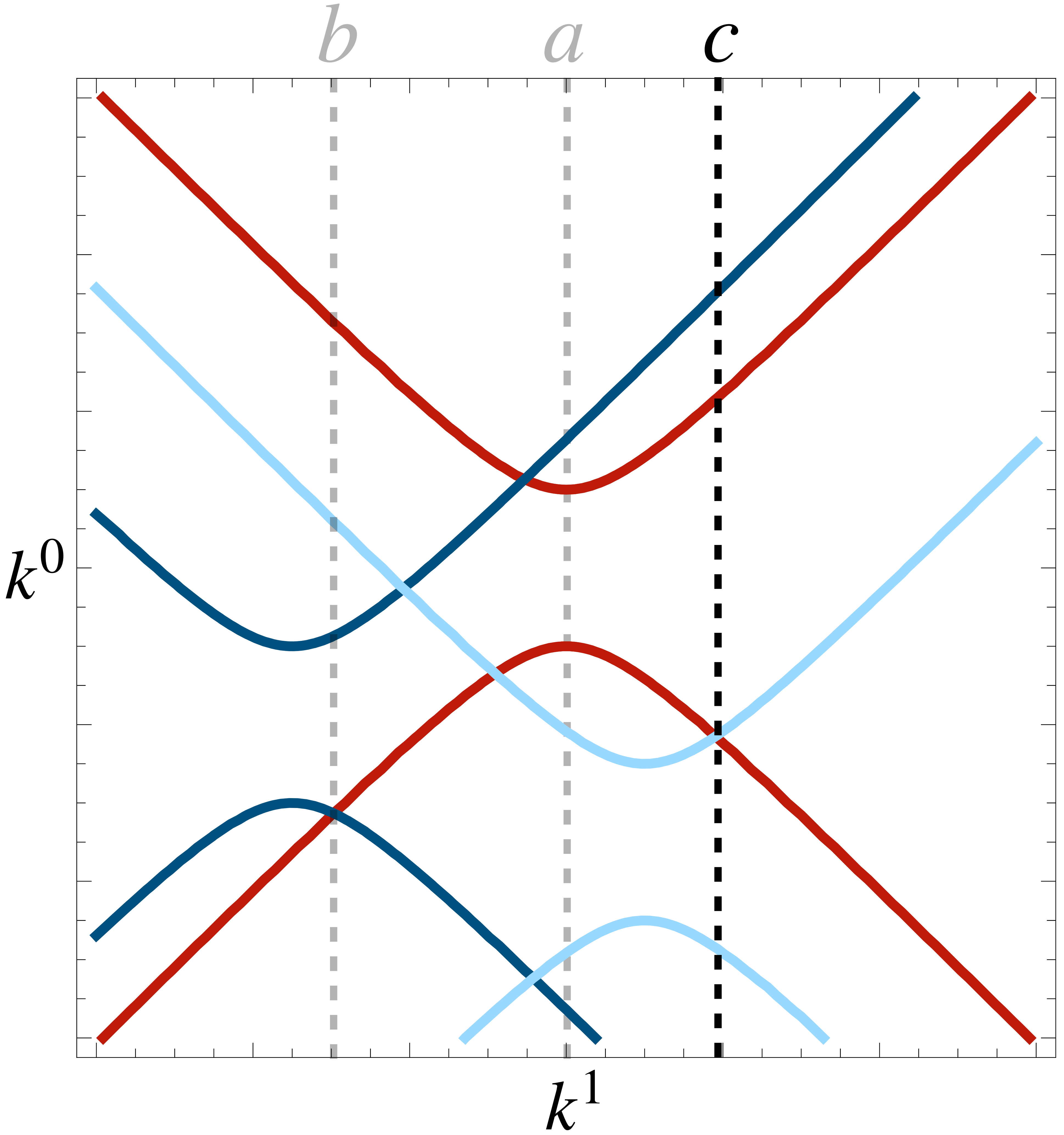

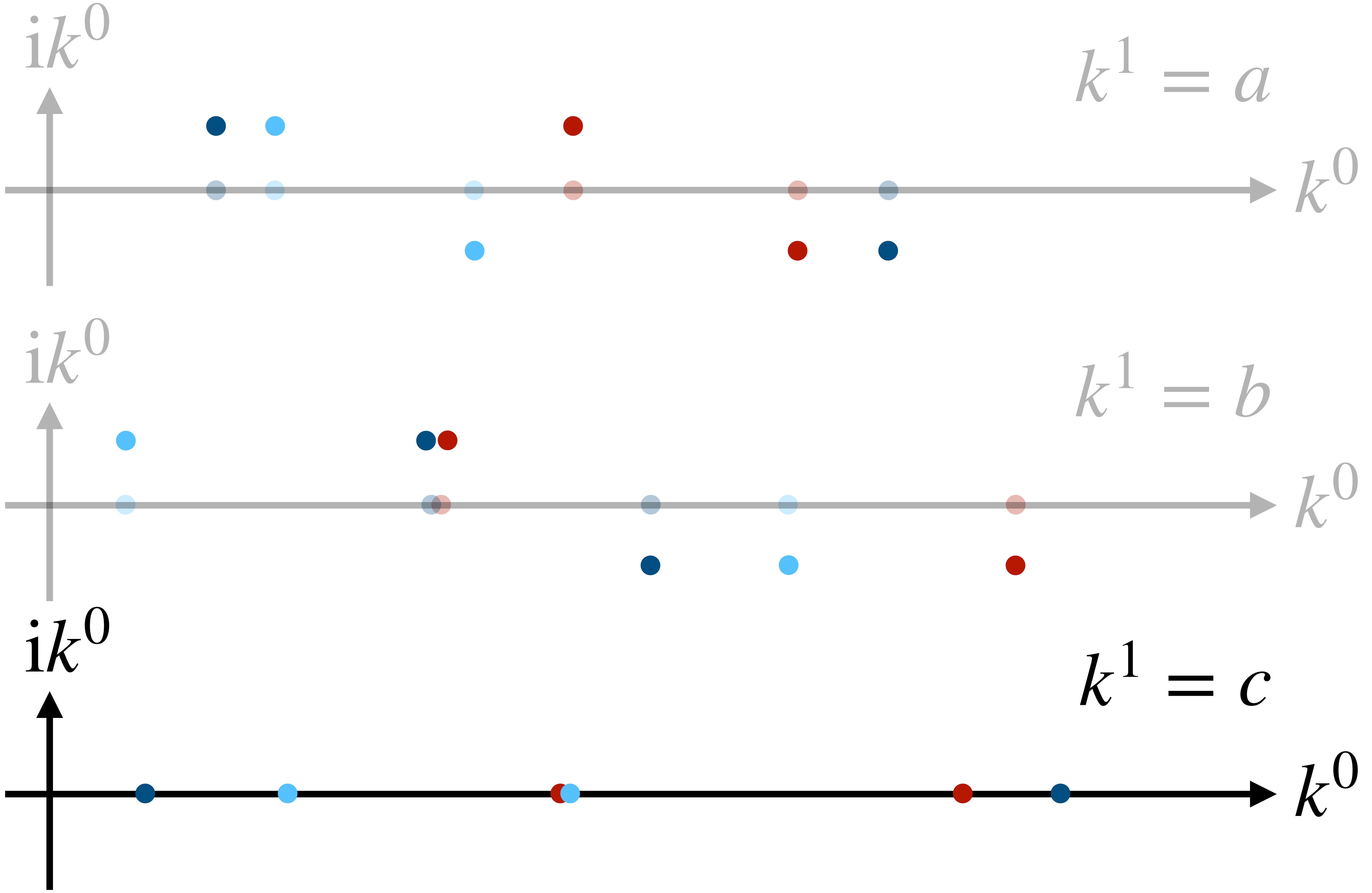

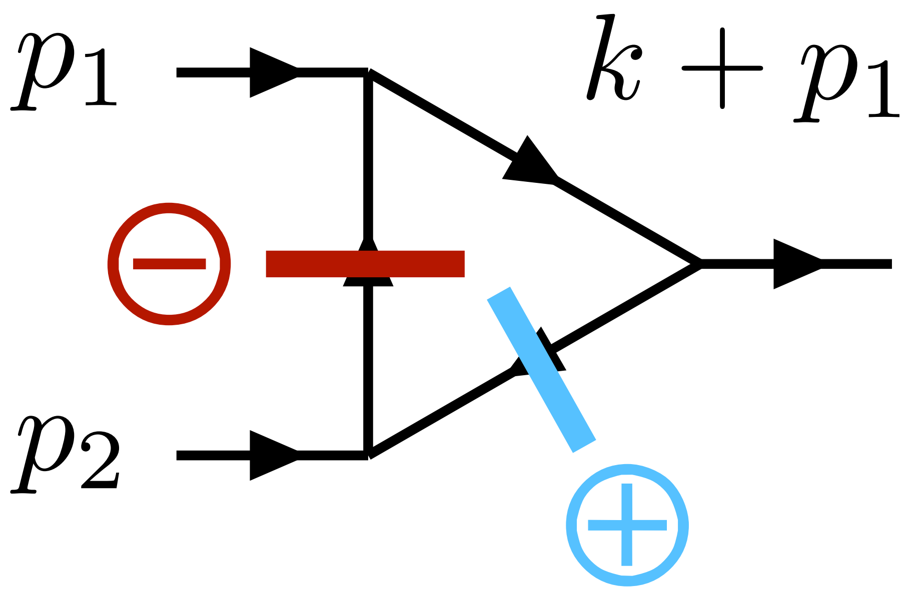

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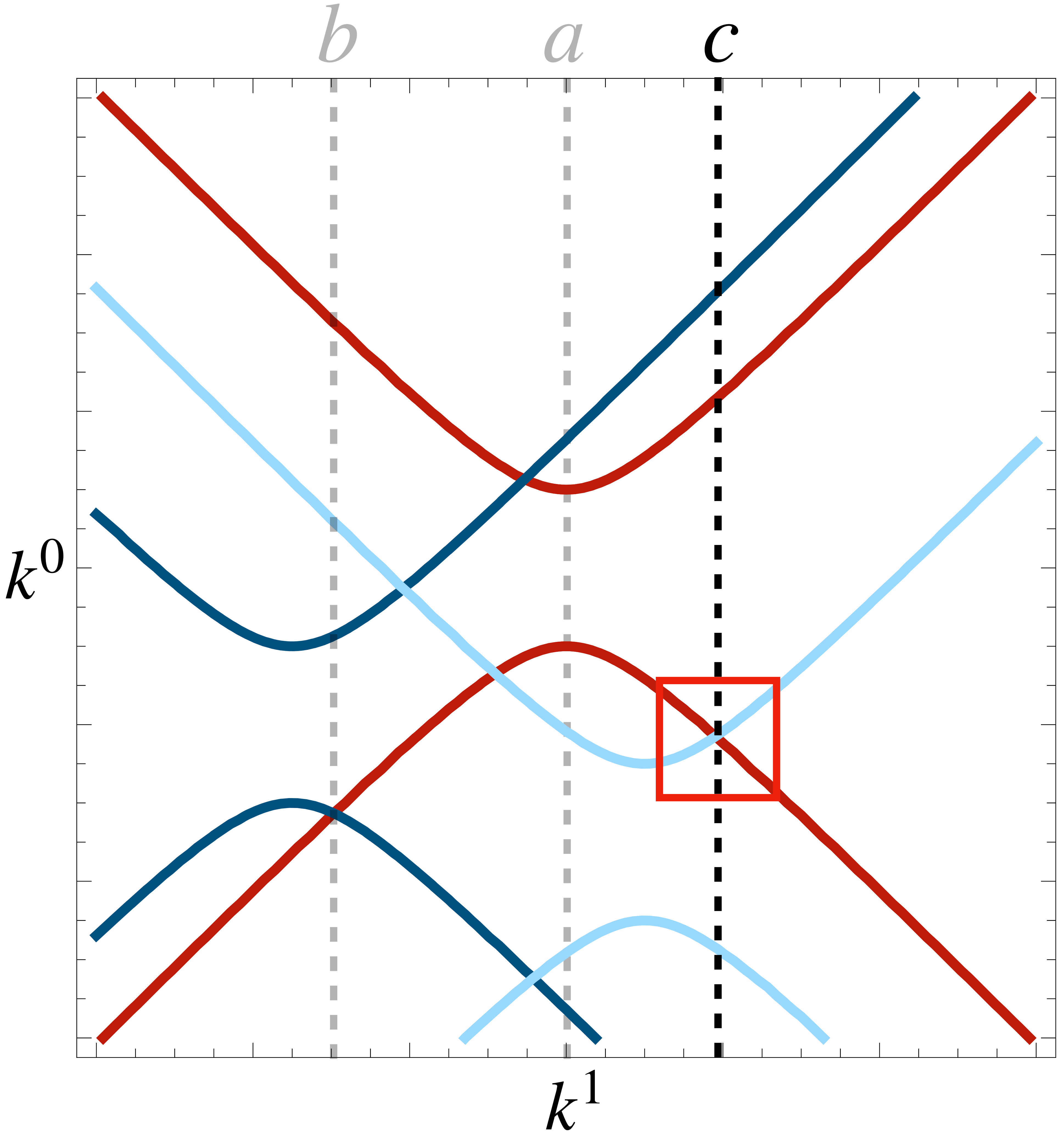

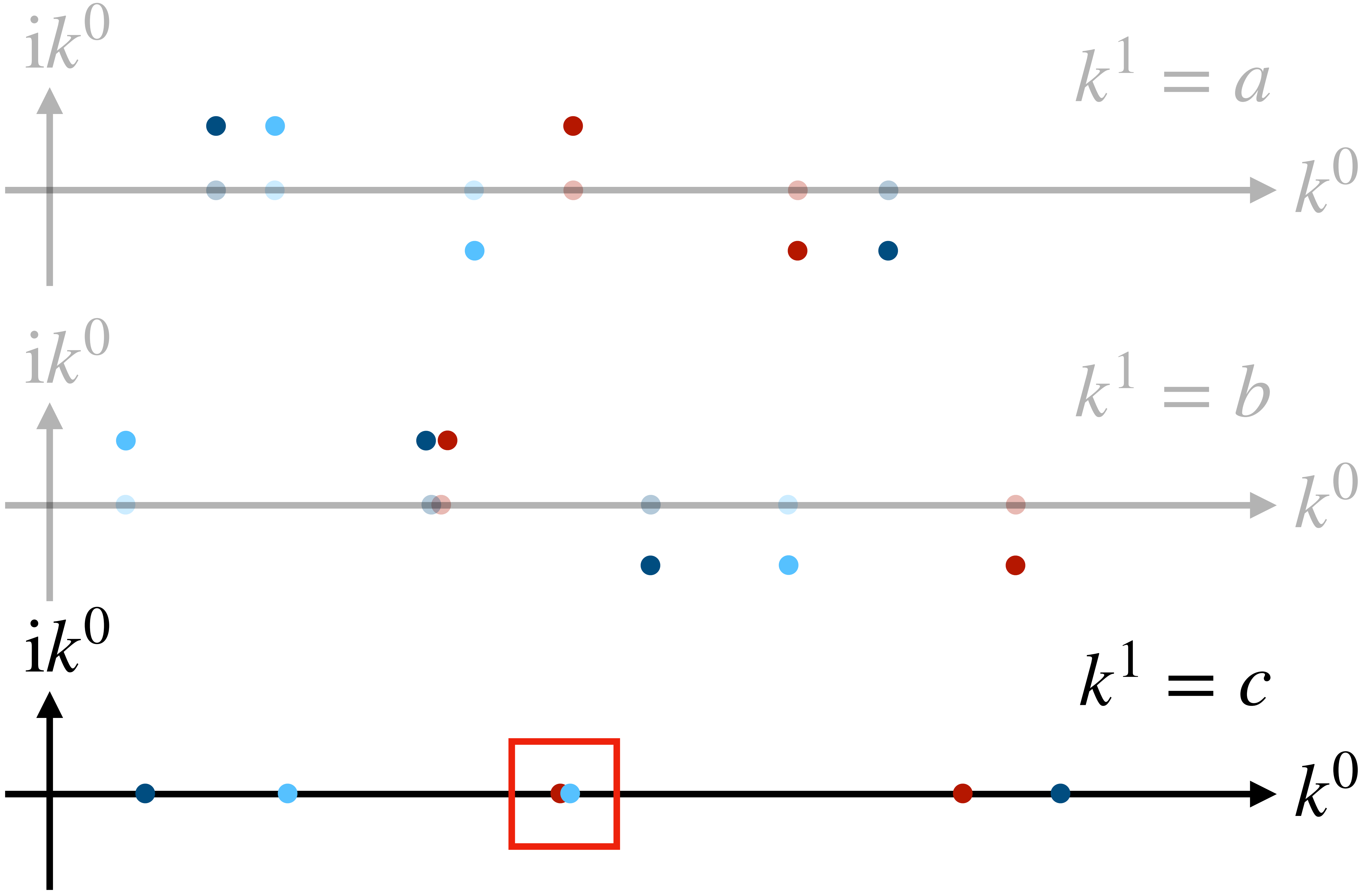

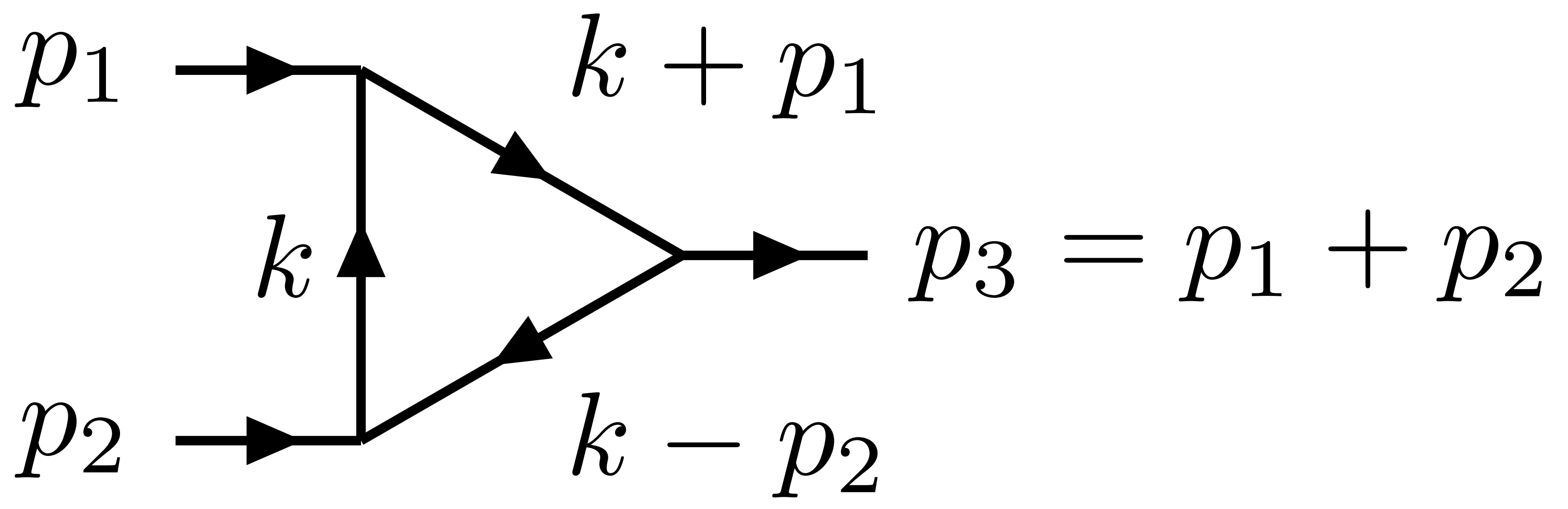

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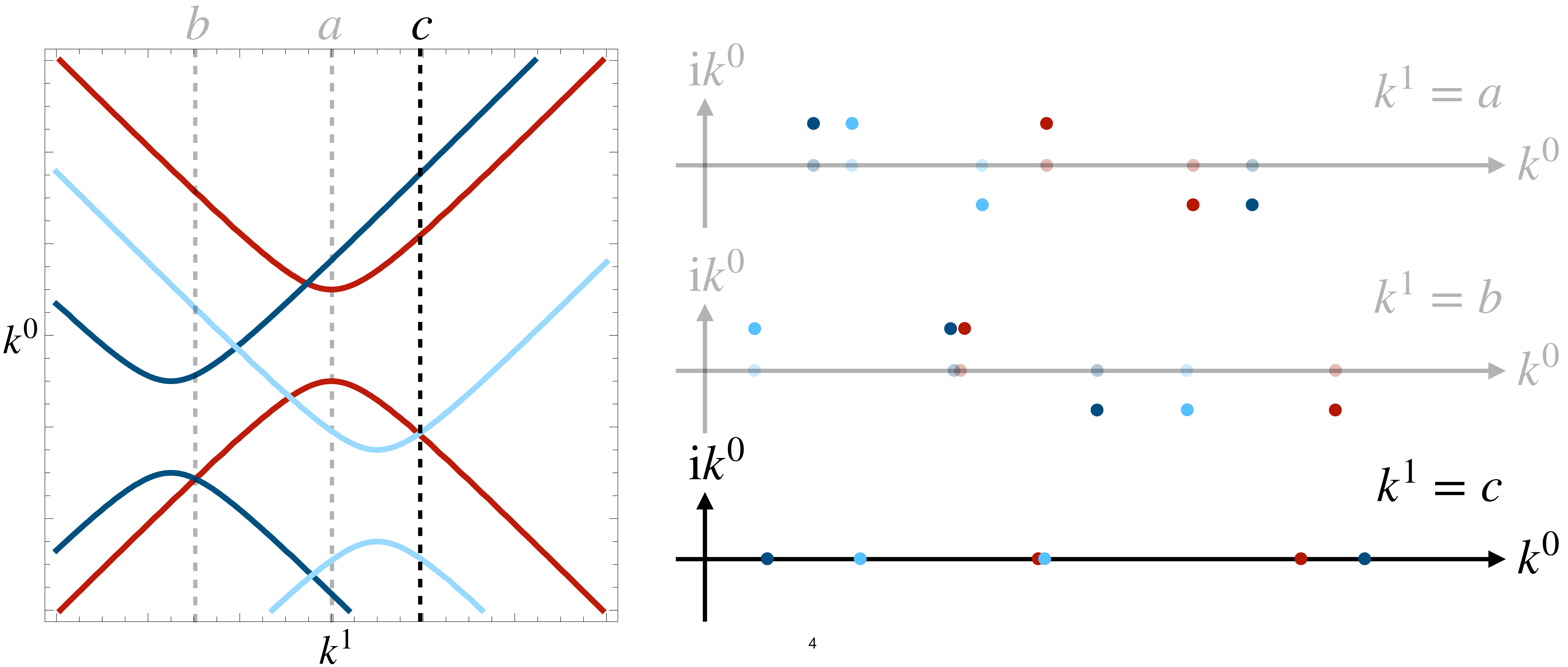

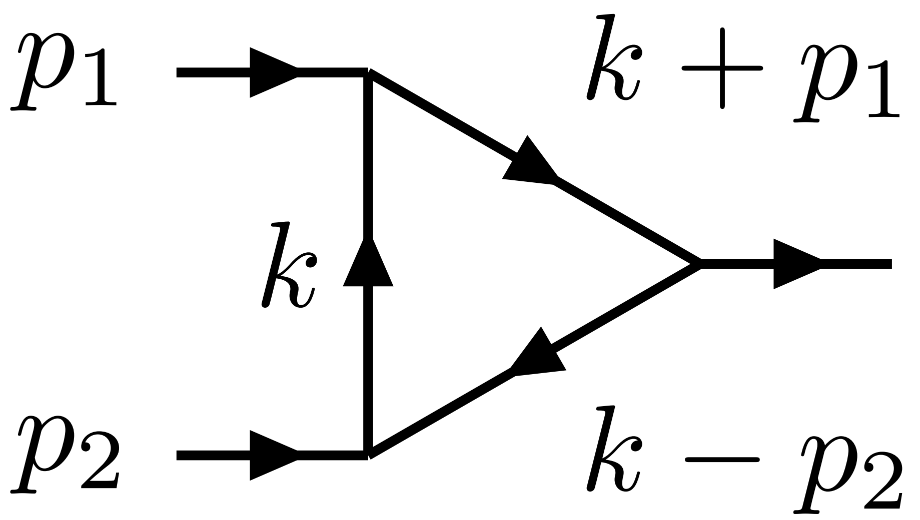

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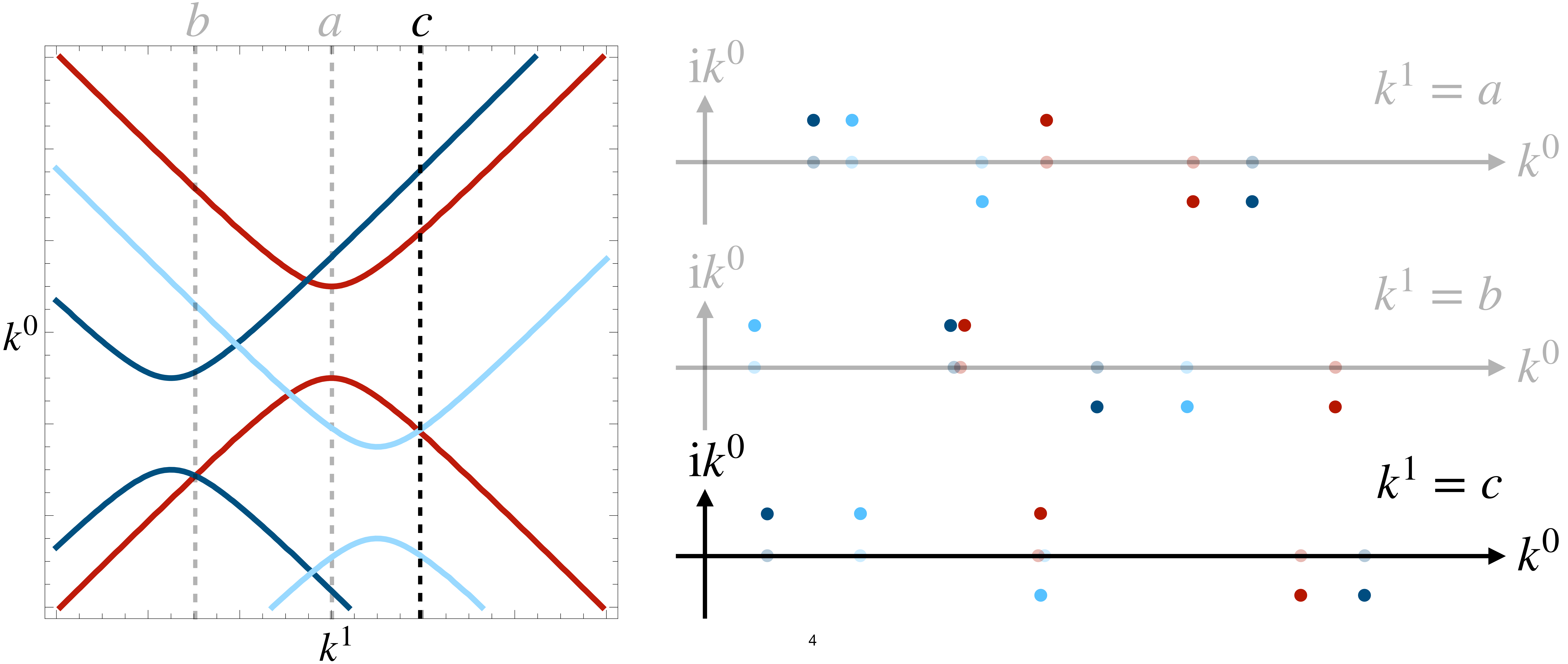

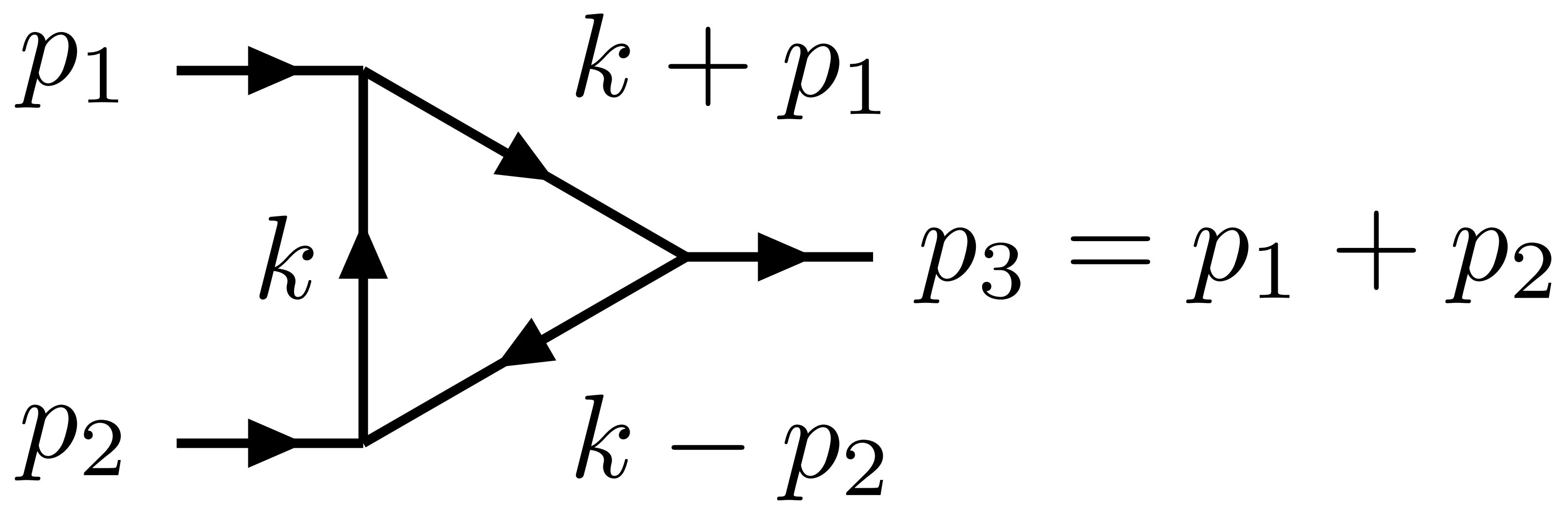

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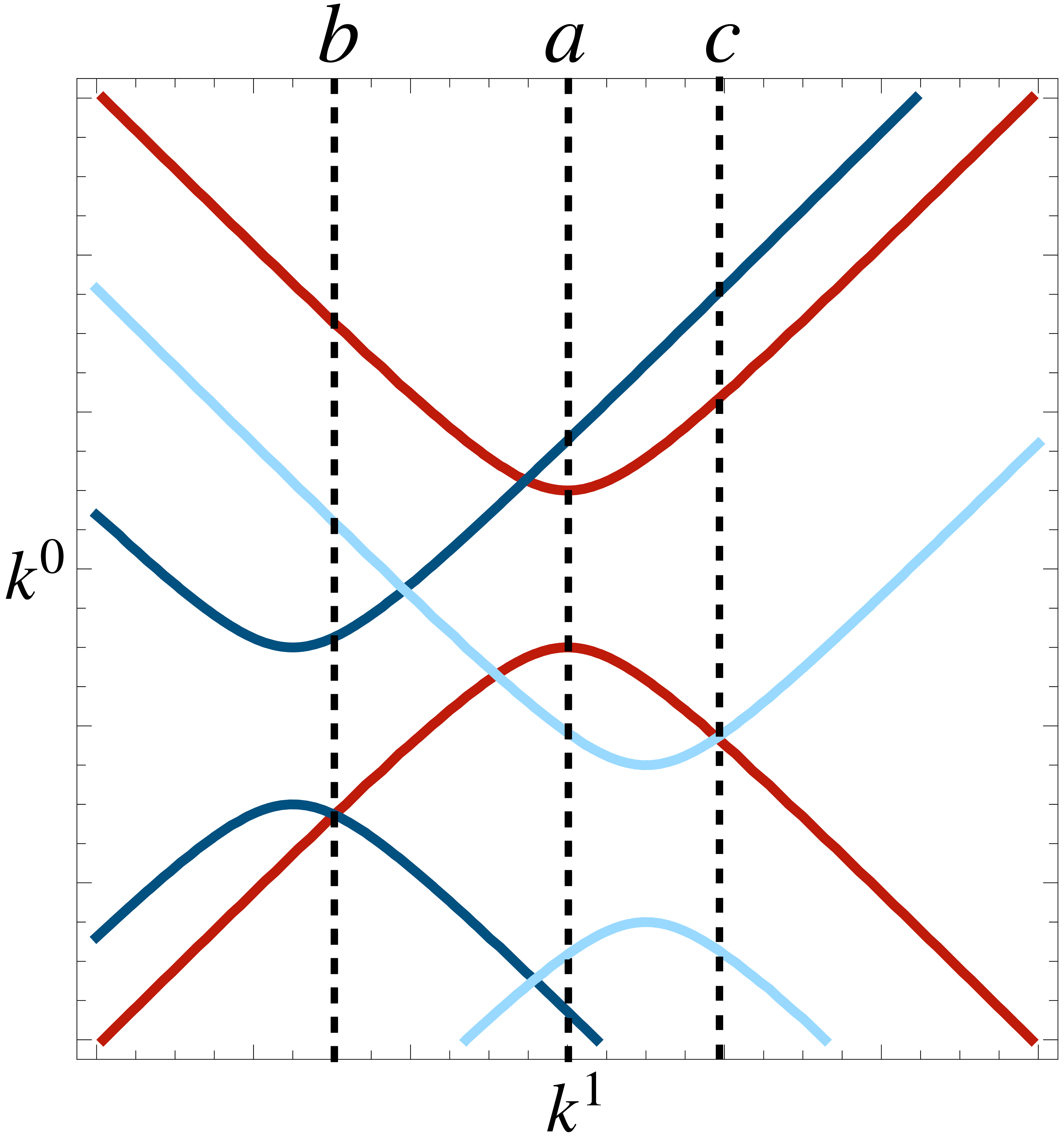

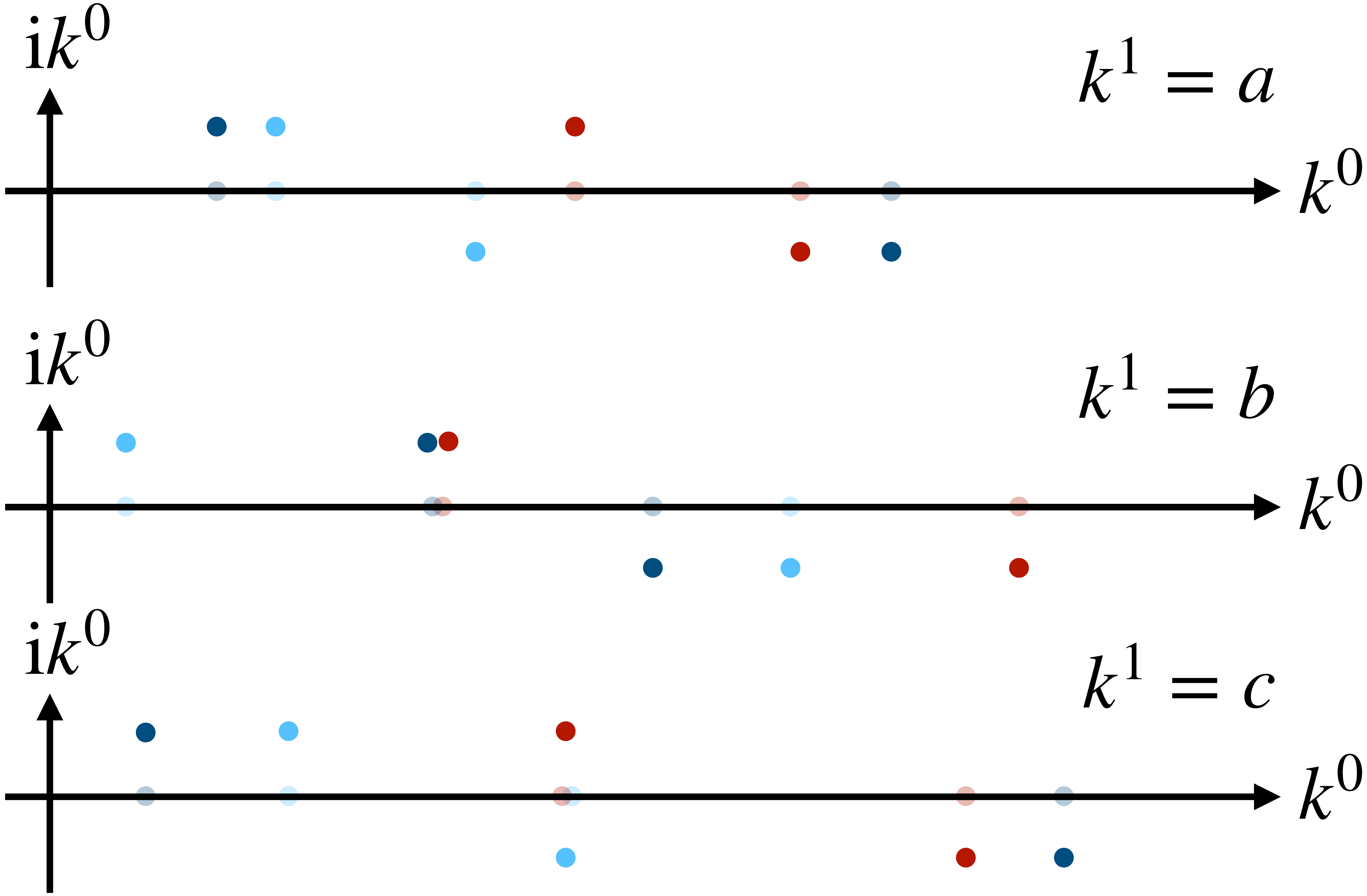

Idea

subtract poles in one dimension at a time

$$I = \lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}x \, \mathscr{I}(x)$$

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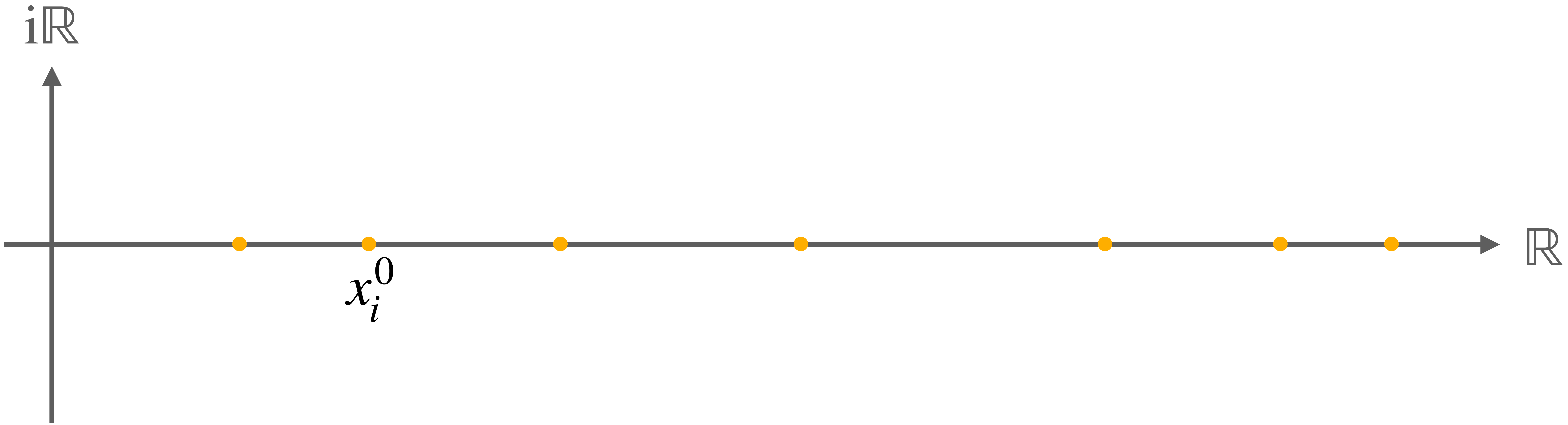
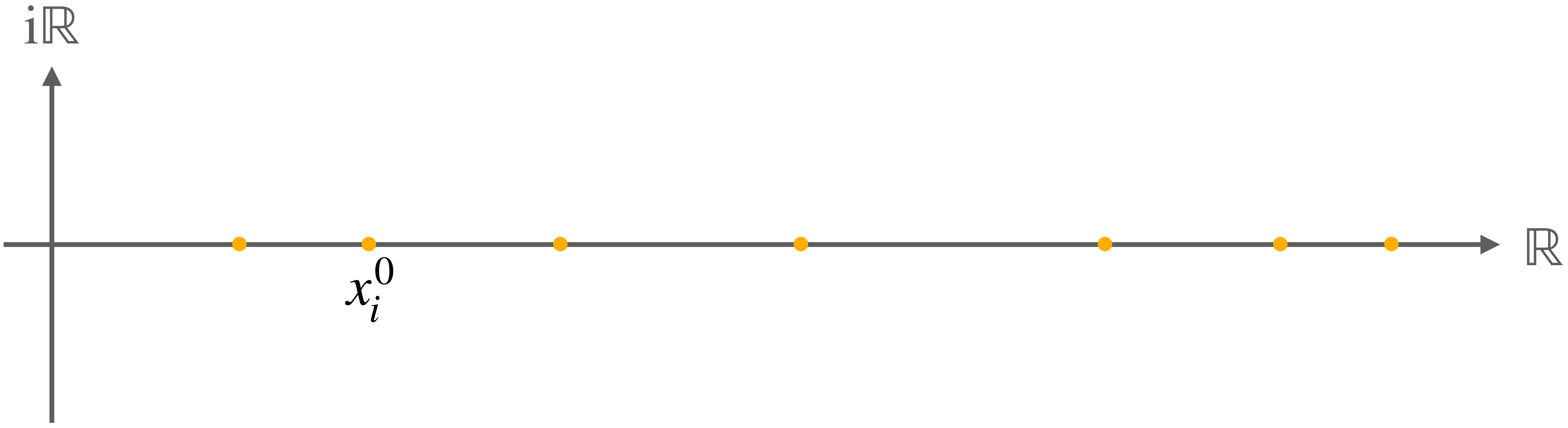


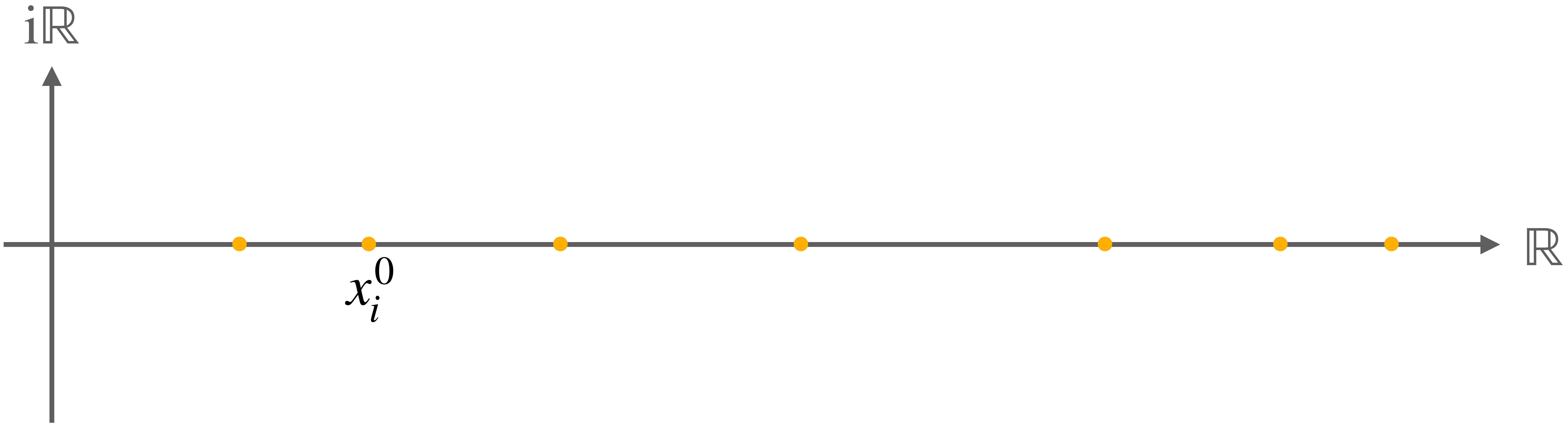
$$I = \lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}x \, \mathscr{I}(x)$$

Monte Carlo integration



$$I = \lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}x \, \mathcal{I}(x)$$

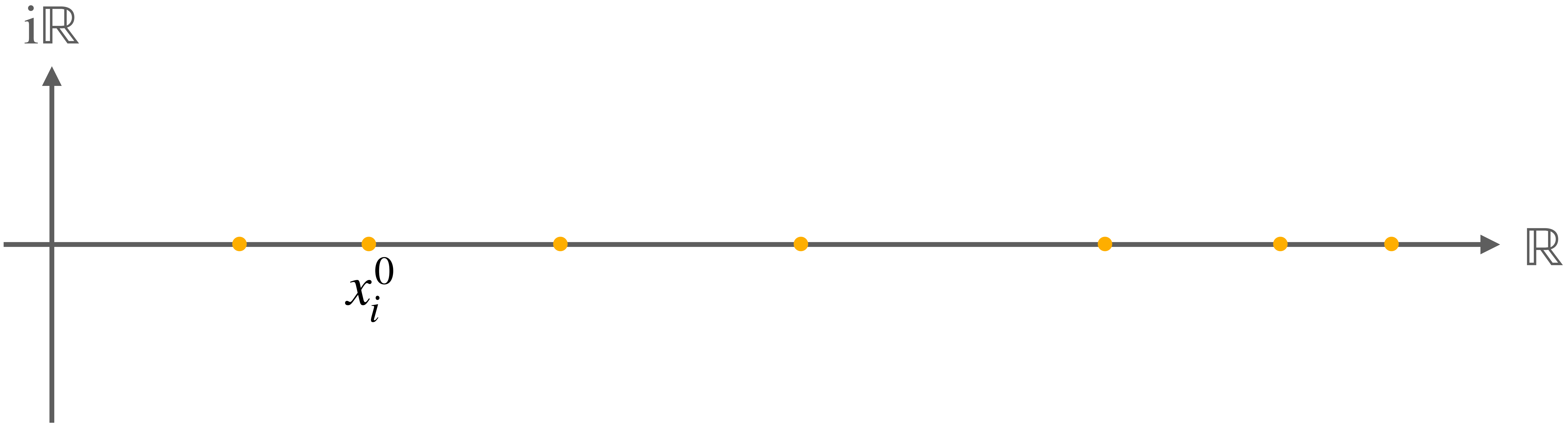
~~Monte Carlo integration~~



$$I = \lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}x \, \mathscr{I}(x)$$

~~**Monte Carlo integration**~~

$\Rightarrow$ **remove poles first**



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behavior of integrand around pole

$$\mathcal{I}(x) = \frac{\mathrm{Res}[\mathcal{I}(y), y = x_i]}{x - x_i} + \mathcal{O}((x - x_i)^0)$$

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behavior of integrand around pole

$$\mathscr{I}(x) = \frac{\mathrm{Res}[\mathscr{I}(y), y = x_i]}{x - x_i} + \mathcal{O}((x - x_i)^0)$$

introduce counterterm

$$I = \lim_{\delta \to 0} \int_{\mathbb{R}} \mathrm{d}x \, \mathscr{I}(x)$$

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$$\mathscr{I}(x) = \frac{\mathrm{Res}[\mathscr{I}(y), y = x_i]}{x - x_i} + \mathcal{O}((x - x_i)^0)$$

introduce counterterm

$$\mathrm{CT}_i(x) = \frac{\mathrm{Res}[\mathscr{I}(y), y = x_i]}{x - x_i} S_{\mathrm{UV}}(x - x_i)$$

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behavior of integrand around pole

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symmetric UV suppression S_{UV}

$$I = \lim_{\delta\to 0}\int_{\mathbb{R}} \mathrm{d}x\, \mathscr{I}(x)$$

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introduce counterterm

independent of x

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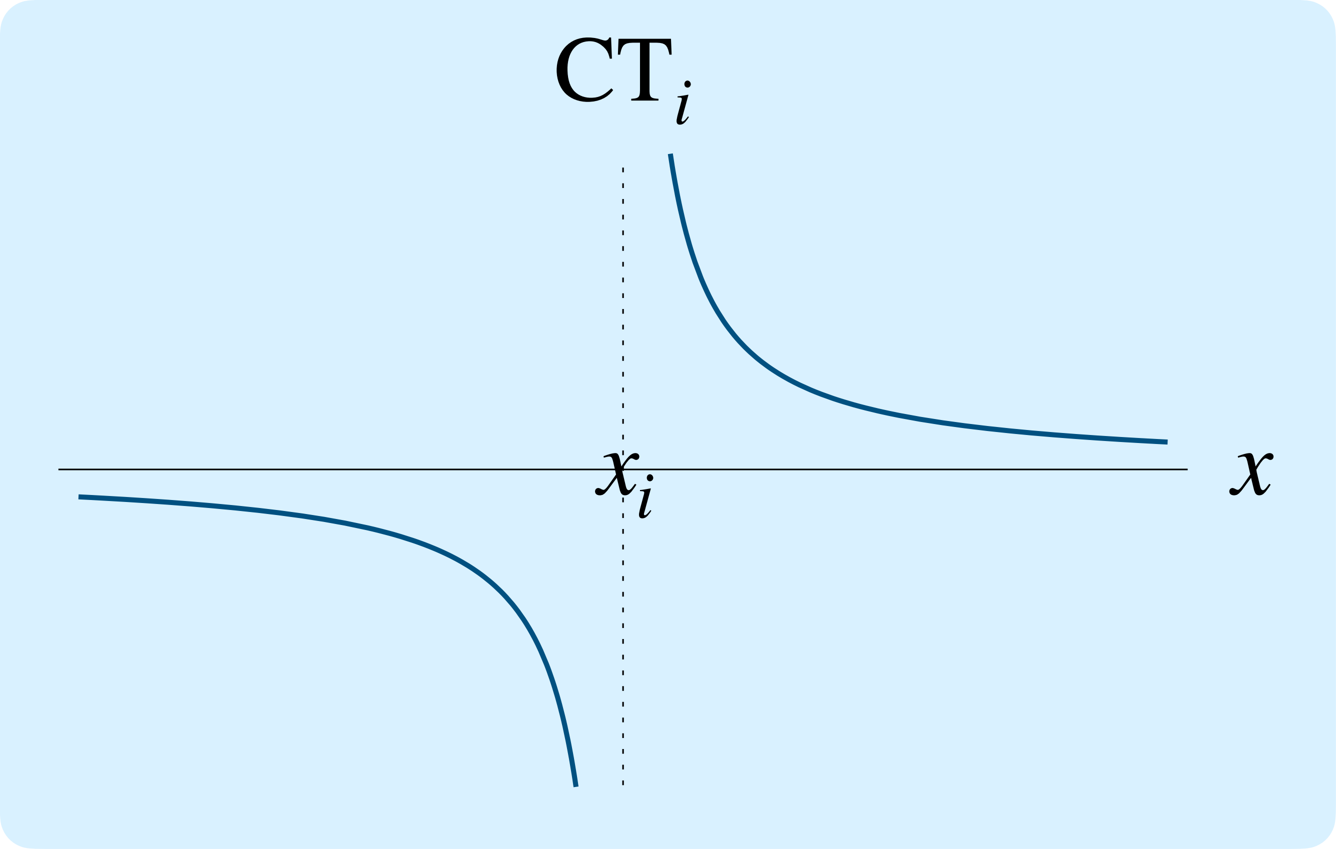

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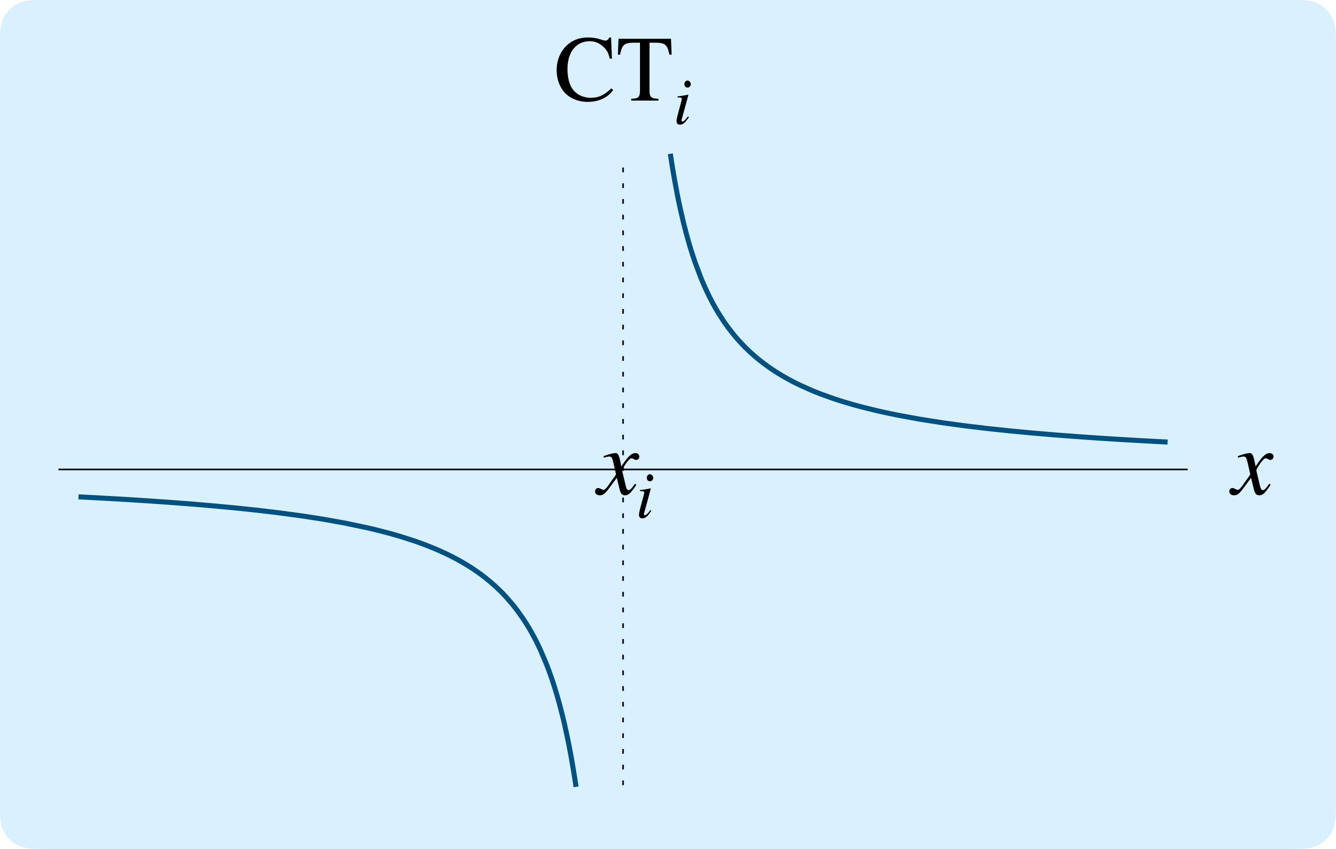
introduce counterterm



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anti-symmetric around x_i

symmetric UV suppression S_{UV}

integrate back using

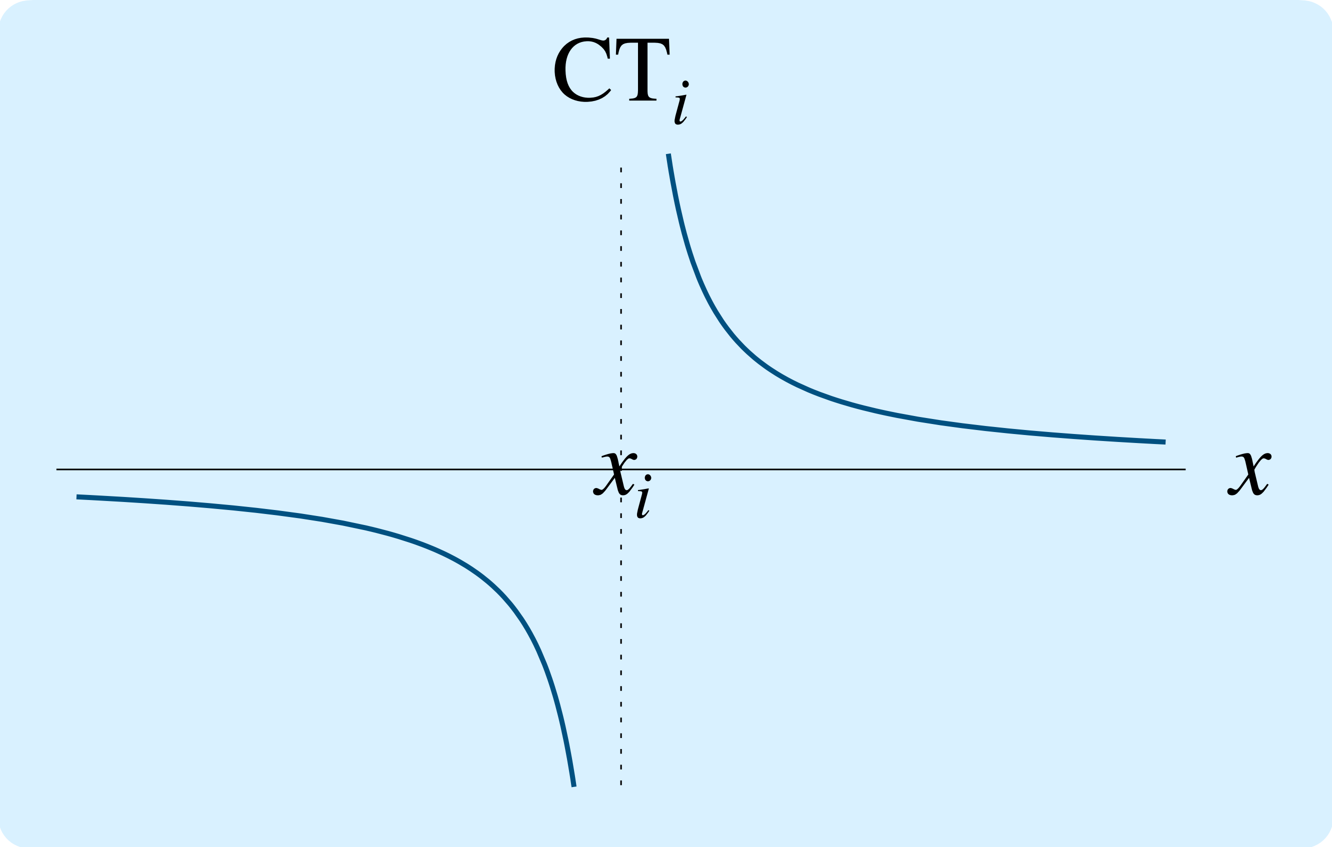
$$\lim_{\delta\to 0}\frac{1}{x \mp \mathrm{i}\delta} = \mathrm{PV}\frac{1}{x} \pm \mathrm{i}\pi\delta(x)$$

$$I = \lim_{\delta\to 0}\int_{\mathbb{R}} \mathrm{d}x\,\mathscr{I}(x)$$

behavior of integrand around pole

$$\mathscr{I}(x) = \frac{\mathrm{Res}[\mathscr{I}(y), y = x_i]}{x - x_i} + \mathcal{O}((x - x_i)^0)$$

introduce counterterm



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$$\lim_{\delta\to 0}\int_{\mathbb{R}} \mathrm{CT}_i(x)\,\mathrm{d}x = \mathrm{PV}\int_{\mathbb{R}} \mathrm{CT}_i(x)\,\mathrm{d}x + \mathrm{i}\pi\,\mathrm{sgn}(\mathrm{Im}\,x_i)\mathrm{Res}[\mathscr{I}(y), y = x_i]$$

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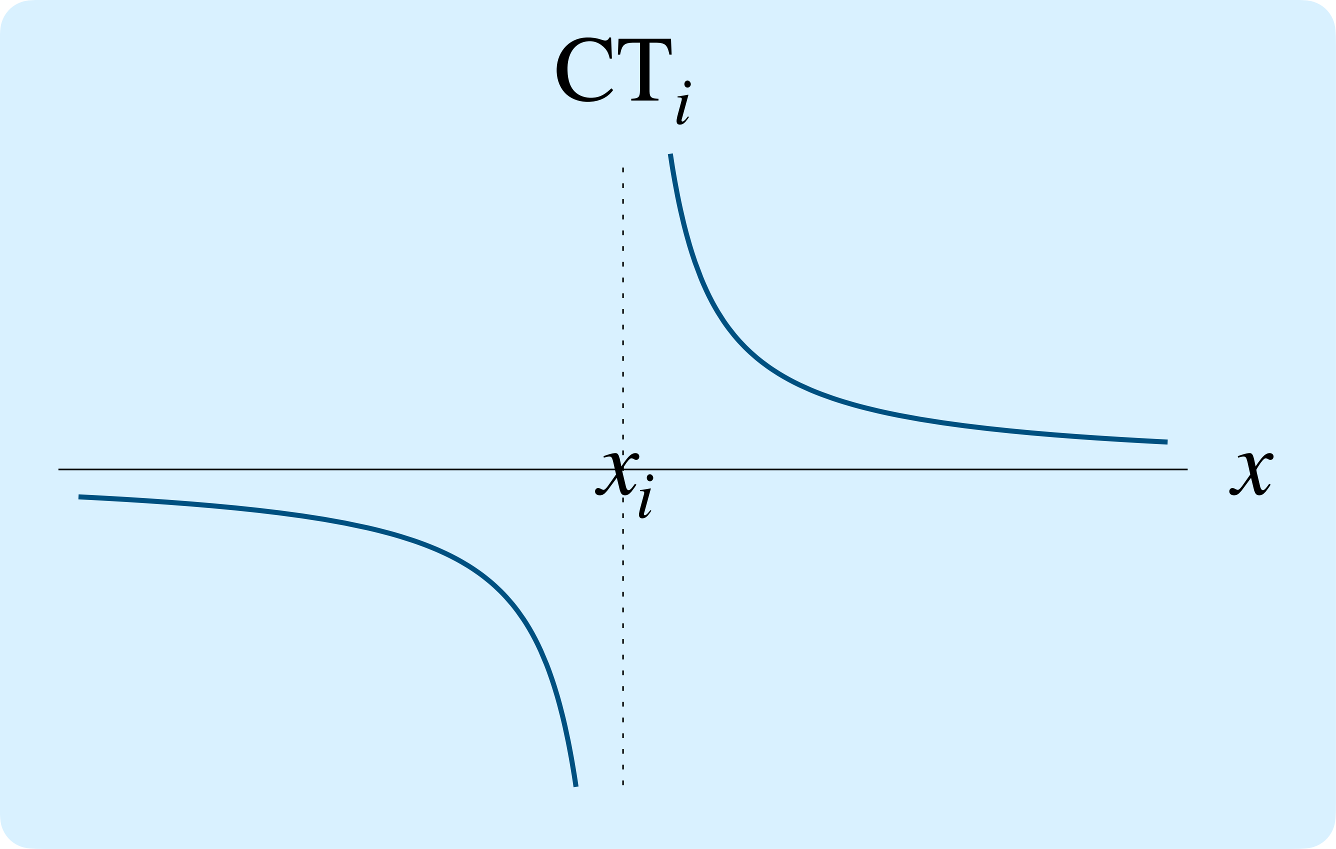

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$$\lim_{\delta\to 0}\int_{\mathbb{R}} \mathrm{CT}_i(x)\,\mathrm{d}x = \underbrace{\cancel{\mathrm{PV}\int_{\mathbb{R}} \mathrm{CT}_i(x)\,\mathrm{d}x}}_{=0} + \mathrm{i}\pi\,\mathrm{sgn}(\mathrm{Im}\,x_i)\mathrm{Res}[\mathscr{I}(y), y = x_i]$$

$$\mathrm{Re}\, I = \int_{\mathbb{R}} \mathrm{d}x \left(\mathcal{I}(x) - \sum_i \mathrm{CT}_i(x) \right)$$

$$\mathrm{Im}\, I = \pi \sum_i \mathrm{sgn}(\mathrm{Im}\, x_i) \mathrm{Res}[\mathcal{I}(y), y = x_i]$$

real and imaginary part separated

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no more poles in integrand

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$\mathrm{i}\delta$ causal prescription removed

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reduced integration dimension

$\mathrm{i}\delta$ causal prescription removed

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no more poles in integrand

$$\mathrm{Re}\, I = \int_{\mathbb{R}} \mathrm{d}x \left(\mathcal{I}(x) - \sum_i \mathrm{CT}_i(x) \right)$$

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reduced integration dimension

$\mathrm{i}\delta$ causal prescription removed

Monte Carlo integration

Higher-order poles?

Higher-order poles?

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Higher-order poles?

Higher-order poles?

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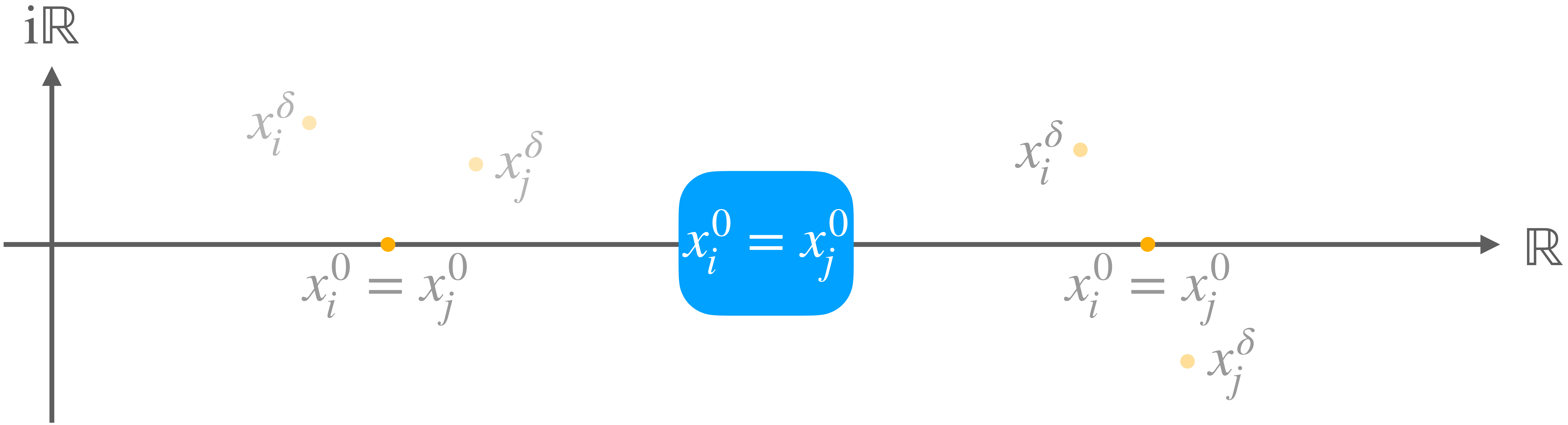
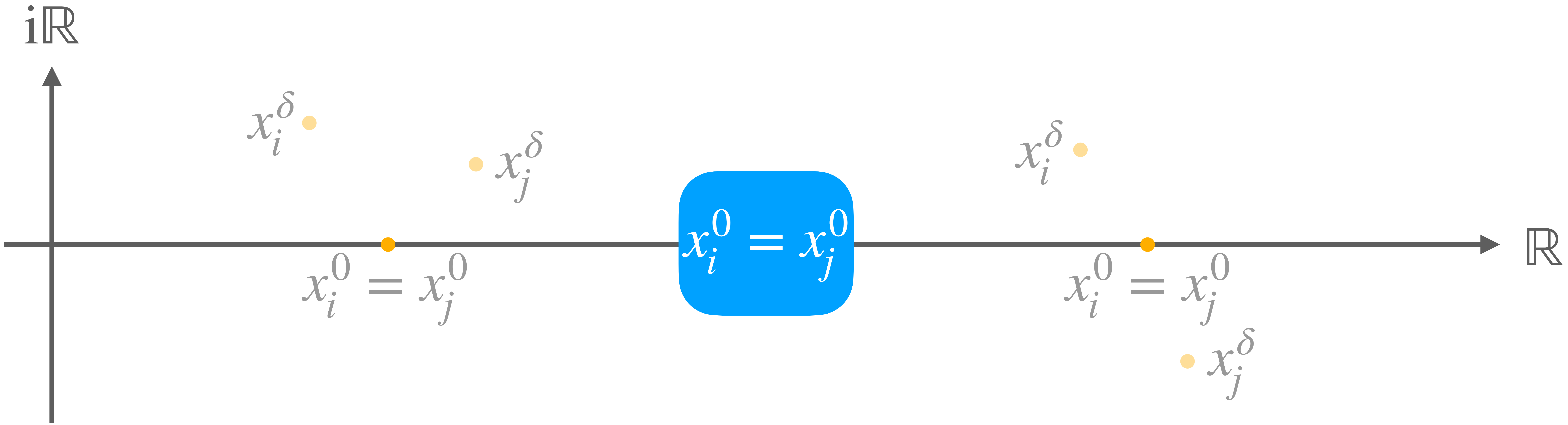


Higher-order poles?



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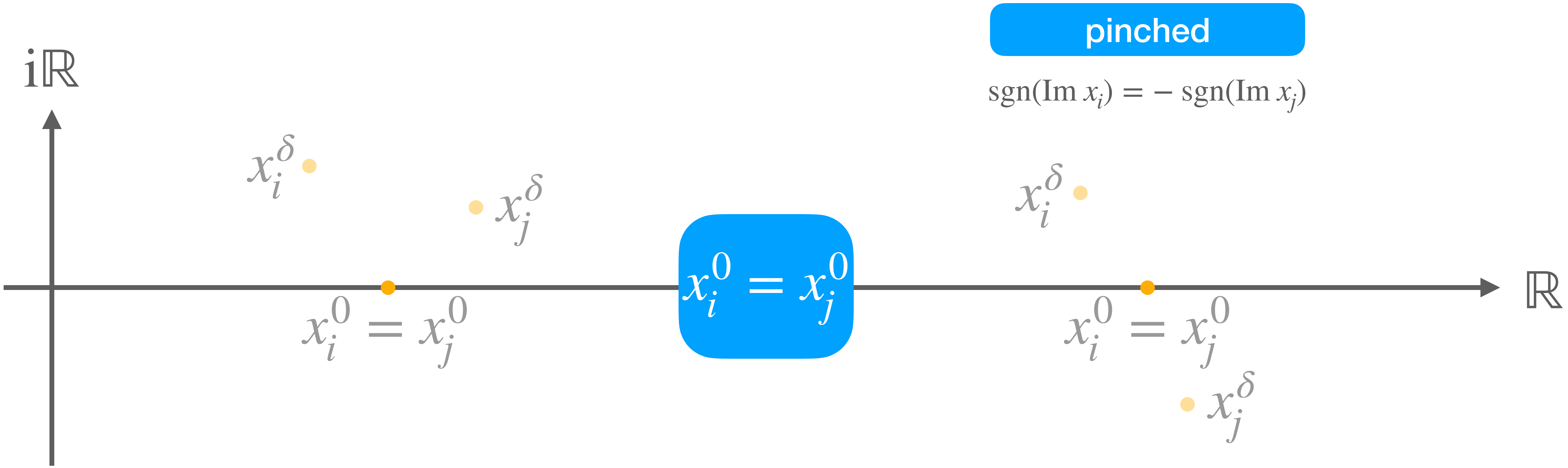

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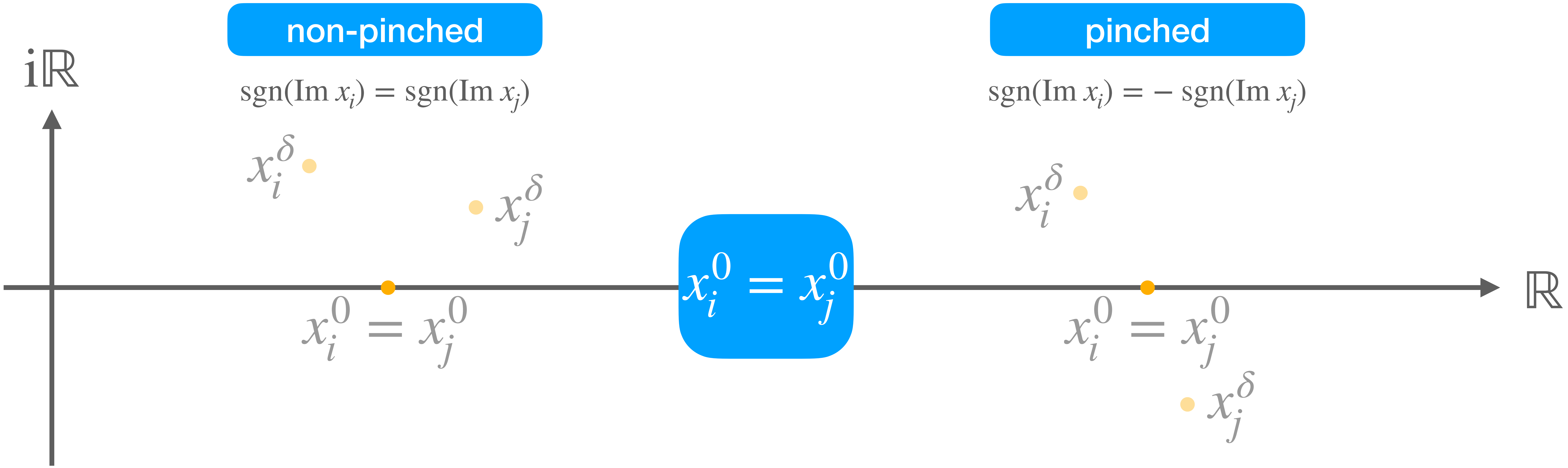

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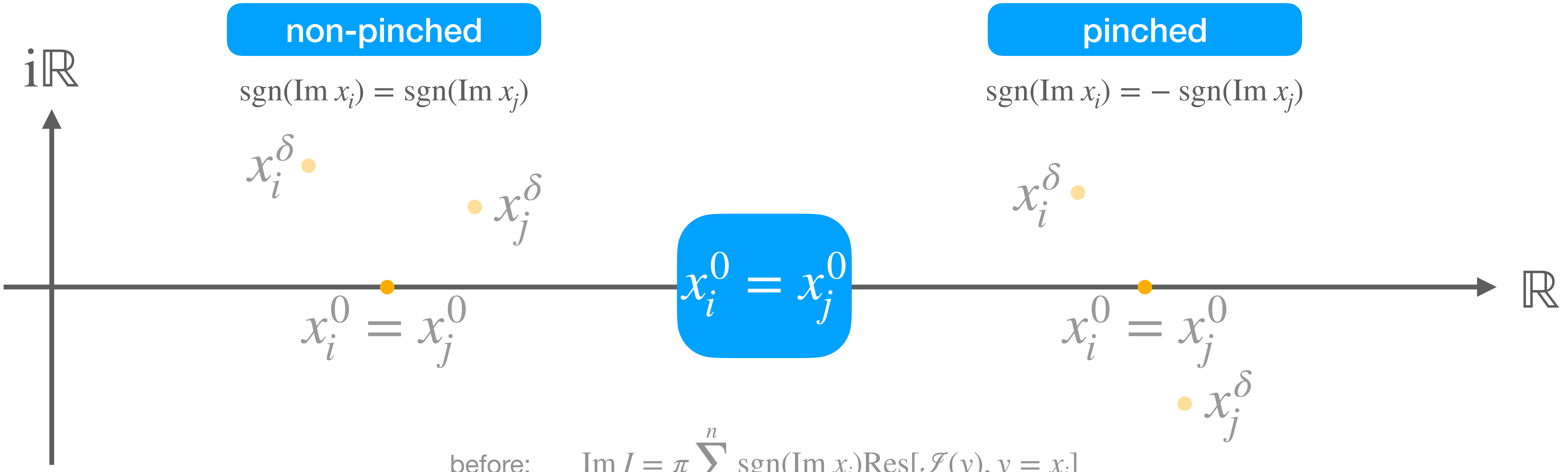
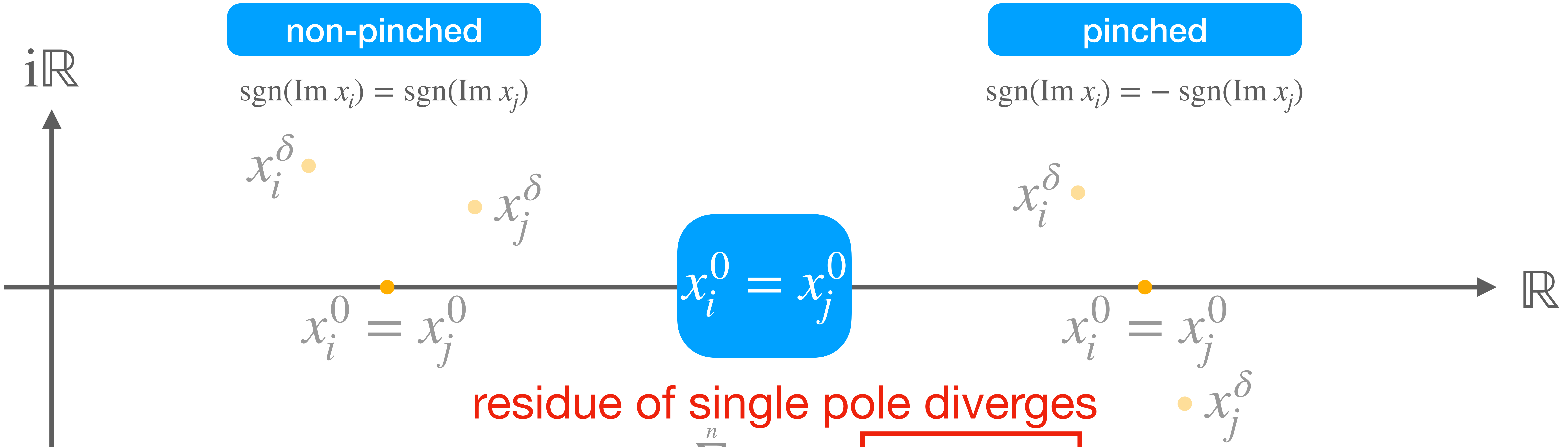


before: $\mathrm{Im}\, I = \pi \sum_{i=1}^{n} \mathrm{sgn}(\mathrm{Im}\, x_i) \mathrm{Res}[\mathcal{I}(y), y = x_i]$

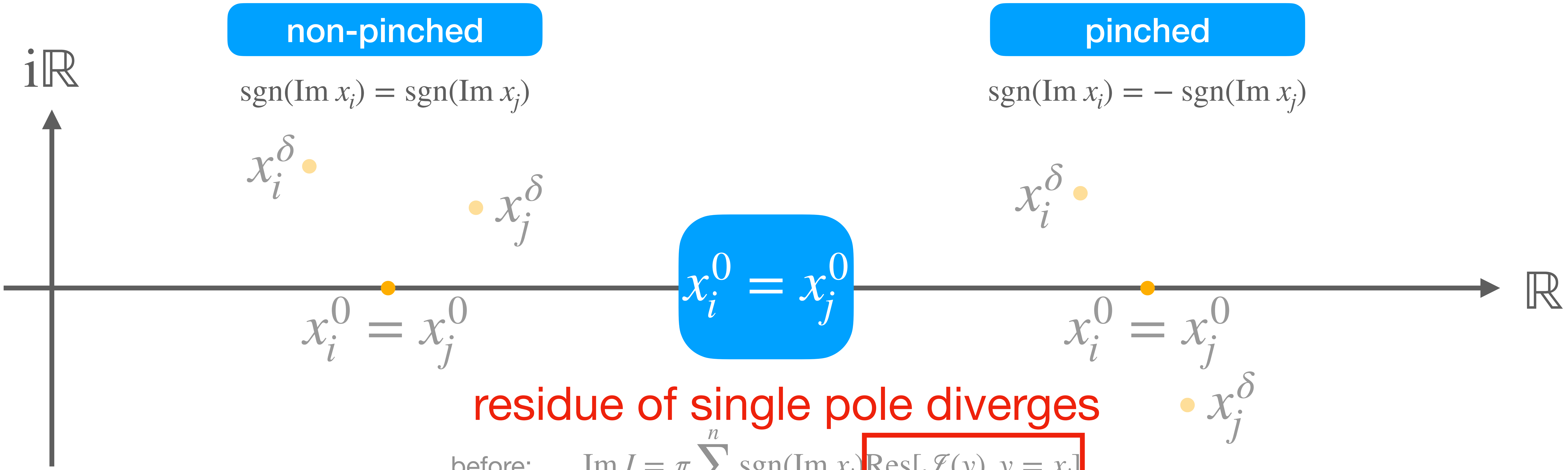
Higher-order poles?



residue of single pole diverges

before: $\mathrm{Im}\, I = \pi \sum_{i=1}^{n} \mathrm{sgn}(\mathrm{Im}\, x_i) \boxed{\mathrm{Res}[\mathscr{I}(y), y = x_i]}$

Higher-order poles?



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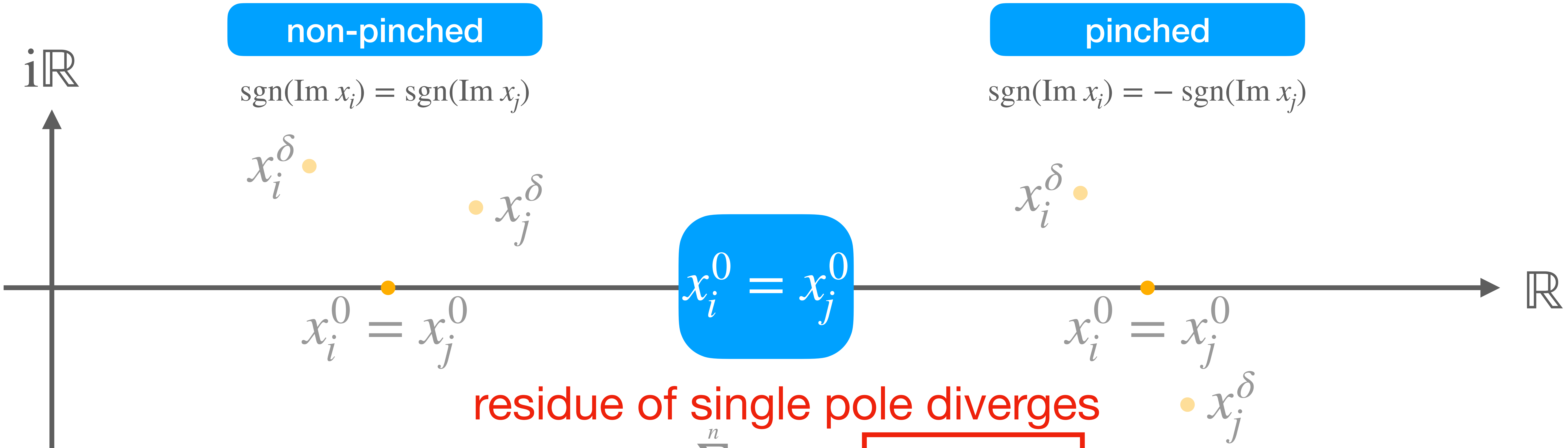
$$\mathrm{Res}[\mathcal{I}(y), y = x_i] + \mathrm{Res}[\mathcal{I}(y), y = x_j] \xrightarrow{\delta \to 0} \text{fin}\,.$$

Higher-order poles?



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residue of double pole

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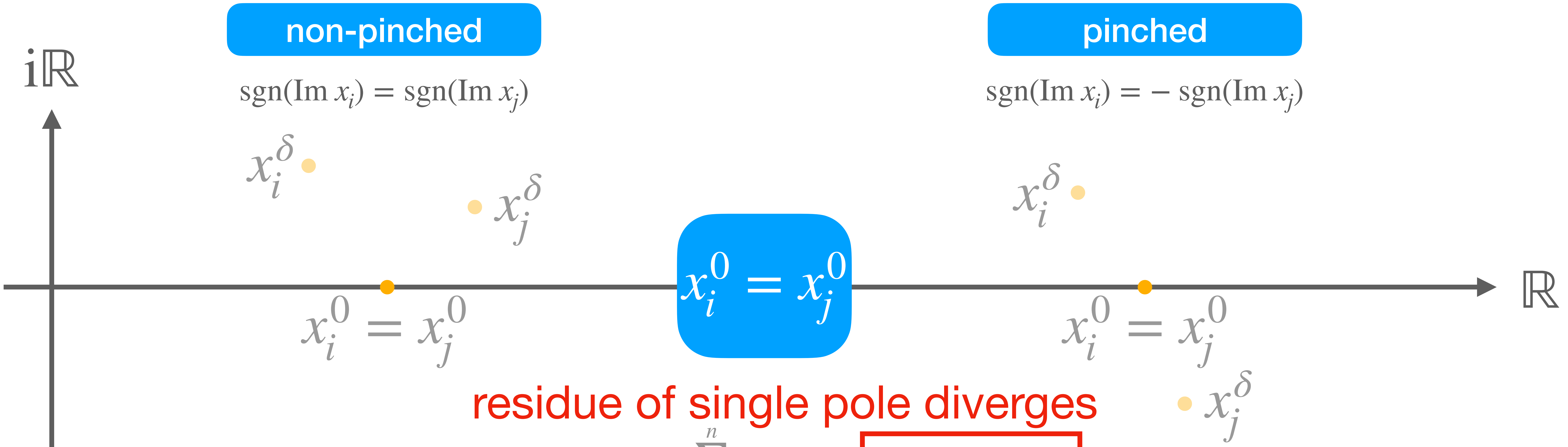

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locally cancelling (remove δ)

$$\mathrm{Res}[\mathcal{I}(y), y = x_i] + \mathrm{Res}[\mathcal{I}(y), y = x_j] \xrightarrow{\delta \to 0} \boxed{\mathrm{fin}}$$

residue of double pole

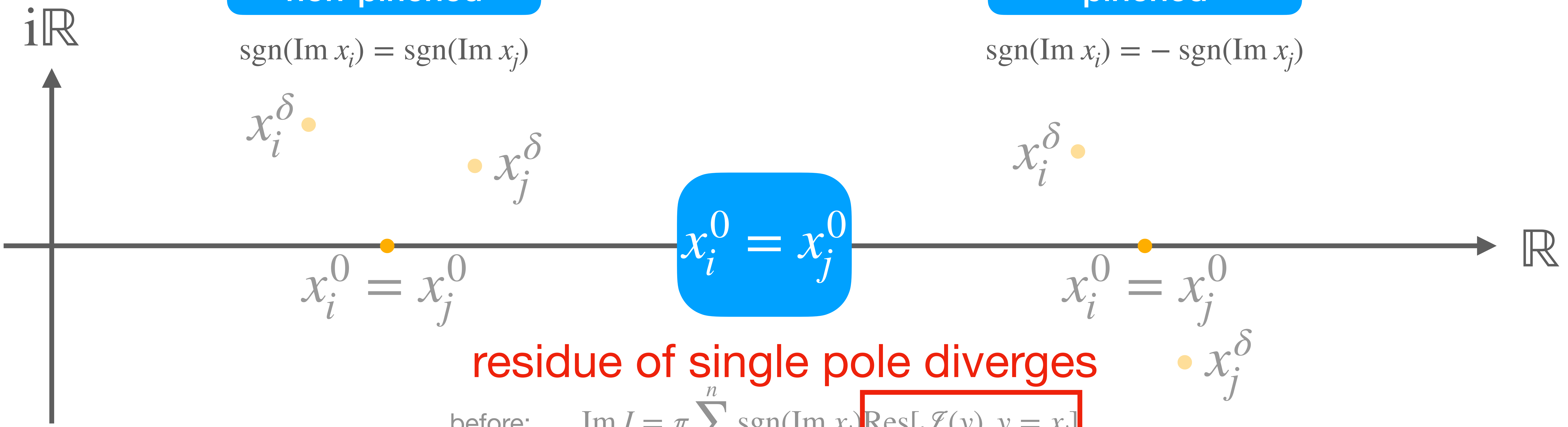
Higher-order poles?

non-pinched

$\mathrm{sgn}(\mathrm{Im}\, x_i) = \mathrm{sgn}(\mathrm{Im}\, x_j)$

pinched

$\mathrm{sgn}(\mathrm{Im}\, x_i) = -\,\mathrm{sgn}(\mathrm{Im}\, x_j)$



$x_i^0 = x_j^0$

residue of single pole diverges

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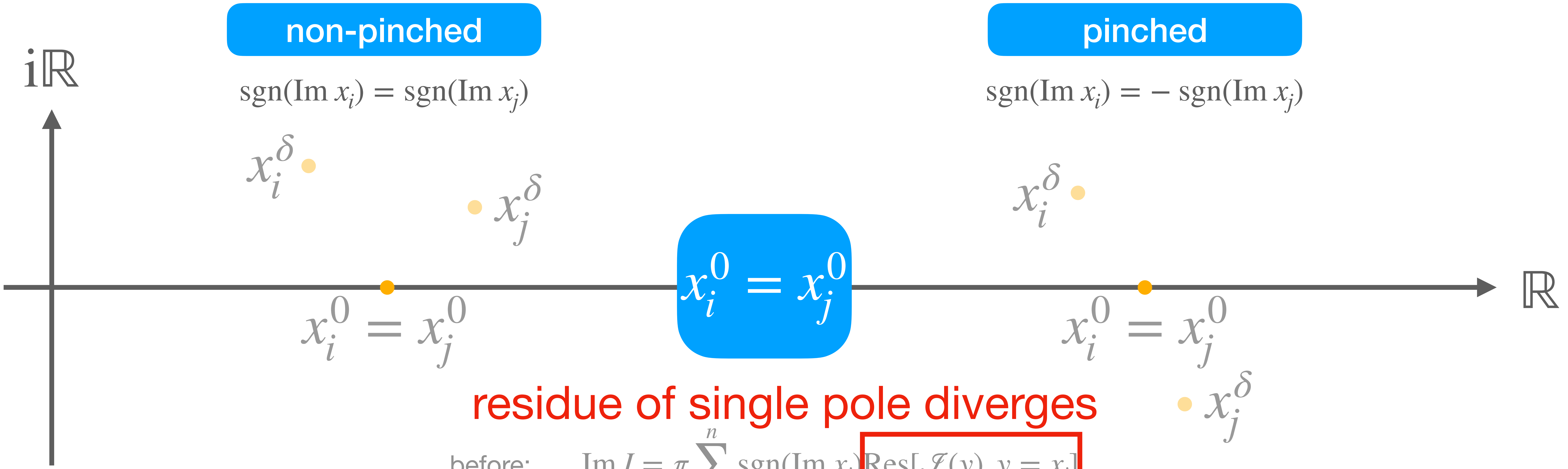
$\mathrm{Res}[\mathcal{I}(y), y = x_i] + \mathrm{Res}[\mathcal{I}(y), y = x_j] \xrightarrow{\delta \to 0} \boxed{\mathrm{fin}}$

residue of double pole

$\mathrm{Res}[\mathcal{I}(y), y = x_i] - \mathrm{Res}[\mathcal{I}(y), y = x_j] \xrightarrow{\delta \to 0} \mathrm{inf}\,.$

Higher-order poles?



residue of single pole diverges

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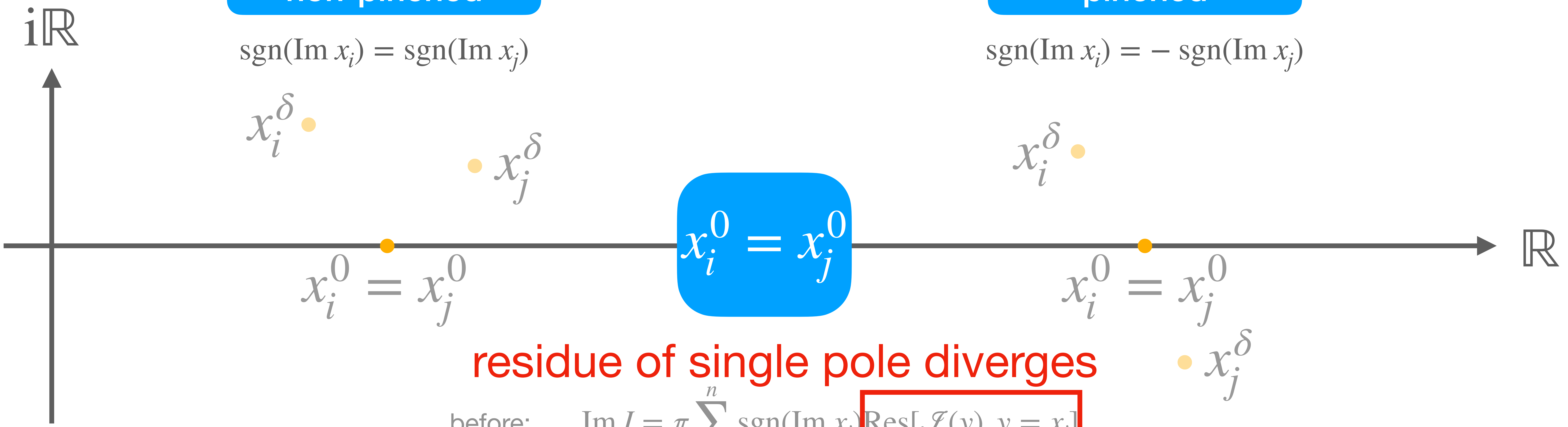

residue of single pole diverges

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locally cancelling (remove δ)

$\mathrm{Res}[\mathcal{I}(y), y = x_i] + \mathrm{Res}[\mathcal{I}(y), y = x_j] \xrightarrow{\delta \to 0} \boxed{\mathrm{fin}}$

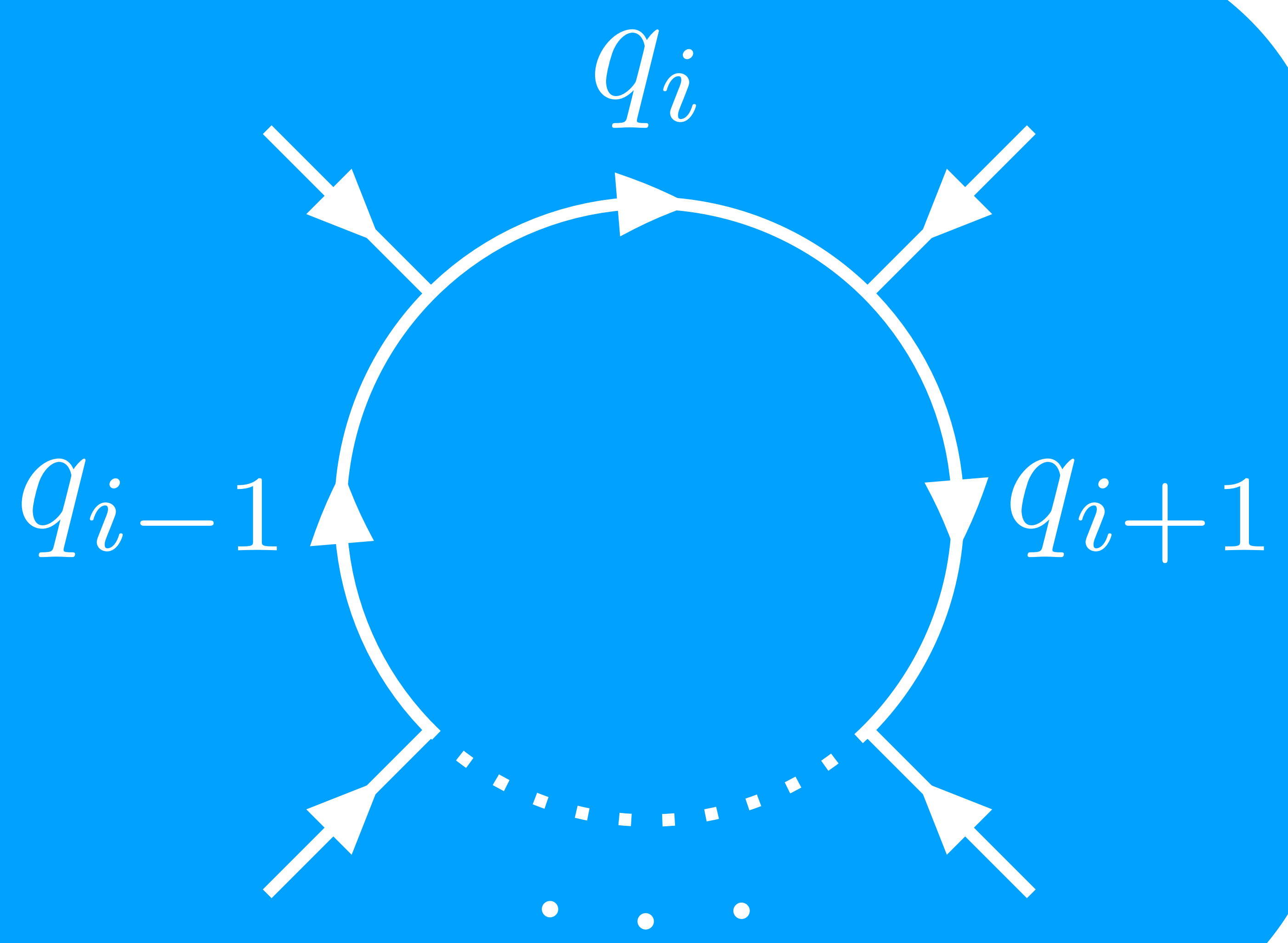
residue of double pole

not cancelling (keep $\delta > 0$)

$\mathrm{Res}[\mathcal{I}(y), y = x_i] - \mathrm{Res}[\mathcal{I}(y), y = x_j] \xrightarrow{\delta \to 0} \boxed{\mathrm{inf}}$

One-loop integrals

$$\mathrm{i}I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathscr{I} =$$



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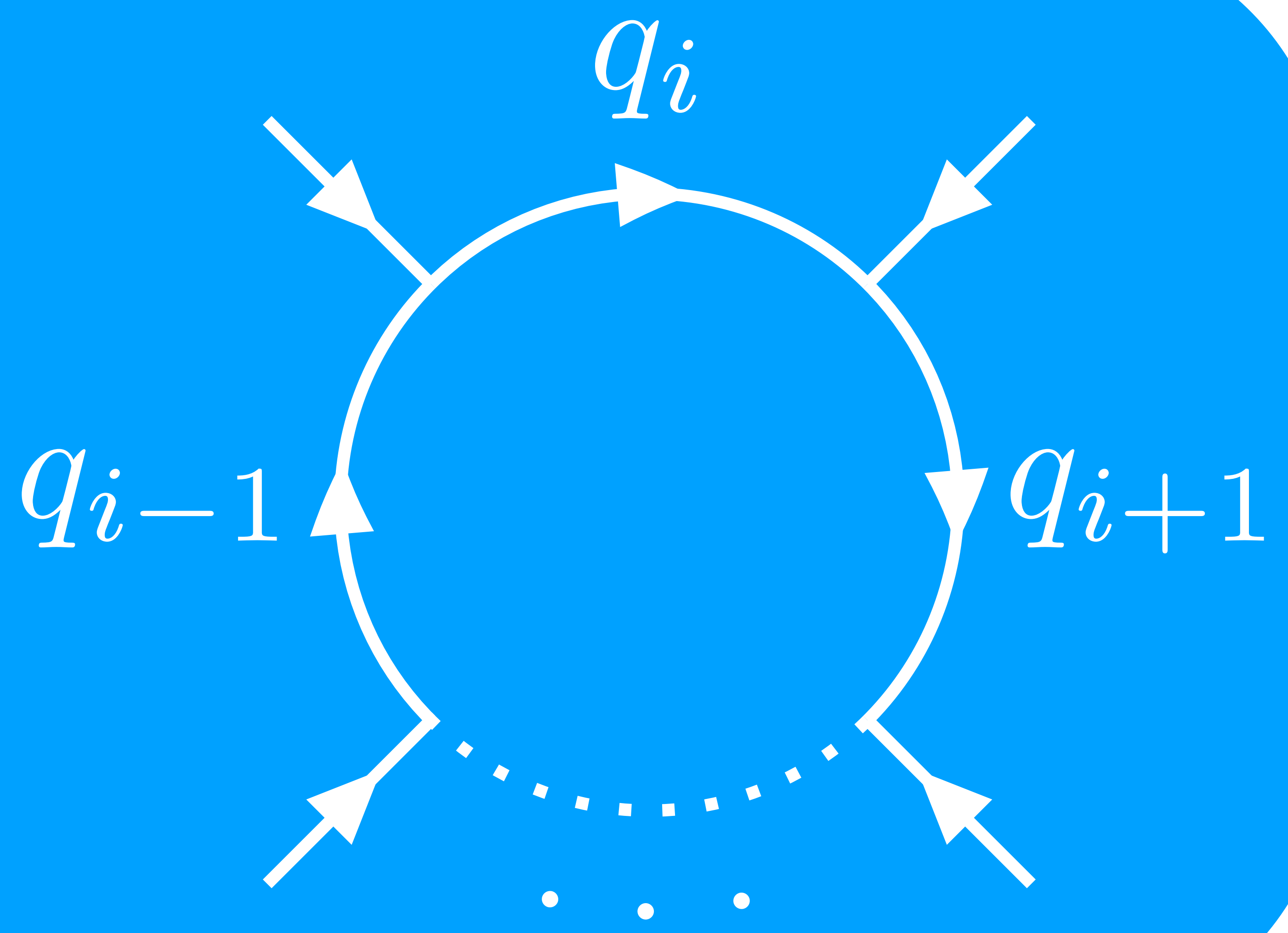
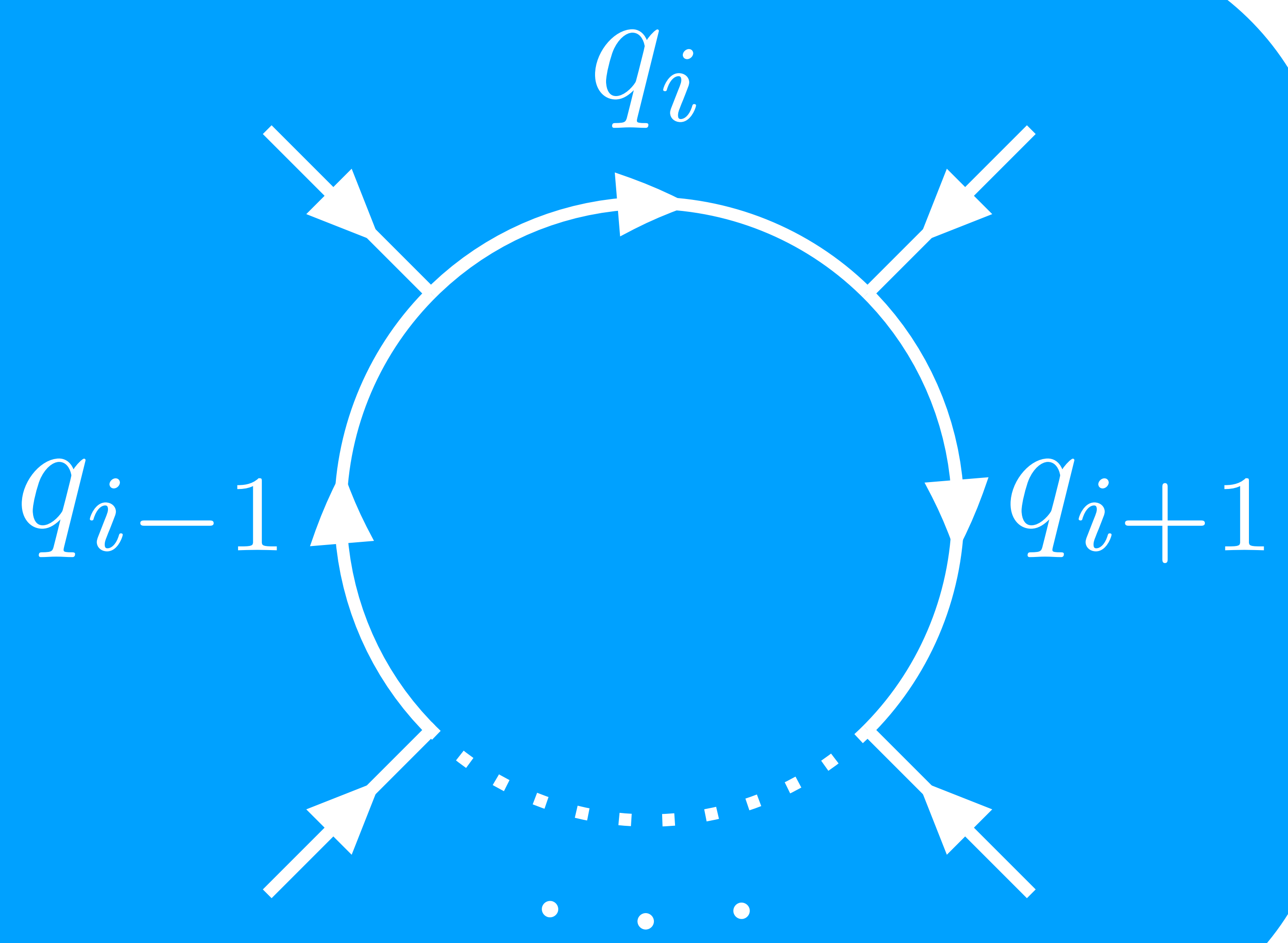


$$\mathrm{i}I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\mathscr{J} =$$



$$\mathscr{J} = \frac{N}{\prod_i D_i}$$

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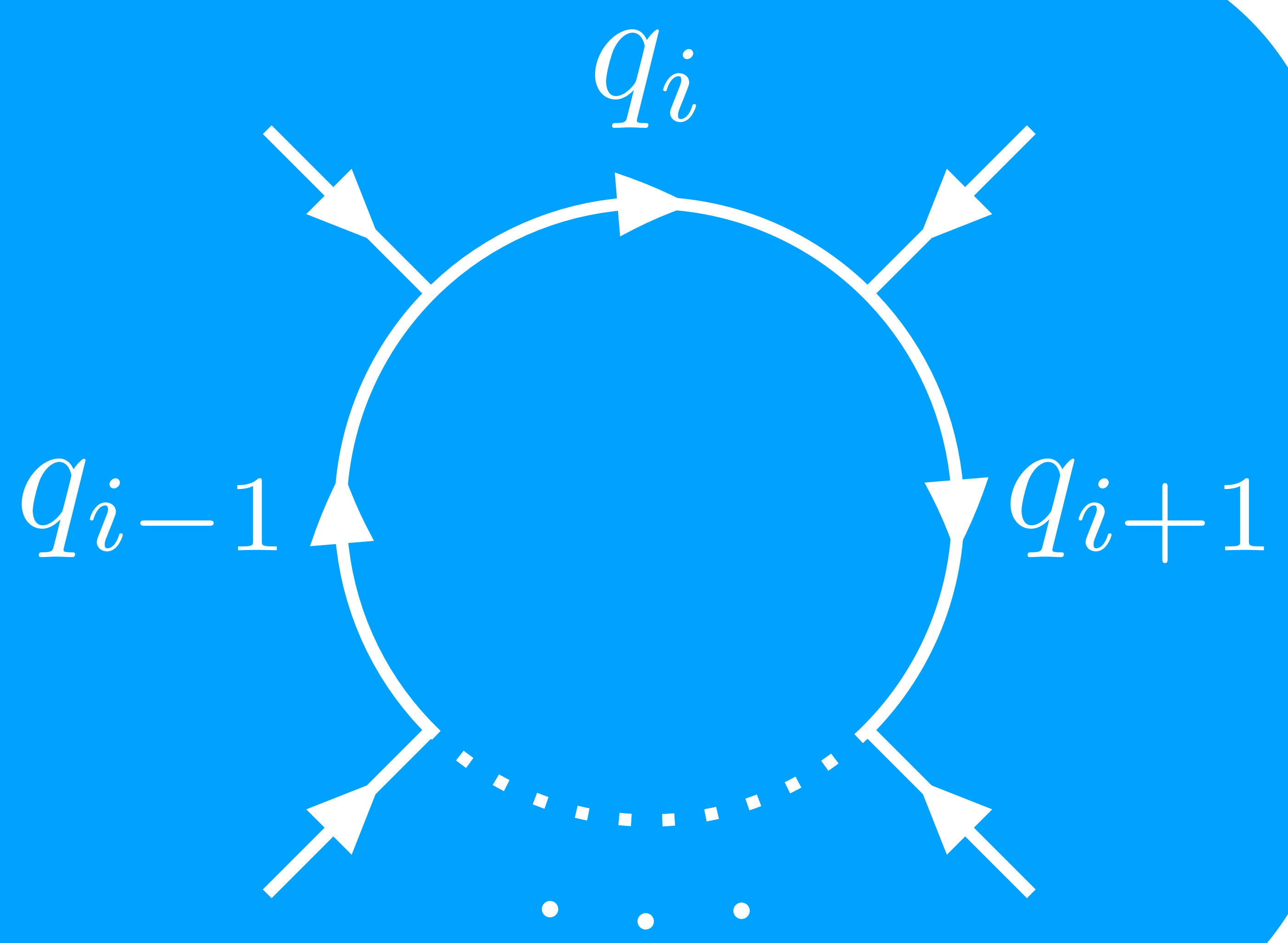

$$\mathscr{J} = \frac{N}{\prod_i D_i}$$

$$D_i = q_i^2 - m_i^2 + \mathrm{i}\delta$$

Integrating out k^0

- separate k^0-integration

$$\mathrm{i}I = \int \frac{\mathrm{d}^3\overrightarrow{k}}{(2\pi)^3} \int_{\mathbb{R}} \frac{\mathrm{d}k^0}{(2\pi)} \mathscr{I}$$

- separate k^0-integration

$$\mathrm{i}I = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}(-\mathrm{i})\mathcal{I}_{\mathrm{LTD}}(\vec{k})$$

$$\mathcal{I}_{\mathrm{LTD}} = \sum_i \mathrm{Res}[\mathcal{I}, k^0 = E_i - p_i^0]$$

Loop-Tree Duality

[Catani, Gleisberg, Krauss, Rodrigo, Winter: 0804.3170]
[Capatti, Hirschi, DK, Ruijl: 1906.06138]
[Runkel, Szor, Vesga, Weinzierl: 1902.02135]

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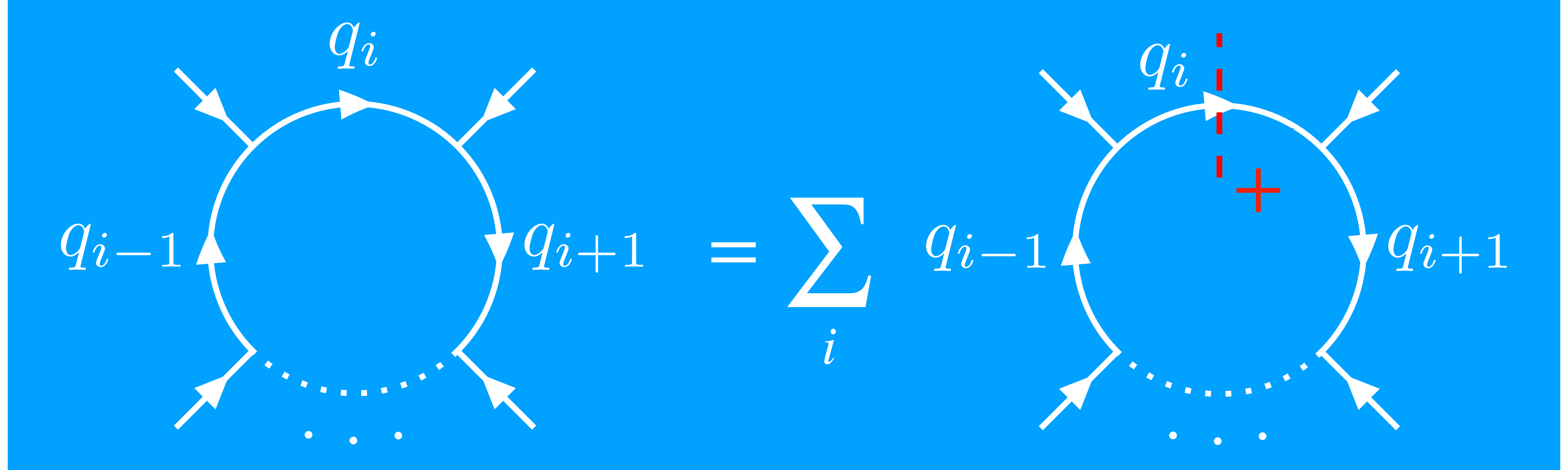

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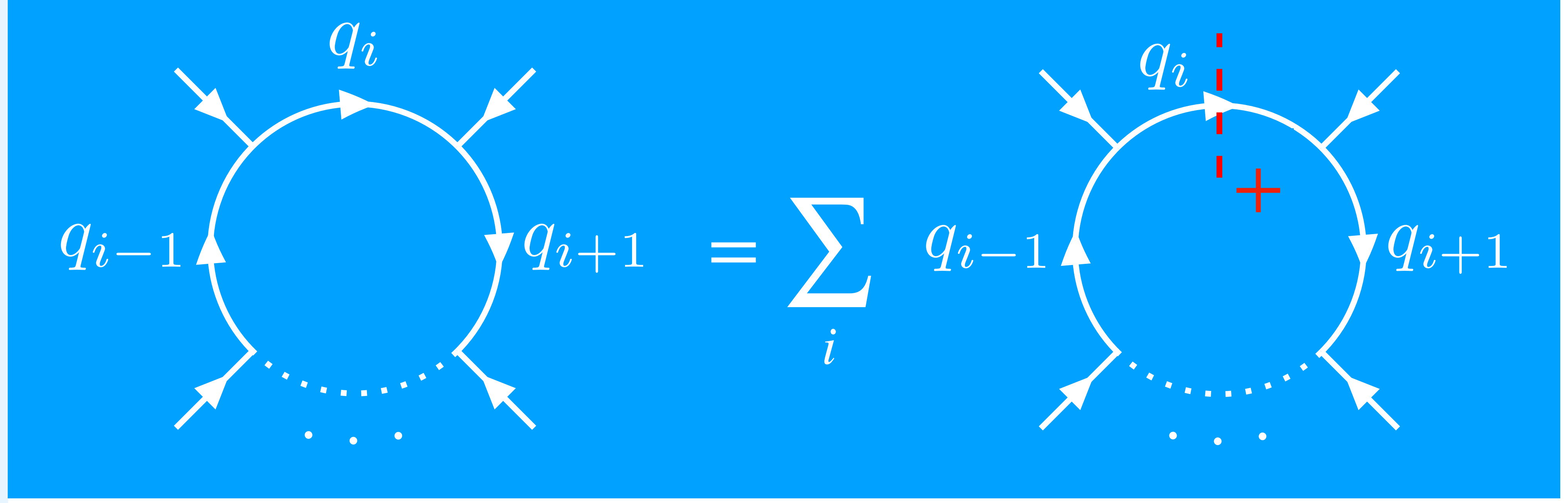

- poles of $\mathcal{I}$

$$E_i = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2 - \mathrm{i}\delta}$$

$$D_i = \left(k^0 + p_i^0 - E_i\right)\left(k^0 + p_i^0 + E_i\right) \Rightarrow k^0 = \pm E_i - p_i^0$$

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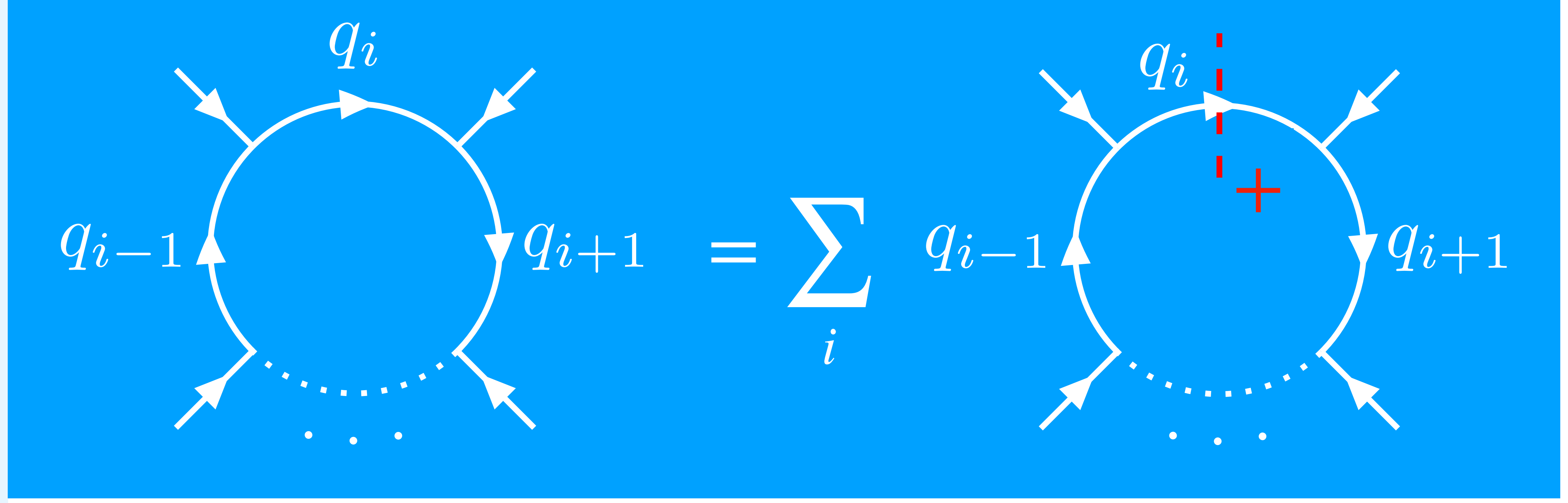
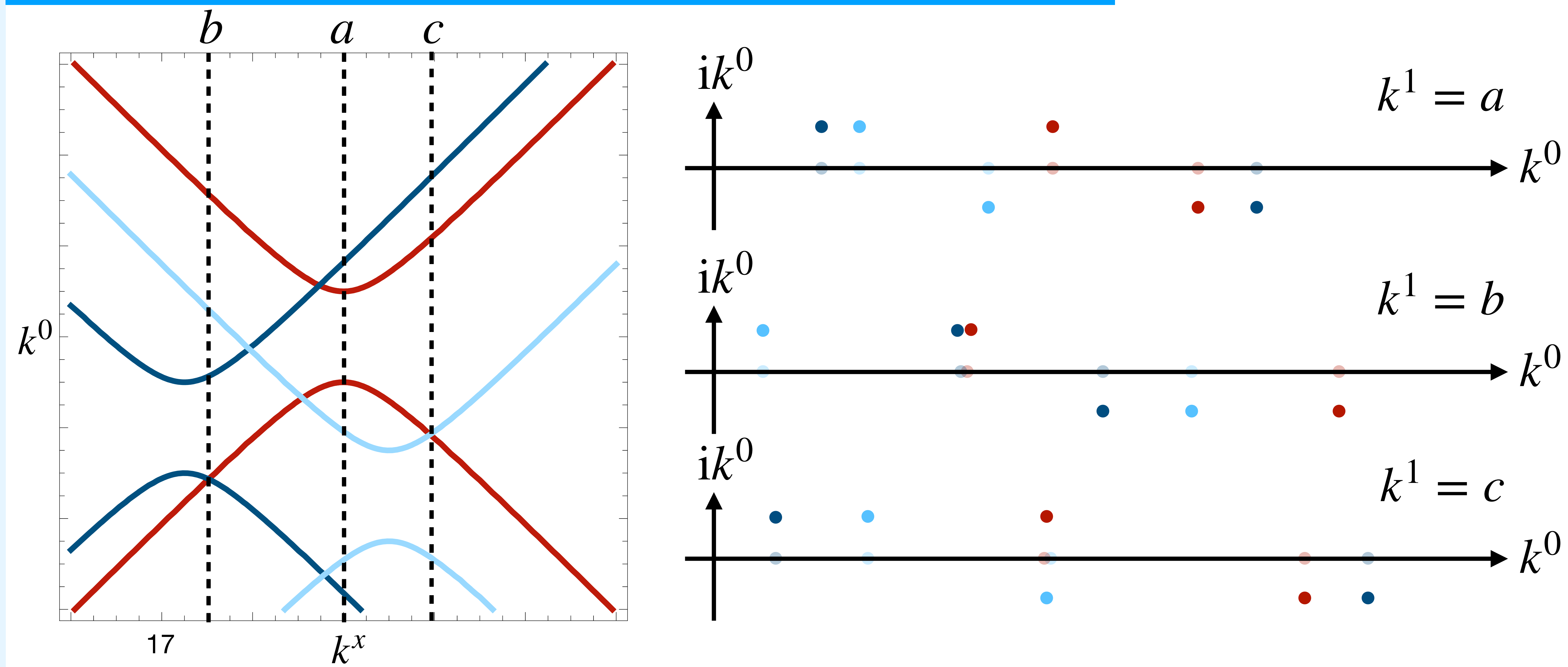


non-pinched thresholds

pinched thresholds



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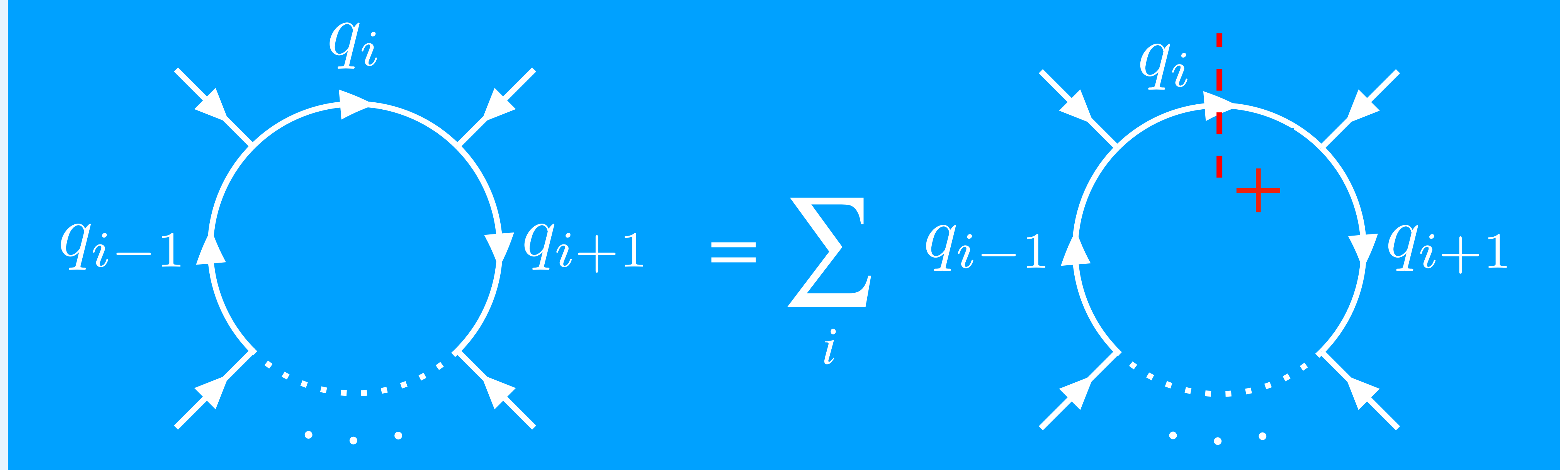

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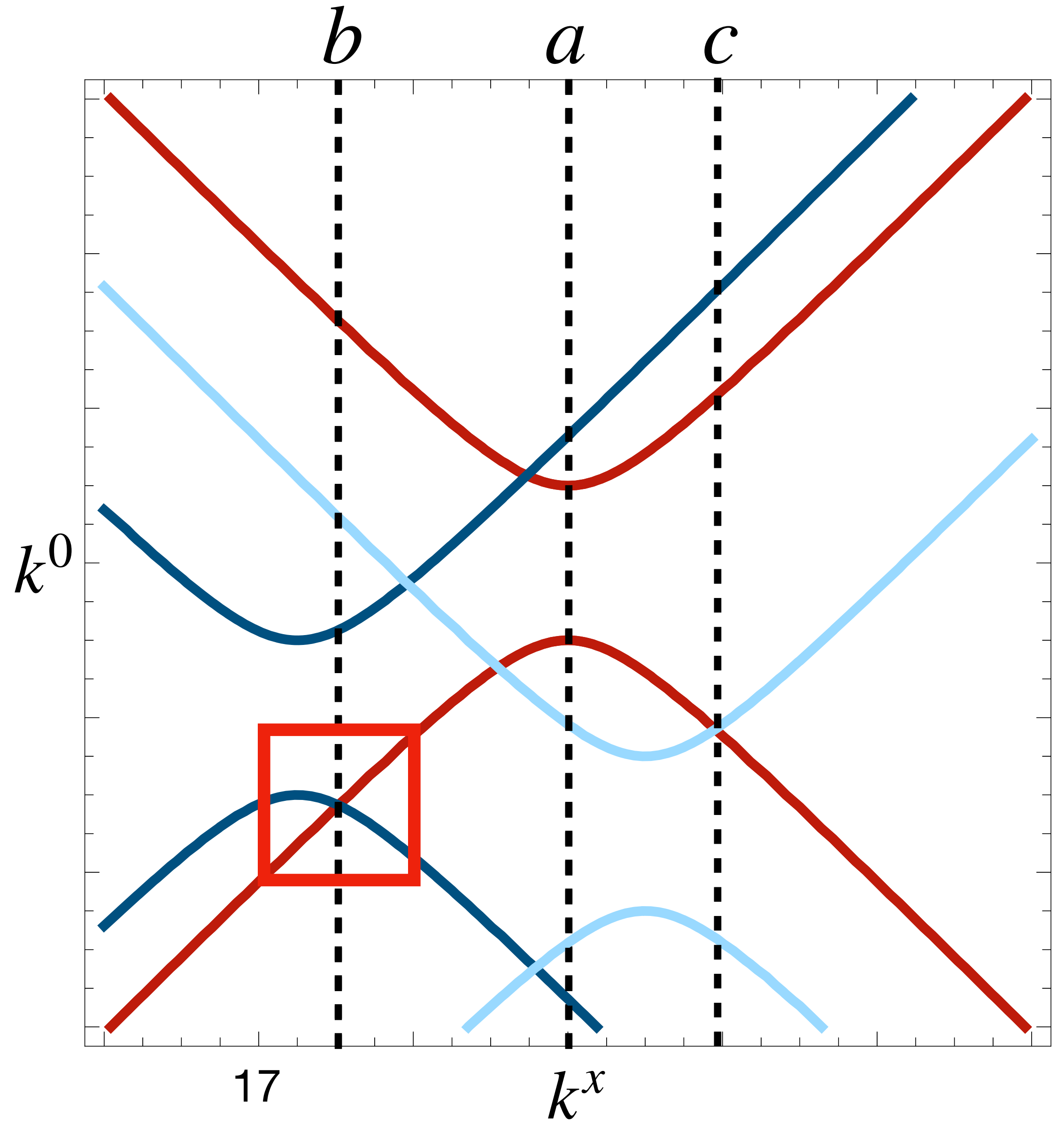
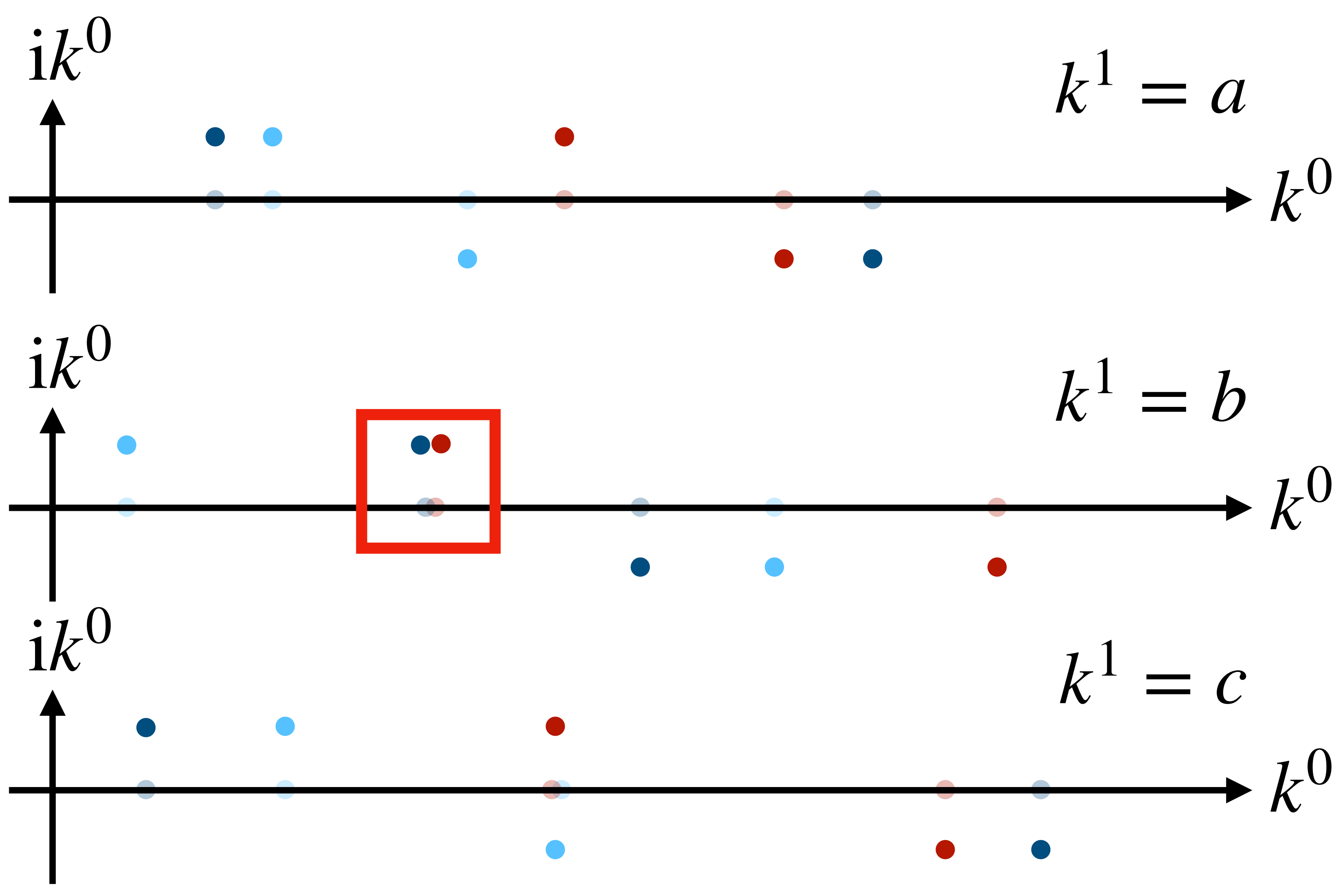
non-pinched thresholds

locally cancelling, spurious poles of $\mathcal{I}_{\mathrm{LTD}}$

$$-E_i - p_i = -E_j - p_j$$

pinched thresholds





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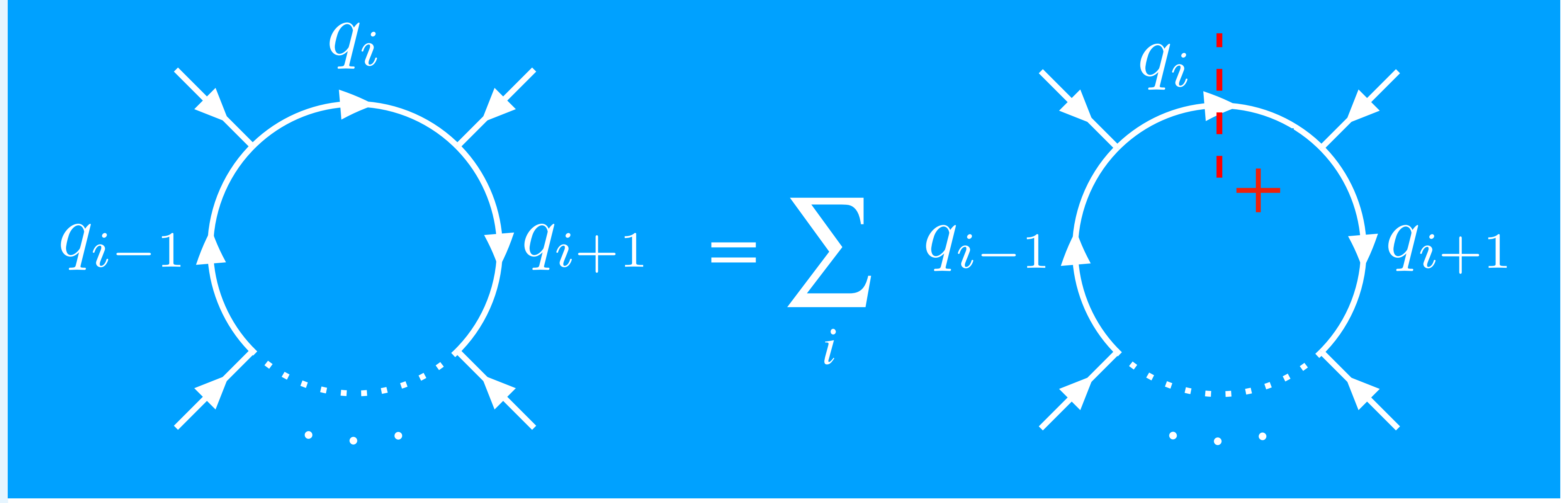

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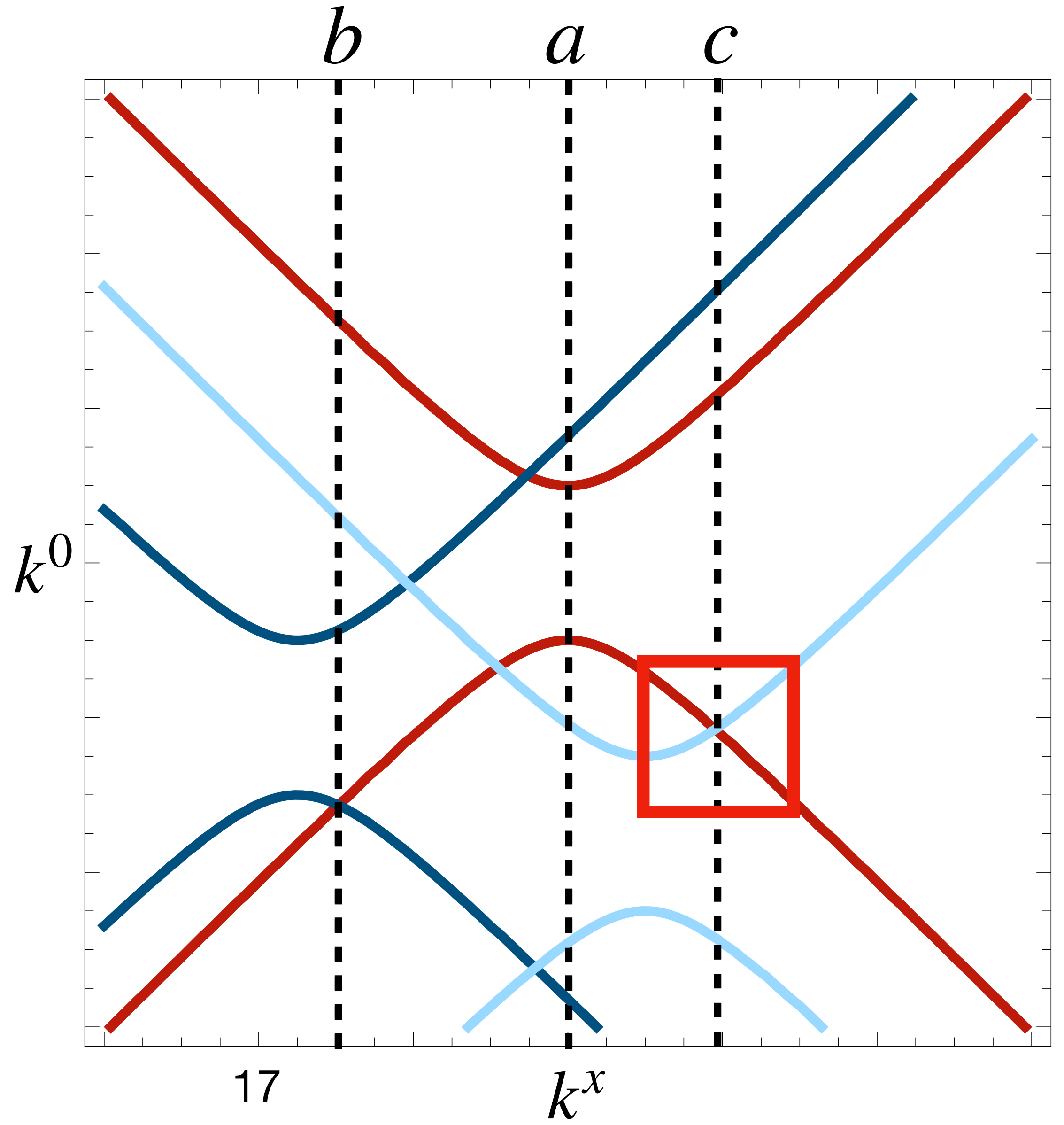

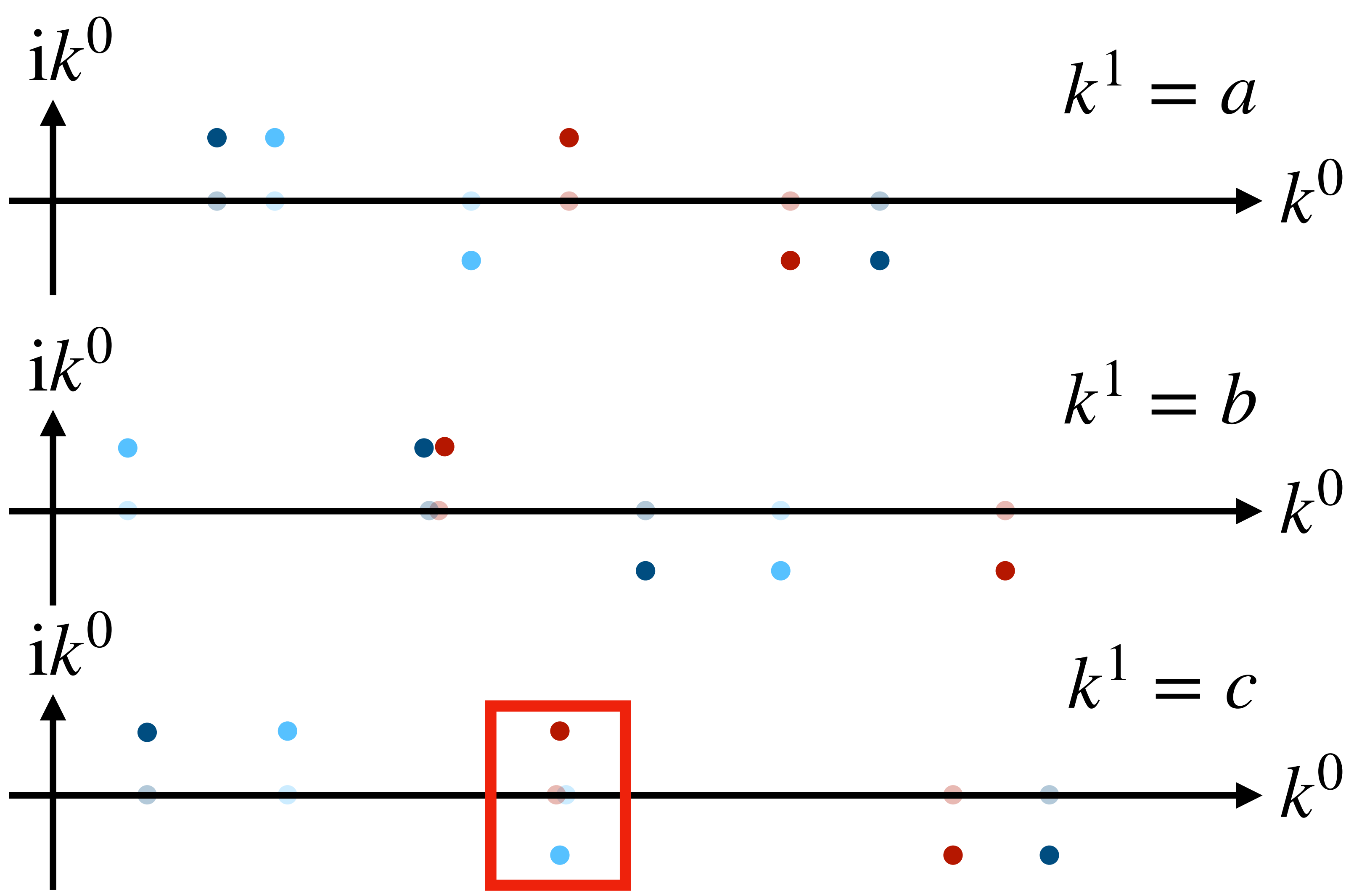

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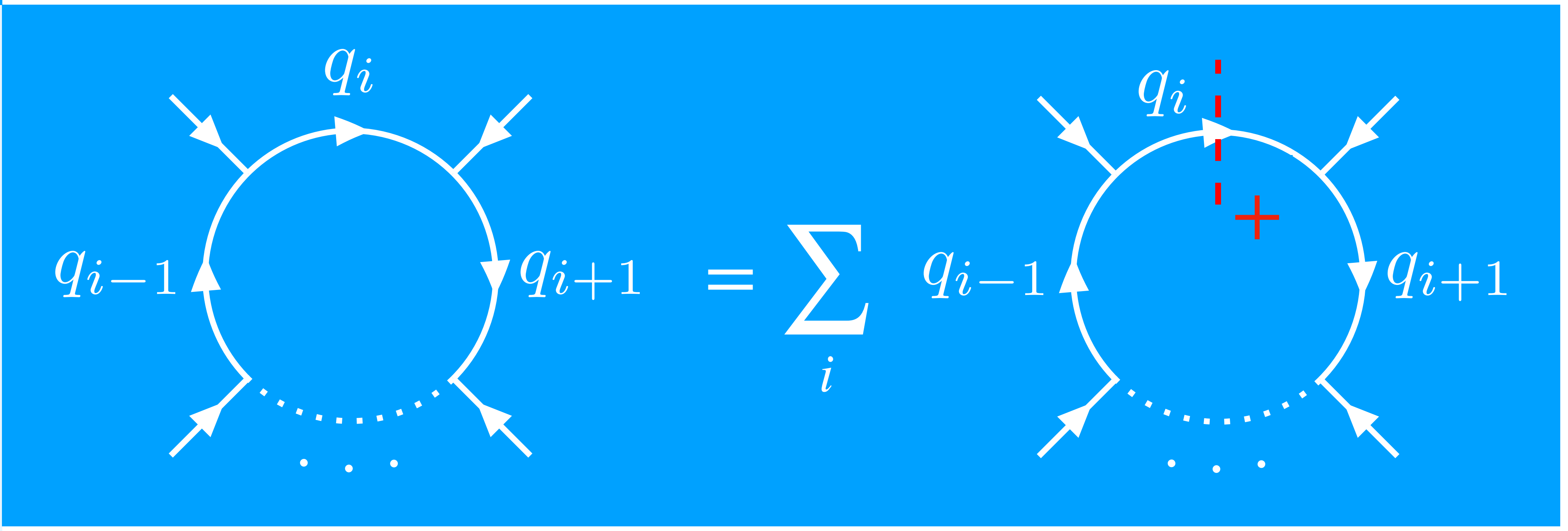

non-pinched thresholds

locally cancelling, spurious poles of $\mathcal{I}_{\mathrm{LTD}}$

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pinched thresholds

poles of $\mathcal{I}_{\mathrm{LTD}}$

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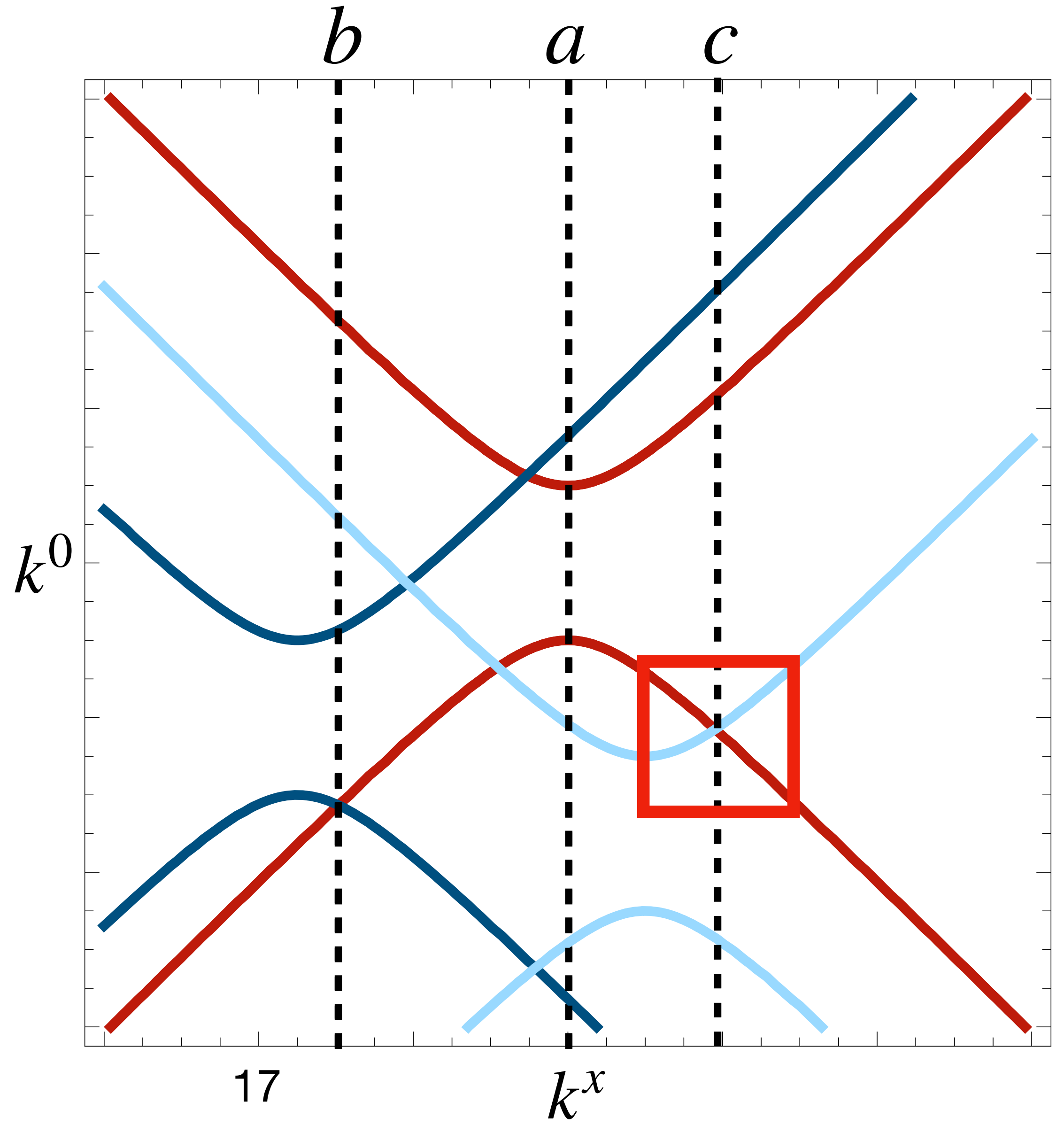
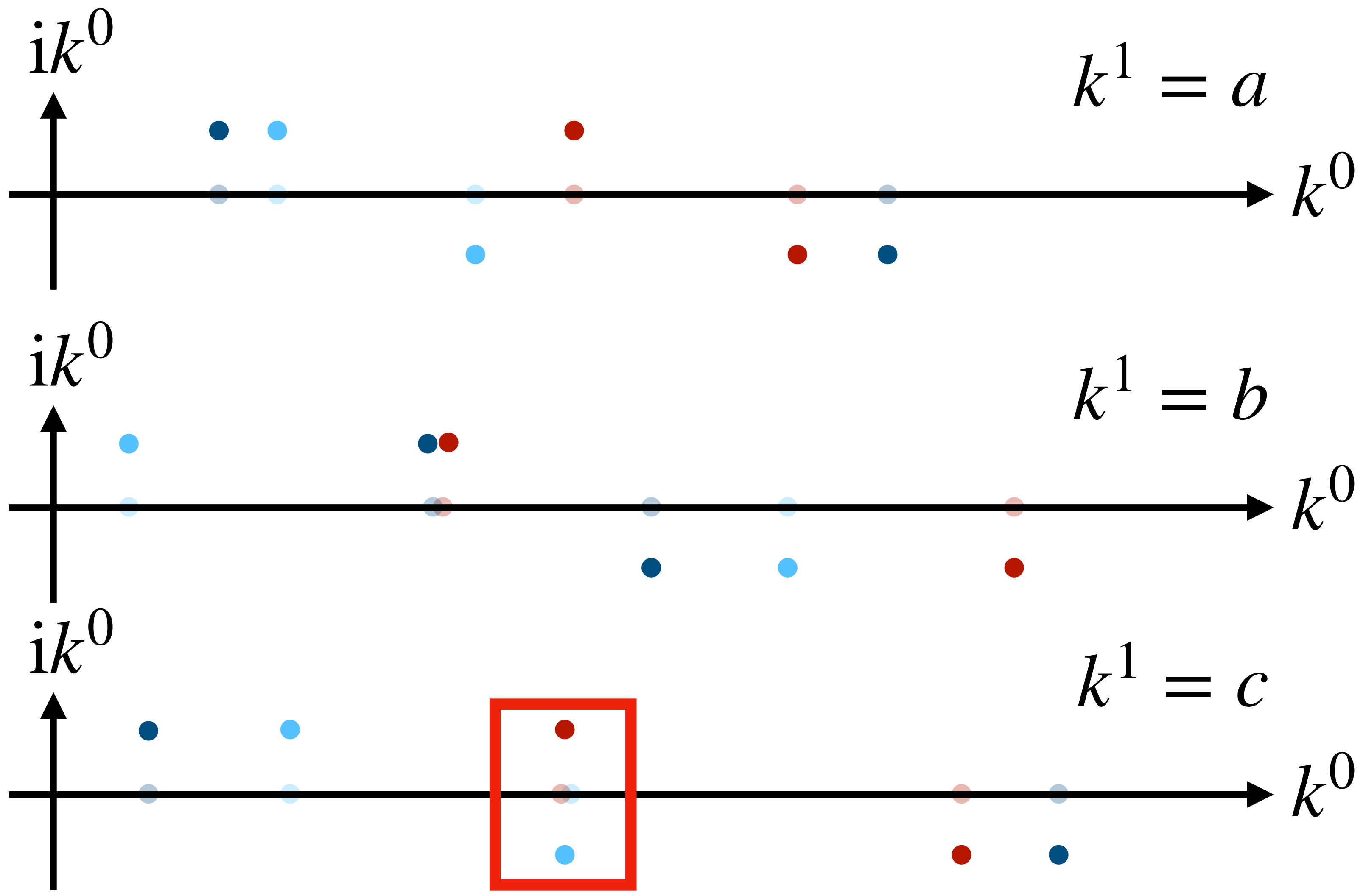
keep $\delta > 0$





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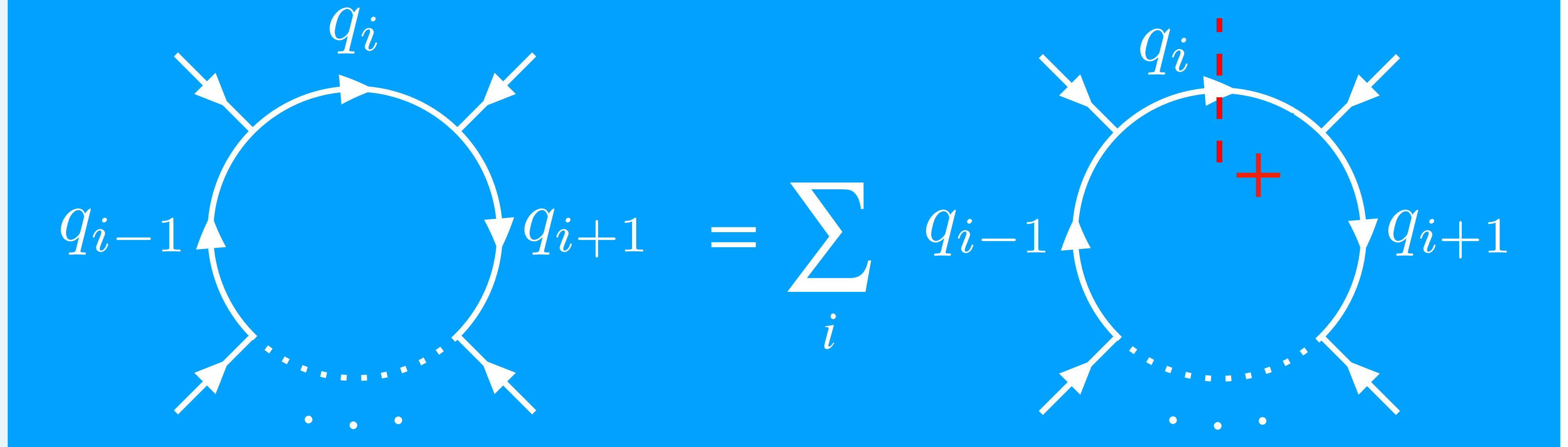

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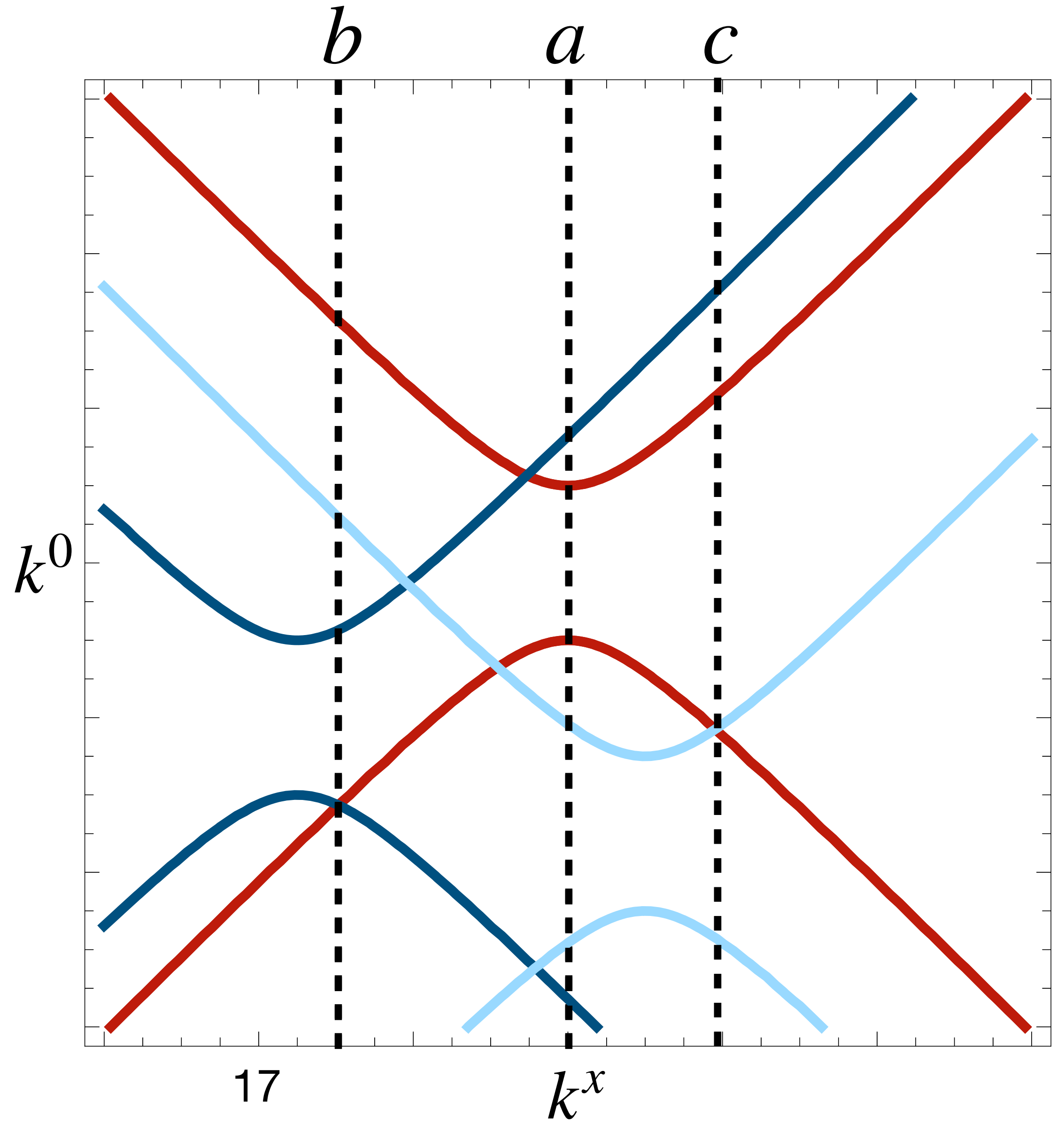

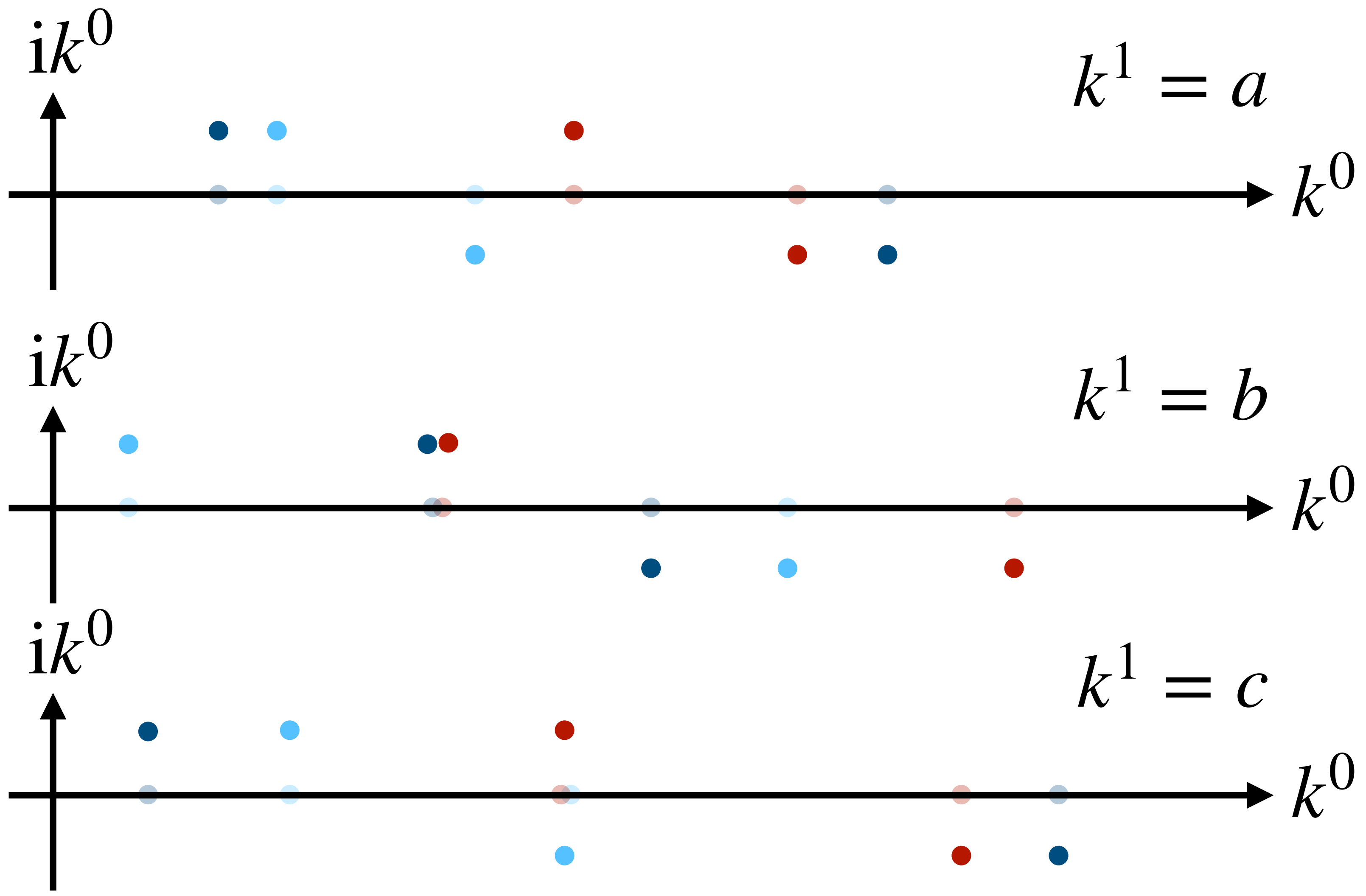

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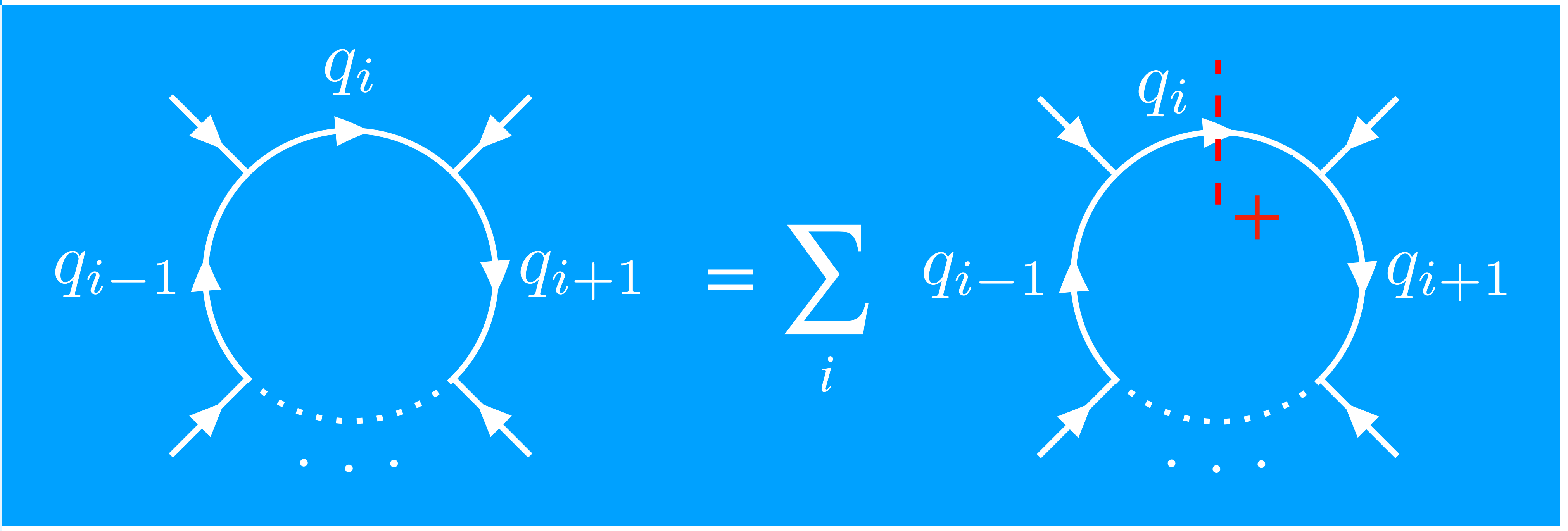

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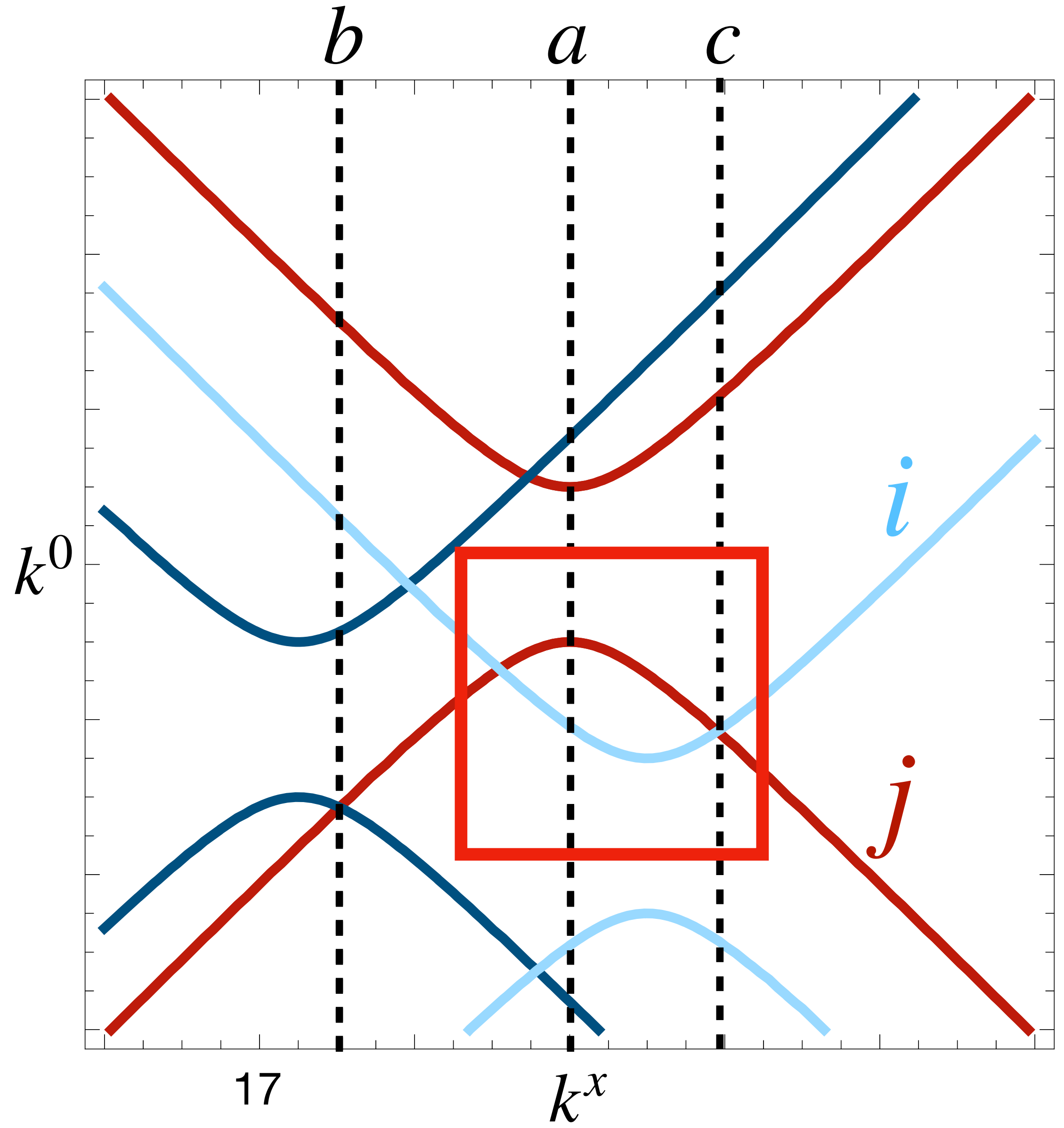
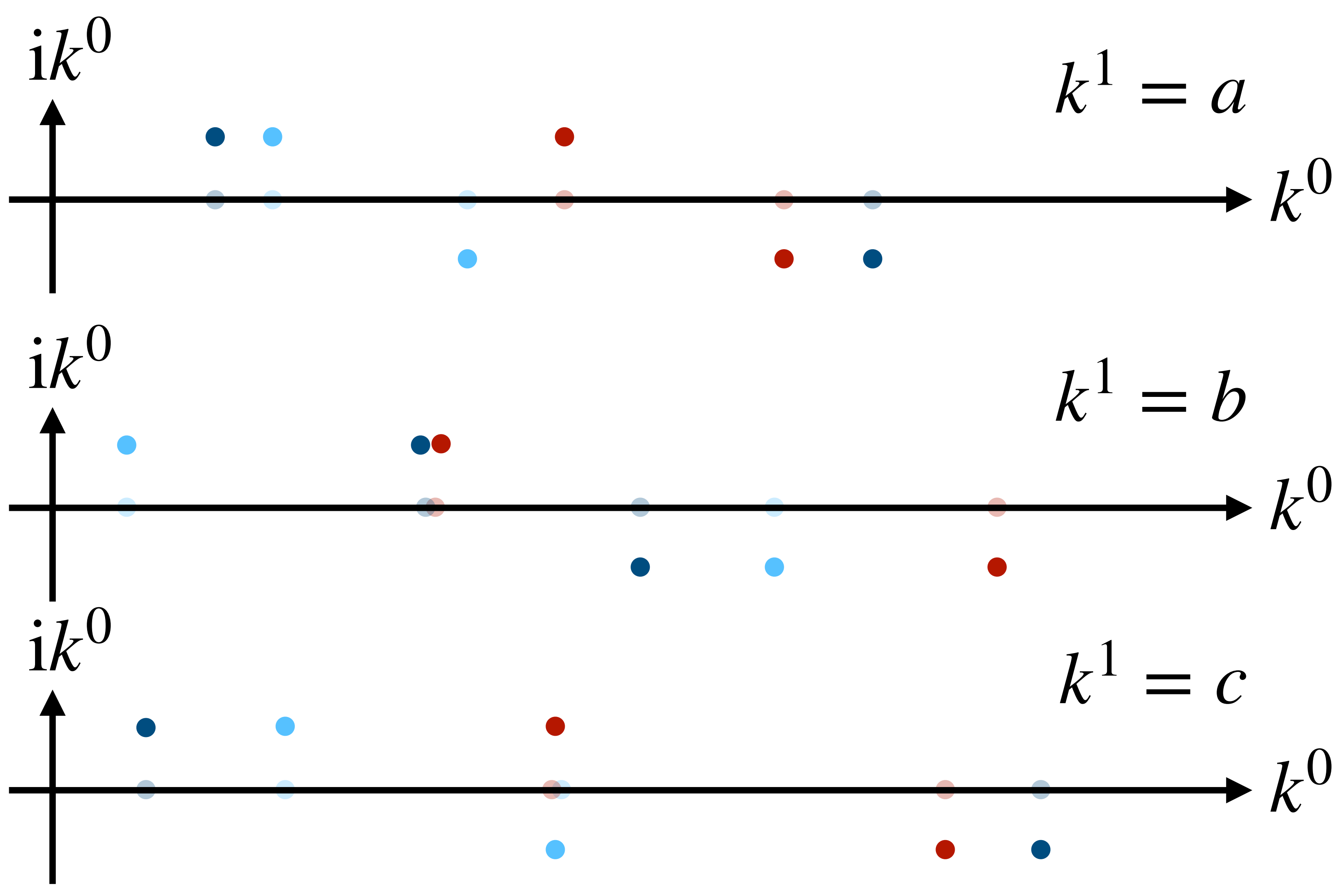
pinched thresholds

poles of $\mathcal{I}_{\mathrm{LTD}}$

keep $\delta > 0$

$$E_i - p_i = -E_j - p_j \quad \Rightarrow \quad E_{ij} : E_i + E_j - p_i + p_j = 0$$





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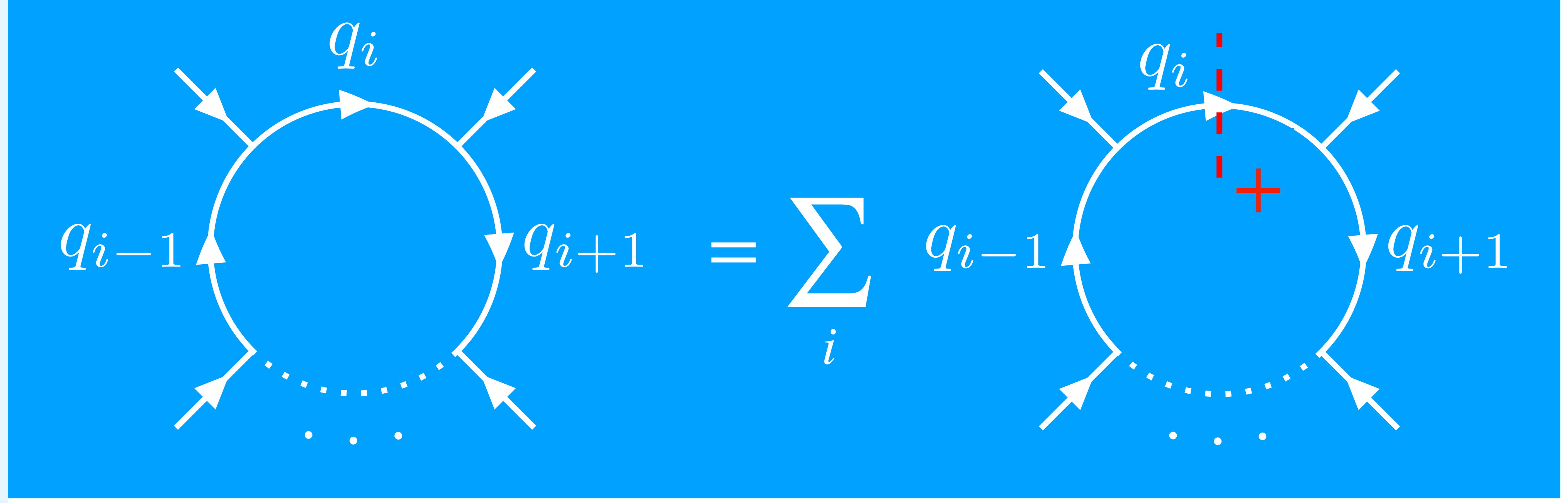


- poles of $\mathscr{I}$

$$E_i = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2 - \mathrm{i}\delta}$$

$$D_i = \left(k^0 + p_i^0 - E_i\right)\left(k^0 + p_i^0 + E_i\right) \Rightarrow k^0 = \pm E_i - p_i^0$$

non-pinched thresholds

locally cancelling, spurious poles of $\mathscr{I}_{\mathrm{LTD}}$

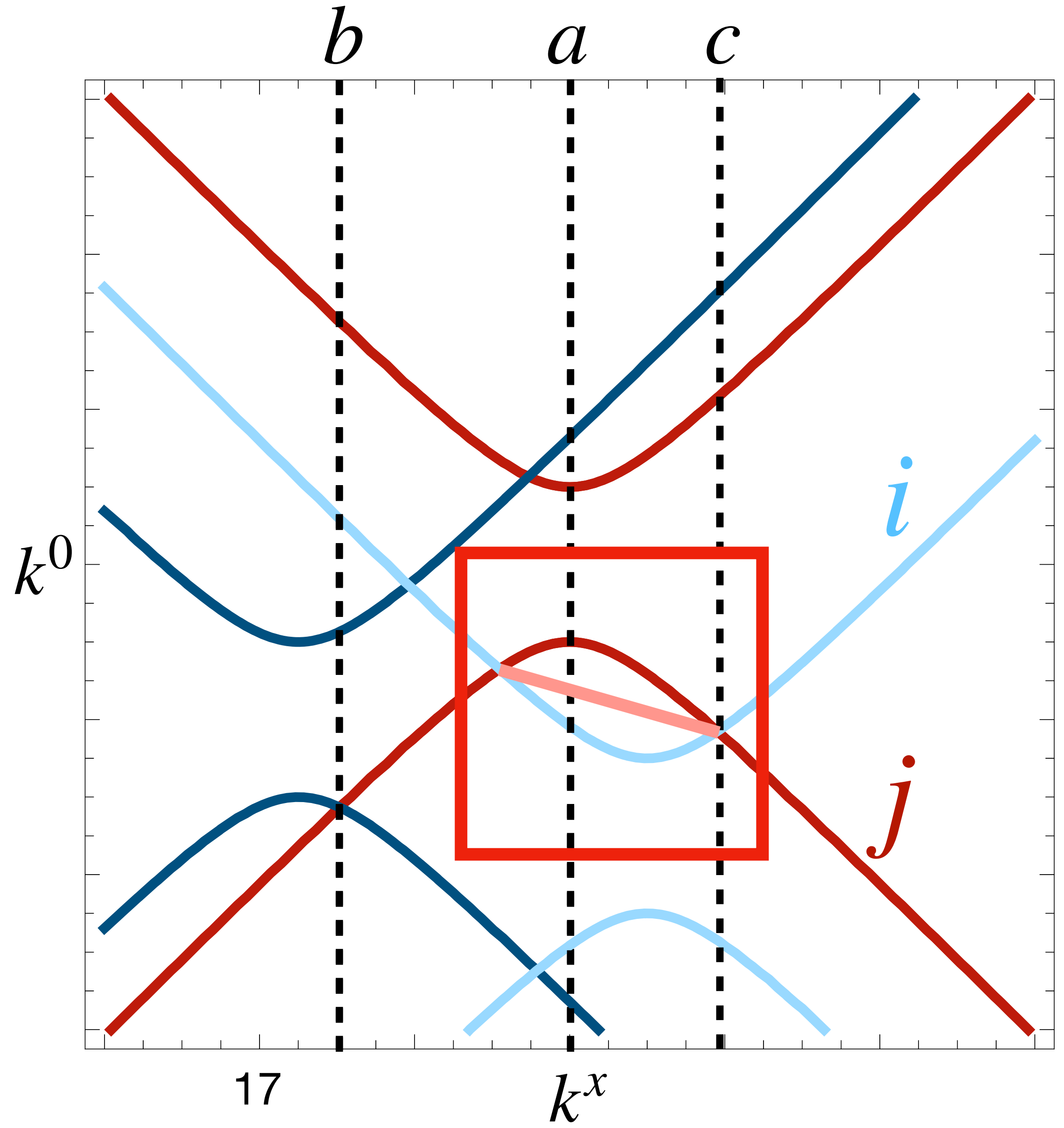
$$-E_i - p_i = -E_j - p_j$$

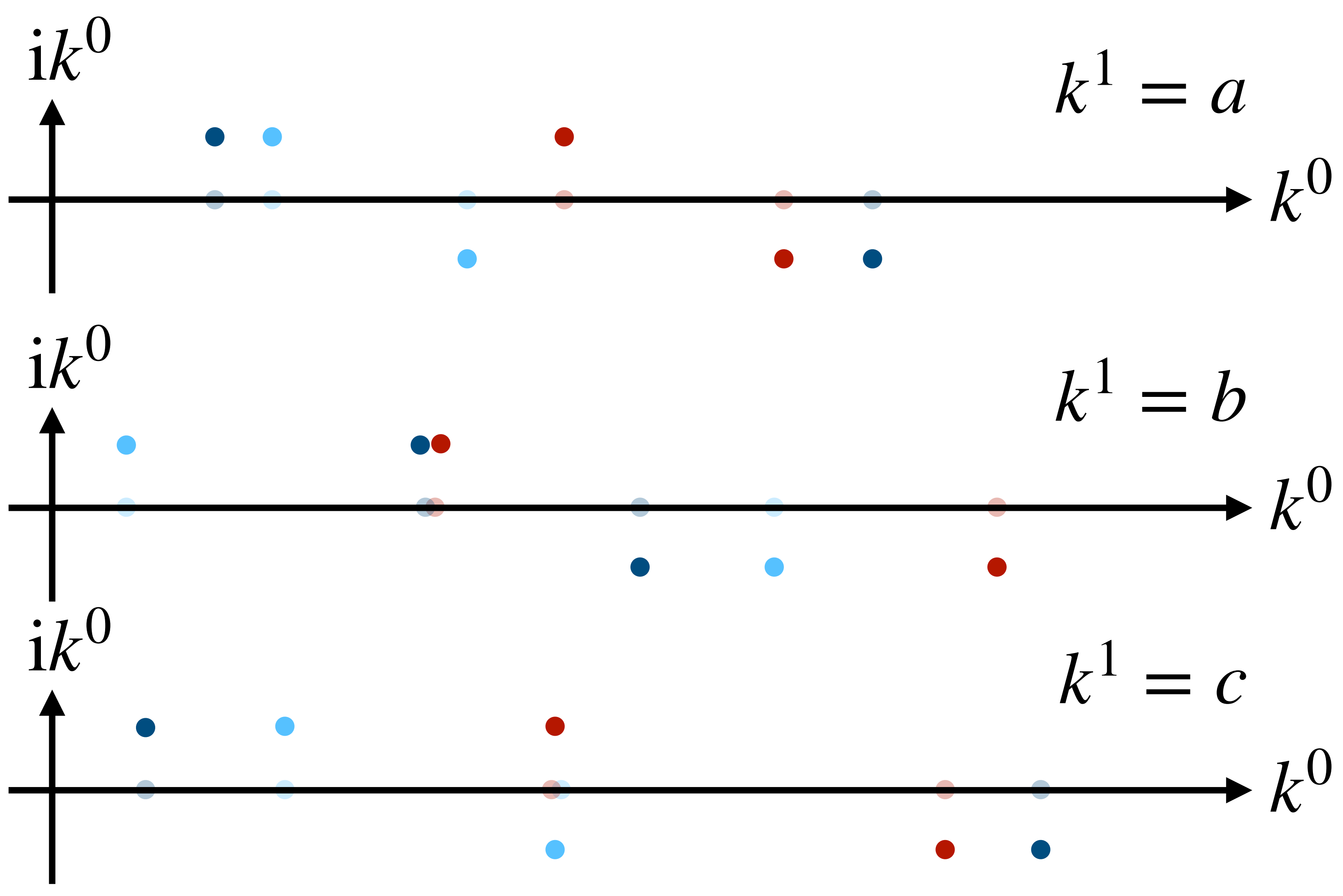
pinched thresholds

poles of $\mathscr{I}_{\mathrm{LTD}}$

keep $\delta > 0$

$$E_i - p_i = -E_j - p_j \quad \Rightarrow \quad E_{ij} : E_i + E_j - p_i + p_j = 0$$





Loop-Tree Duality

[Catani, Gleisberg, Krauss, Rodrigo, Winter: 0804.3170]
[Capatti, Hirschi, DK, Ruijl: 1906.06138]
[Runkel, Szor, Vesga, Weinzierl: 1902.02135]

- separate k^0-integration

$$\mathrm{i}I = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}(-\mathrm{i})\mathcal{I}_{\mathrm{LTD}}(\vec{k})$$

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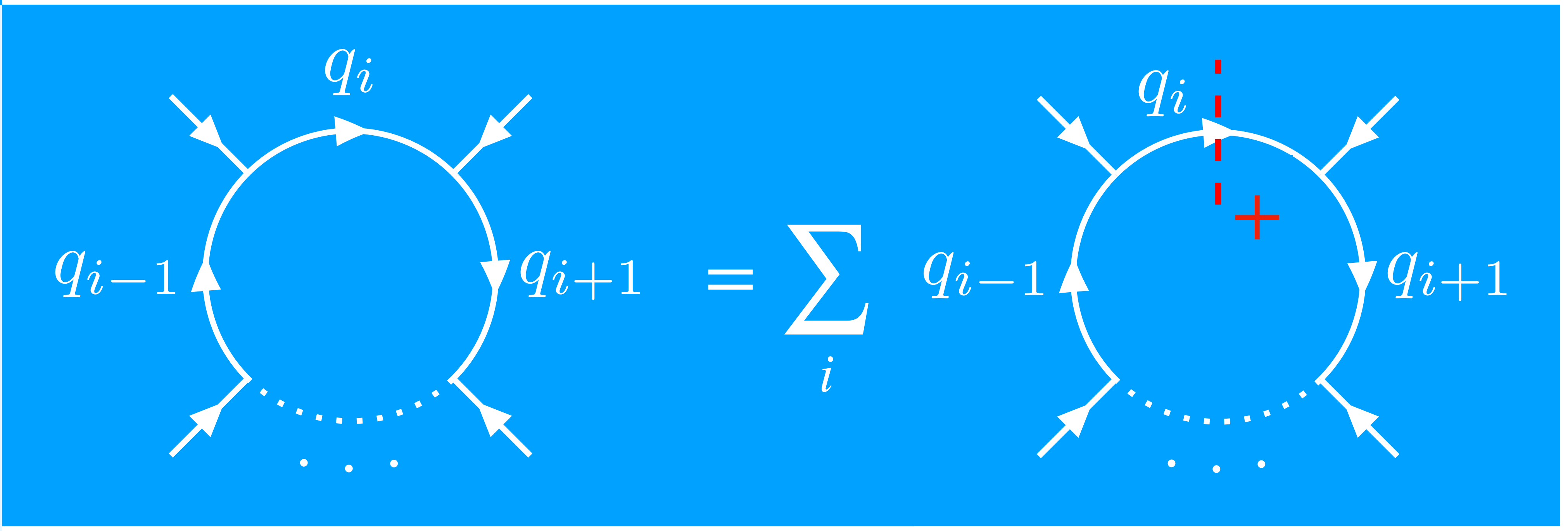

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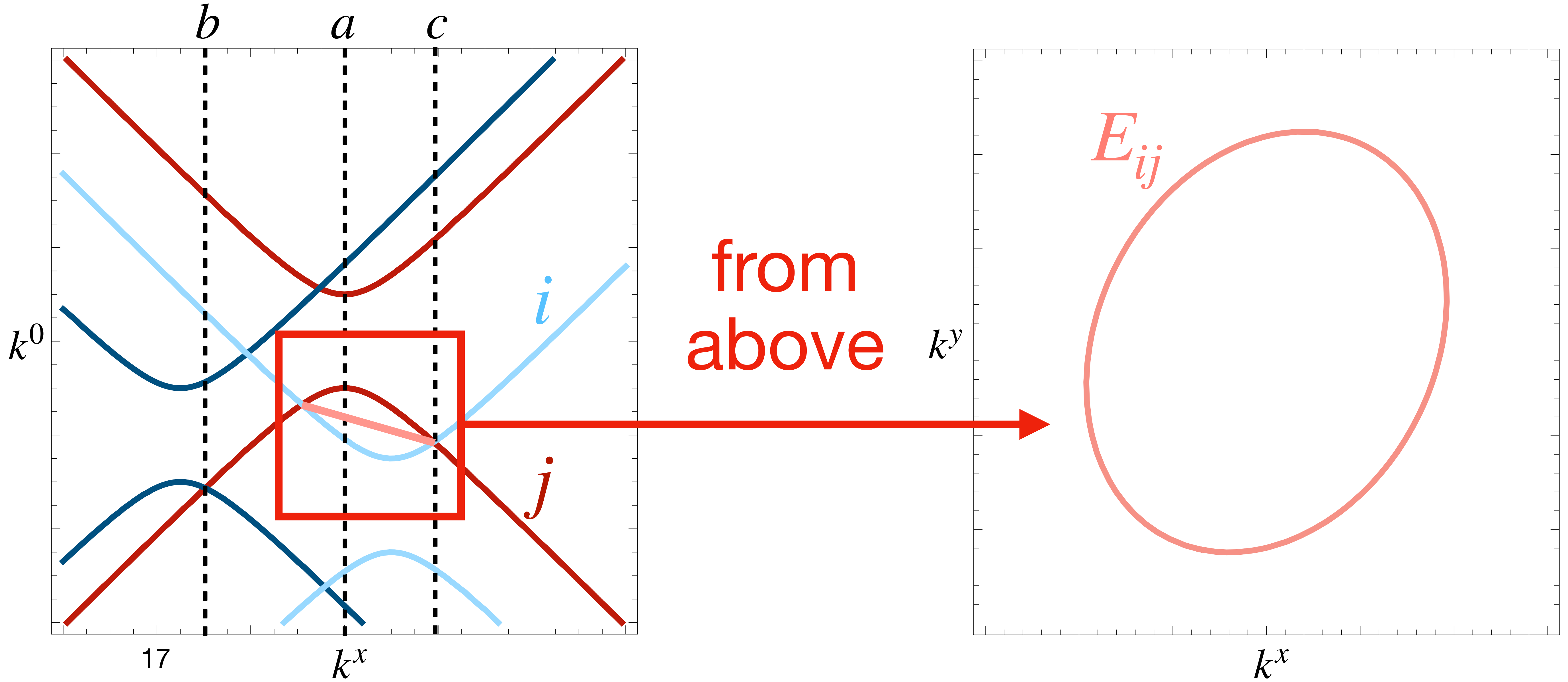

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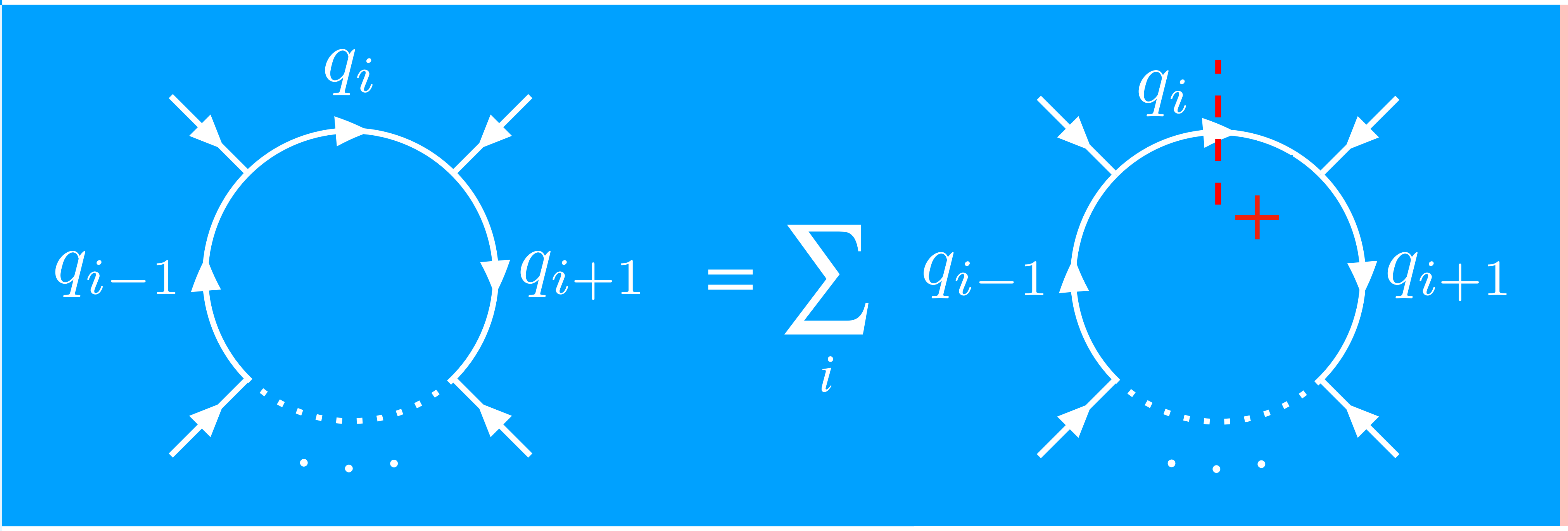

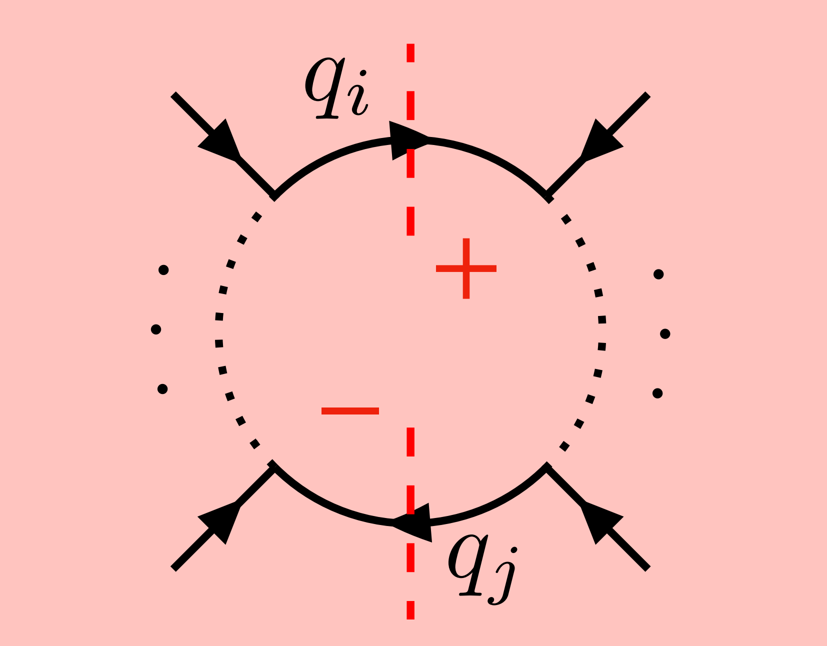

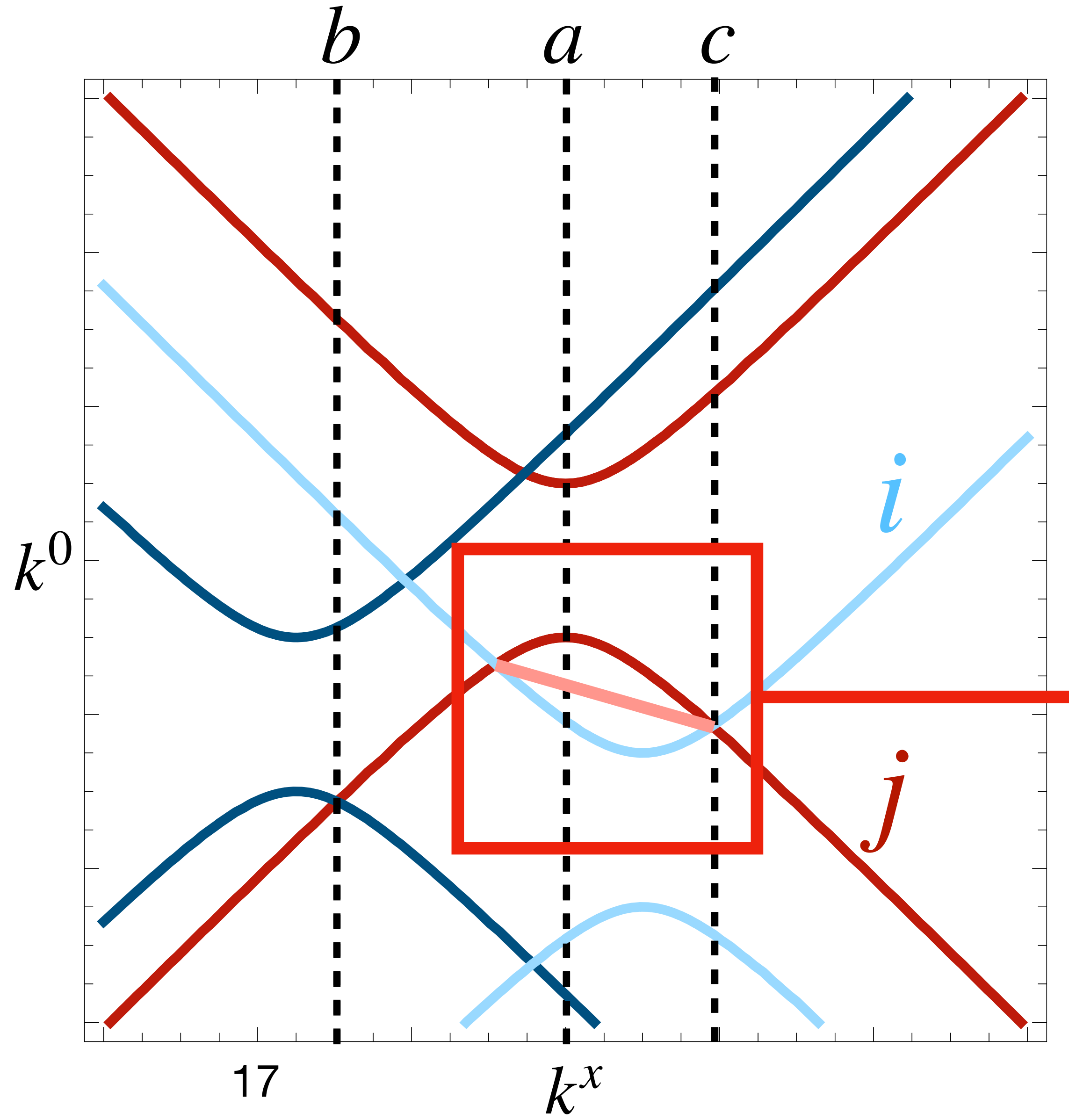

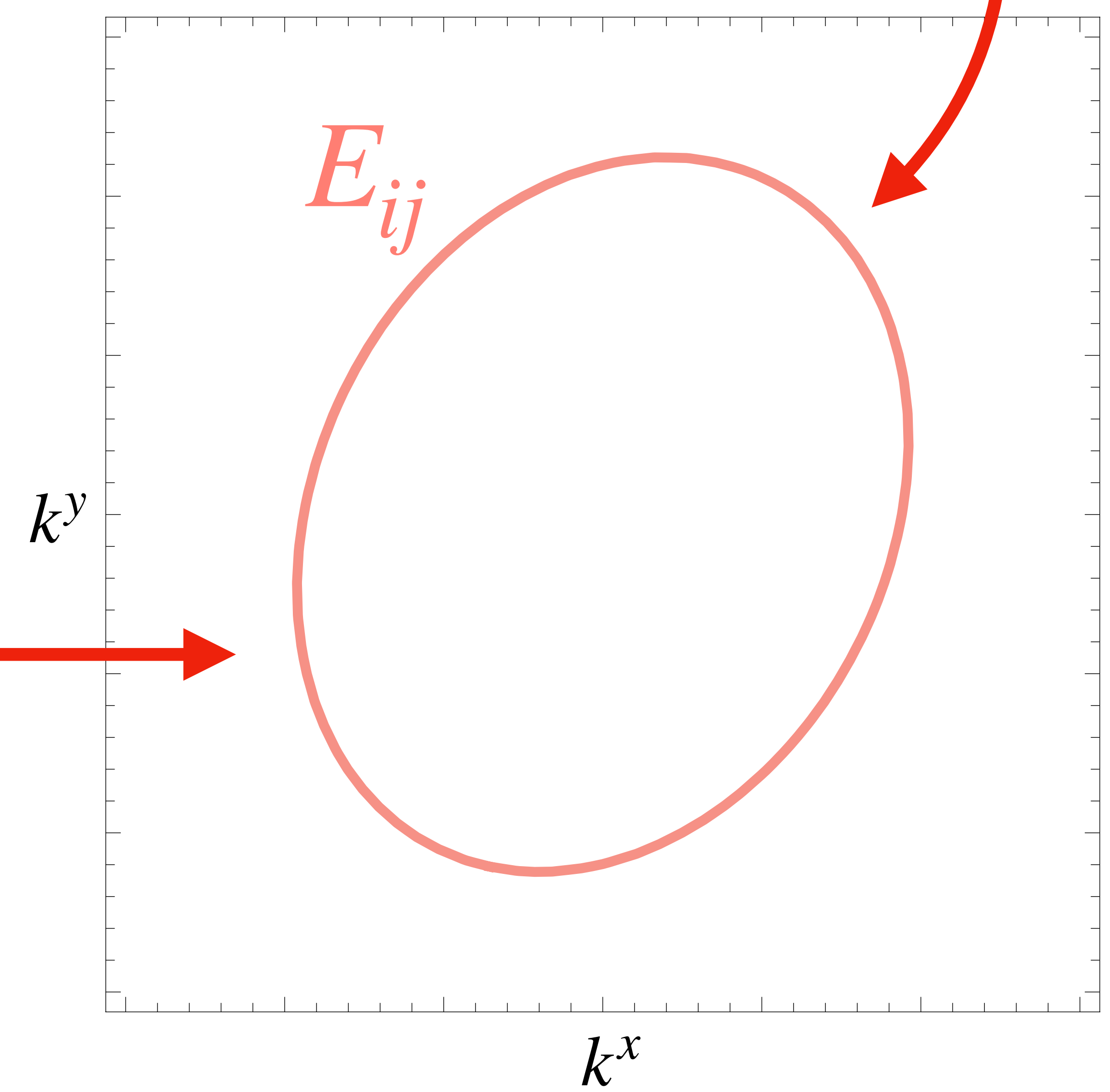

Can we integrate out a spatial dimension?

- separate a spatial integration

$$\mathrm{i}I = -\,\mathrm{i}\int_{H^2} \frac{\mathrm{d}^2\hat{k}}{(2\pi)^3} \int_{\mathbb{R}} \mathrm{d}r\, r^2 \mathcal{I}_{\mathrm{LTD}}(r\hat{k})$$

- separate a spatial integration

$$\mathrm{i}I = -\,\mathrm{i}\int_{H^2}\frac{\mathrm{d}^2\hat{k}}{(2\pi)^3}\int_{\mathbb{R}}\mathrm{d}r\, r^2 \mathscr{I}_{\mathrm{LTD}}(r\hat{k})$$

- identify poles
  parameterise pinched thresholds
  (analytically!)

$$E_{ij} = 0 \quad \Rightarrow \quad r = r_{ij}^{\pm}(\hat{k})$$

- separate a spatial integration

$$\mathrm{i}I = -\,\mathrm{i}\int_{H^2}\frac{\mathrm{d}^2\hat{k}}{(2\pi)^3}\int_{\mathbb{R}}\mathrm{d}r\, r^2\mathscr{I}_{\mathrm{LTD}}(r\hat{k})$$

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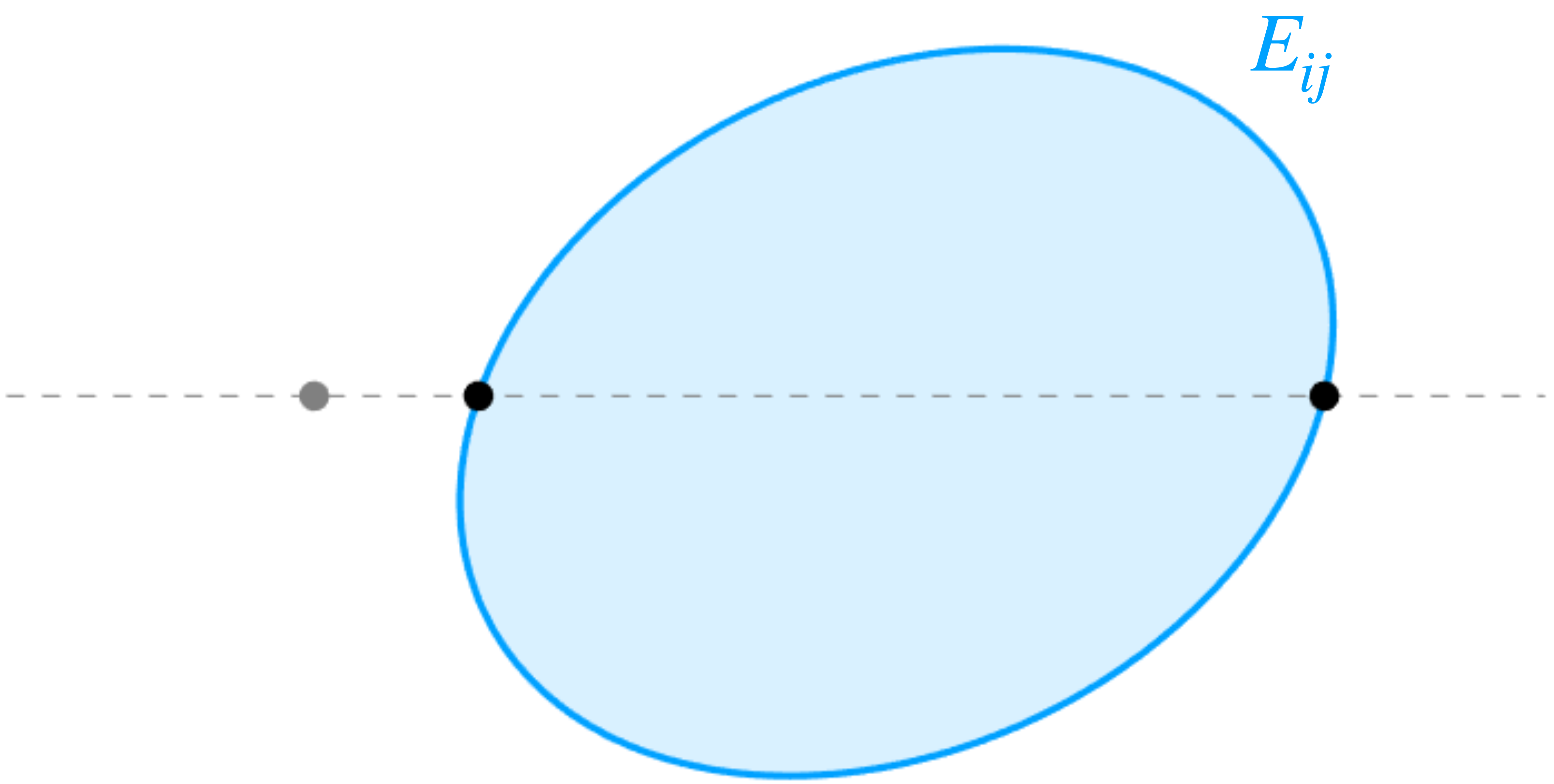

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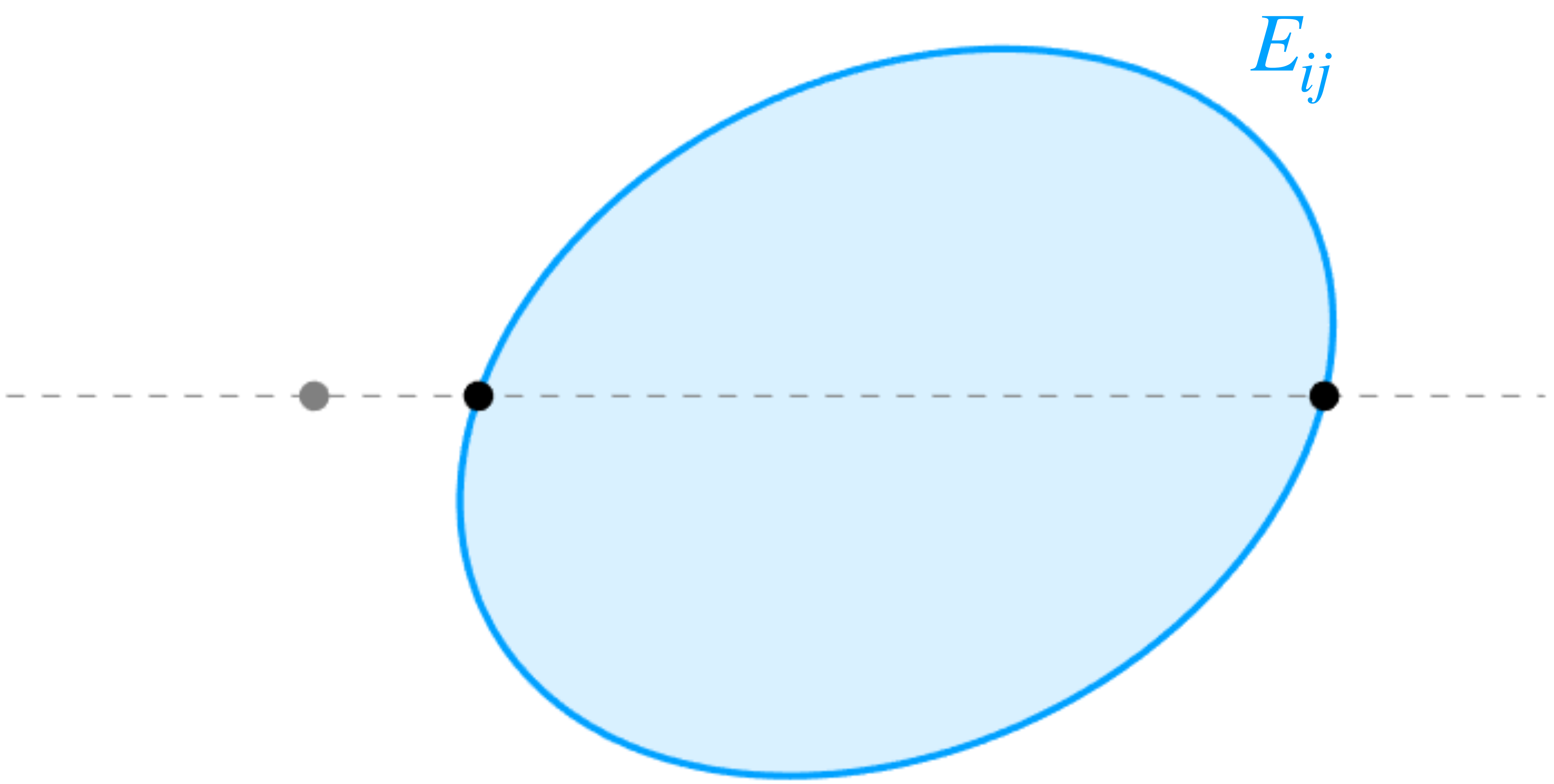
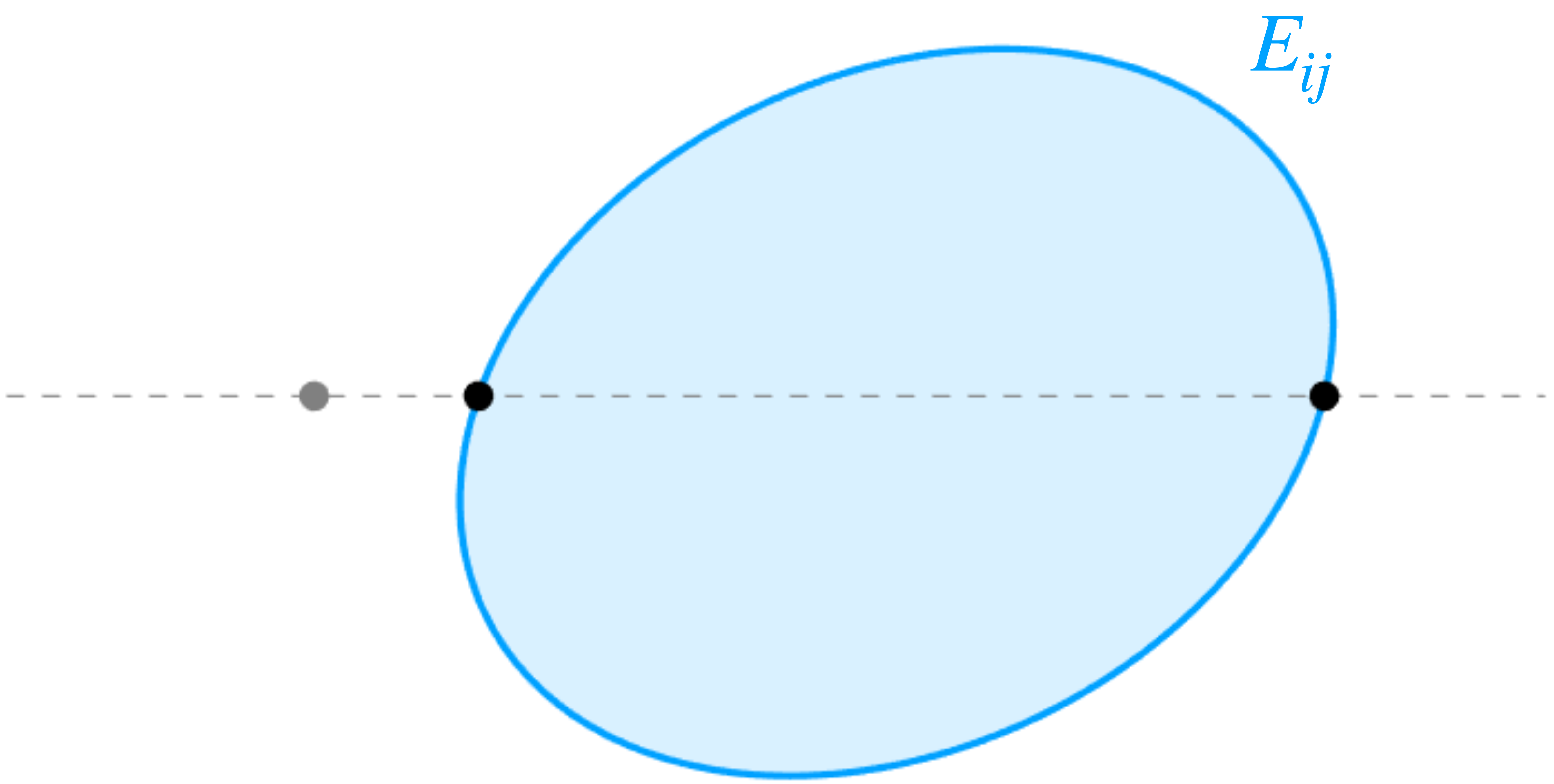


- construct counterterms

$$\mathrm{CT}_{ij}^{\pm}(r,\hat{k}) = \frac{R_{ij}^{\pm}(\hat{k})}{r - r_{ij}^{\pm}(\hat{k})}\exp\left(-\frac{(r - r_{ij}^{\pm}(\hat{k}))^2}{E_{\mathrm{cm}}^2}\right)$$

- separate a spatial integration

$$\mathrm{i}I = -\mathrm{i}\int_{H^2}\frac{\mathrm{d}^2\hat{k}}{(2\pi)^3}\int_{\mathbb{R}}\mathrm{d}r\, r^2\mathcal{I}_{\mathrm{LTD}}(r\hat{k})$$



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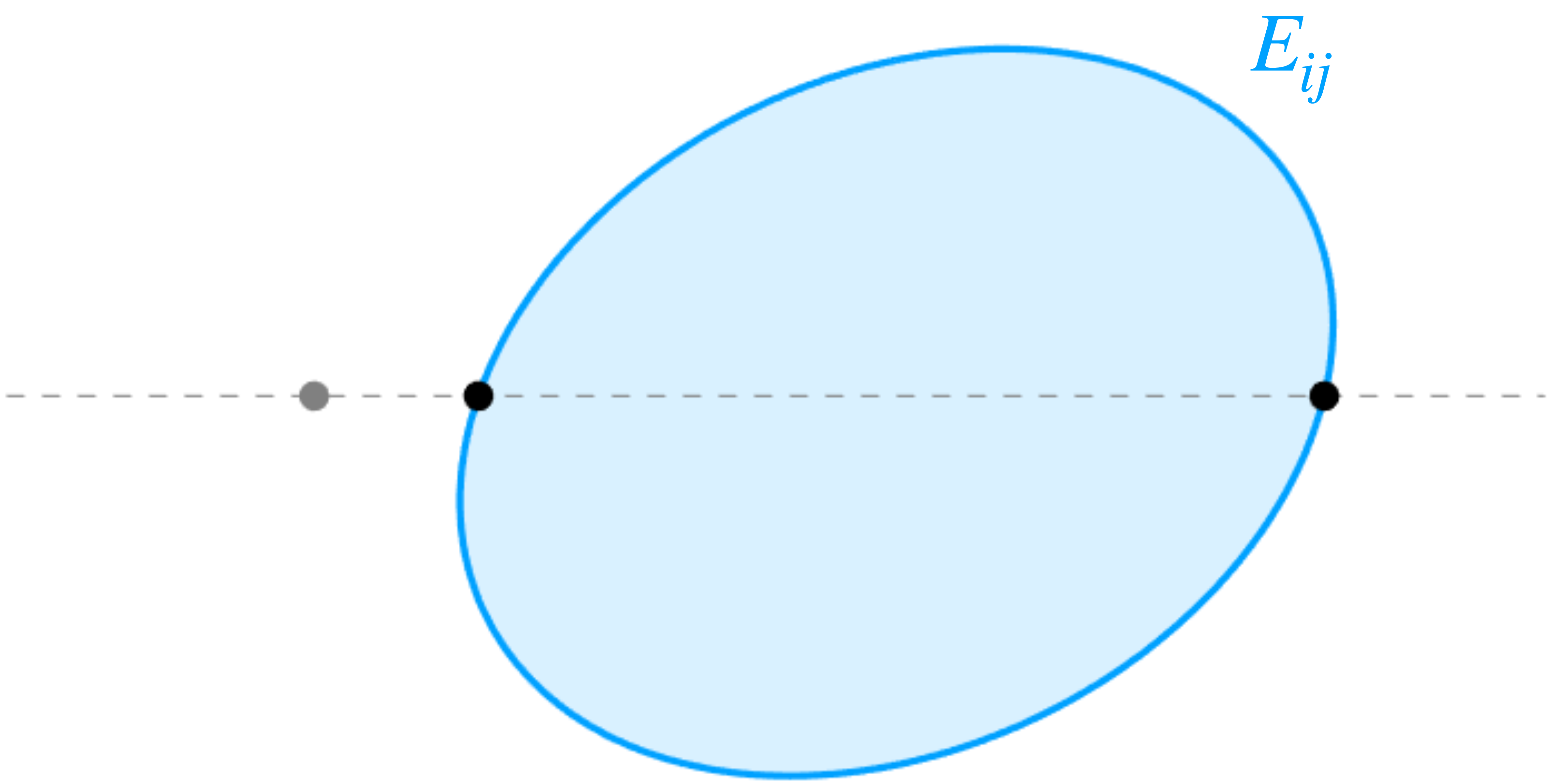
before:

$$\mathrm{CT}_i(x) = \frac{\mathrm{Res}[\mathcal{I}(y), y=x_i]}{x-x_i}S_{\mathrm{UV}}(x-x_i)$$

- separate a spatial integration

$$\mathrm{i}I = -\mathrm{i}\int_{H^2} \frac{\mathrm{d}^2\hat{k}}{(2\pi)^3} \int_{\mathbb{R}} \mathrm{d}r\, r^2 \mathcal{I}_{\mathrm{LTD}}(r\hat{k})$$



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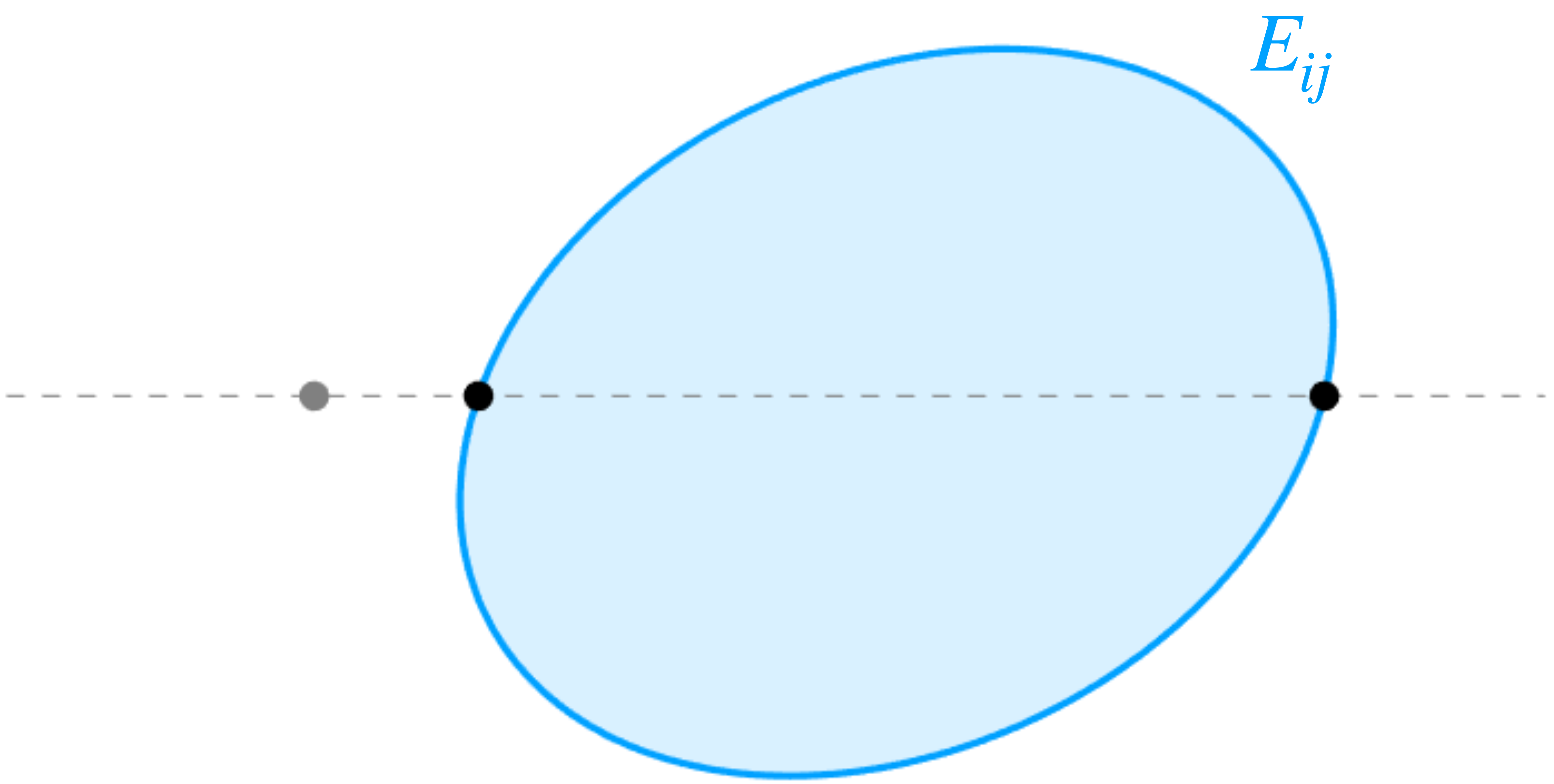

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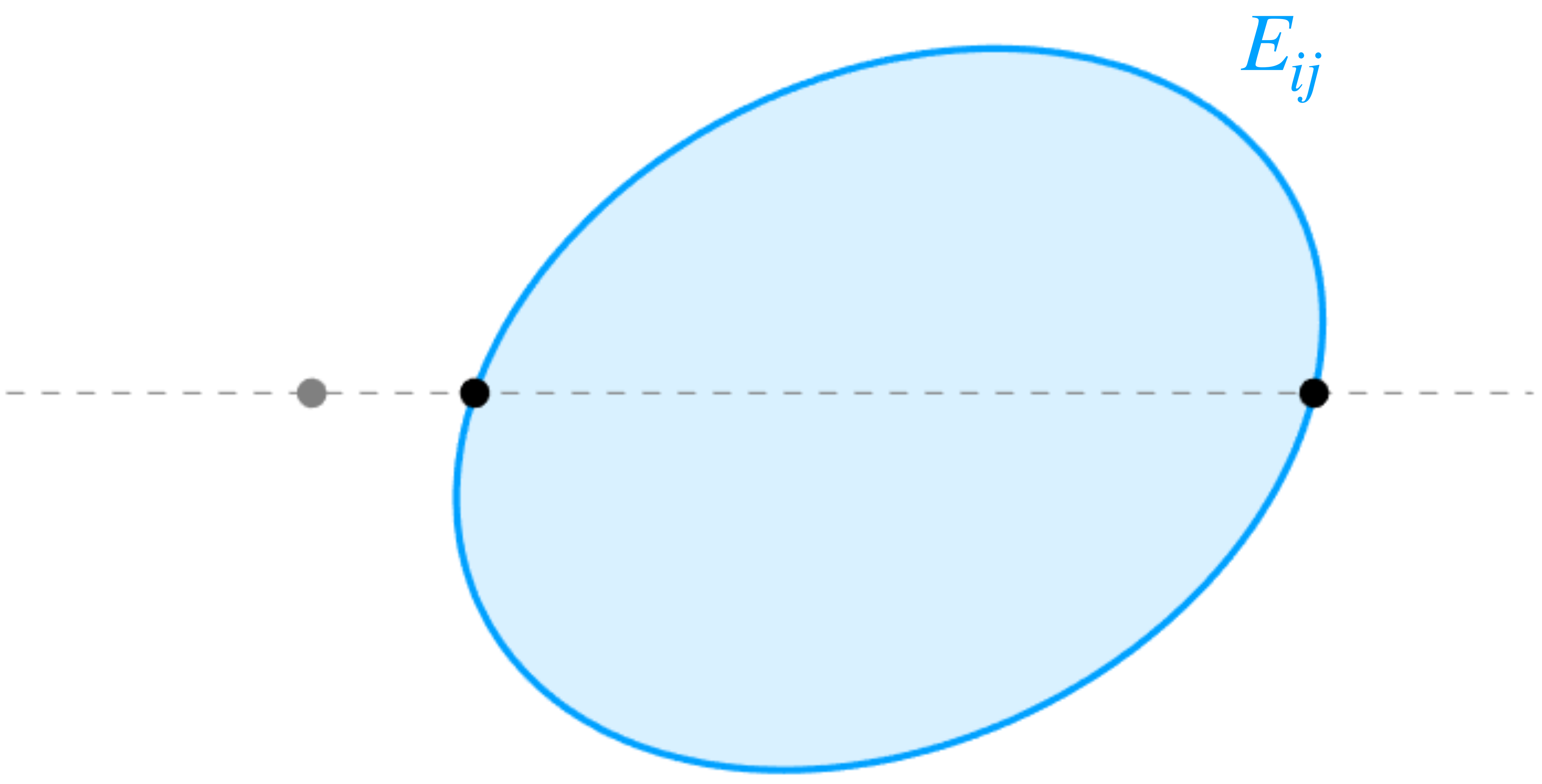
$$R^{\pm}_{ij}(\hat{k}) := \mathrm{Res}[r^2 \mathcal{I}_{\mathrm{LTD}}(r\hat{k}), r = r^{\pm}_{ij}(\hat{k})]$$

- separate a spatial integration

$$\mathrm{i}I = -\,\mathrm{i}\int_{H^2} \frac{\mathrm{d}^2\hat{k}}{(2\pi)^3} \int_{\mathbb{R}} \mathrm{d}r\, r^2 \mathcal{I}_{\mathrm{LTD}}(r\hat{k})$$



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$$\operatorname{Re} I = -\int_{H^2} \frac{\mathrm{d}^2\hat{k}}{(2\pi)^3} \int_{\mathbb{R}} \mathrm{d}r \left(r^2 \mathscr{I}_{\mathrm{LTD}}(r\hat{k}) - \sum_{(i,j)\in\mathscr{E}} \sum_{\rho\in\{\pm 1\}} \mathrm{CT}_{ij}^{\rho} \right)$$

subtracted integral

$$\operatorname{Im} I = -\frac{1}{2} \int_{H^2} \frac{\mathrm{d}^2\hat{k}}{(2\pi)^2} \sum_{(i,j)\in\mathscr{E}} \sum_{\rho\in\{\pm 1\}} \rho\, R_{ij}^{\rho}$$

sum of residues

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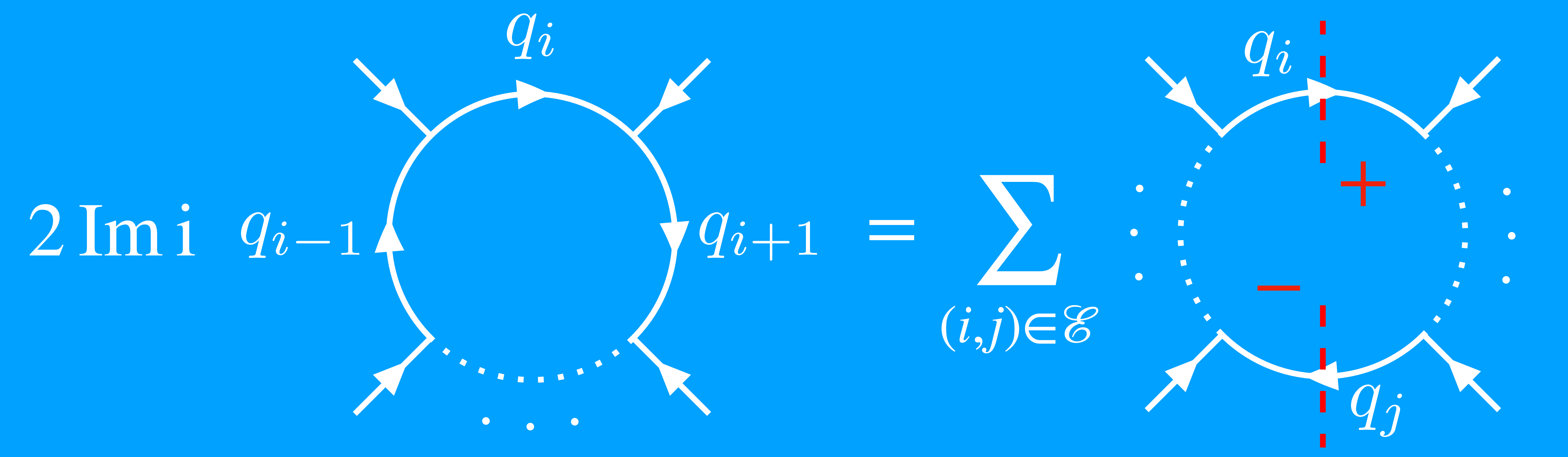

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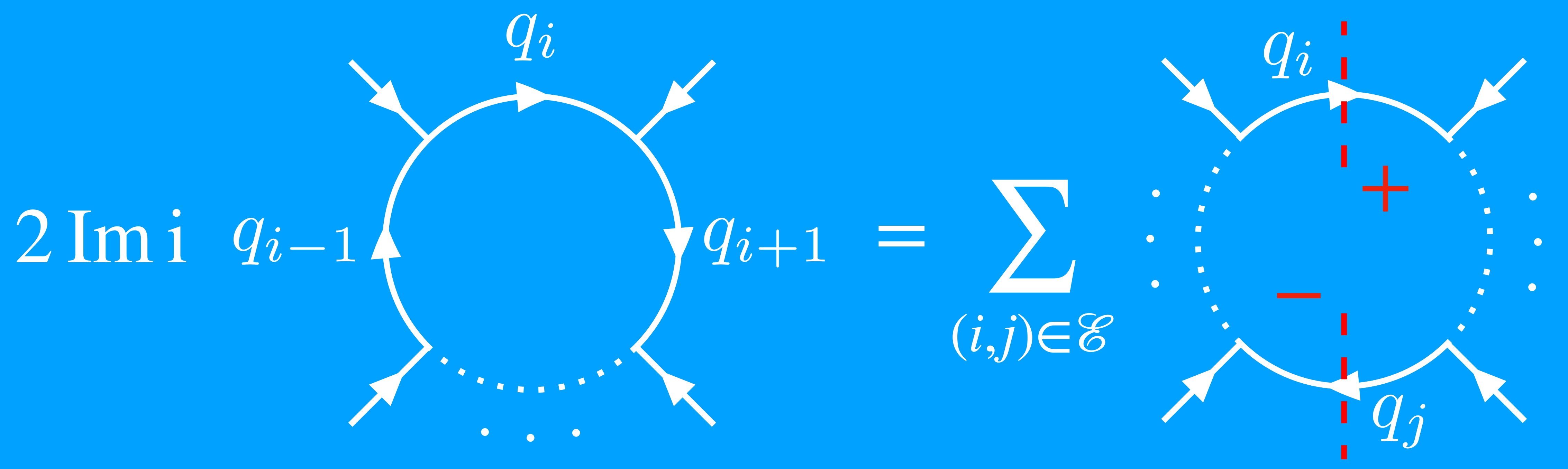

locally finite representation of generalised unitarity

$$2\,\mathrm{Im}\, A(i \to f) = \sum_x \int \mathrm{d}\Pi_x\, A(i \to x)\, A^*(f \to x)$$

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sum of residues

only finite without pinches

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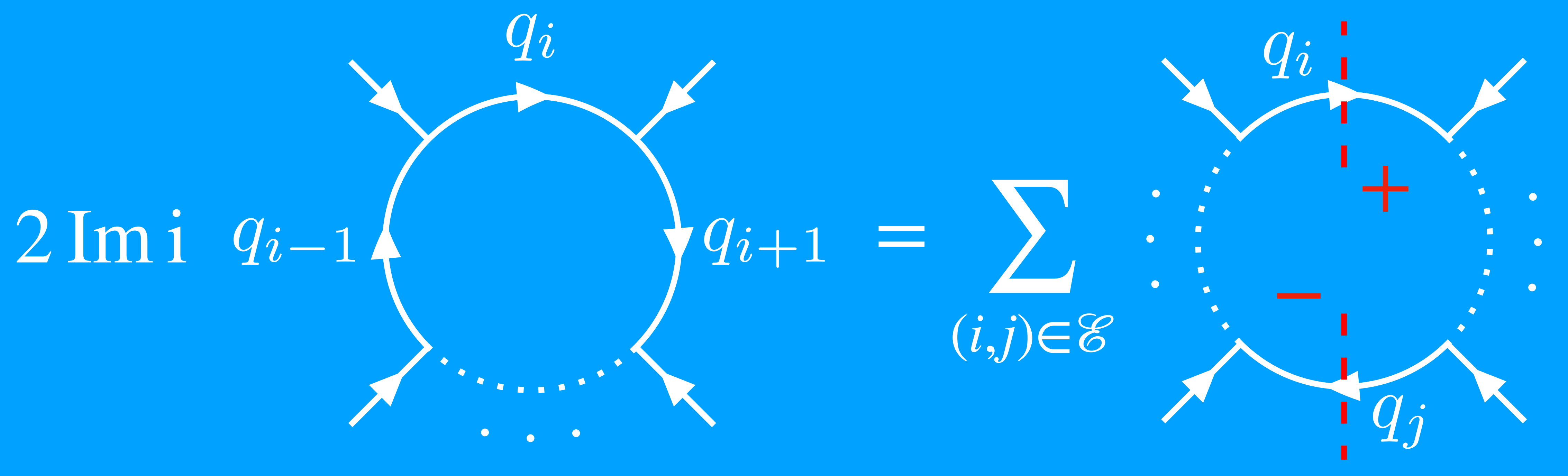

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subtracted integral

always free of poles

$$\mathrm{Im}\, I = -\frac{1}{2} \int_{H^2} \frac{\mathrm{d}^2\hat{k}}{(2\pi)^2} \sum_{(i,j)\in\mathscr{E}} \sum_{\rho\in\{\pm 1\}} \rho\, R_{ij}^{\rho}$$

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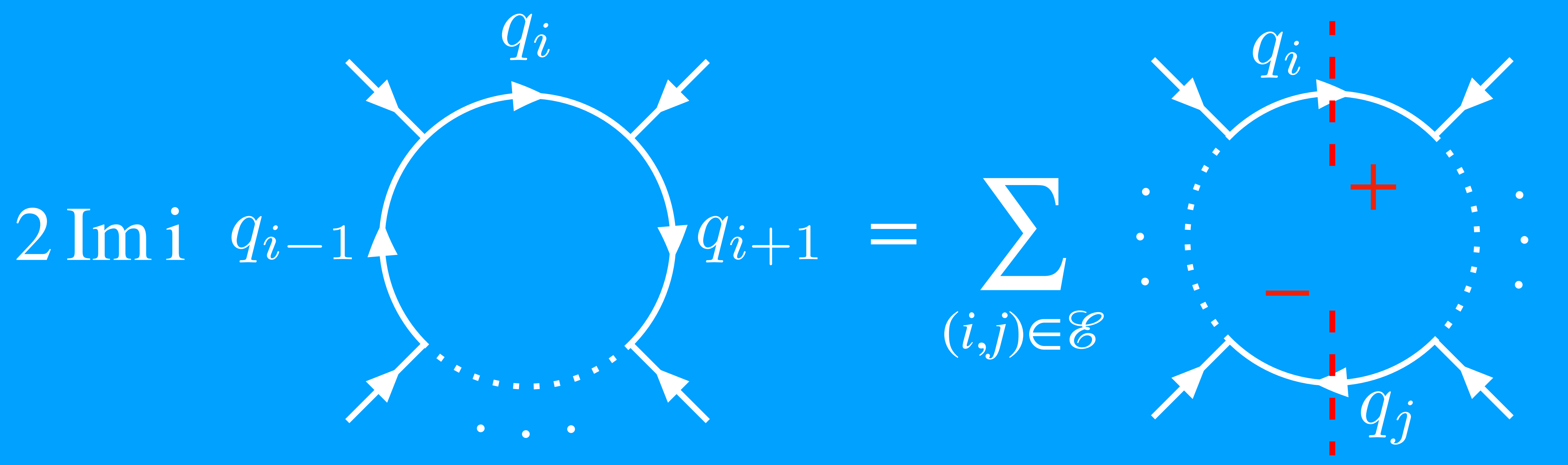

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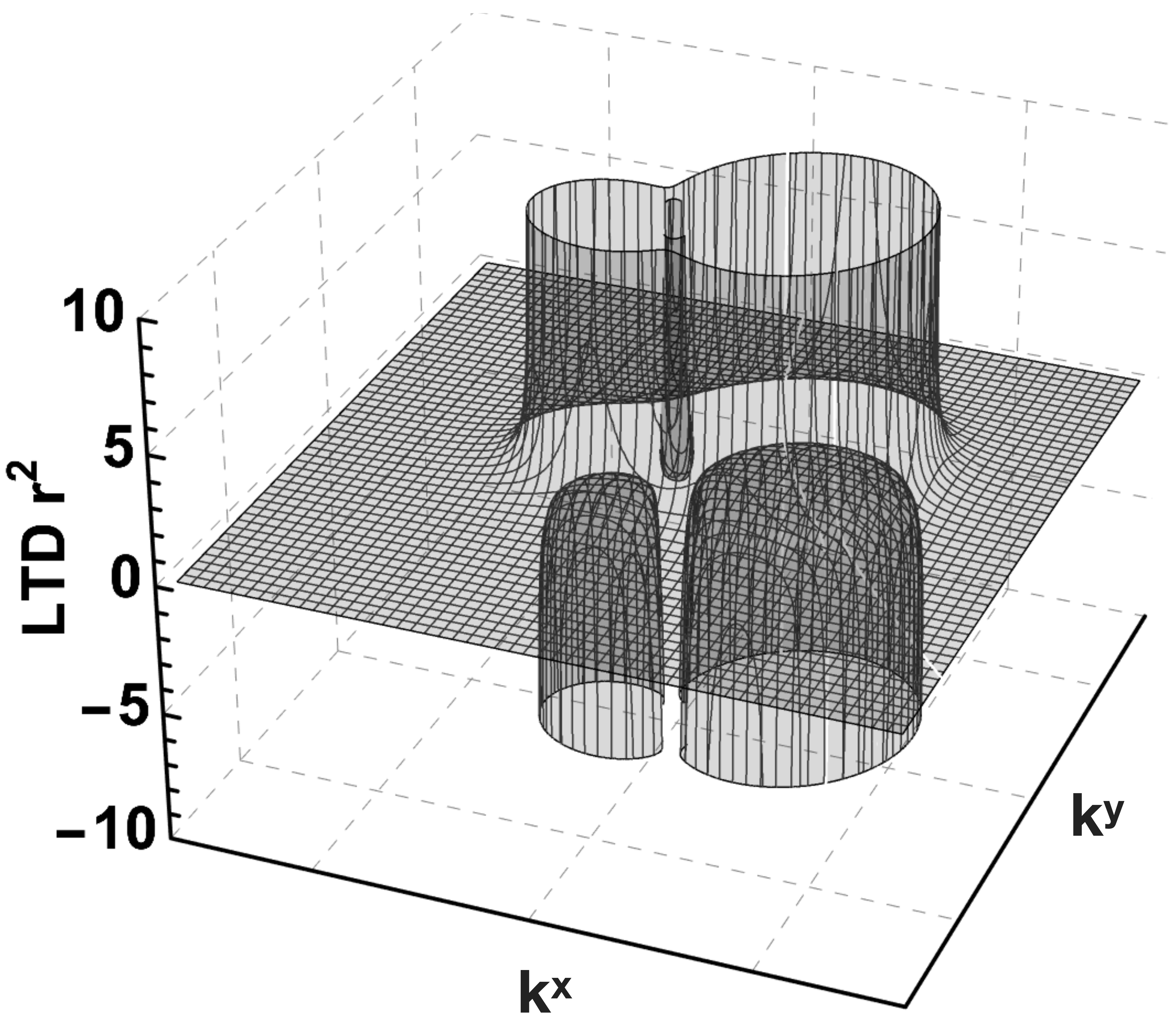
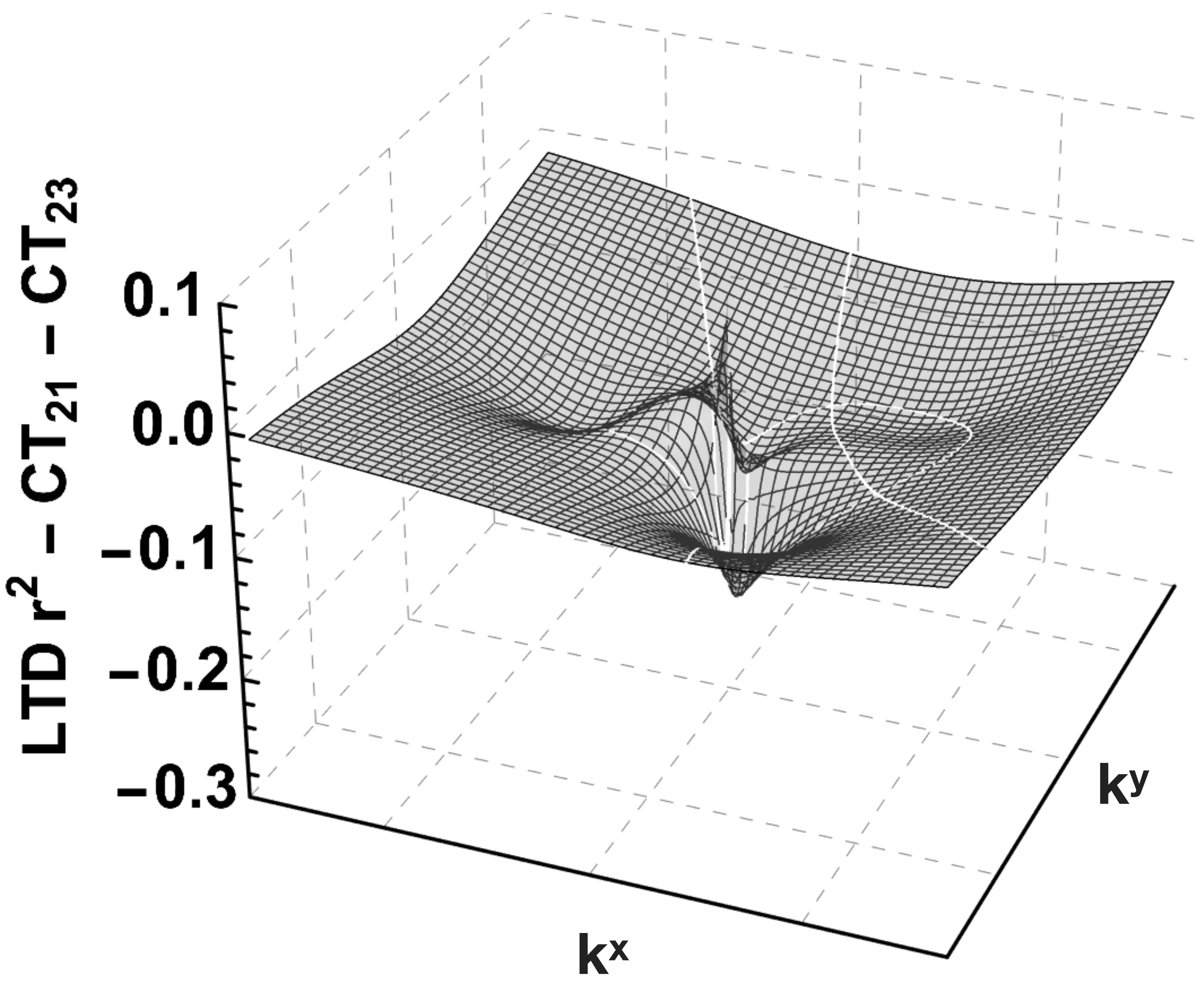
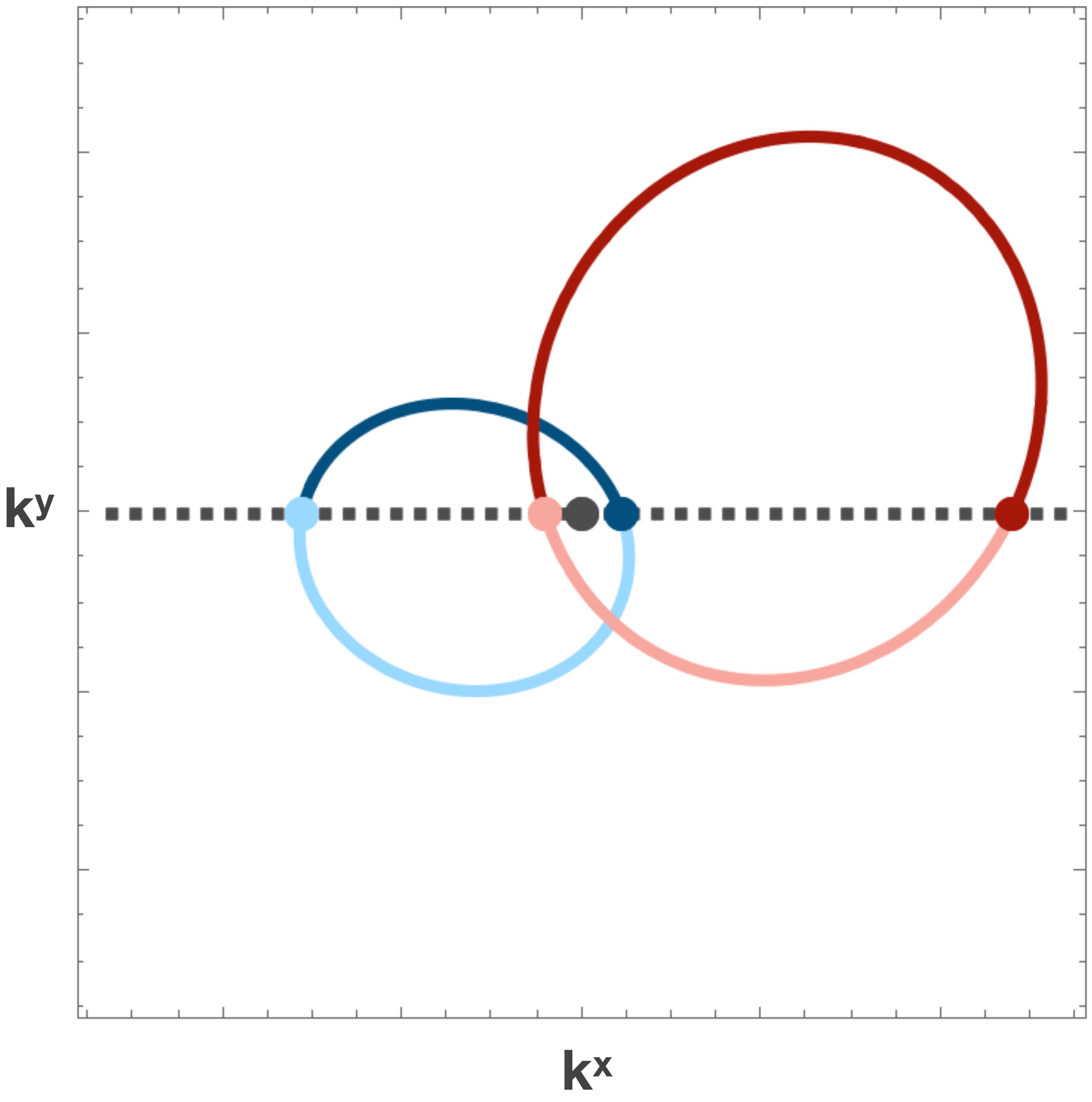
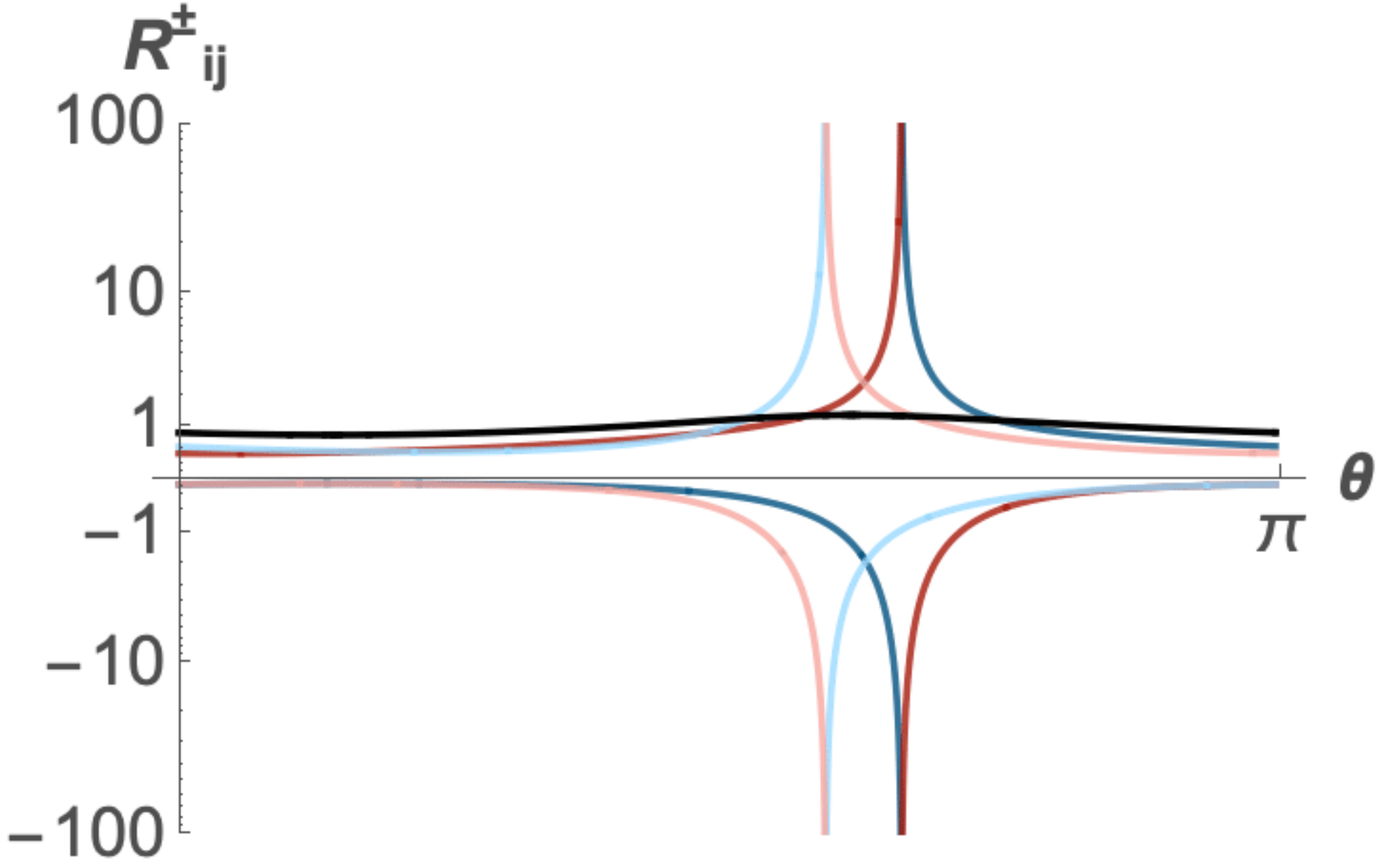




LTD integrand

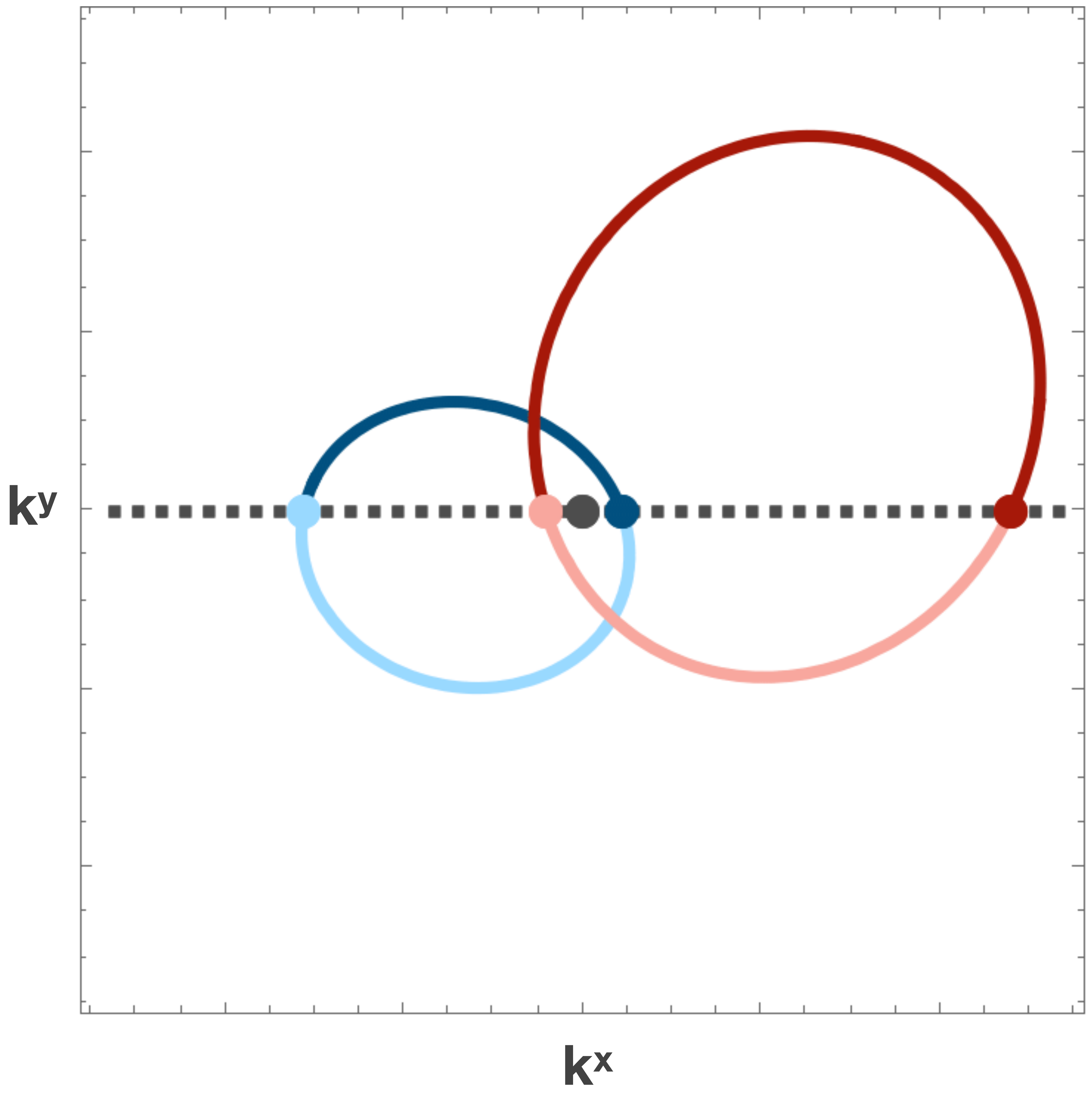
subtracted integrand





momentum space

residues



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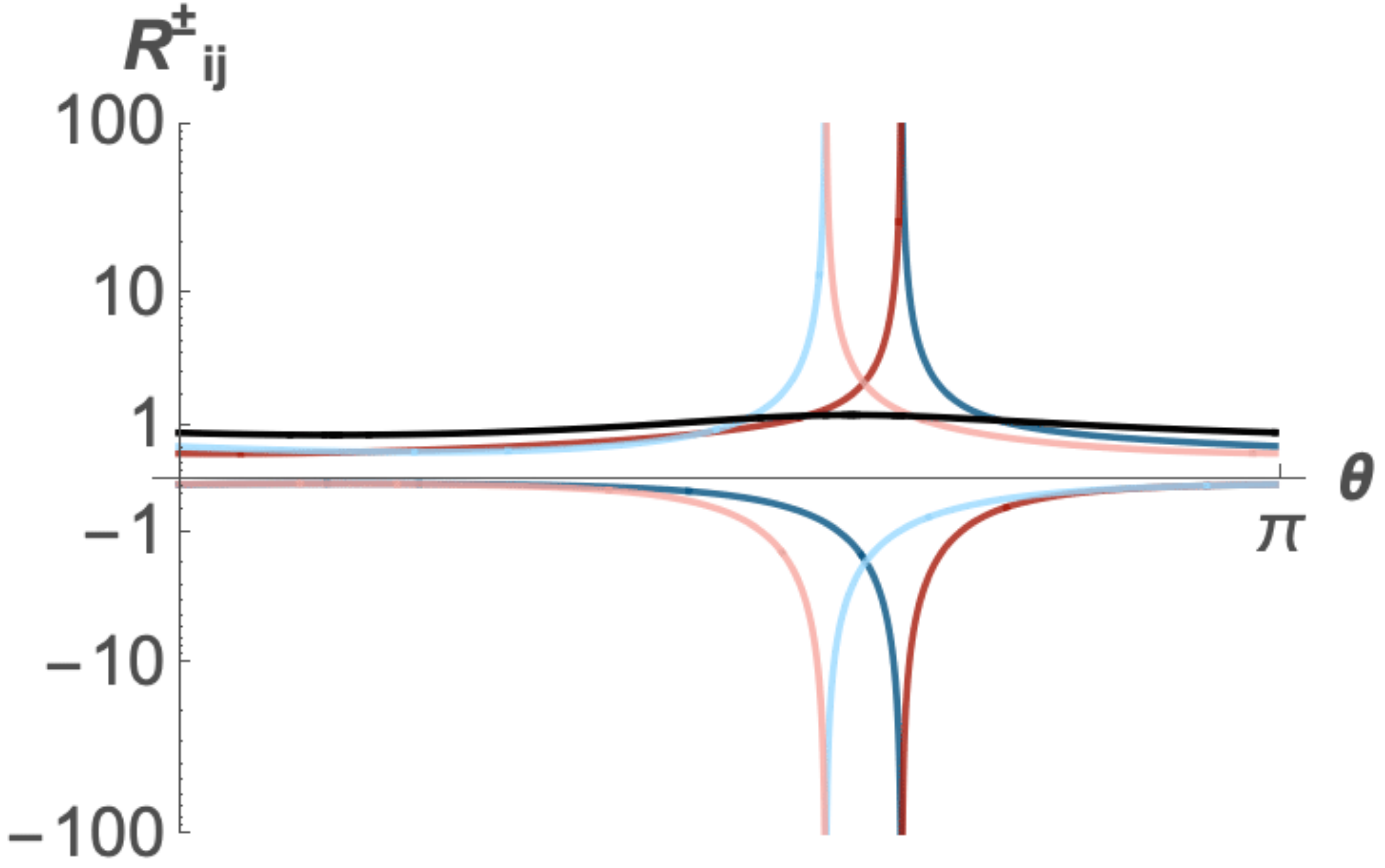

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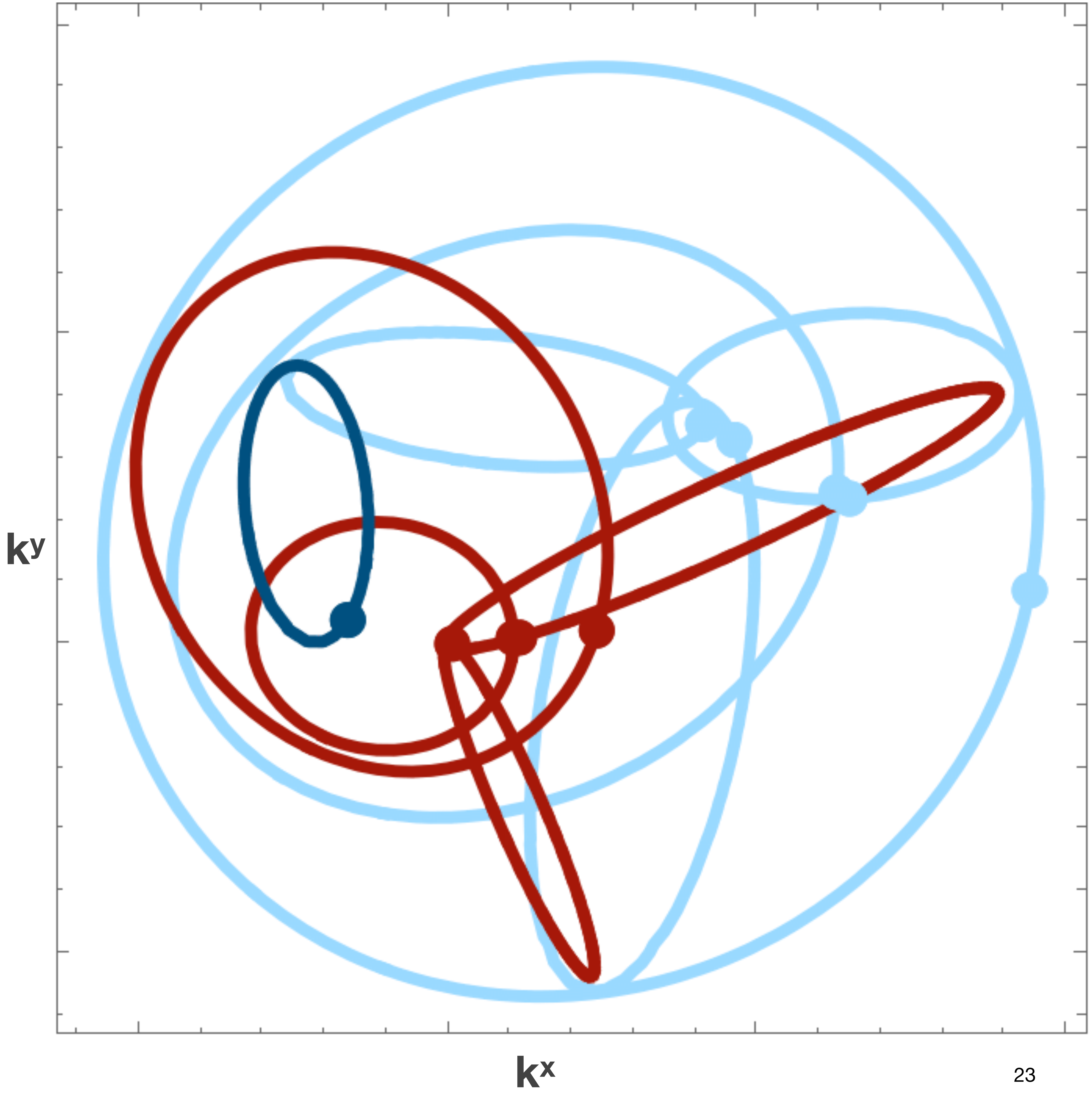
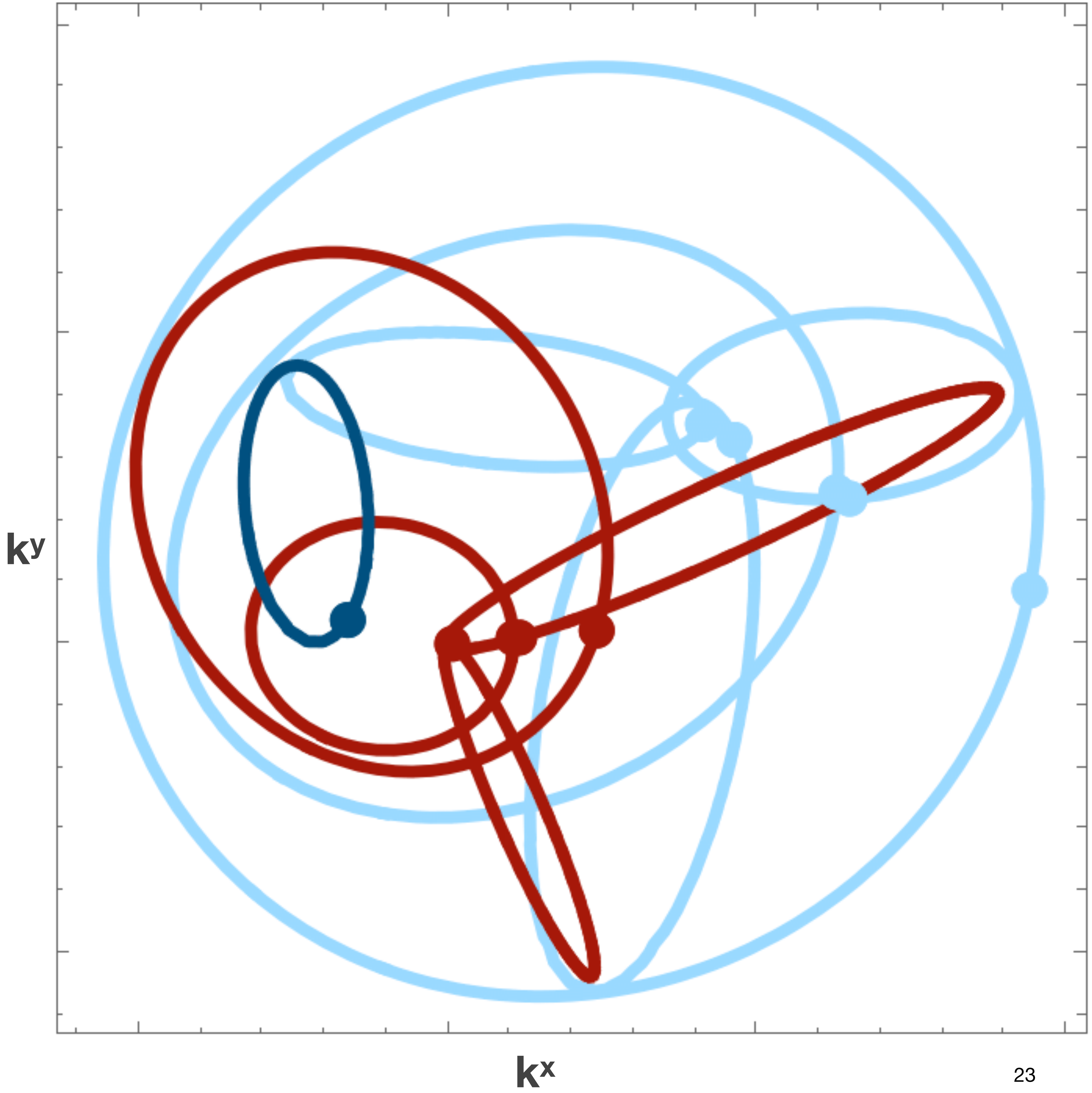


thresholds can be grouped

not all intersections lead to higher-order poles



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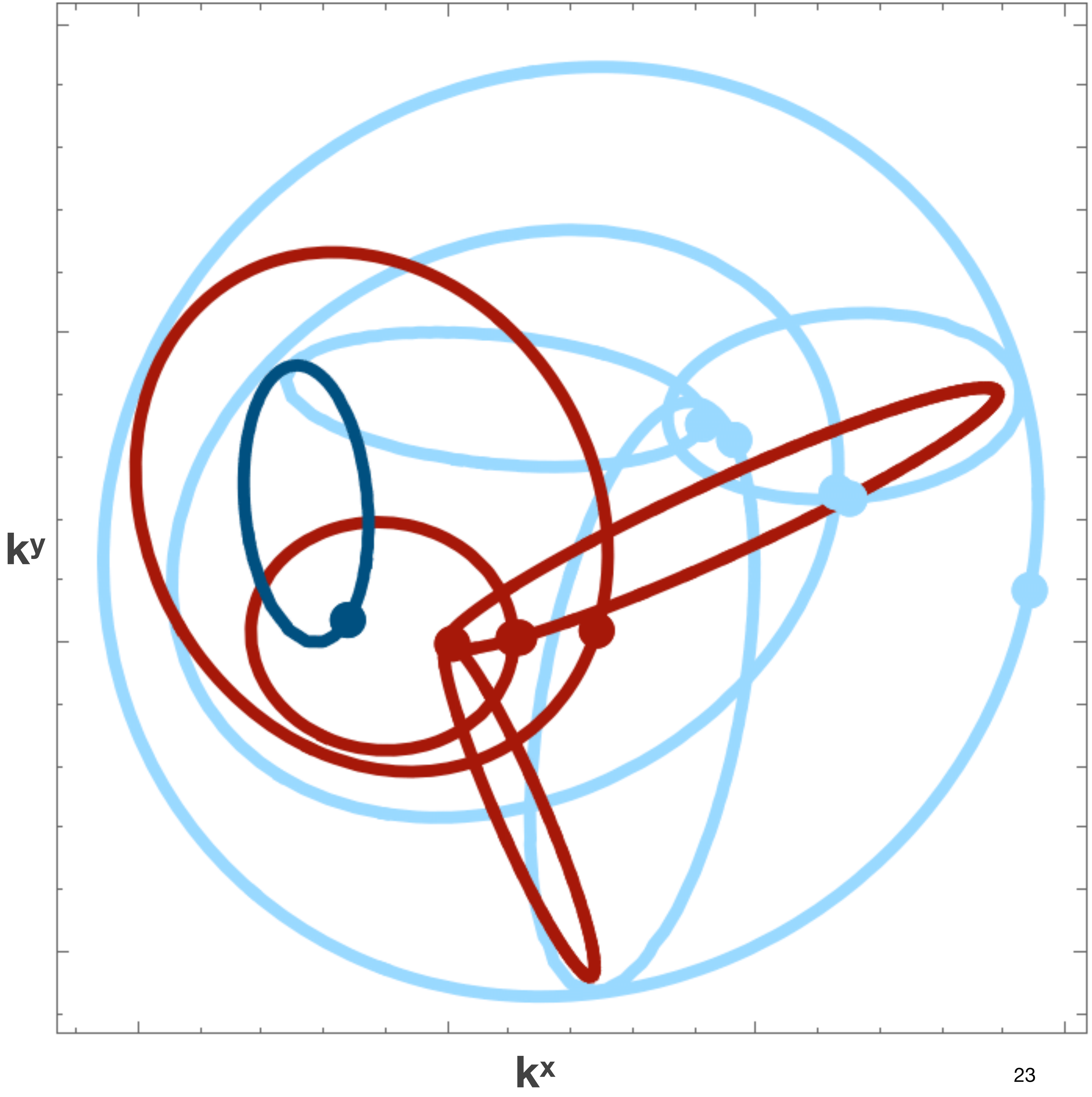

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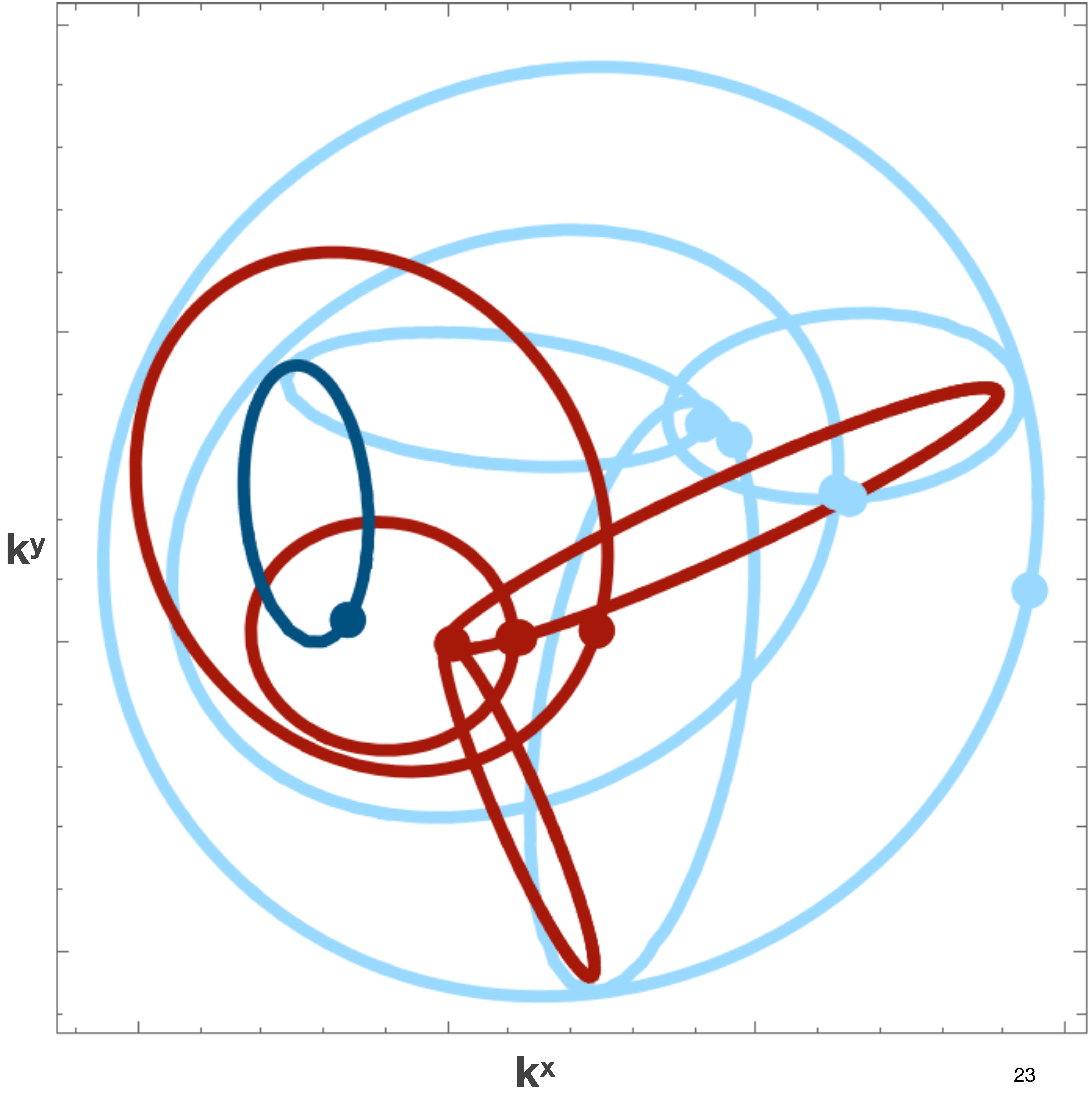
$$R_{ij}^{\pm} = \Theta\left(r_{ij}^{\pm} \in \mathbb{R}\right) \frac{1}{-4E_iE_j} \frac{r^2}{\left(\frac{\vec{q}_i}{E_i} + \frac{\vec{q}_j}{E_j}\right)\cdot \hat{k}}\Bigg|_{r=r_{ij}^{\pm}} \frac{N}{\prod_{l\neq i,j} D_l}\Bigg|_{ij}$$



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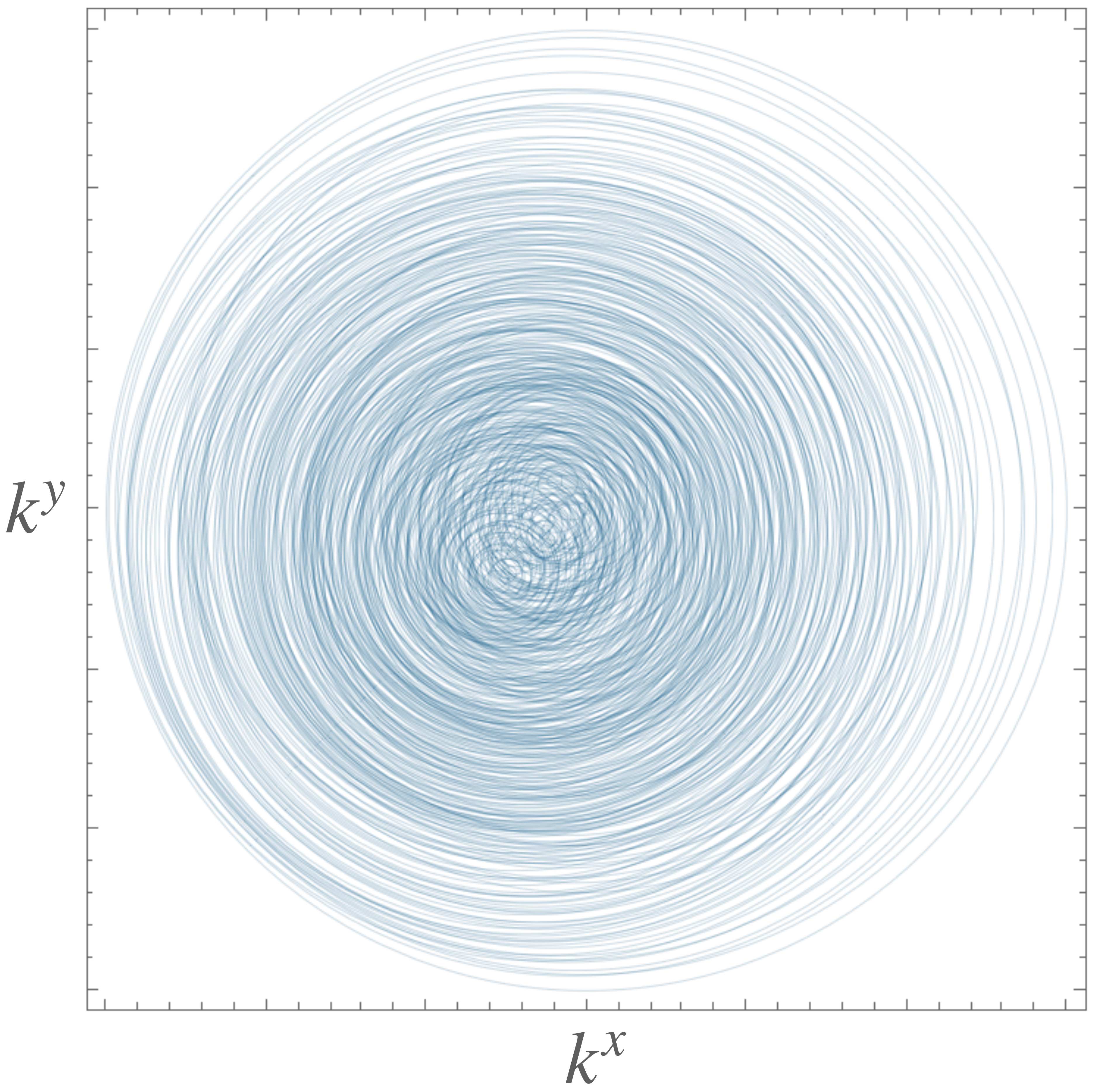
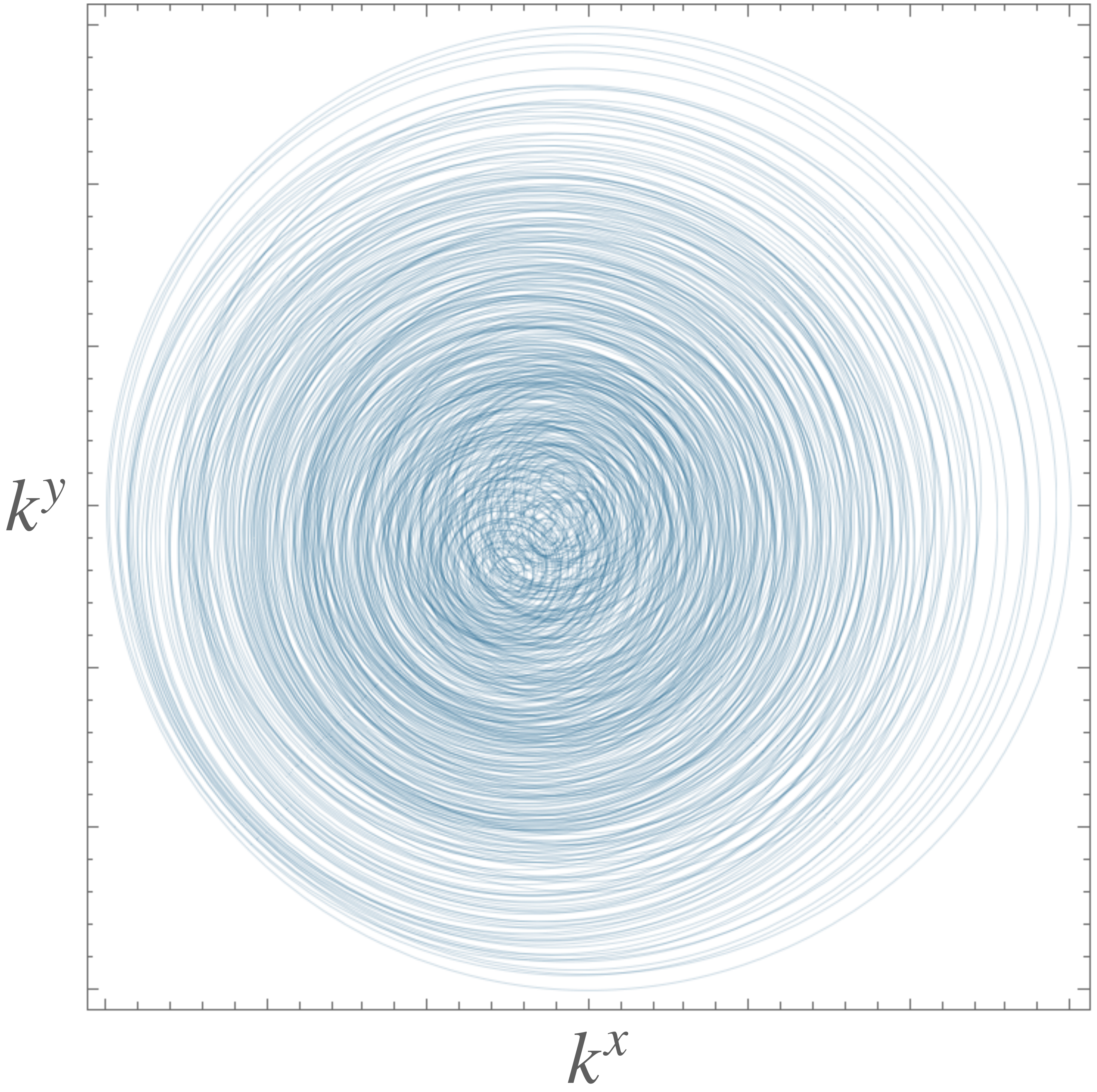
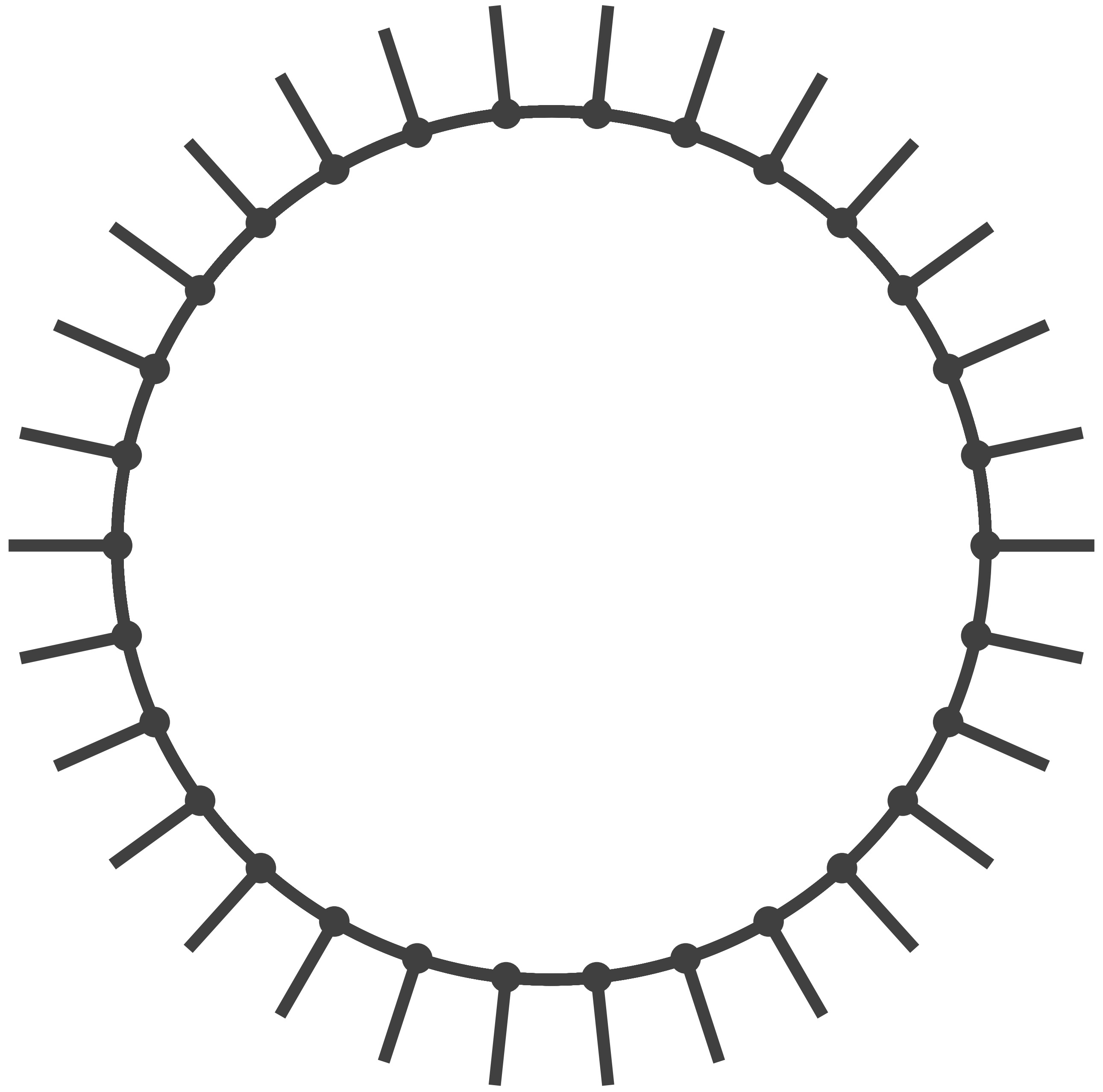
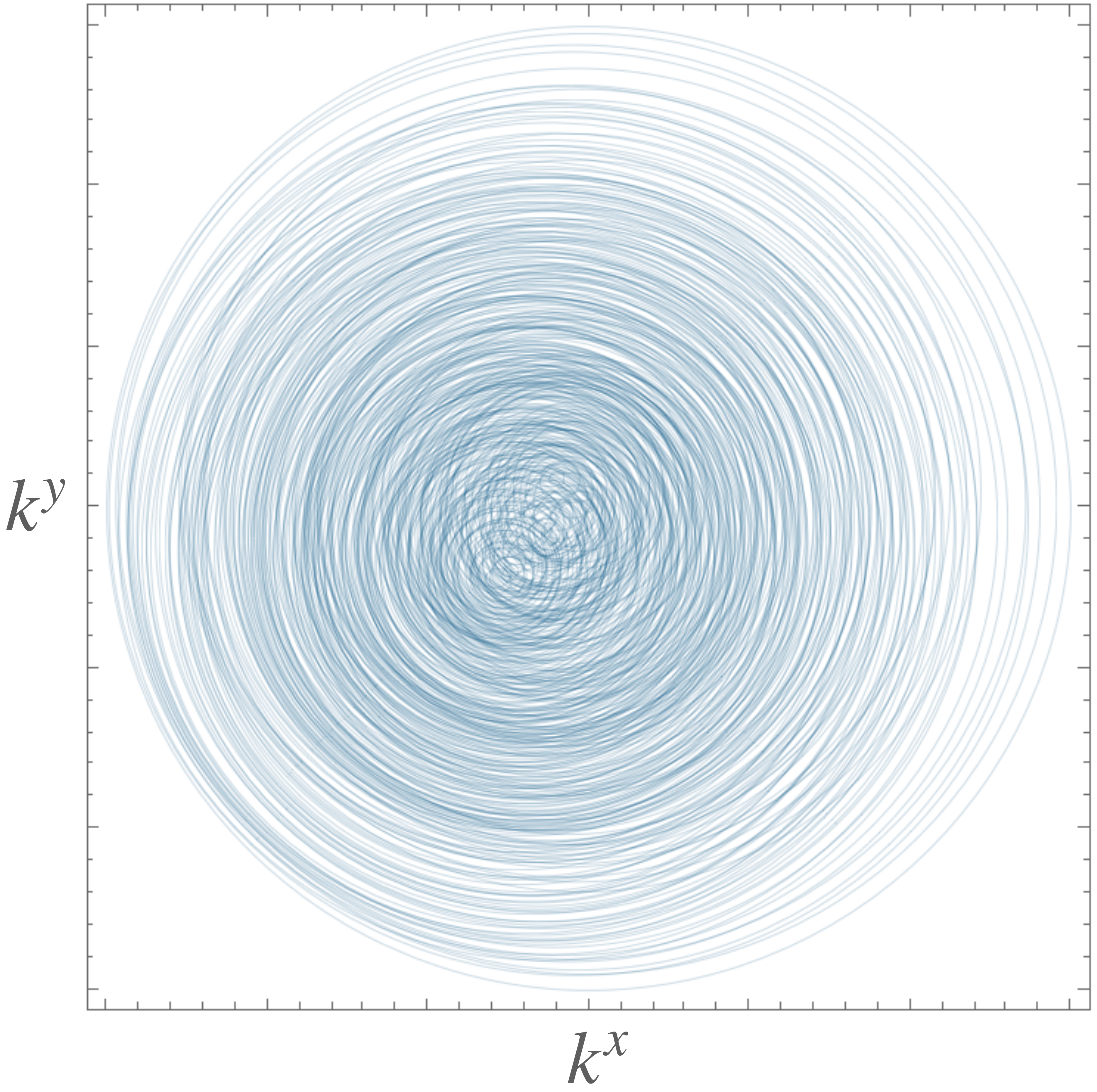
How far can we push threshold subtraction?

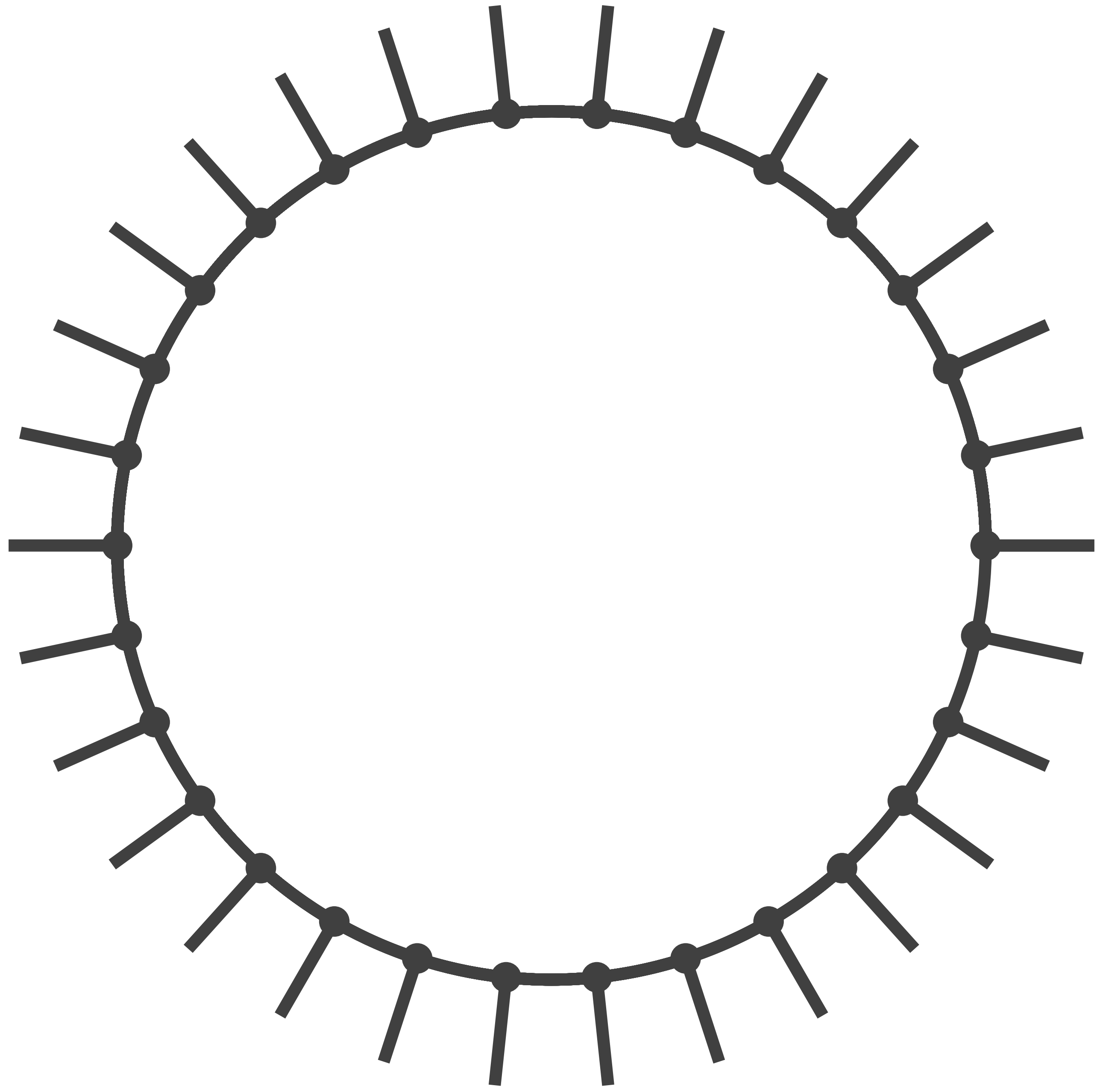

k^y
k^x





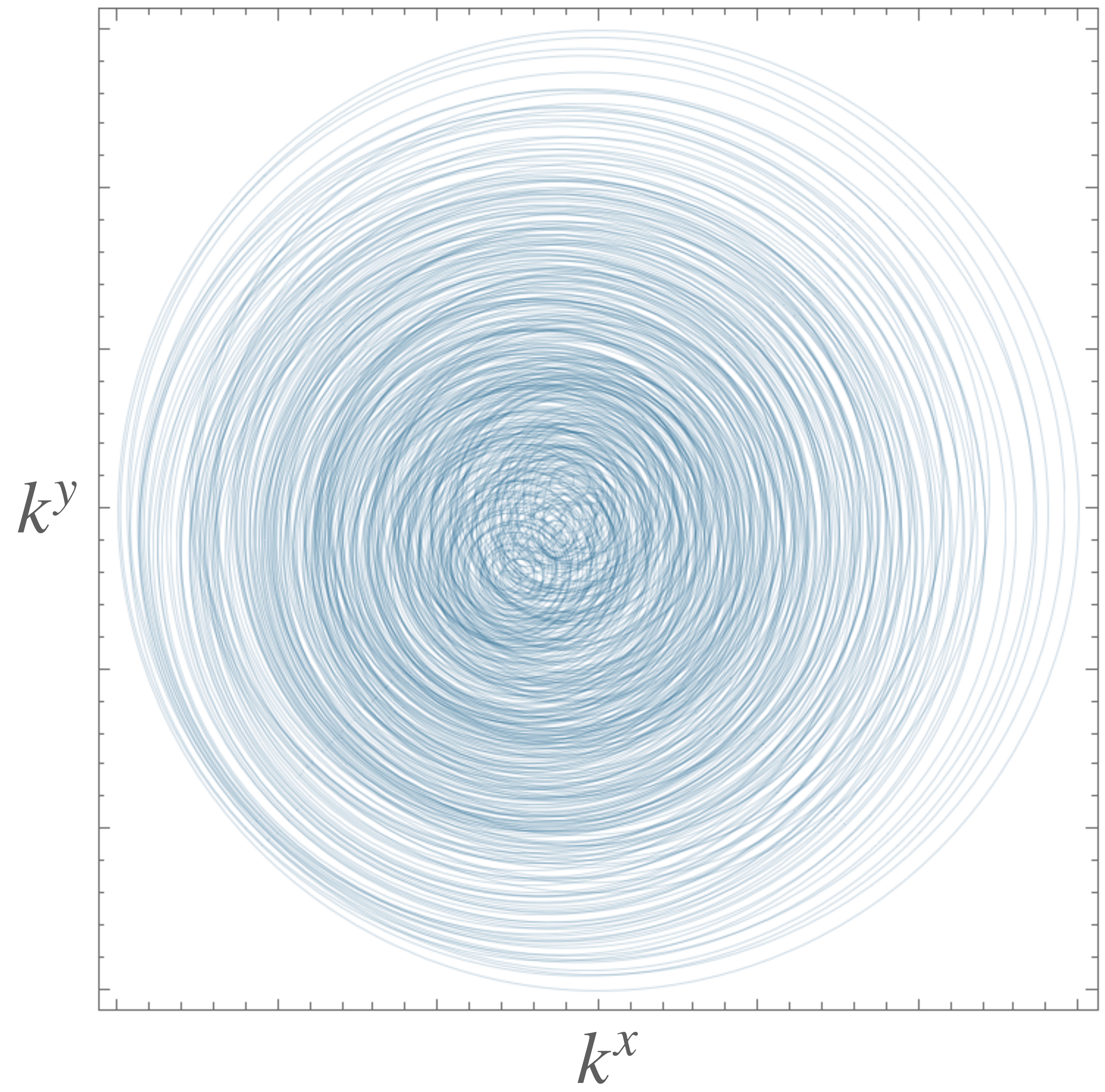
Triacontagon

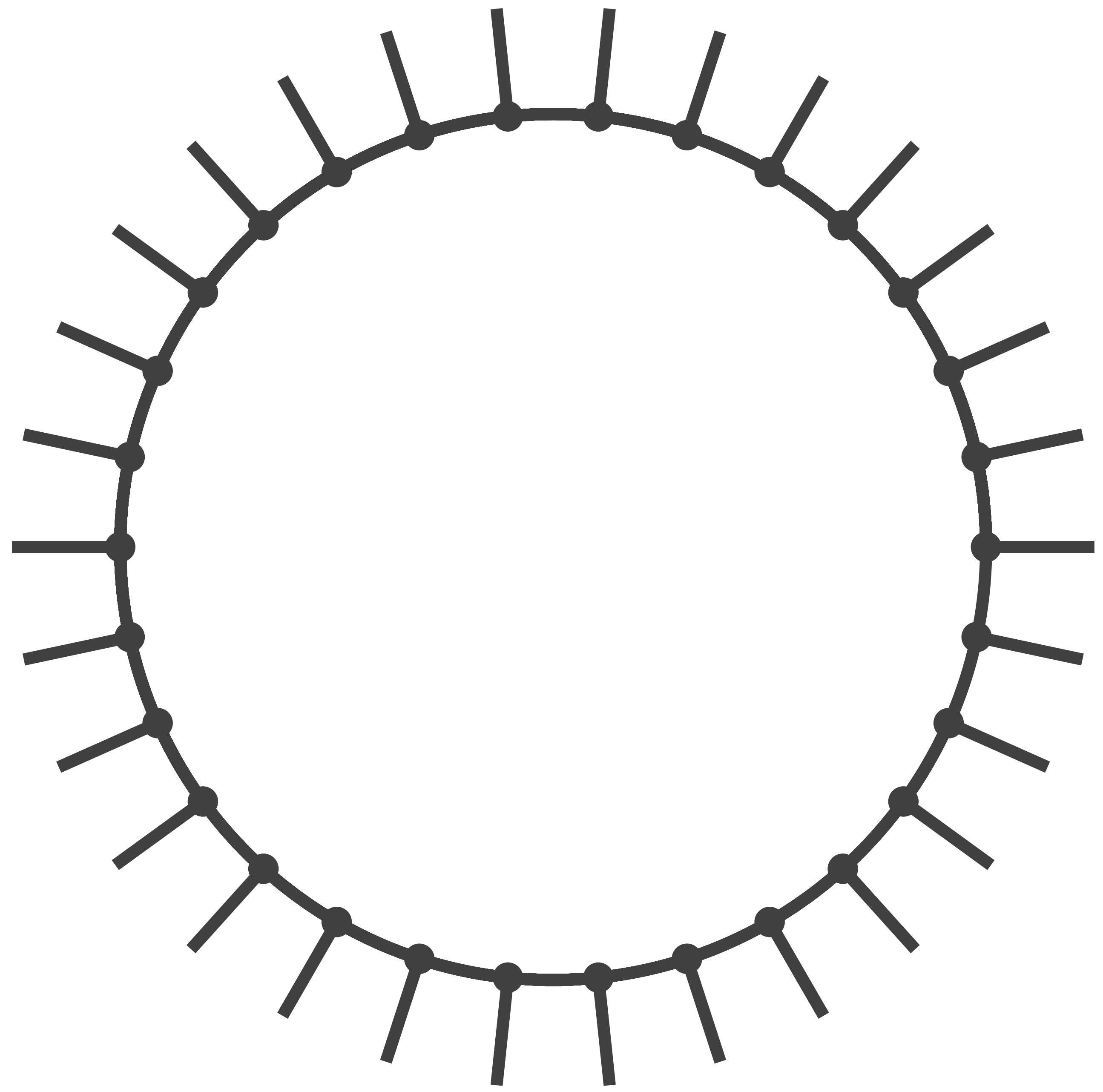




Triacontagon

Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	$\Delta\ [\sigma]$	$\Delta\ [\%]$	$\Delta\ [\%]\lvert\cdot\rvert$
1L30P.I	5	1	1	10^9	Re	-02	-1.007398	-1.007449 +/- 0.001467	0.035	0.005	0.002
				10^9	Im		3.175180	3.175183 +/- 0.000085	0.030	8e-05	
1L30P.II	6	1	1	10^9	Re	-12	-4.166377	-4.165527 +/- 0.006697	0.127	0.020	0.016
				10^9	Im		3.413930	3.413917 +/- 0.000075	0.182	4e-04	
1L30P.III	408	15	354	10^9	Re	-09	-2.991654	-2.984733 +/- 0.026977	0.257	0.231	0.231
				10^9	Im		-0.000000	-0.000001 +/- 0.003831	3e-04		
1L30P.IV	408	15	354	10^9	Re	-07	-1.757748	-1.757913 +/- 0.002169	0.076	0.009	0.009
				10^9	Im		-0.000000	0.000001 +/- 0.000199	0.007		





Triacontagon

Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ $[\sigma]$	Δ [%]	Δ [%]$\lvert\cdot\rvert$
1L30P.I	5	1	1	10^9	Re	-02	-1.007398	-1.007449 +/- 0.001467	0.035	0.005	0.002
				10^9	Im		3.175180	3.175183 +/- 0.000085	0.030	8e-05	
1L30P.II	6	1	1	10^9	Re	-12	-4.166377	-4.165527 +/- 0.006697	0.127	0.020	0.016
				10^9	Im		3.413930	3.413917 +/- 0.000075	0.182	4e-04	
1L30P.III	408	15	354	10^9	Re	-09	-2.991654	-2.984733 +/- 0.026977	0.257	0.231	0.231
				10^9	Im		-0.000000	-0.000001 +/- 0.003831	3e-04		
1L30P.IV	408	15	354	10^9	Re	-07	-1.757748	-1.757913 +/- 0.002169	0.076	0.009	0.009
				10^9	Im		-0.000000	0.000001 +/- 0.000199	0.007		

Reproducing finite scalar integrals

Reproducing finite scalar integrals

Kinematics from: [1510.00187: Buchta, Chachamis, Draggiotis, Rodrigo]
[1912.09291: Capatti, Hirschi, DK, Pelloni, Ruijl]

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ $[\sigma]$	Δ [%]	Δ [%]$\|\cdot\|$
Triangle	P3	1	1	1	10^9	Re	-04	5.372715	5.372733 +/- 0.000378	0.048	3e-04	2e-04
					10^9	Im		-6.681071	-6.681071 +/- 0.000103	5e-04	7e-07	
	P4	1	1	1	10^9	Re	-06	-0.561372	-0.561372 +/- 0.000014	0.014	3e-05	2e-05
					10^9	Im		-1.016658	-1.016658 +/- 0.000006	6e-05	4e-08	
Box	P7	4	1	2	10^9	Re	-10	-2.387669	-2.387670 +/- 0.000087	0.018	7e-05	5e-05
					10^9	Im		-3.030811	-3.030812 +/- 0.000022	0.048	4e-05	
	P8	2	1	1	10^9	Re	-11	-4.271184	-4.271188 +/- 0.000117	0.039	1e-04	1e-04
					10^9	Im		4.493049	4.493044 +/- 0.000065	0.078	1e-04	
	P9	3	1	2	10^9	Re	-10	-0.737898	-0.737896 +/- 0.000027	0.065	2e-04	1e-04
					10^9	Im		-1.196570	-1.196570 +/- 0.000003	0.018	4e-06	
	P10	2	1	1	10^9	Re	-10	-1.855449	-1.855447 +/- 0.000093	0.018	9e-05	8e-05
					10^9	Im		2.135544	2.135545 +/- 0.000037	0.039	7e-05	
	1L4P.K1	5	1	3	10^9	Re	-03	-0.554879	-0.554876 +/- 0.000055	0.050	5e-04	2e-04
					10^9	Im		-1.131167	-1.131167 +/- 0.000017	0.010	2e-05	
	1L4P.K2	5	1	3	10^9	Re	-05	-7.240038	-7.239970 +/- 0.000596	0.114	0.001	0.001
					10^9	Im		-5.719278	-5.719281 +/- 0.000148	0.020	5e-05	
	1L4P.K3	5	1	3	10^9	Re	-06	-2.069838	-2.069871 +/- 0.000168	0.194	0.002	0.001
					10^9	Im		-1.553783	-1.553779 +/- 0.000035	0.116	3e-04	
	1L4P.K3*	3	1	3	10^9	Re	-06	-2.276333	-2.276333 +/- 0.000102	0.002	1e-05	1e-05
					10^9	Im		0.179910	0.179910 +/- 0.000024	2e-05	3e-07	
Pentagon	P13	4	1	1	10^9	Re	-11	1.023505	1.023505 +/- 0.000052	0.001	8e-06	9e-05
					10^9	Im		1.403822	1.403823 +/- 0.000011	0.152	1e-04	
	P14	7	1	2	10^9	Re	-14	-0.153898	-0.153897 +/- 0.000010	0.086	0.001	9e-05
					10^9	Im		-1.037575	-1.037575 +/- 0.000008	0.016	1e-05	
	P15	2	1	1	10^9	Re	-14	-0.429464	-0.429465 +/- 0.000492	0.002	2e-04	2e-04
					10^9	Im		-6.554400	-6.554412 +/- 0.000051	0.229	2e-04	
	1L5P.I	8	1	4	10^9	Re	-12	-2.564923	-2.564918 +/- 0.000489	0.009	2e-04	0.001
					10^9	Im		3.443330	3.443352 +/- 0.000239	0.094	0.001	
	1L5P.II	10	1	10	10^9	Re	-13	5.971432	5.971441 +/- 0.000223	0.039	1e-04	1e-04
					10^9	Im		0	0.000000 +/- 0.000137	1e-04		
Hexagon	1L6P.I	12	1	6	10^9	Re	-13	-1.176800	-1.176797 +/- 0.000080	0.028	2e-04	2e-04
					10^9	Im		-0.030396	-0.030397 +/- 0.000022	0.055	0.004	

Reproducing finite scalar integrals

Kinematics from: [1510.00187: Buchta, Chachamis, Draggiotis, Rodrigo]
[1912.09291: Capatti, Hirschi, DK, Pelloni, Ruijl]

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ $[\sigma]$	Δ [%]	Δ [%]$\|\cdot\|$
Box	1L4P.K1*	5	2	5	10^9	Re	-03	-2.184002	-2.183949 +/- 0.000102	0.525	0.002	0.002
					10^9	Im		-1.852246	-1.852246 +/- 0.000016	3e-04	3e-07	
	1L4P.K2*	5	2	5	10^9	Re	-04	-1.081252	-1.081228 +/- 0.000052	0.459	0.002	0.002
					10^9	Im		-0.302700	-0.302700 +/- 0.000009	3e-04	9e-07	
Pentagon	1L5P.III	8	2	4	10^9	Re	-12	-1.713406	-1.713344 +/- 0.000218	0.286	0.004	0.003
					10^9	Im		0.839048	0.839054 +/- 0.000051	0.114	0.001	
	1L5P.IV	8	2	4	10^9	Re	-12	-3.900140	-3.899798 +/- 0.000921	0.371	0.009	0.007
					10^9	Im		3.489942	3.489974 +/- 0.000222	0.142	0.001	
	1L5P.VI	8	3	4	10^9	Re	-13	-2.180572	-2.180520 +/- 0.000242	0.214	0.002	0.002
					10^9	Im		-0.041187	-0.041197 +/- 0.000070	0.145	0.025	
	1L5P.K1	8	2	4	10^9	Re	-05	-6.453455	-6.453245 +/- 0.001131	0.186	0.003	0.003
					10^9	Im		1.908472	1.908451 +/- 0.000325	0.063	0.001	
	1L5P.K2	8	2	4	10^9	Re	-06	-1.806787	-1.806439 +/- 0.000669	0.520	0.019	0.019
					10^9	Im		0.151074	0.151080 +/- 0.000106	0.054	0.004	
	1L5P.K3	8	2	4	10^9	Re	-09	-1.235314	-1.235031 +/- 0.000516	0.549	0.023	0.020
					10^9	Im		0.662398	0.662402 +/- 0.000106	0.032	0.001	
	1L5P.K1*	8	3	5	10^9	Re	-05	-7.949164	-7.948492 +/- 0.000765	0.879	0.008	0.008
					10^9	Im		-2.603962	-2.603963 +/- 0.000226	0.008	7e-05	
	1L5P.K2*	8	4	6	10^9	Re	-06	-3.276952	-3.276448 +/- 0.000546	0.923	0.015	0.015
					10^9	Im		0.483021	0.483015 +/- 0.000104	0.060	0.001	
	1L5P.K3*	6	2	6	10^9	Re	-09	-1.531295	-1.531188 +/- 0.000365	0.294	0.007	0.005
					10^9	Im		1.214974	1.214974 +/- 0.000093	1e-04	8e-07	
Hexagon	1L6P.II	6	2	4	10^9	Re	+01	0.423433	0.423433 +/- 0.000082	0.003	6e-05	2e-04
					10^9	Im		2.070140	2.070144 +/- 0.000078	0.055	2e-04	
	1L6P.III	12	2	6	10^9	Re	-15	-2.259010	-2.258576 +/- 0.000623	0.698	0.019	0.016
					10^9	Im		-1.369185	-1.369196 +/- 0.000186	0.062	0.001	
	1L6P.IV	12	2	6	10^9	Re	-15	-2.165894	-2.170801 +/- 0.012222	0.401	0.227	0.194
					10^9	Im		-1.297703	-1.297724 +/- 0.001807	0.011	0.002	
	1L6P.VII	10	2	2	10^9	Re	-17	-7.733373	-7.733390 +/- 0.000743	0.023	2e-04	2e-04
					10^9	Im		3.019394	3.019388 +/- 0.000711	0.008	2e-04	
	1L6P.VIII	10	2	4	10^9	Re	-02	0.640326	0.643323 +/- 0.006342	0.473	0.468	0.143
					10^9	Im		-2.119278	-2.120308 +/- 0.001200	0.859	0.049	
	1L6P.IX	12	3	6	10^9	Re	-14	-1.152818	-1.152711 +/- 0.000139	0.768	0.009	0.009
					10^9	Im		-0.007940	-0.007937 +/- 0.000034	0.083	0.036	
	1L6P.X	10	3	4	10^9	Re	+00	2.473271	2.473398 +/- 0.000228	0.558	0.005	0.003
					10^9	Im		2.814754	2.814756 +/- 0.000032	0.060	7e-05	
	1L6P.K1	12	2	6	10^9	Re	-06	-1.547556	-1.547394 +/- 0.000255	0.634	0.010	0.010
					10^9	Im		-0.510252	-0.510248 +/- 0.000069	0.067	0.001	
	1L6P.K2	12	2	6	10^9	Re	-08	-6.963373	-6.960169 +/- 0.007156	0.448	0.046	0.046
					10^9	Im		-0.604400	-0.604415 +/- 0.000446	0.033	0.002	
	1L6P.K3	12	2	6	10^9	Re	-12	-2.519558	-2.519262 +/- 0.002864	0.103	0.012	0.012
					10^9	Im		-0.406595	-0.406598 +/- 0.000198	0.013	0.001	
	1L6P.K1*	12	4	8	10^9	Re	-06	-2.273543	-2.273682 +/- 0.000416	0.334	0.006	0.005
					10^9	Im		-1.302132	-1.302148 +/- 0.000118	0.130	0.001	
	1L6P.K2*	12	3	8	10^9	Re	-08	-6.379295	-6.378242 +/- 0.003305	0.319	0.017	0.016
					10^9	Im		2.199344	2.199348 +/- 0.000544	0.008	2e-04	
	1L6P.K3*	10	4	10	10^9	Re	-12	-2.216660	-2.216821 +/- 0.005975	0.027	0.007	0.006
					10^9	Im		1.281615	1.281615 +/- 0.000671	2e-04	9e-06	
Octagon	1L8P.K1	23	3	13	10^9	Re	-10	-1.627993	-1.624523 +/- 0.005629	0.616	0.213	0.065
					10^9	Im		-5.099169	-5.098951 +/- 0.001718	0.127	0.004	
	1L8P.K2	23	4	13	10^9	Re	-12	-1.952888	-1.941883 +/- 0.018752	0.587	0.563	0.237
					10^9	Im		-4.209151	-4.209013 +/- 0.002236	0.062	0.003	
	1L8P.K3	23	4	13	10^9	Re	-19	-0.825671	-0.817215 +/- 0.070178	0.120	1.024	0.656
					10^9	Im		-1.273792	-1.268531 +/- 0.070178	0.075	0.413	
	1L8P.K1*	23	4	16	10^9	Re	-09	-1.468062	-1.467656 +/- 0.000472	0.860	0.028	0.027
					10^9	Im		0.356928	0.356952 +/- 0.000141	0.170	0.007	
	1L8P.K2*	23	5	16	10^9	Re	-12	-2.705869	-2.699826 +/- 0.010242	0.590	0.223	0.206
					10^9	Im		1.147180	1.147210 +/- 0.001610	0.019	0.003	
	1L8P.K3*	21	4	21	10^9	Re	-08	-4.042207	-4.034787 +/- 0.007951	0.933	0.184	0.182
					10^9	Im		0.575146	0.575152 +/- 0.001044	0.005	0.001	

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ $[\sigma]$	Δ [%]	Δ [%] $\lvert\cdot\rvert$
Triangle	P3	1	1	1	10^9	Re	-04	5.372715	5.372733 +/- 0.000378	0.048	3e-04	2e-04
					10^9	Im		-6.681071	-6.681071 +/- 0.000103	5e-04	7e-07	
	P4	1	1	1	10^9	Re	-06	-0.561372	-0.561372 +/- 0.000014	0.014	3e-05	2e-05
					10^9	Im		-1.016658	-1.016658 +/- 0.000006	6e-05	4e-08	
Box	P7	4	1	2	10^9	Re	-10	-2.387669	-2.387670 +/- 0.000087	0.018	7e-05	5e-05
					10^9	Im		-3.030811	-3.030812 +/- 0.000022	0.048	4e-05	
	P8	2	1	1	10^9	Re	-11	-4.271184	-4.271188 +/- 0.000117	0.039	1e-04	1e-04
					10^9	Im		4.493049	4.493044 +/- 0.000065	0.078	1e-04	
	P9	3	1	2	10^9	Re	-10	-0.737898	-0.737896 +/- 0.000027	0.065	2e-04	1e-04
					10^9	Im		-1.196570	-1.196570 +/- 0.000003	0.018	4e-06	
	P10	2	1	1	10^9	Re	-10	-1.855449	-1.855447 +/- 0.000093	0.018	9e-05	8e-05
					10^9	Im		2.135544	2.135545 +/- 0.000037	0.039	7e-05	
	1L4P.K1	5	1	3	10^9	Re	-03	-0.554879	-0.554876 +/- 0.000055	0.050	5e-04	2e-04
					10^9	Im		-1.131167	-1.131167 +/- 0.000017	0.010	2e-05	
	1L4P.K2	5	1	3	10^9	Re	-05	-7.240038	-7.239970 +/- 0.000596	0.114	0.001	0.001
					10^9	Im		-5.719278	-5.719281 +/- 0.000148	0.020	5e-05	
	1L4P.K3	5	1	3	10^9	Re	-06	-2.069838	-2.069871 +/- 0.000168	0.194	0.002	0.001
					10^9	Im		-1.553783	-1.553779 +/- 0.000035	0.116	3e-04	
	1L4P.K3*	3	1	3	10^9	Re	-06	-2.276333	-2.276333 +/- 0.000102	0.002	1e-05	1e-05
					10^9	Im		0.179910	0.179910 +/- 0.000024	2e-05	3e-07	
Pentagon	P13	4	1	1	10^9	Re	-11	1.023505	1.023505 +/- 0.000052	0.001	8e-06	9e-05
					10^9	Im		1.403822	1.403823 +/- 0.000011	0.152	1e-04	
	P14	7	1	2	10^9	Re	-14	-0.153898	-0.153897 +/- 0.000010	0.086	0.001	9e-05
					10^9	Im		-1.037575	-1.037575 +/- 0.000008	0.016	1e-05	
	P15	2	1	1	10^9	Re	-14	-0.429464	-0.429465 +/- 0.000492	0.002	2e-04	2e-04
					10^9	Im		-6.554400	-6.554412 +/- 0.000051	0.229	2e-04	
	1L5P.I	8	1	4	10^9	Re	-12	-2.564923	-2.564918 +/- 0.000489	0.009	2e-04	0.001
					10^9	Im		3.443330	3.443352 +/- 0.000239	0.094	0.001	
	1L5P.II	10	1	10	10^9	Re	-13	5.971432	5.971441 +/- 0.000223	0.039	1e-04	1e-04
					10^9	Im		0	0.000000 +/- 0.000137	1e-04		
Hexagon	1L6P.I	12	1	6	10^9	Re	-13	-1.176800	-1.176797 +/- 0.000080	0.028	2e-04	2e-04
					10^9	Im		-0.030396	-0.030397 +/- 0.000022	0.055	0.004	

Reproducing finite scalar integrals

Kinematics from: [1510.00187: Buchta, Chachamis, Draggiotis, Rodrigo]
[1912.09291: Capatti, Hirschi, DK, Pelloni, Ruijl]

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ $[\sigma]$	Δ [%]	Δ [%] $\lvert\cdot\rvert$
Box	1L4P.K1*	5	2	5	10^9	Re	-03	-2.184002	-2.183949 +/- 0.000102	0.525	0.002	0.002
					10^9	Im		-1.852246	-1.852246 +/- 0.000016	3e-04	3e-07	
	1L4P.K2*	5	2	5	10^9	Re	-04	-1.081252	-1.081228 +/- 0.000052	0.459	0.002	0.002
					10^9	Im		-0.302700	-0.302700 +/- 0.000009	3e-04	9e-07	
Pentagon	1L5P.III	8	2	4	10^9	Re	-12	-1.713406	-1.713344 +/- 0.000218	0.286	0.004	0.003
					10^9	Im		0.839048	0.839054 +/- 0.000051	0.114	0.001	
	1L5P.IV	8	2	4	10^9	Re	-12	-3.900140	-3.899798 +/- 0.000921	0.371	0.009	0.007
					10^9	Im		3.489942	3.489974 +/- 0.000222	0.142	0.001	
	1L5P.VI	8	3	4	10^9	Re	-13	-2.180572	-2.180520 +/- 0.000242	0.214	0.002	0.002
					10^9	Im		-0.041187	-0.041197 +/- 0.000070	0.145	0.025	
	1L5P.K1	8	2	4	10^9	Re	-05	-6.453455	-6.453245 +/- 0.001131	0.186	0.003	0.003
					10^9	Im		1.908472	1.908451 +/- 0.000325	0.063	0.001	
	1L5P.K2	8	2	4	10^9	Re	-06	-1.806787	-1.806439 +/- 0.000669	0.520	0.019	0.019
					10^9	Im		0.151074	0.151080 +/- 0.000106	0.054	0.004	
	1L5P.K3	8	2	4	10^9	Re	-09	-1.235314	-1.235031 +/- 0.000516	0.549	0.023	0.020
					10^9	Im		0.662398	0.662402 +/- 0.000106	0.032	0.001	
	1L5P.K1*	8	3	5	10^9	Re	-05	-7.949164	-7.948492 +/- 0.000765	0.879	0.008	0.008
					10^9	Im		-2.603962	-2.603963 +/- 0.000226	0.008	7e-05	
	1L5P.K2*	8	4	6	10^9	Re	-06	-3.276952	-3.276448 +/- 0.000546	0.923	0.015	0.015
					10^9	Im		0.483021	0.483015 +/- 0.000104	0.060	0.001	
	1L5P.K3*	6	2	6	10^9	Re	-09	-1.531295	-1.531188 +/- 0.000365	0.294	0.007	0.005
					10^9	Im		1.214974	1.214974 +/- 0.000093	1e-04	8e-07	
Hexagon	1L6P.II	6	2	4	10^9	Re	+01	0.423433	0.423433 +/- 0.000082	0.003	6e-05	2e-04
					10^9	Im		2.070140	2.070144 +/- 0.000078	0.055	2e-04	
	1L6P.III	12	2	6	10^9	Re	-15	-2.259010	-2.258576 +/- 0.000623	0.698	0.019	0.016
					10^9	Im		-1.369185	-1.369196 +/- 0.000186	0.062	0.001	
	1L6P.IV	12	2	6	10^9	Re	-15	-2.165894	-2.170801 +/- 0.012222	0.401	0.227	0.194
					10^9	Im		-1.297703	-1.297724 +/- 0.001807	0.011	0.002	
	1L6P.VII	10	2	2	10^9	Re	-17	-7.733373	-7.733390 +/- 0.000743	0.023	2e-04	2e-04
					10^9	Im		3.019394	3.019388 +/- 0.000711	0.008	2e-04	
	1L6P.VIII	10	2	4	10^9	Re	-02	0.640326	0.643323 +/- 0.006342	0.473	0.468	0.143
					10^9	Im		-2.119278	-2.120308 +/- 0.001200	0.859	0.049	
	1L6P.IX	12	3	6	10^9	Re	-14	-1.152818	-1.152711 +/- 0.000139	0.768	0.009	0.009
					10^9	Im		-0.007940	-0.007937 +/- 0.000034	0.083	0.036	
	1L6P.X	10	3	4	10^9	Re	+00	2.473271	2.473398 +/- 0.000228	0.558	0.005	0.003
					10^9	Im		2.814754	2.814756 +/- 0.000032	0.060	7e-05	
	1L6P.K1	12	2	6	10^9	Re	-06	-1.547556	-1.547394 +/- 0.000255	0.634	0.010	0.010
					10^9	Im		-0.510252	-0.510248 +/- 0.000069	0.067	0.001	
	1L6P.K2	12	2	6	10^9	Re	-08	-6.963373	-6.960169 +/- 0.007156	0.448	0.046	0.046
					10^9	Im		-0.604400	-0.604415 +/- 0.000446	0.033	0.002	
	1L6P.K3	12	2	6	10^9	Re	-12	-2.519558	-2.519262 +/- 0.002864	0.103	0.012	0.012
					10^9	Im		-0.406595	-0.406598 +/- 0.000198	0.013	0.001	
	1L6P.K1*	12	4	8	10^9	Re	-06	-2.273543	-2.273682 +/- 0.000416	0.334	0.006	0.005
					10^9	Im		-1.302132	-1.302148 +/- 0.000118	0.130	0.001	
	1L6P.K2*	12	3	8	10^9	Re	-08	-6.379295	-6.378242 +/- 0.003305	0.319	0.017	0.016
					10^9	Im		2.199344	2.199348 +/- 0.000544	0.008	2e-04	
	1L6P.K3*	10	4	10	10^9	Re	-12	-2.216660	-2.216821 +/- 0.005975	0.027	0.007	0.006
					10^9	Im		1.281615	1.281615 +/- 0.000671	2e-04	9e-06	
Octagon	1L8P.K1	23	3	13	10^9	Re	-10	-1.627993	-1.624523 +/- 0.005629	0.616	0.213	0.065
					10^9	Im		-5.099169	-5.098951 +/- 0.001718	0.127	0.004	
	1L8P.K2	23	4	13	10^9	Re	-12	-1.952888	-1.941883 +/- 0.018752	0.587	0.563	0.237
					10^9	Im		-4.209151	-4.209013 +/- 0.002236	0.062	0.003	
	1L8P.K3	23	4	13	10^9	Re	-19	-0.825671	-0.817215 +/- 0.070178	0.120	1.024	0.656
					10^9	Im		-1.273792	-1.268531 +/- 0.070178	0.075	0.413	
	1L8P.K1*	23	4	16	10^9	Re	-09	-1.468062	-1.467656 +/- 0.000472	0.860	0.028	0.027
					10^9	Im		0.356928	0.356952 +/- 0.000141	0.170	0.007	
	1L8P.K2*	23	5	16	10^9	Re	-12	-2.705869	-2.699826 +/- 0.010242	0.590	0.223	0.206
					10^9	Im		1.147180	1.147210 +/- 0.001610	0.019	0.003	
	1L8P.K3*	21	4	21	10^9	Re	-08	-4.042207	-4.034787 +/- 0.007951	0.933	0.184	0.182
					10^9	Im		0.575146	0.575152 +/- 0.001044	0.005	0.001	

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ [σ]	Δ [%]	Δ [%] |·|
Triangle	P3	1	1	1	10^9	Re	-04	5.372715	5.372733 +/- 0.000378	0.048	3e-04	2e-04
					10^9	Im		-6.681071	-6.681071 +/- 0.000103	5e-04	7e-07	
	P4	1	1	1	10^9	Re	-06	-0.561372	-0.561372 +/- 0.000014	0.014	3e-05	2e-05
					10^9	Im		-1.016658	-1.016658 +/- 0.000006	6e-05	4e-08	
Box	P7	4	1	2	10^9	Re	-10	-2.387669	-2.387670 +/- 0.000087	0.018	7e-05	5e-05
					10^9	Im		-3.030811	-3.030812 +/- 0.000022	0.048	4e-05	
	P8	2	1	1	10^9	Re	-11	-4.271184	-4.271188 +/- 0.000117	0.039	1e-04	1e-04
					10^9	Im		4.493049	4.493044 +/- 0.000065	0.078	1e-04	
	P9	3	1	2	10^9	Re	-10	-0.737898	-0.737896 +/- 0.000027	0.065	2e-04	1e-04
					10^9	Im		-1.196570	-1.196570 +/- 0.000003	0.018	4e-06	
	P10	2	1	1	10^9	Re	-10	-1.855449	-1.855447 +/- 0.000093	0.018	9e-05	8e-05
					10^9	Im		2.135544	2.135545 +/- 0.000037	0.039	7e-05	
	1L4P.K1	5	1	3	10^9	Re	-03	-0.554879	-0.554876 +/- 0.000055	0.050	5e-04	2e-04
					10^9	Im		-1.131167	-1.131167 +/- 0.000017	0.010	2e-05	
	1L4P.K2	5	1	3	10^9	Re	-05	-7.240038	-7.239970 +/- 0.000596	0.114	0.001	0.001
					10^9	Im		-5.719278	-5.719281 +/- 0.000148	0.020	5e-05	
	1L4P.K3	5	1	3	10^9	Re	-06	-2.069838	-2.069871 +/- 0.000168	0.194	0.002	0.001
					10^9	Im		-1.553783	-1.553779 +/- 0.000035	0.116	3e-04	
	1L4P.K3*	3	1	3	10^9	Re	-06	-2.276333	-2.276333 +/- 0.000102	0.002	1e-05	1e-05
					10^9	Im		0.179910	0.179910 +/- 0.000024	2e-05	3e-07	
Pentagon	P13	4	1	1	10^9	Re	-11	1.023505	1.023505 +/- 0.000052	0.001	8e-06	9e-05
					10^9	Im		1.403822	1.403823 +/- 0.000011	0.152	1e-04	
	P14	7	1	2	10^9	Re	-14	-0.153898	-0.153897 +/- 0.000010	0.086	0.001	9e-05
					10^9	Im		-1.037575	-1.037575 +/- 0.000008	0.016	1e-05	
	P15	2	1	1	10^9	Re	-14	-0.429464	-0.429465 +/- 0.000492	0.002	2e-04	2e-04
					10^9	Im		-6.554400	-6.554412 +/- 0.000051	0.229	2e-04	
	1L5P.I	8	1	4	10^9	Re	-12	-2.564923	-2.564918 +/- 0.000489	0.009	2e-04	0.001
					10^9	Im		3.443330	3.443352 +/- 0.000239	0.094	0.001	
	1L5P.II	10	1	10	10^9	Re	-13	5.971432	5.971441 +/- 0.000223	0.039	1e-04	1e-04
					10^9	Im		0	0.000000 +/- 0.000137	1e-04		
Hexagon	1L6P.I	12	1	6	10^9	Re	-13	-1.176800	-1.176797 +/- 0.000080	0.028	2e-04	2e-04
					10^9	Im		-0.030396	-0.030397 +/- 0.000022	0.055	0.004	

Reproducing finite scalar integrals

Kinematics from: [1510.00187: Buchta, Chachamis, Draggiotis, Rodrigo] [1912.09291: Capatti, Hirschi, DK, Pelloni, Ruijl]

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ [σ]	Δ [%]	Δ [%] |·|
Box	1L4P.K1*	5	2	5	10^9	Re	-03	-2.184002	-2.183949 +/- 0.000102	0.525	0.002	0.002
					10^9	Im		-1.852246	-1.852246 +/- 0.000016	3e-04	3e-07	
	1L4P.K2*	5	2	5	10^9	Re	-04	-1.081252	-1.081228 +/- 0.000052	0.459	0.002	0.002
					10^9	Im		-0.302700	-0.302700 +/- 0.000009	3e-04	9e-07	
Pentagon	1L5P.III	8	2	4	10^9	Re	-12	-1.713406	-1.713344 +/- 0.000218	0.286	0.004	0.003
					10^9	Im		0.839048	0.839054 +/- 0.000051	0.114	0.001	
	1L5P.IV	8	2	4	10^9	Re	-12	-3.900140	-3.899798 +/- 0.000921	0.371	0.009	0.007
					10^9	Im		3.489942	3.489974 +/- 0.000222	0.142	0.001	
	1L5P.VI	8	3	4	10^9	Re	-13	-2.180572	-2.180520 +/- 0.000242	0.214	0.002	0.002
					10^9	Im		-0.041187	-0.041197 +/- 0.000070	0.145	0.025	
	1L5P.K1	8	2	4	10^9	Re	-05	-6.453455	-6.453245 +/- 0.001131	0.186	0.003	0.003
					10^9	Im		1.908472	1.908451 +/- 0.000325	0.063	0.001	
	1L5P.K2	8	2	4	10^9	Re	-06	-1.806787	-1.806439 +/- 0.000669	0.520	0.019	0.019
					10^9	Im		0.151074	0.151080 +/- 0.000106	0.054	0.004	
	1L5P.K3	8	2	4	10^9	Re	-09	-1.235314	-1.235031 +/- 0.000516	0.549	0.023	0.020
					10^9	Im		0.662398	0.662402 +/- 0.000106	0.032	0.001	
	1L5P.K1*	8	3	5	10^9	Re	-05	-7.949164	-7.948492 +/- 0.000765	0.879	0.008	0.008
					10^9	Im		-2.603962	-2.603963 +/- 0.000226	0.008	7e-05	
	1L5P.K2*	8	4	6	10^9	Re	-06	-3.276952	-3.276448 +/- 0.000546	0.923	0.015	0.015
					10^9	Im		0.483021	0.483015 +/- 0.000104	0.060	0.001	
	1L5P.K3*	6	2	6	10^9	Re	-09	-1.531295	-1.531188 +/- 0.000365	0.294	0.007	0.005
					10^9	Im		1.214972	1.214974 +/- 0.000093	1e-04	8e-07	
Hexagon	1L6P.II	6	2	4	10^9	Re	+01	0.423433	0.423433 +/- 0.000082	0.003	6e-05	2e-04
					10^9	Im		2.070140	2.070144 +/- 0.000078	0.055	2e-04	
	1L6P.III	12	2	6	10^9	Re	-15	-2.259010	-2.258576 +/- 0.000623	0.698	0.019	0.016
					10^9	Im		-1.369185	-1.369196 +/- 0.000186	0.062	0.001	
	1L6P.IV	12	2	6	10^9	Re	-15	-2.165894	-2.170801 +/- 0.012222	0.401	0.227	0.194
					10^9	Im		-1.297703	-1.297724 +/- 0.001807	0.011	0.002	
	1L6P.VII	10	2	2	10^9	Re	-17	-7.733373	-7.733390 +/- 0.000743	0.023	2e-04	2e-04
					10^9	Im		3.019394	3.019388 +/- 0.000711	0.008	2e-04	
	1L6P.VIII	10	2	4	10^9	Re	-02	0.640326	0.643323 +/- 0.006342	0.473	0.468	0.143
					10^9	Im		-2.119278	-2.120308 +/- 0.001200	0.859	0.049	
	1L6P.IX	12	3	6	10^9	Re	-14	-1.152818	-1.152711 +/- 0.000139	0.768	0.009	0.009
					10^9	Im		-0.007940	-0.007937 +/- 0.000034	0.083	0.036	
	1L6P.X	10	3	4	10^9	Re	+00	2.473271	2.473398 +/- 0.000228	0.558	0.005	0.003
					10^9	Im		2.814754	2.814756 +/- 0.000032	0.060	7e-05	
	1L6P.K1	12	2	6	10^9	Re	-06	-1.547556	-1.547394 +/- 0.000255	0.634	0.010	0.010
					10^9	Im		-0.510252	-0.510248 +/- 0.000069	0.067	0.001	
	1L6P.K2	12	2	6	10^9	Re	-08	-6.963373	-6.960169 +/- 0.007156	0.448	0.046	0.046
					10^9	Im		-0.604400	-0.604415 +/- 0.000446	0.033	0.002	
	1L6P.K3	12	2	6	10^9	Re	-12	-2.519558	-2.519262 +/- 0.002864	0.103	0.012	0.012
					10^9	Im		-0.406595	-0.406598 +/- 0.000198	0.013	0.001	
	1L6P.K1*	12	4	8	10^9	Re	-06	-2.273543	-2.273682 +/- 0.000416	0.334	0.006	0.005
					10^9	Im		-1.302132	-1.302148 +/- 0.000118	0.130	0.001	
	1L6P.K2*	12	3	8	10^9	Re	-08	-6.379295	-6.378242 +/- 0.003305	0.319	0.017	0.016
					10^9	Im		2.199344	2.199348 +/- 0.000544	0.008	2e-04	
	1L6P.K3*	10	4	10	10^9	Re	-12	-2.216660	-2.216821 +/- 0.005975	0.027	0.007	0.006
					10^9	Im		1.281615	1.281615 +/- 0.000671	2e-04	9e-06	
Octagon	1L8P.K1	23	3	13	10^9	Re	-10	-1.627993	-1.624523 +/- 0.005629	0.616	0.213	0.065
					10^9	Im		-5.099169	-5.098951 +/- 0.001718	0.127	0.004	
	1L8P.K2	23	4	13	10^9	Re	-12	-1.952888	-1.941883 +/- 0.018752	0.587	0.563	0.237
					10^9	Im		-4.209151	-4.209013 +/- 0.002236	0.062	0.003	
	1L8P.K3	23	4	13	10^9	Re	-19	-0.825671	-0.817215 +/- 0.070178	0.120	1.024	0.656
					10^9	Im		-1.273792	-1.268531 +/- 0.070178	0.075	0.413	
	1L8P.K1*	23	4	16	10^9	Re	-09	-1.468062	-1.467656 +/- 0.000472	0.860	0.028	0.027
					10^9	Im		0.356928	0.356952 +/- 0.000141	0.170	0.007	
	1L8P.K2*	23	5	16	10^9	Re	-12	-2.705869	-2.699826 +/- 0.010242	0.590	0.223	0.206
					10^9	Im		1.147180	1.147210 +/- 0.001610	0.019	0.003	
	1L8P.K3*	21	4	21	10^9	Re	-08	-4.042207	-4.034787 +/- 0.007951	0.933	0.184	0.182
					10^9	Im		0.575146	0.575152 +/- 0.001044	0.005	0.001	

IR divergent scalar topologies with IR counterterms (triangle diagrams)

[Anastasiou, Sterman: 1812.03753]

[Capatti, Hirschi, DK, Pelloni, Ruijl: 1912.09291]

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	Δ [σ]	Δ [%]	Δ [%]$\lvert\cdot\rvert$
Box	4S4C_IR_sub	1	1	1	10^9	Re	-03	0.380313	0.380322 +/- 0.000552	0.016	0.002	0.001
					10^9	Im		-3.447431	-3.447458 +/- 0.000248	0.106	0.001	
Pentagon	1S2C_IR_sub	6	2	6	10^9	Re	+00	-4.801140	-4.800655 +/- 0.000736	0.658	0.010	0.010
					10^9	Im		-0.414486	-0.414435 +/- 0.000209	0.243	0.012	

One-loop amplitudes interfered with tree level (with UV and IR counterterms)

	Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Numerical
$u\bar{u} \to W^+W^-$	Box	uubar_WpWm	3	1	3	10^9	Re	+00	-1.596350 +/- 0.000682
						10^9	Im		-0.059046 +/- 0.000065
$u\bar{u} \to W^+W^-Z$	Pentagon	uubar_WpWmZ	4	1	2	10^9	Re	-04	1.004428 +/- 0.000529
						10^9	Im		1.878321 +/- 0.000090

Conclusion

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$\Rightarrow$ **new robust method for numerical calculation of loop integrals in momentum space**

Outlook

- extension to multi-loop integrals

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- construction of locally finite integrand
- no need for contour deformation
- separation of real and imaginary part
- partially reducing integration dimension to two

$\Rightarrow$ **new robust method for numerical calculation of loop integrals in momentum space**

Outlook

- extension to multi-loop integrals

$\Rightarrow$ **combining loop and PS integration**
(threshold residue = PS integral, local cancellation of real and virtual, local unitarity)

Backup slides

Local (mis)cancellations

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pinched (singular)

$$r_{ij}^{\pm}(\hat{k}) = r_{ab}^{\mp}(\hat{k})$$

Local (mis)cancellations

pinched (singular) $\boxed{r_{ij}^{\pm}(\hat{k}) = r_{ab}^{\mp}(\hat{k})}$

non-pinched (locally cancelling) $\boxed{r_{ij}^{\pm}(\hat{k}) = r_{ab}^{\pm}(\hat{k})}$

Local (mis)cancellations

pinched (singular) $r_{ij}^{\pm}(\hat{k}) = r_{ab}^{\mp}(\hat{k})$

non-pinched (locally cancelling) $r_{ij}^{\pm}(\hat{k}) = r_{ab}^{\pm}(\hat{k})$

Double poles

$$R_{ij}^{\pm} = \Theta\left(r_{ij}^{\pm} \in \mathbb{R}\right) \frac{1}{-4E_iE_j} \frac{r^2}{\left(\frac{\vec{q}_i}{E_i} + \frac{\vec{q}_j}{E_j}\right) \cdot \hat{k}} \Bigg|_{r=r_{ij}^{\pm}} \frac{N}{\prod_{l \neq i,j} D_l} \Bigg|_{ij}$$

Local (mis)cancellations

pinched (singular) $r^{\pm}_{ij}(\hat{k}) = r^{\mp}_{ab}(\hat{k})$

non-pinched (locally cancelling) $r^{\pm}_{ij}(\hat{k}) = r^{\pm}_{ab}(\hat{k})$

Double poles

$$R^{\pm}_{ij} = \Theta\left(r^{\pm}_{ij} \in \mathbb{R}\right) \frac{1}{-4E_iE_j} \frac{r^2}{\left(\frac{\vec{q}_i}{E_i} + \frac{\vec{q}_j}{E_j}\right) \cdot \hat{k}} \Bigg|_{r=r^{\pm}_{ij}} \left. \frac{N}{\boxed{\prod_{l \neq i,j} D_l}} \right|_{ij}$$

Local (mis)cancellations

pinched (singular) $r^{\pm}_{ij}(\hat{k}) = r^{\mp}_{ab}(\hat{k})$

non-pinched (locally cancelling) $r^{\pm}_{ij}(\hat{k}) = r^{\pm}_{ab}(\hat{k})$

for either $a = i$ or $b = j$

Double poles

$$R^{\pm}_{ij} = \Theta\left(r^{\pm}_{ij} \in \mathbb{R}\right) \frac{1}{-4E_iE_j} \frac{r^2}{\left(\frac{\vec{q}_i}{E_i} + \frac{\vec{q}_j}{E_j}\right) \cdot \hat{k}} \Bigg|_{r=r^{\pm}_{ij}} \frac{N}{\boxed{\prod_{l \neq i,j} D_l}} \Bigg|_{ij}$$

Local (mis)cancellations

pinched (singular) $r_{ij}^{\pm}(\hat{k}) = r_{ab}^{\mp}(\hat{k})$

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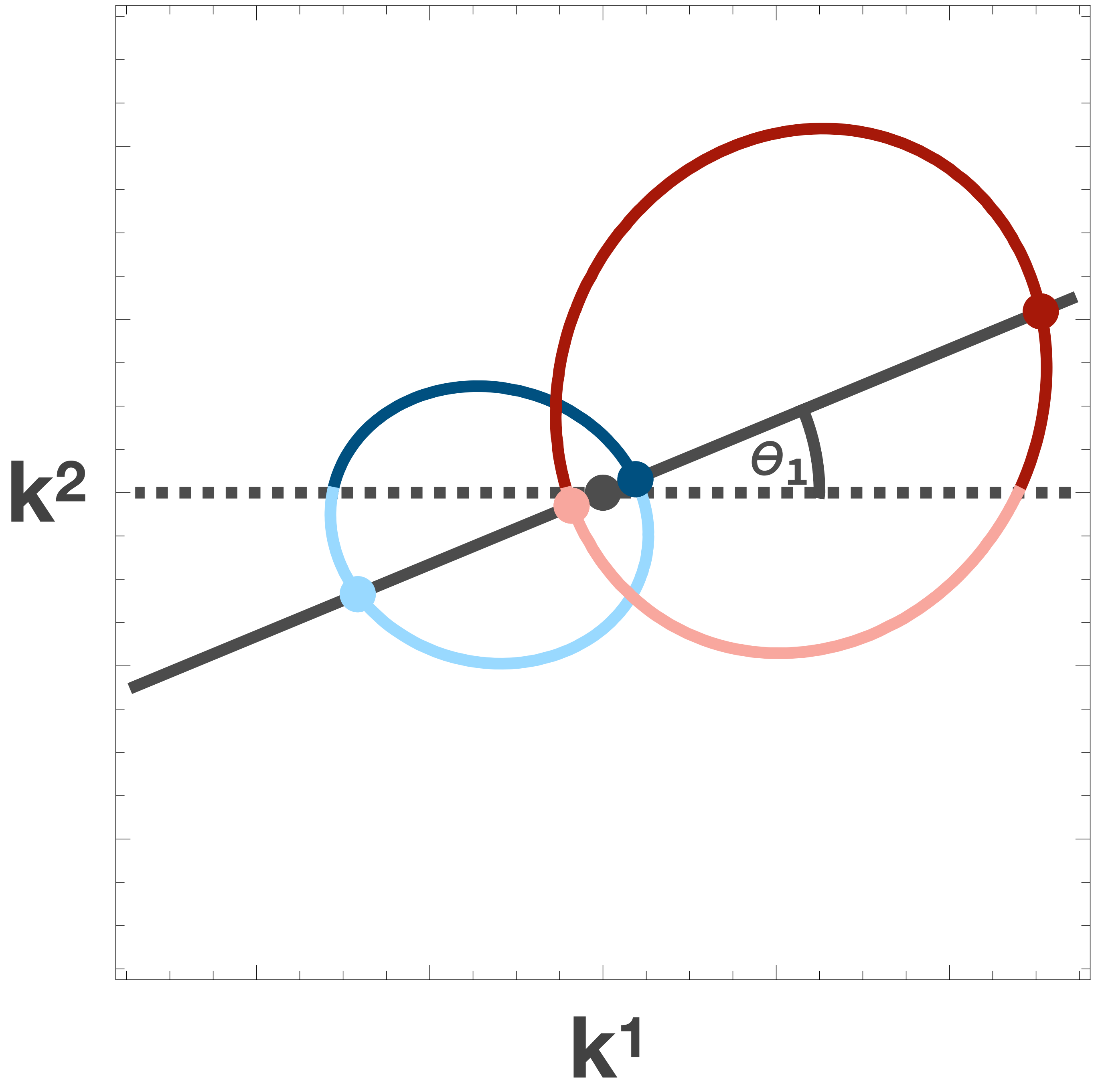
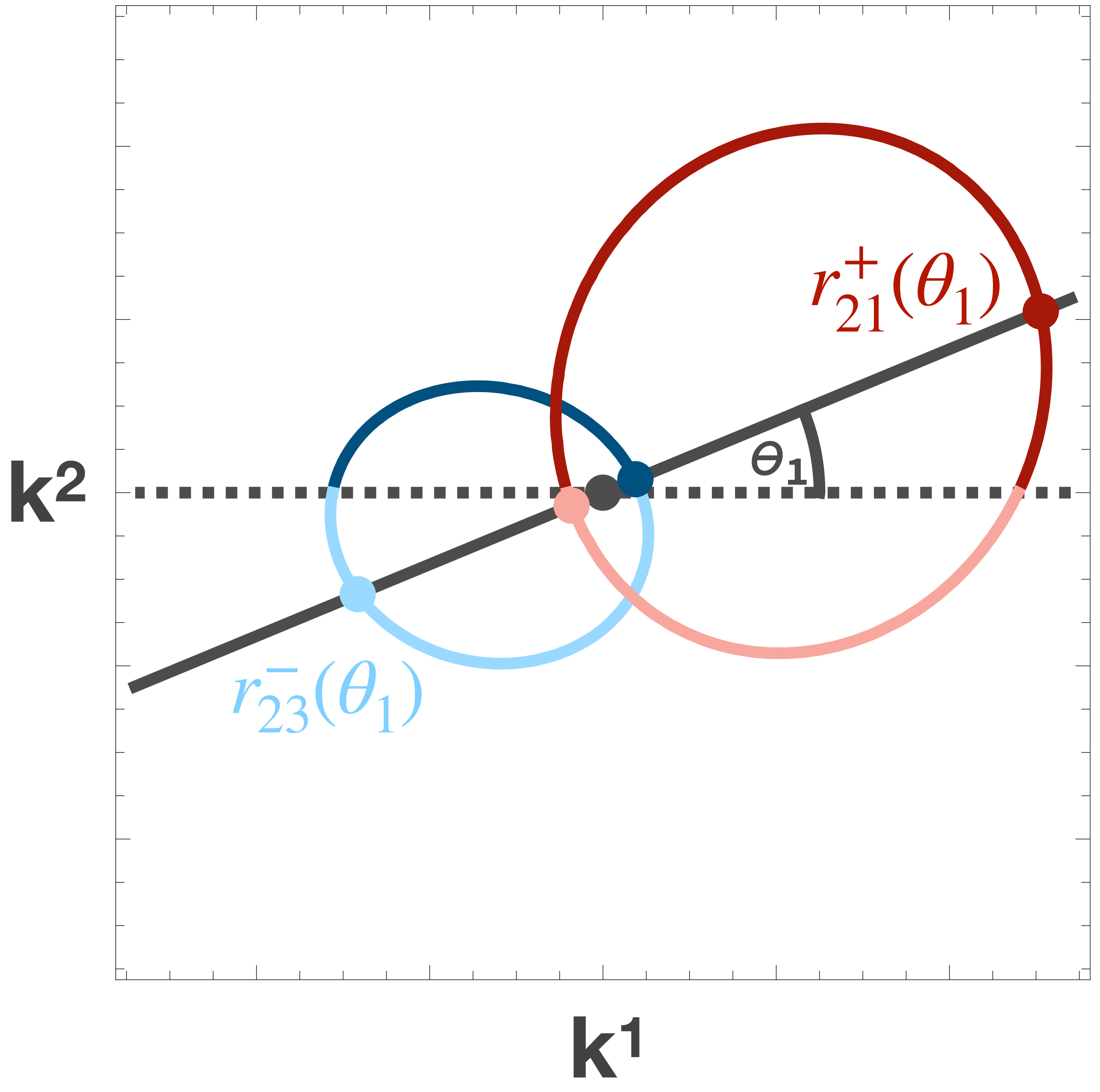
depends on coordinate system



Origin 1 (no pinched poles)



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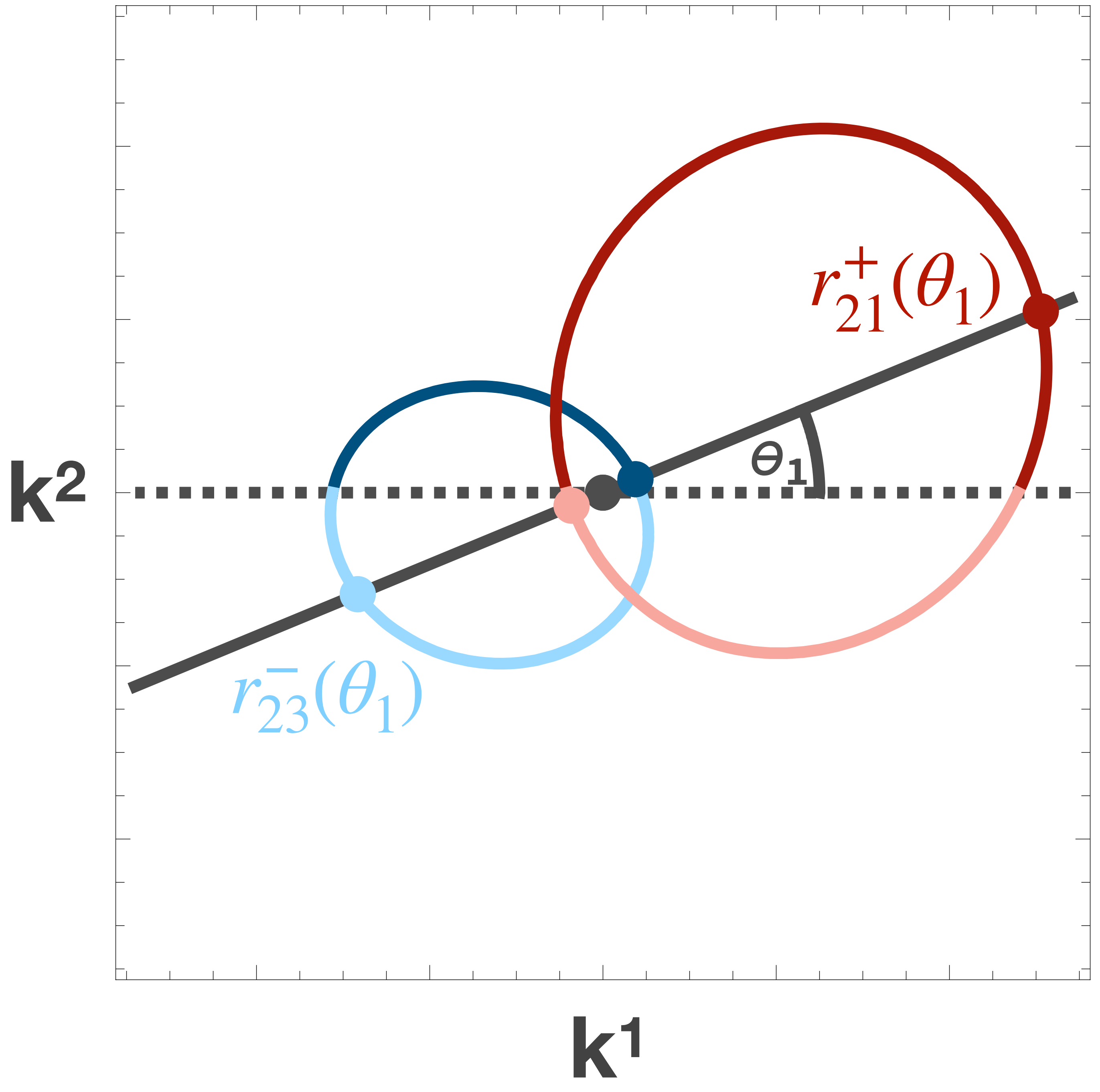

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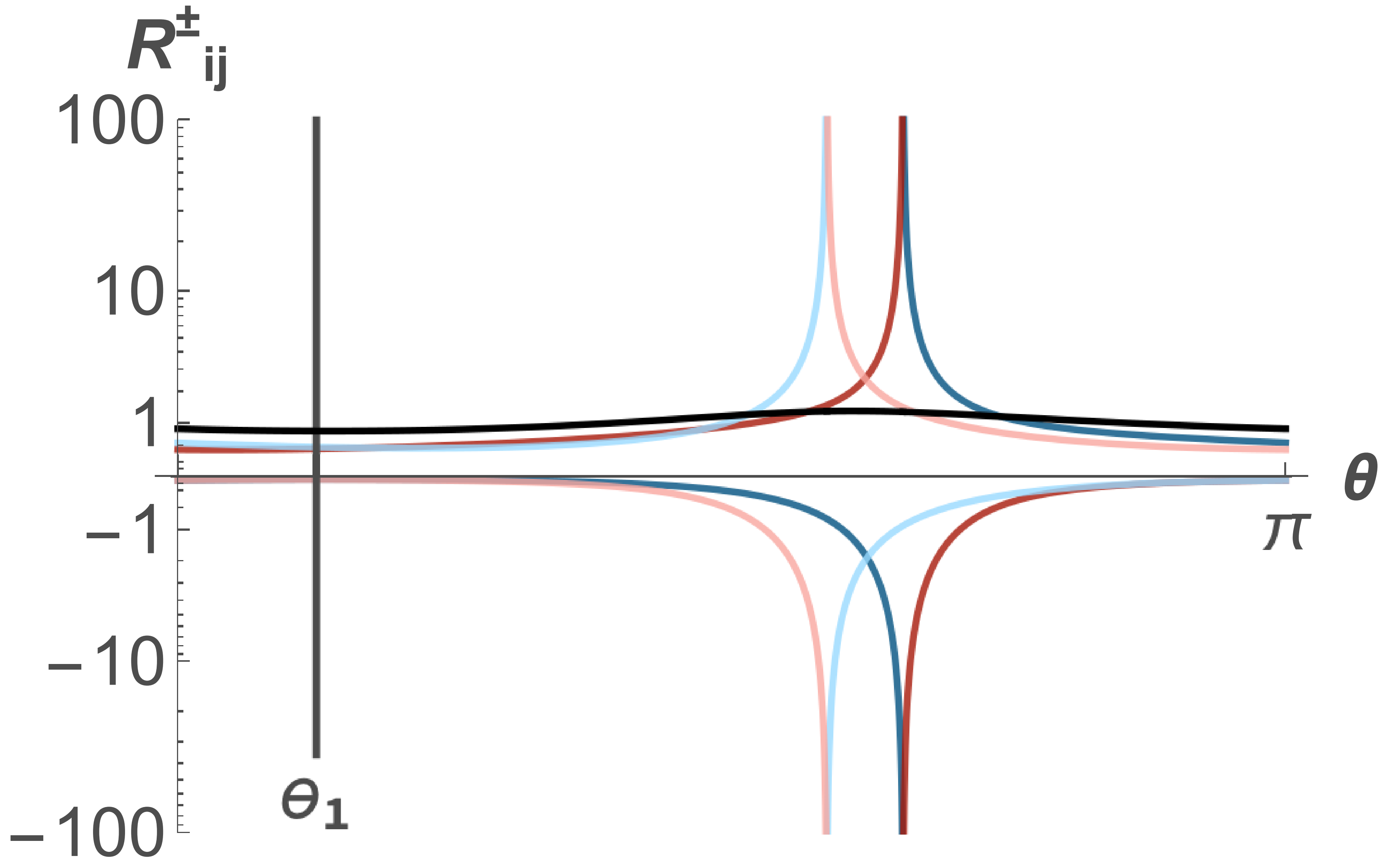
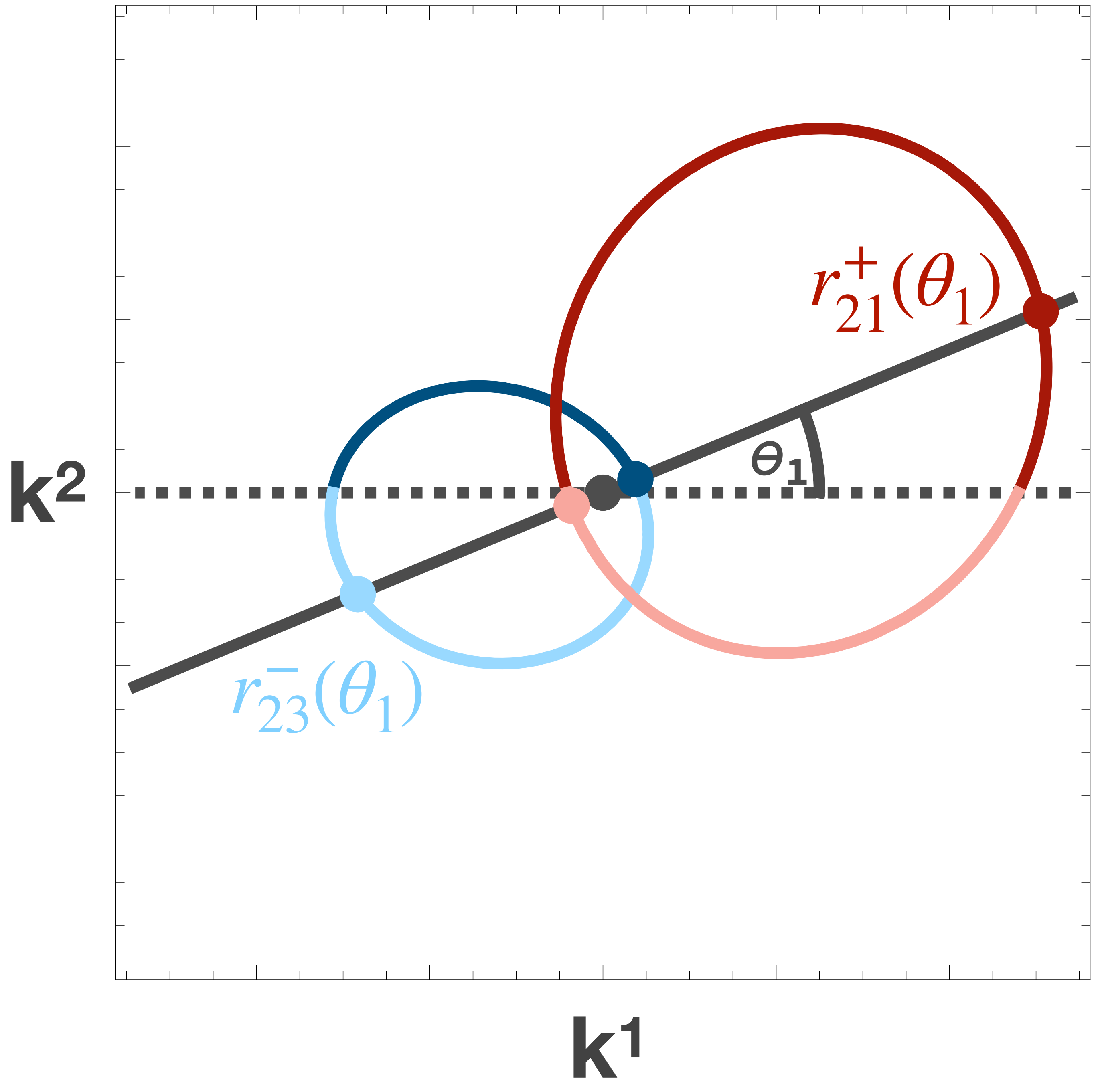
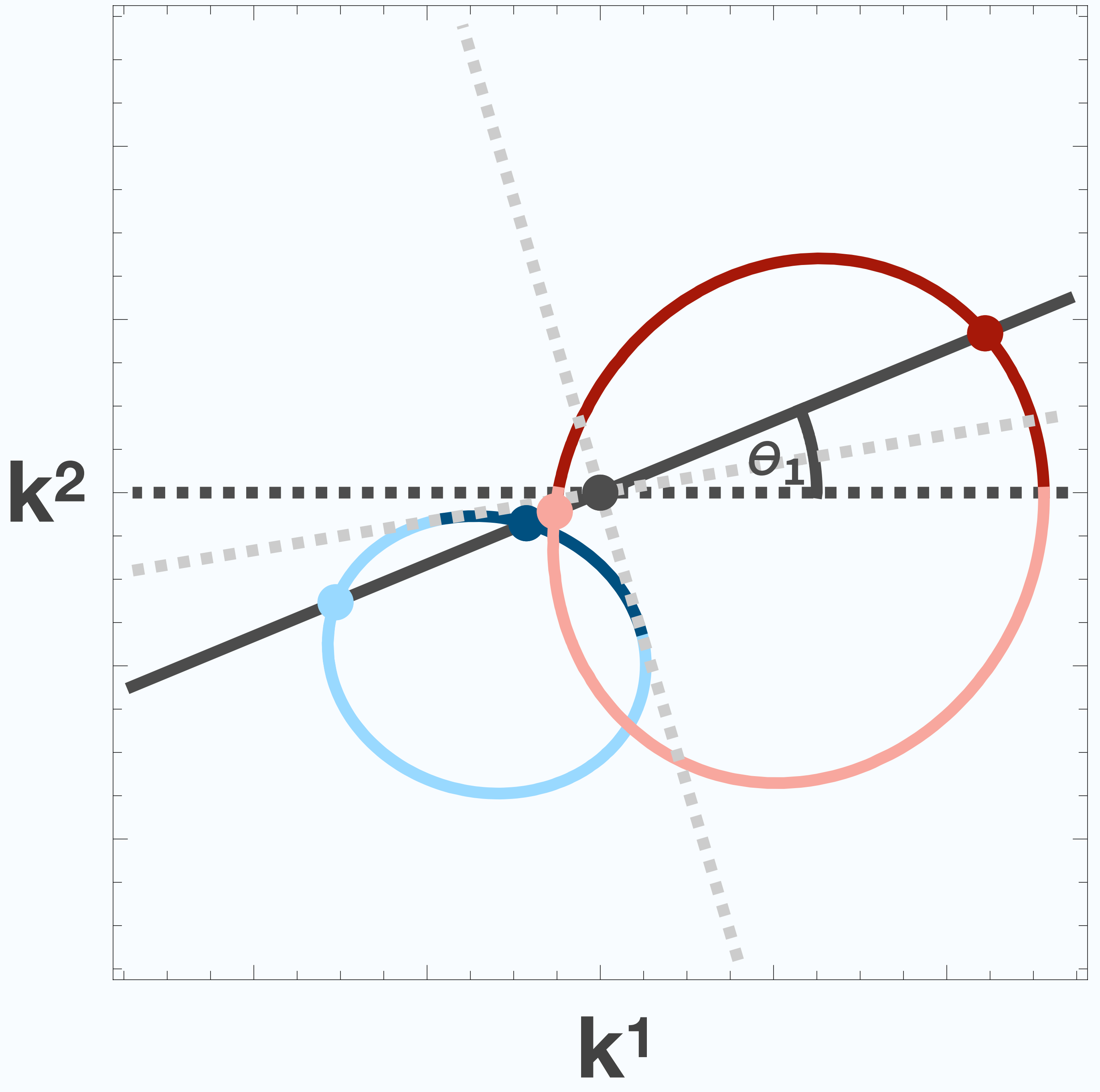
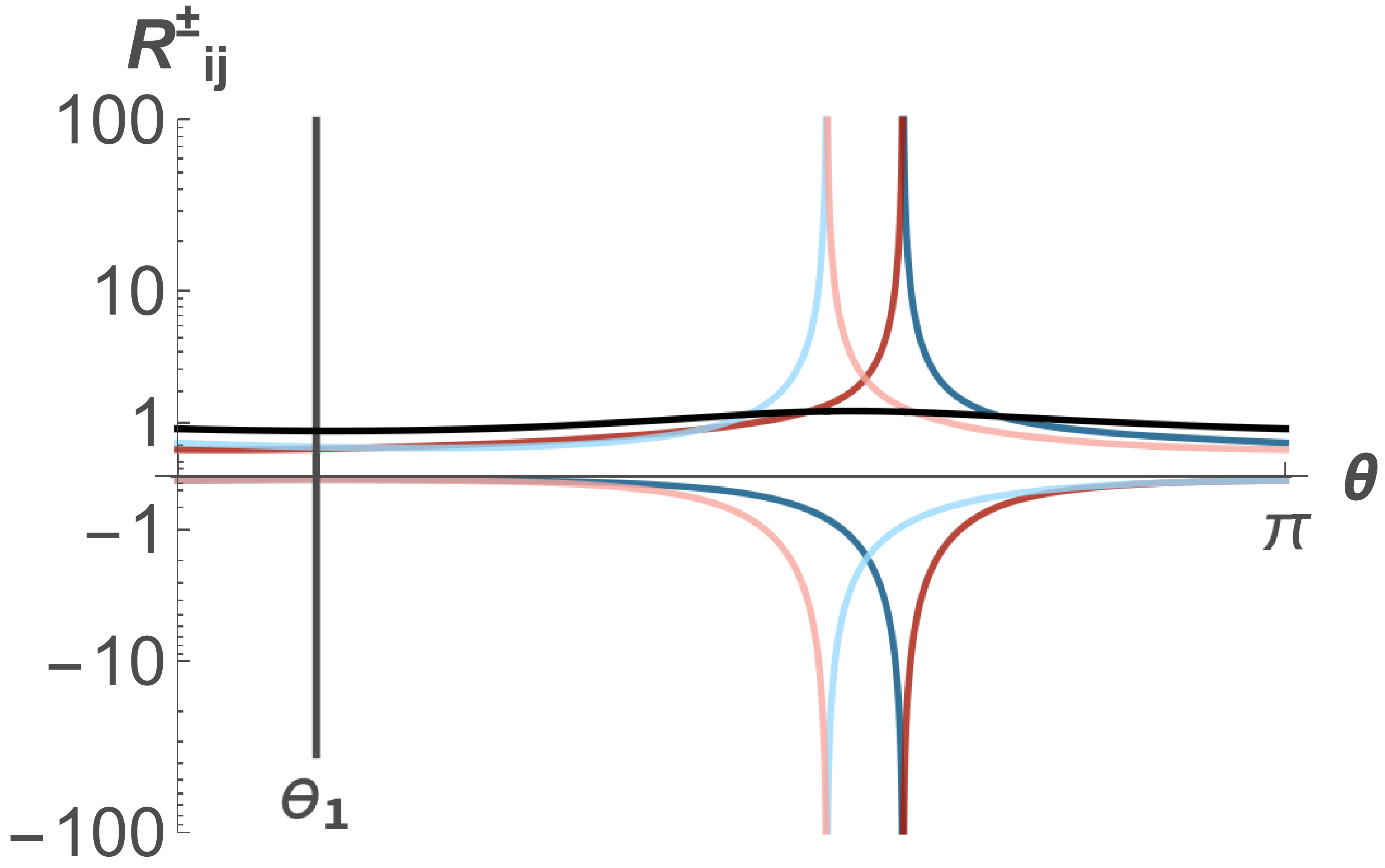


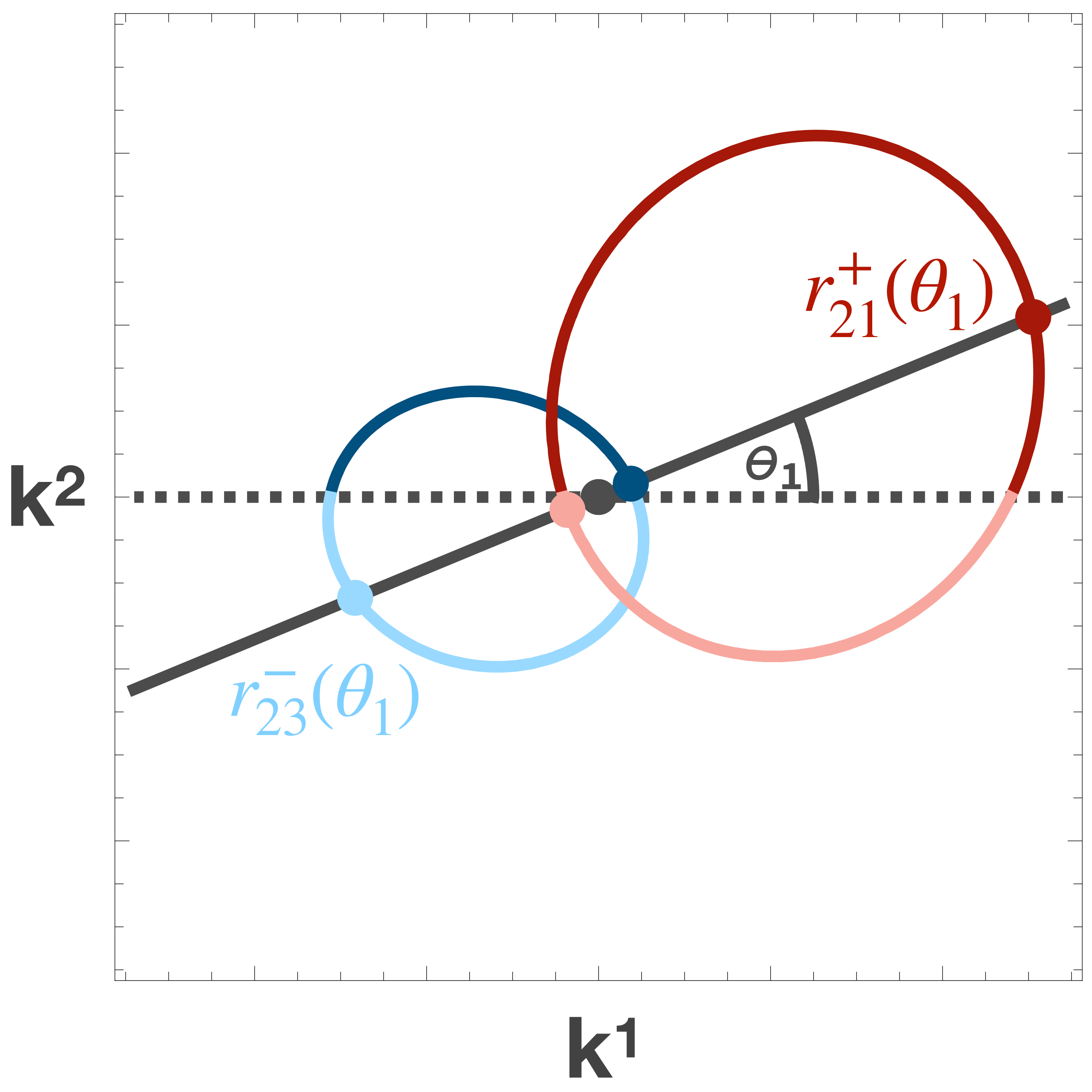


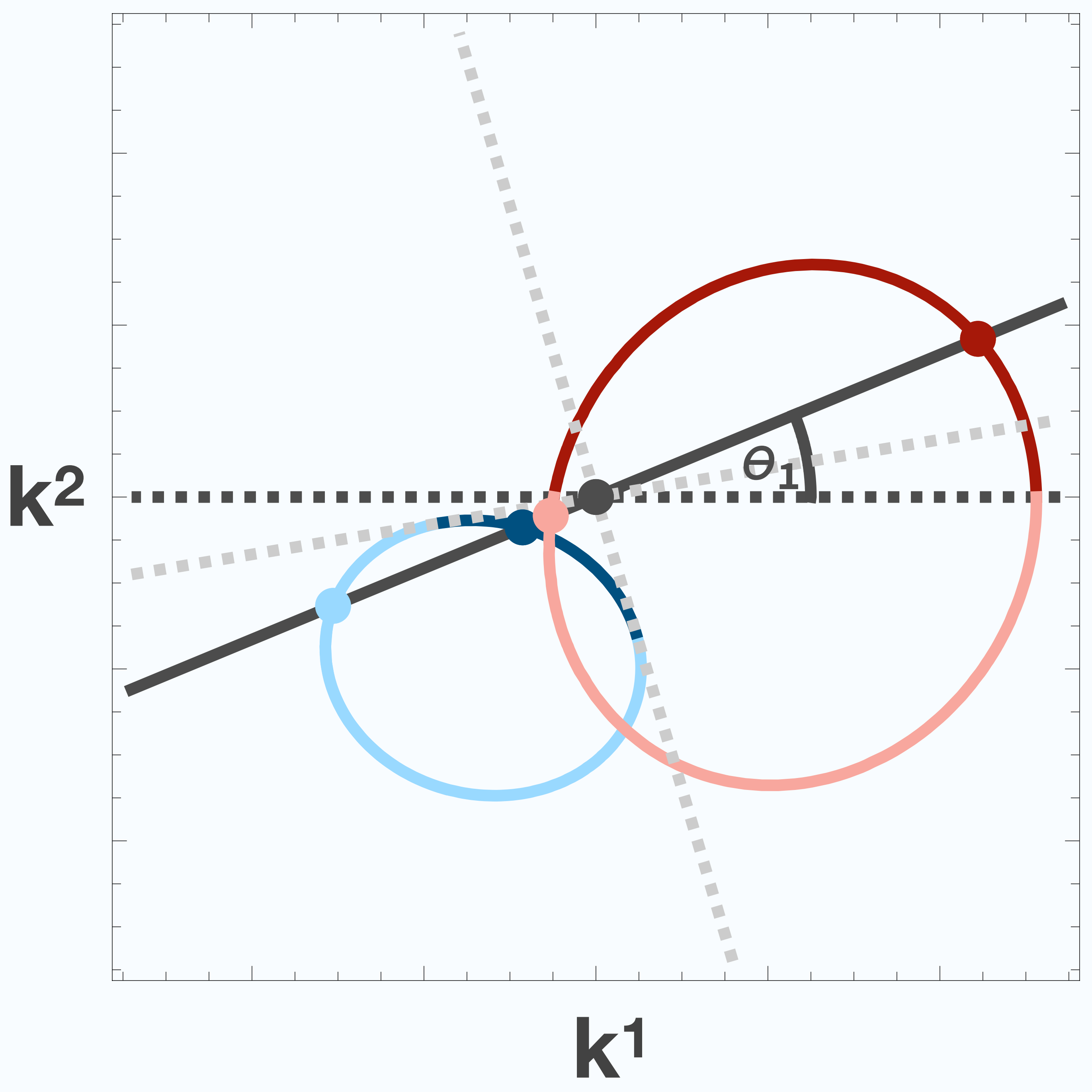


Origin 1 (no pinched poles)

Origin 2 (pinched pole)







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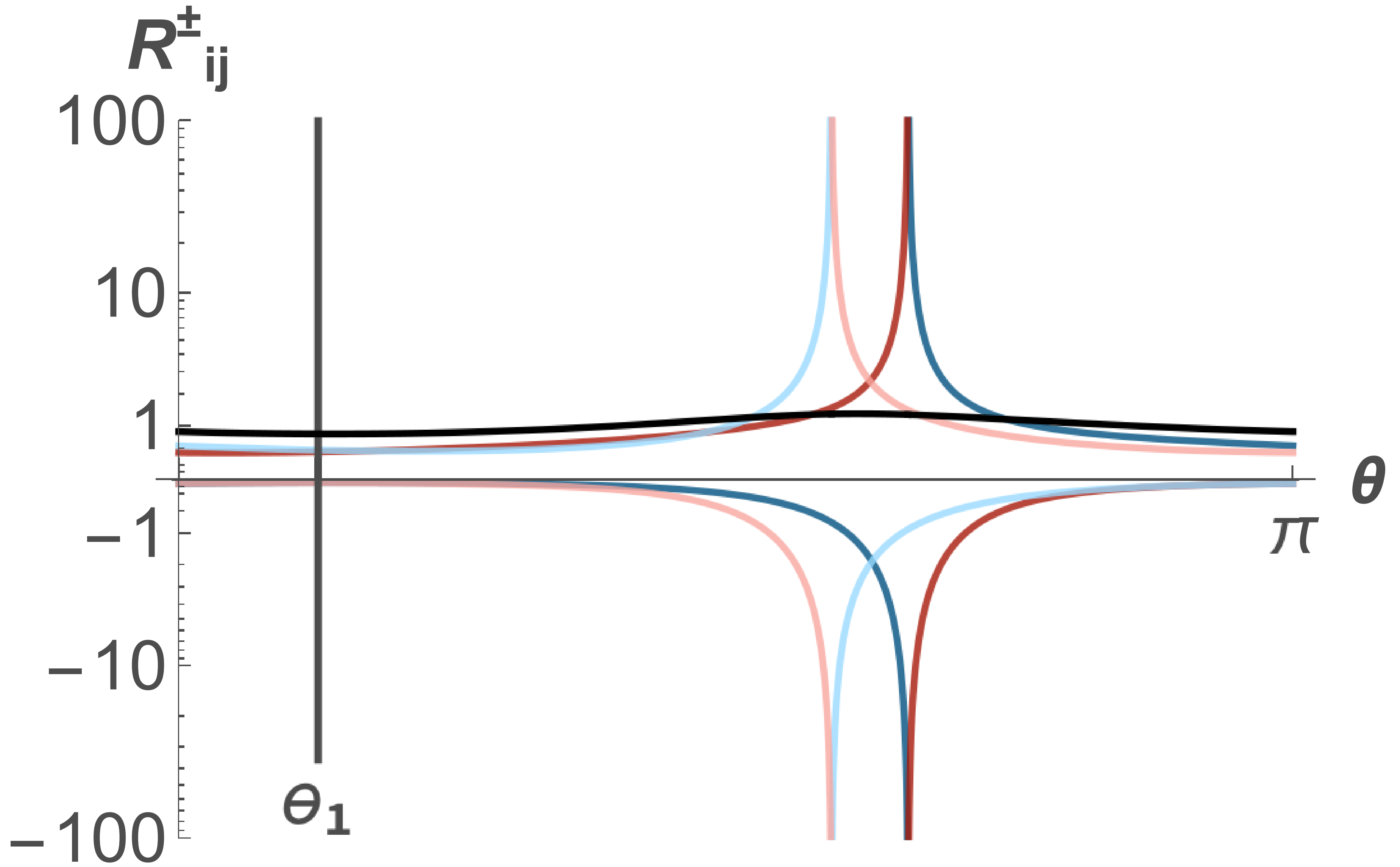

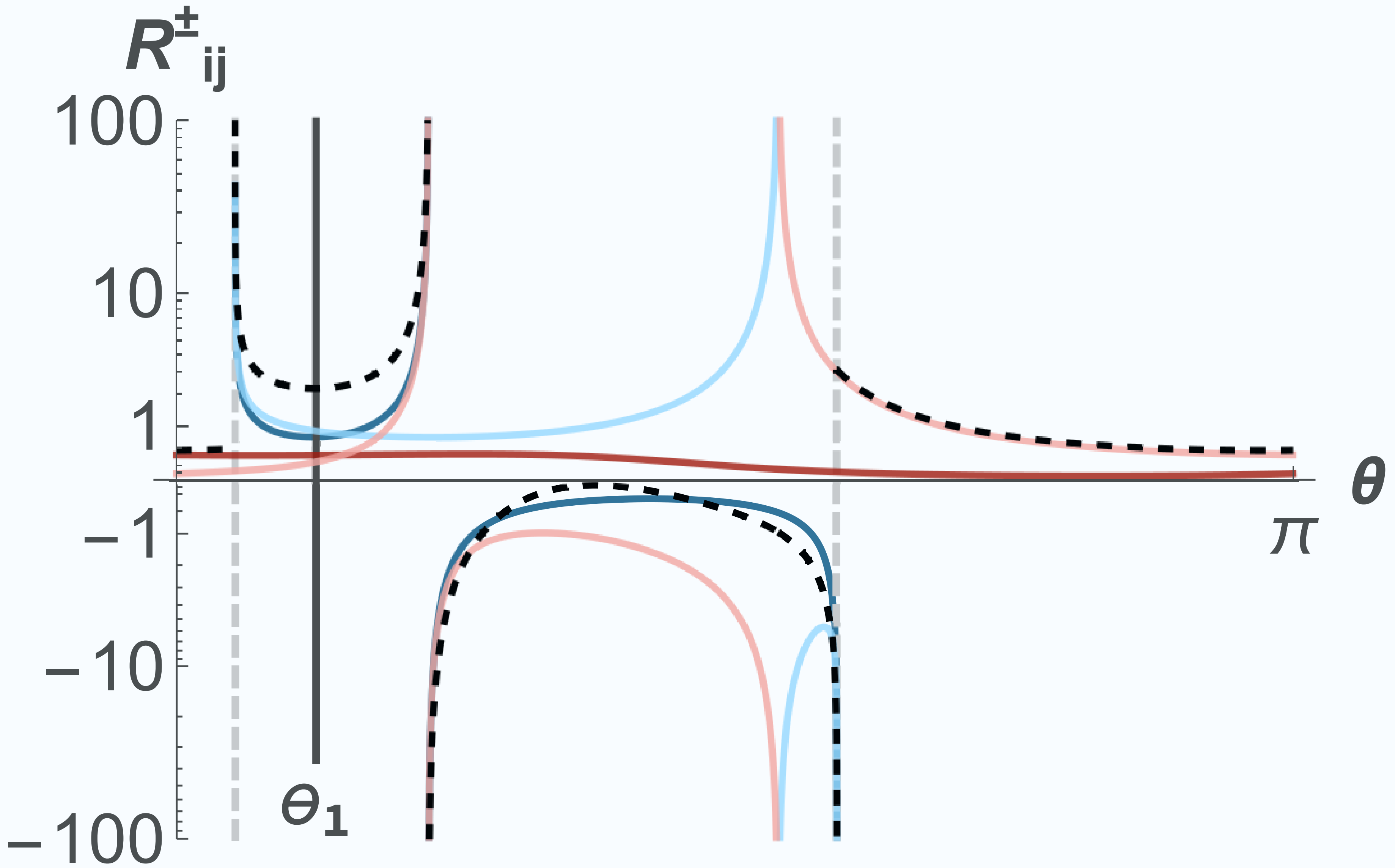

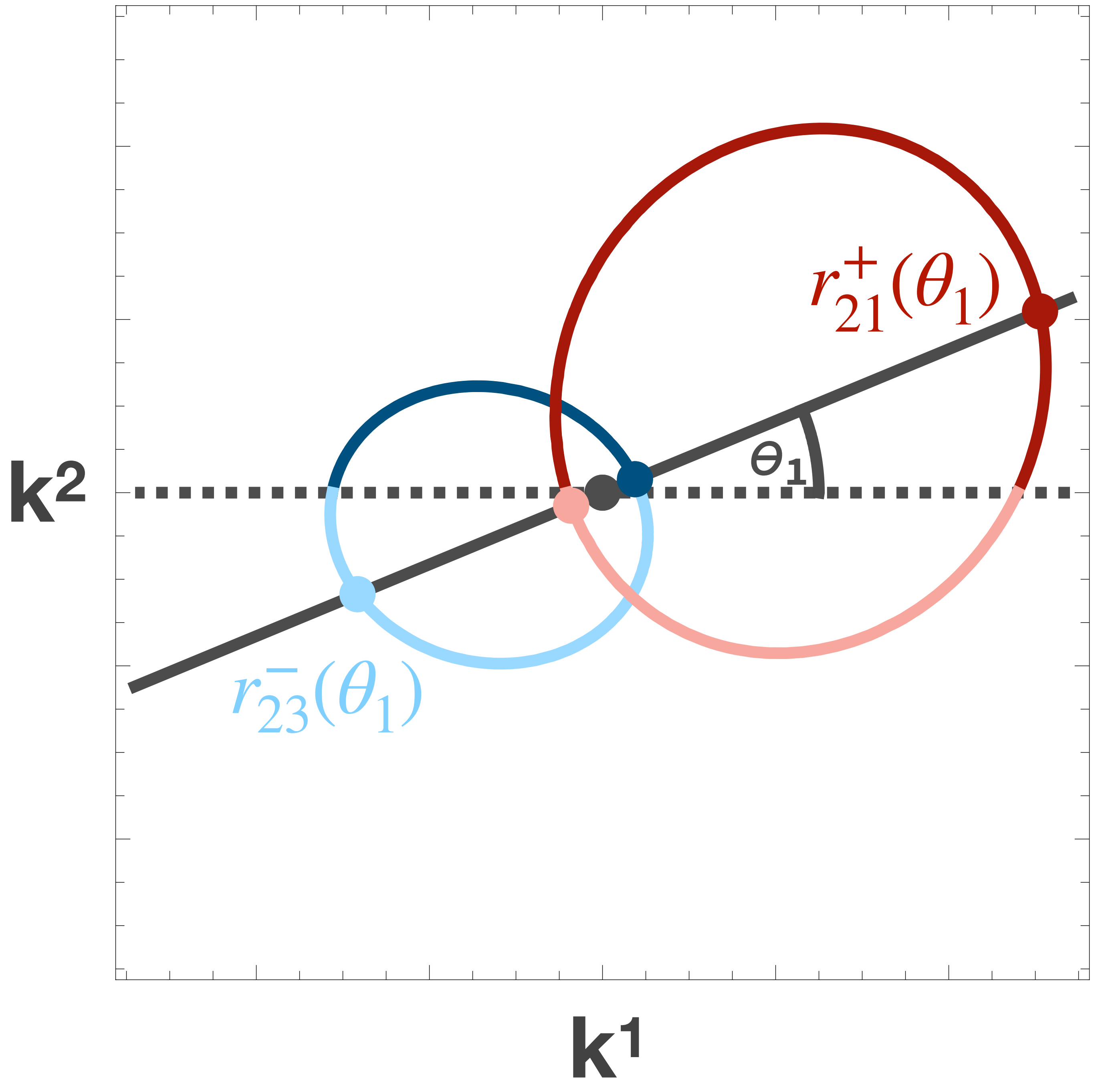

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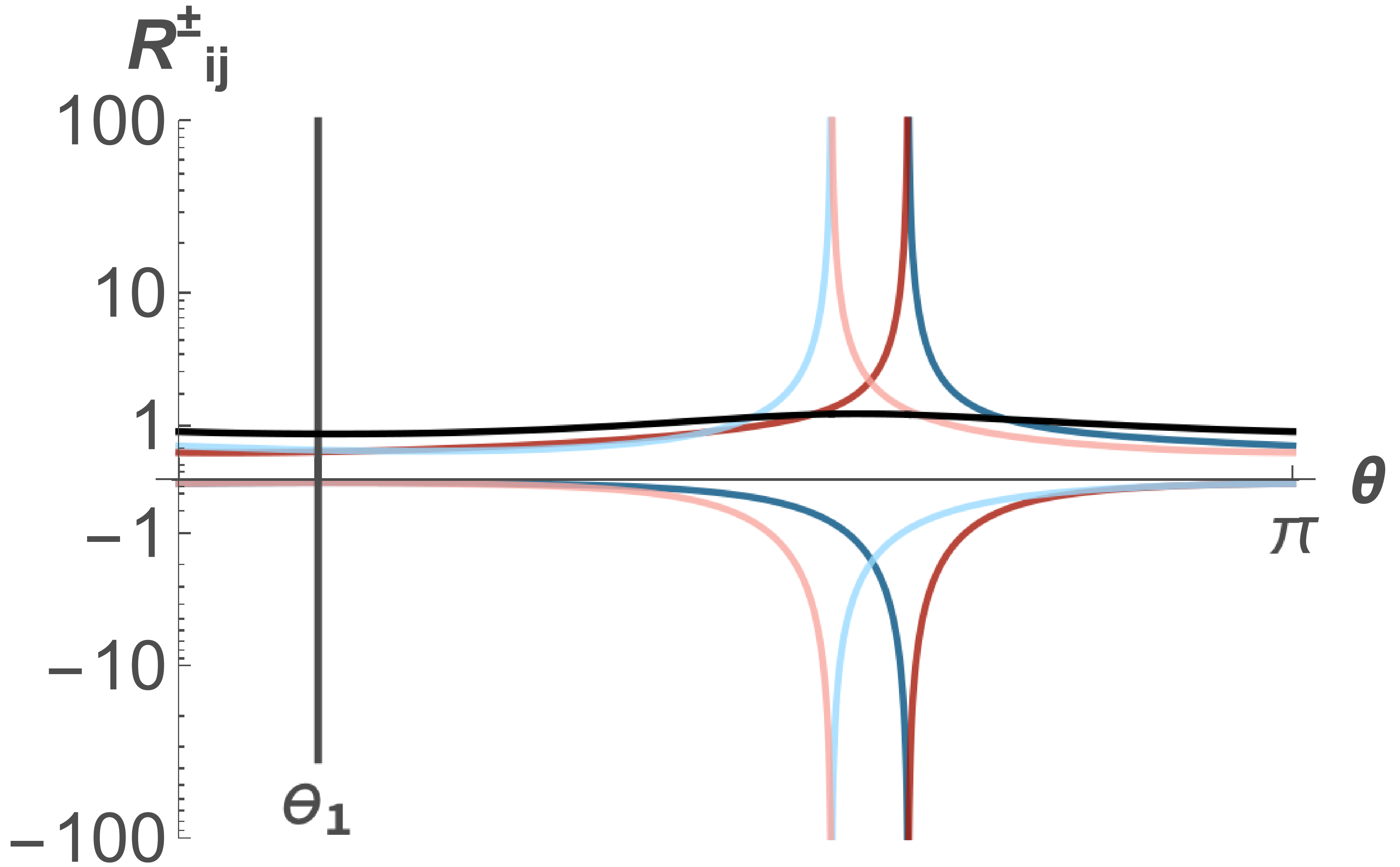
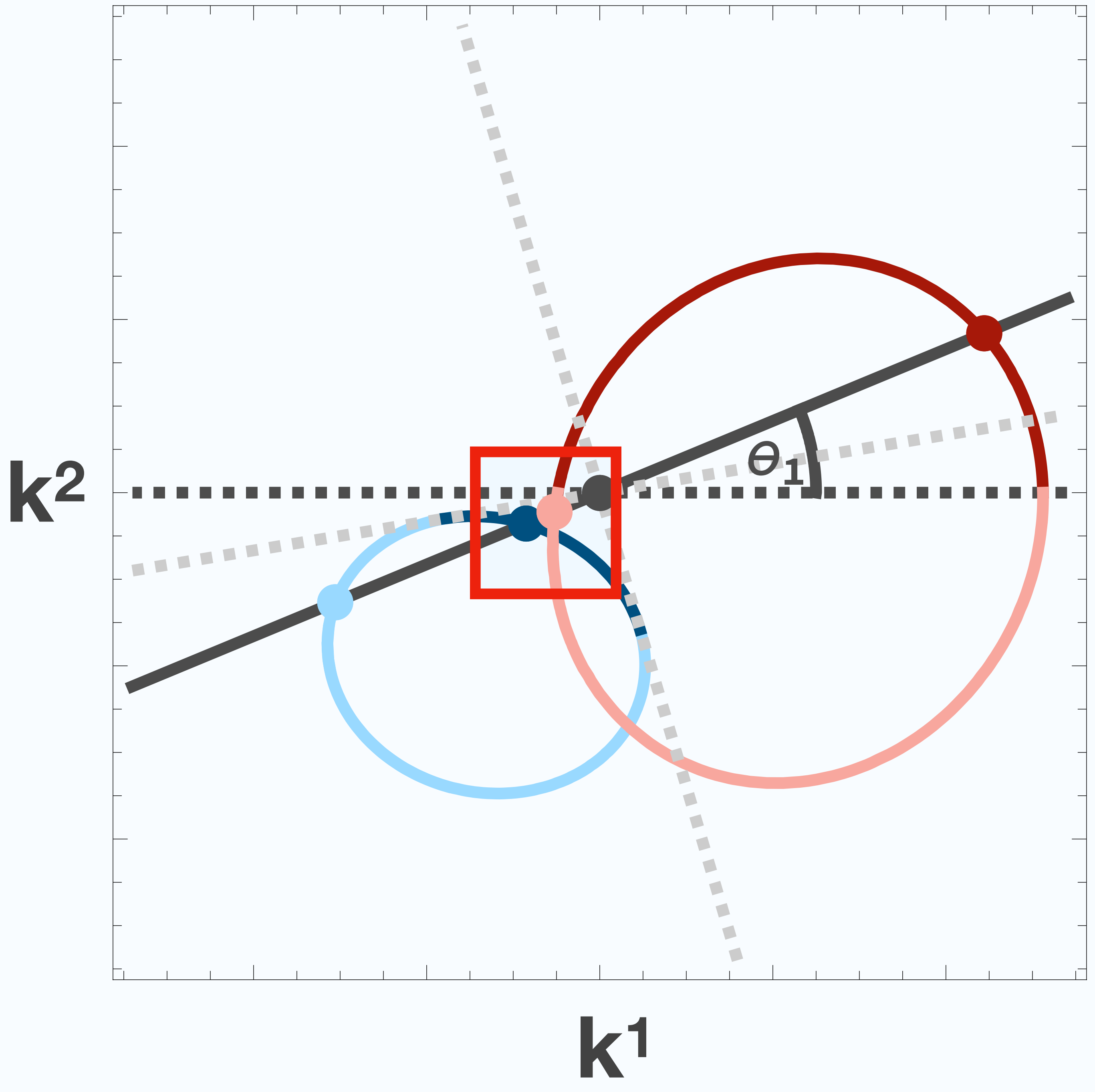
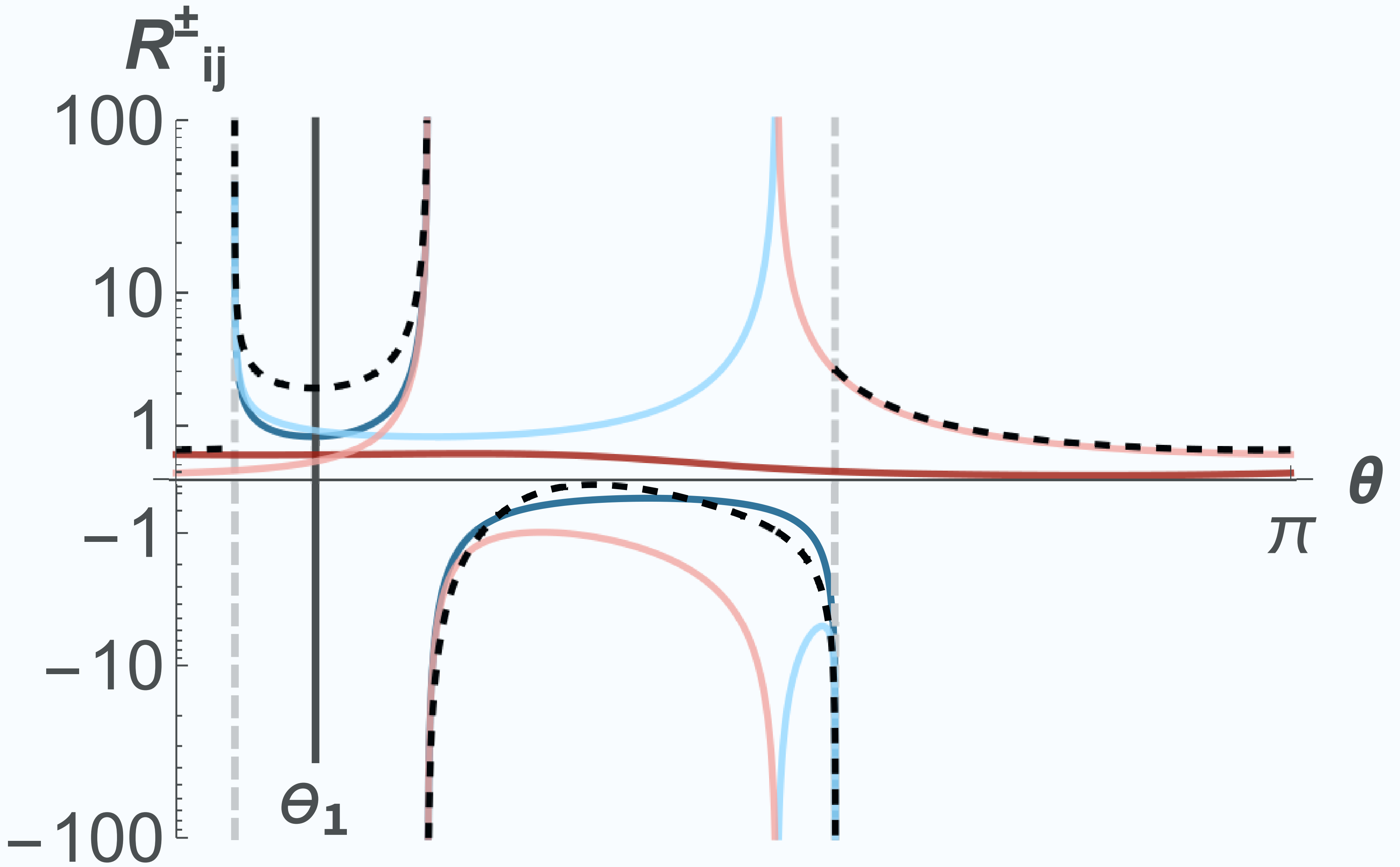
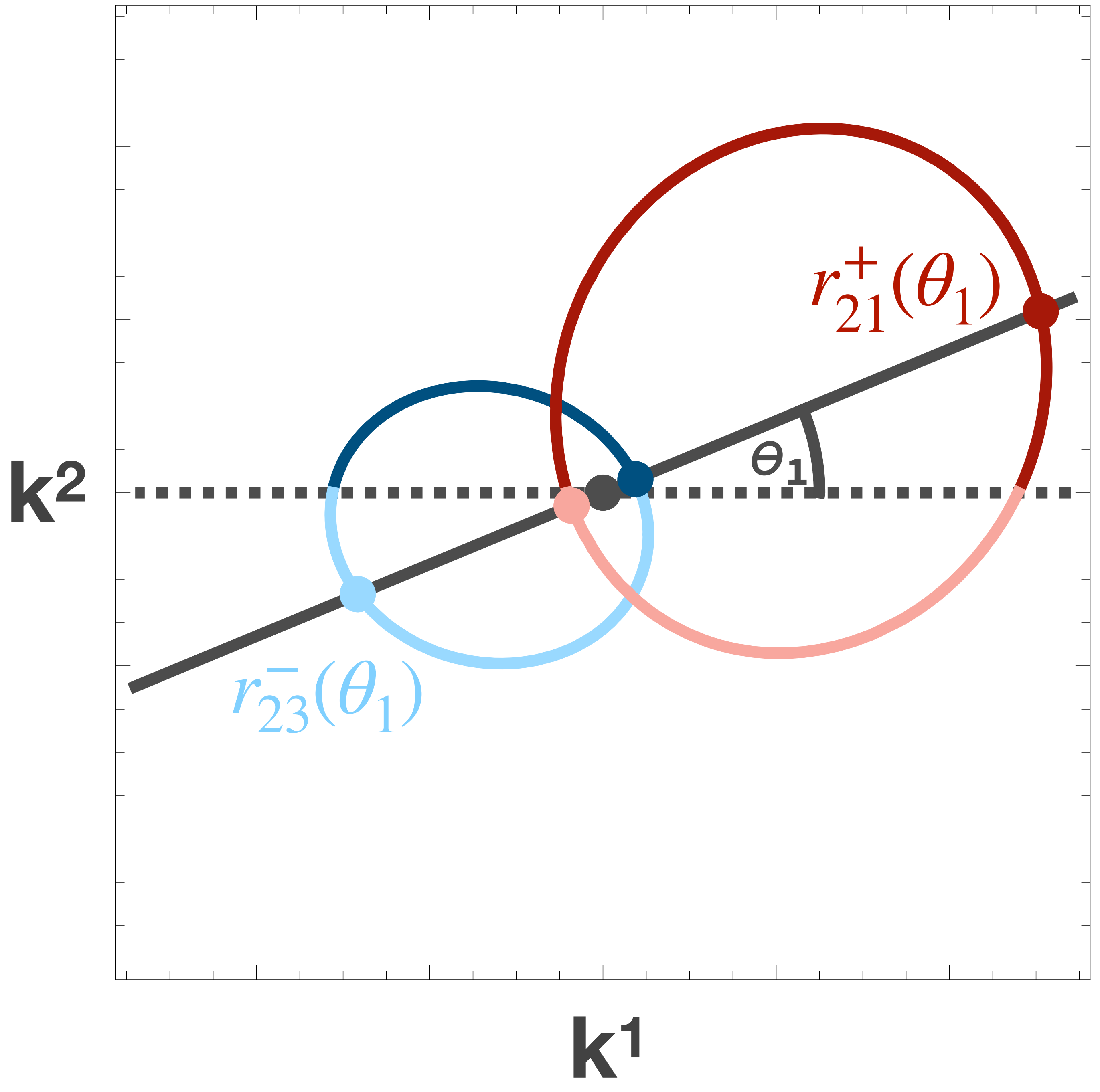
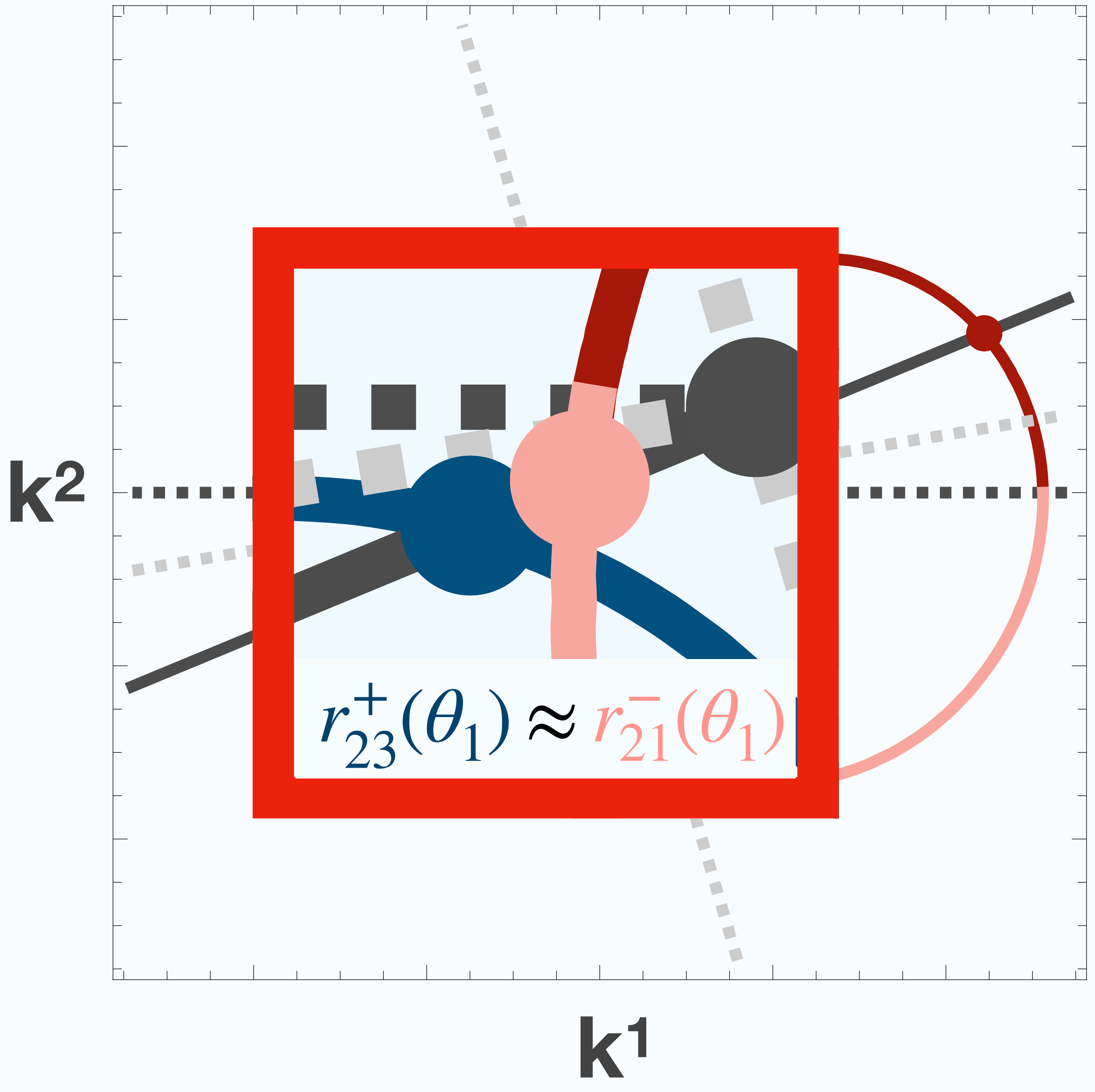




Origin 2 (pinched pole)







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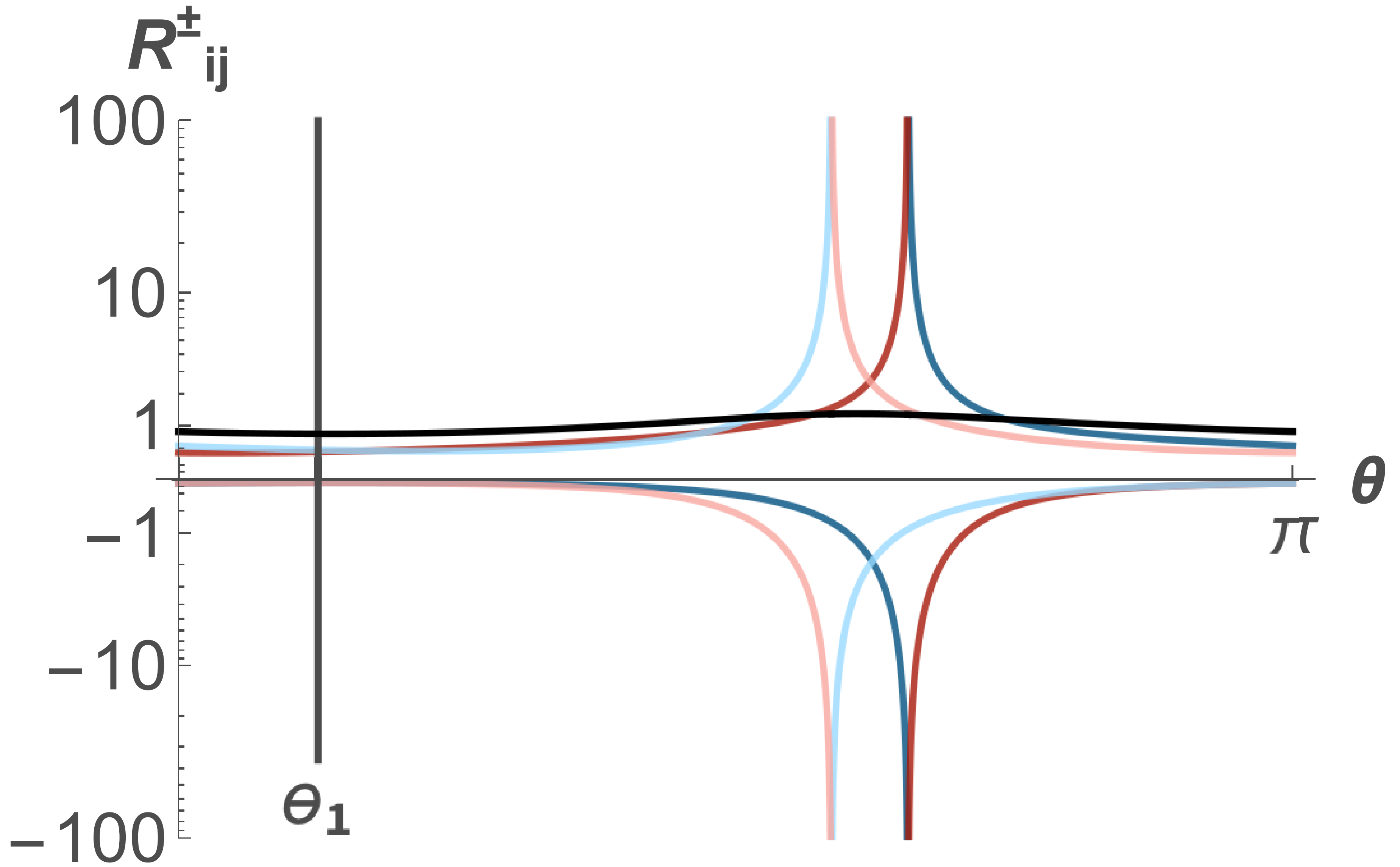
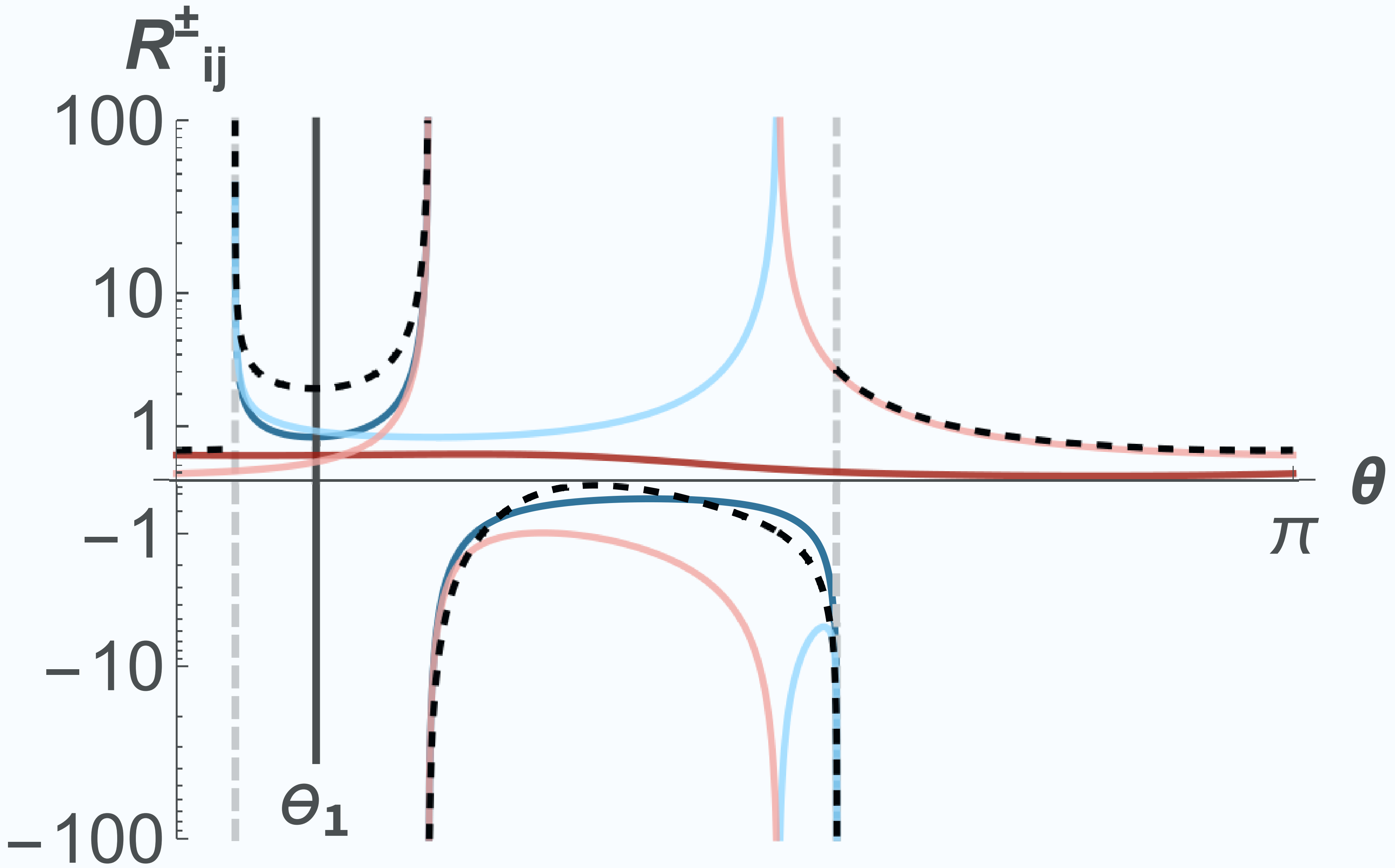
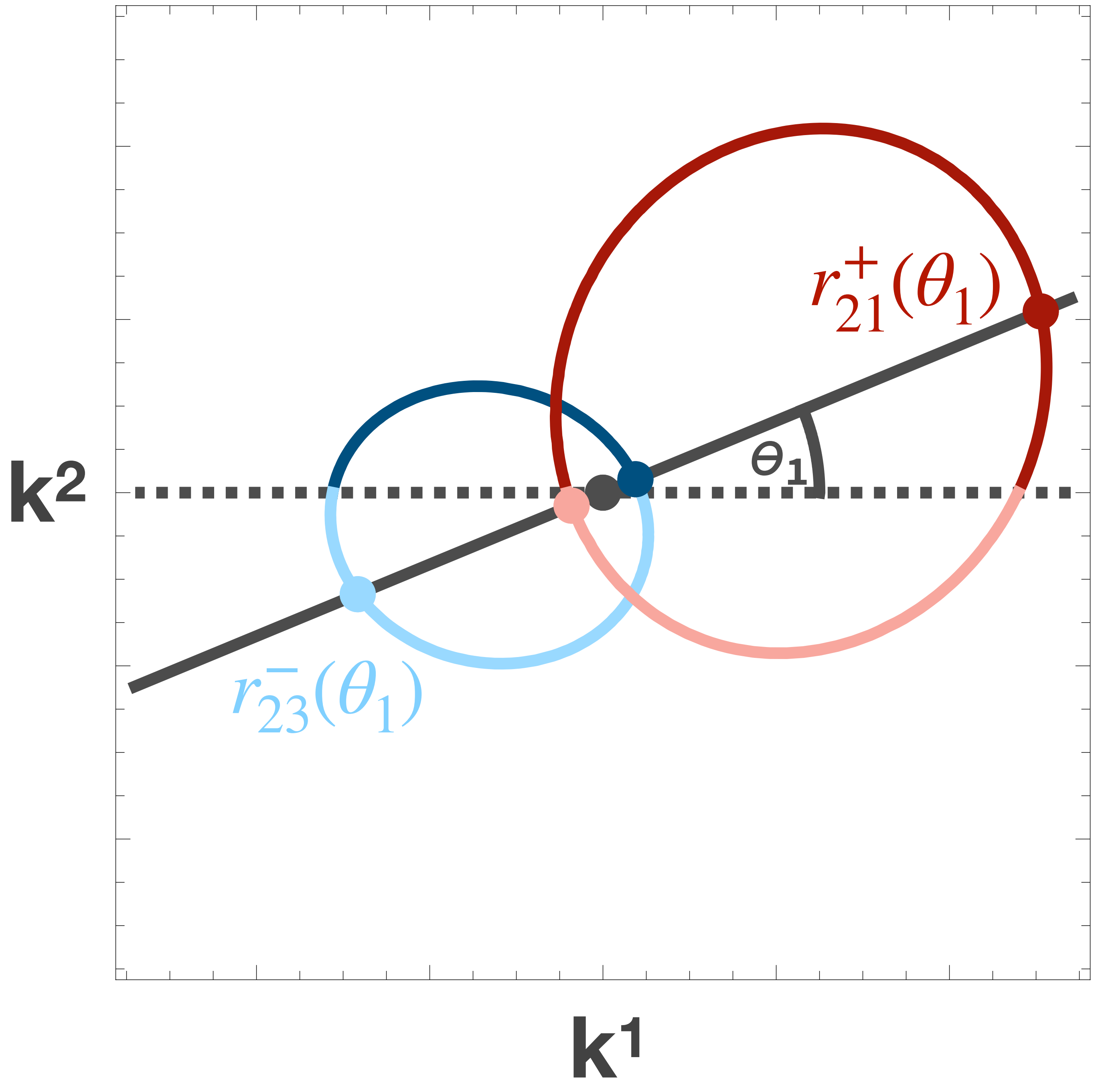
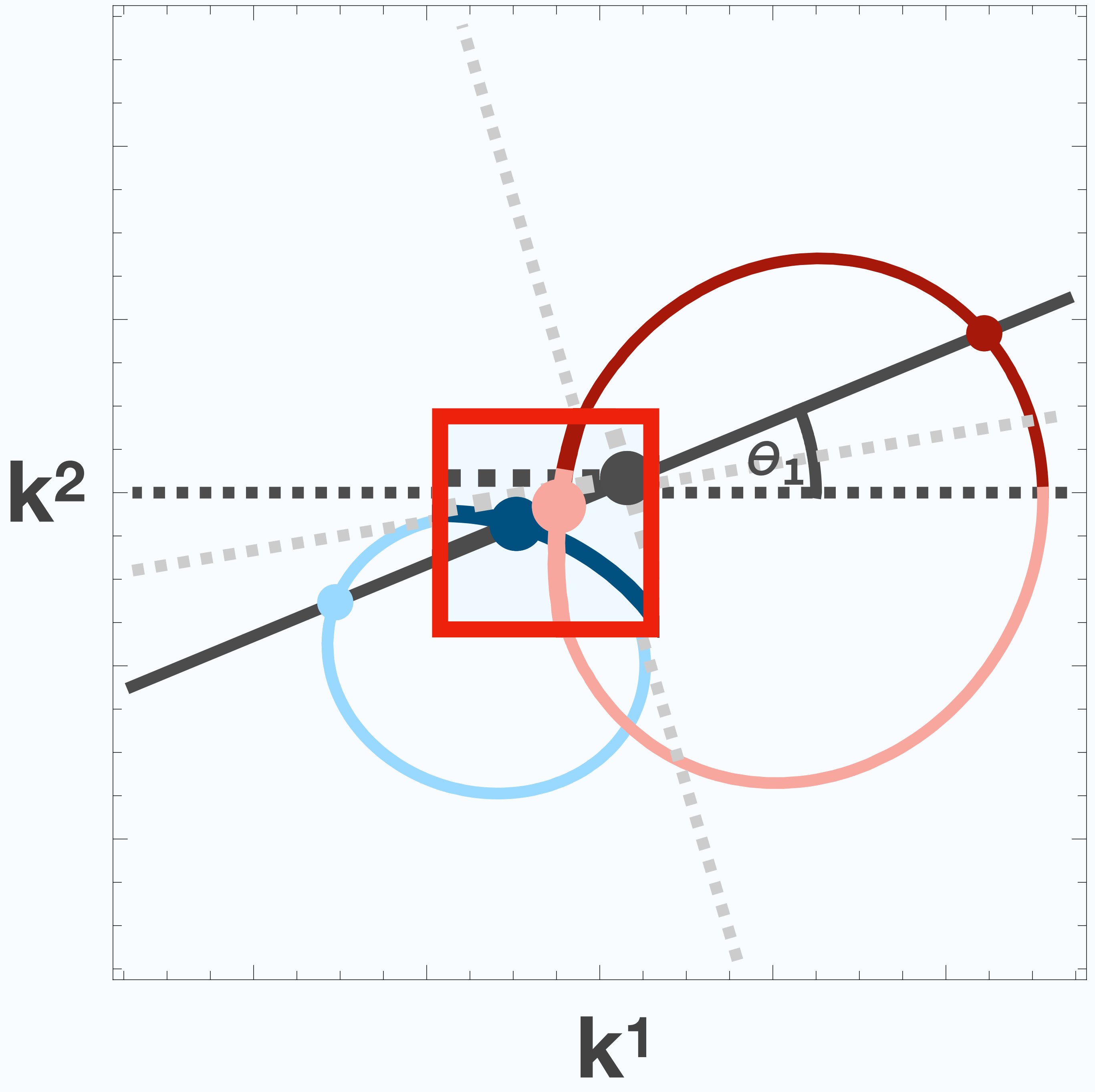
Origin 2 (pinched pole)









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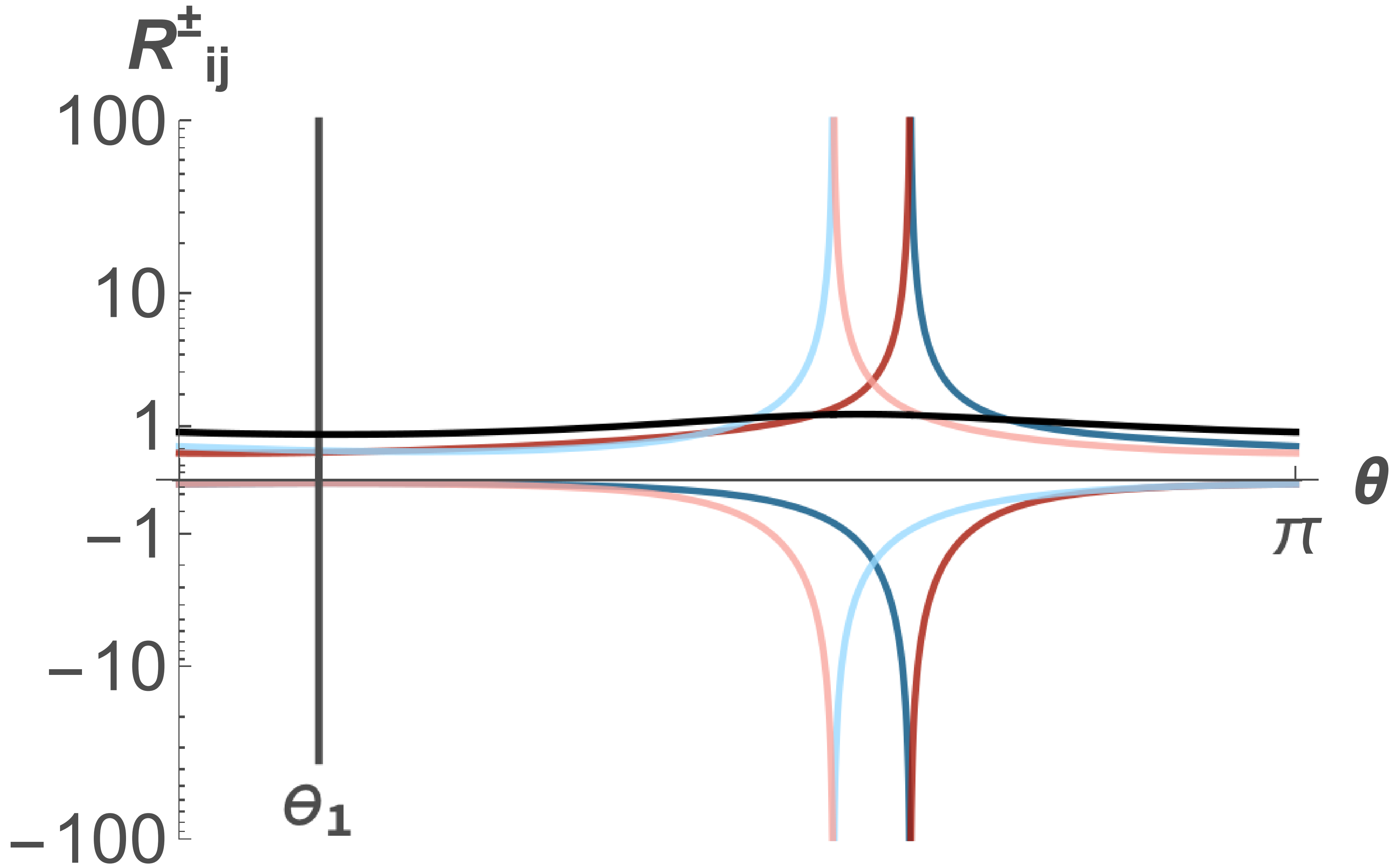

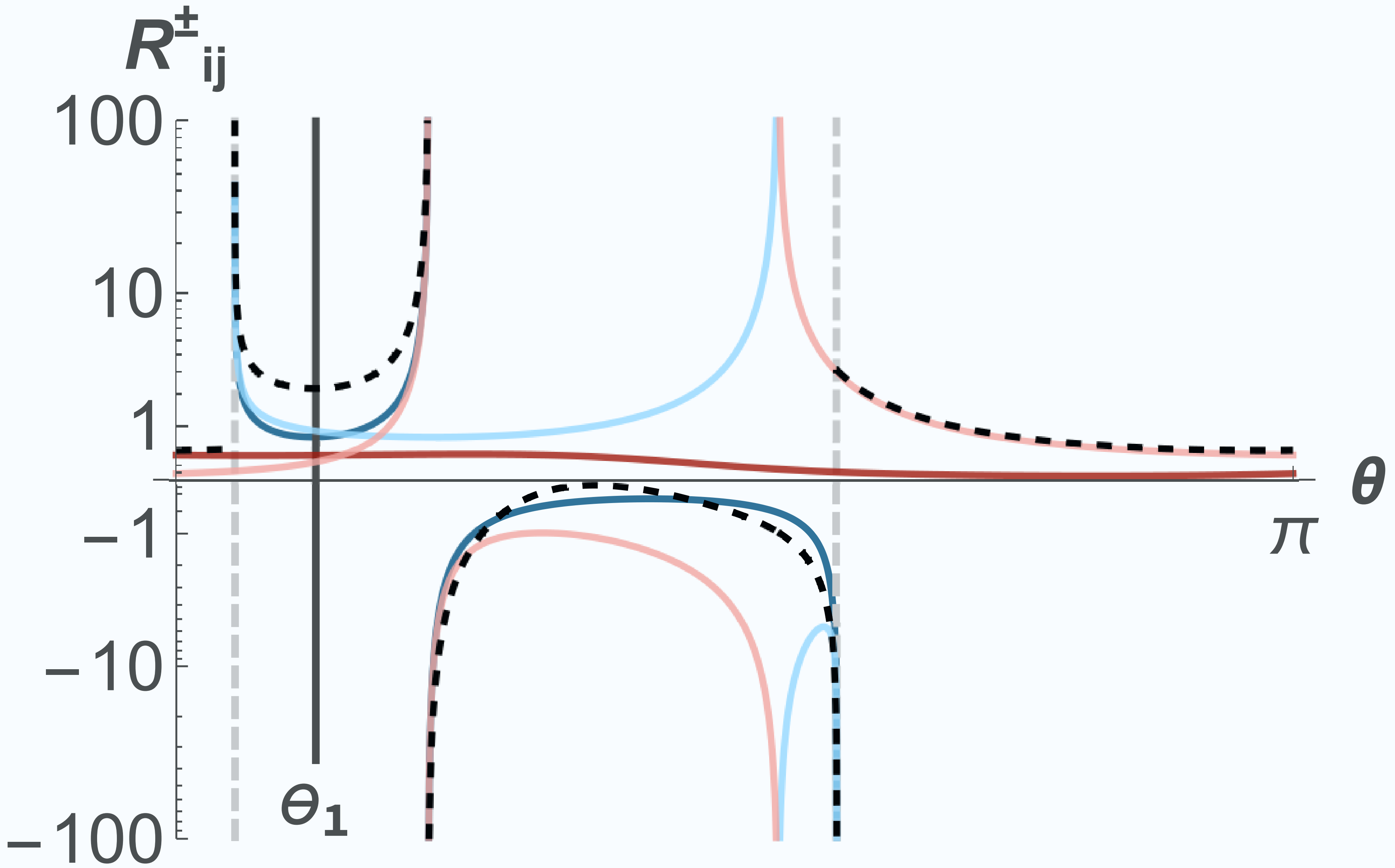

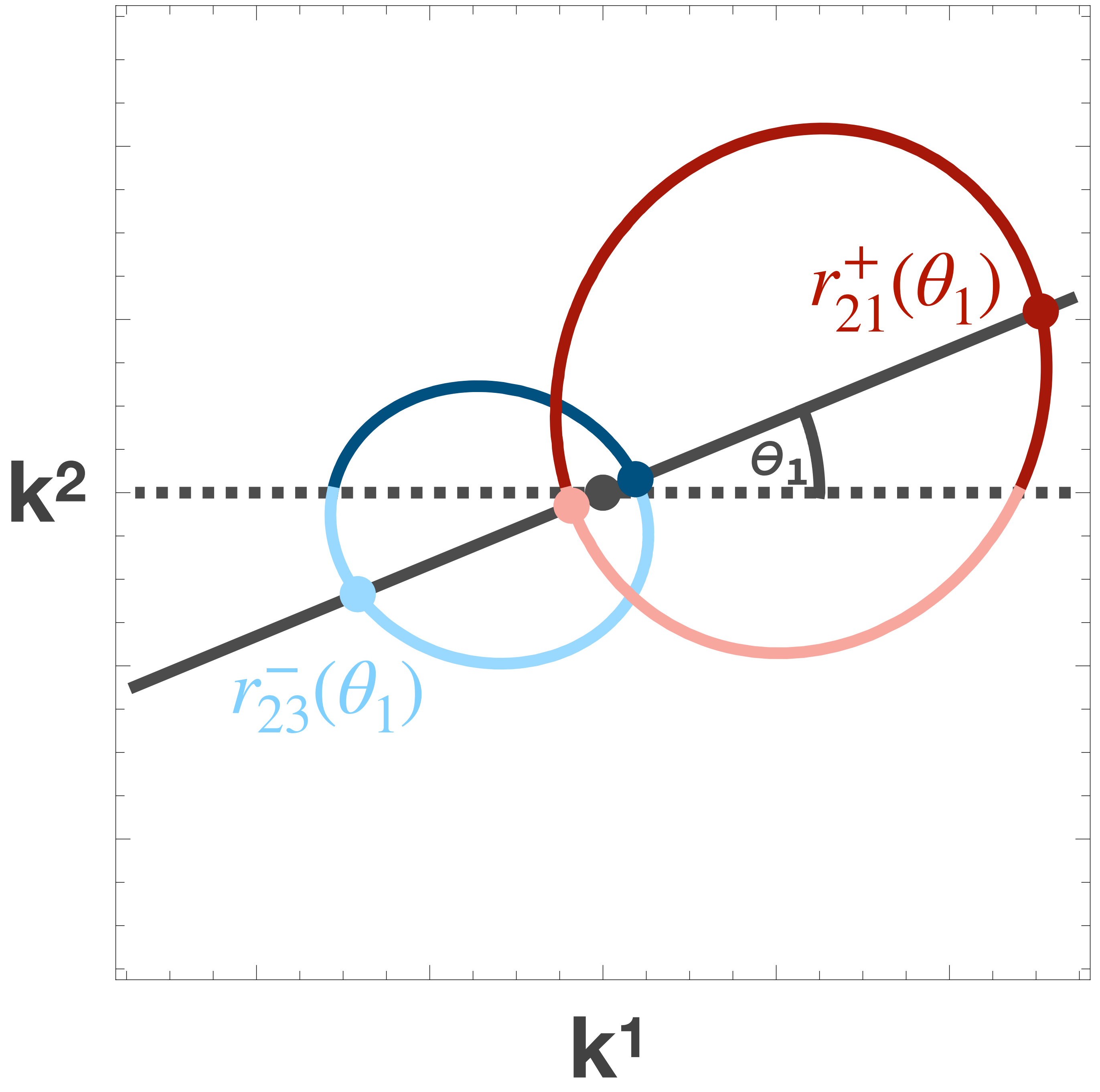

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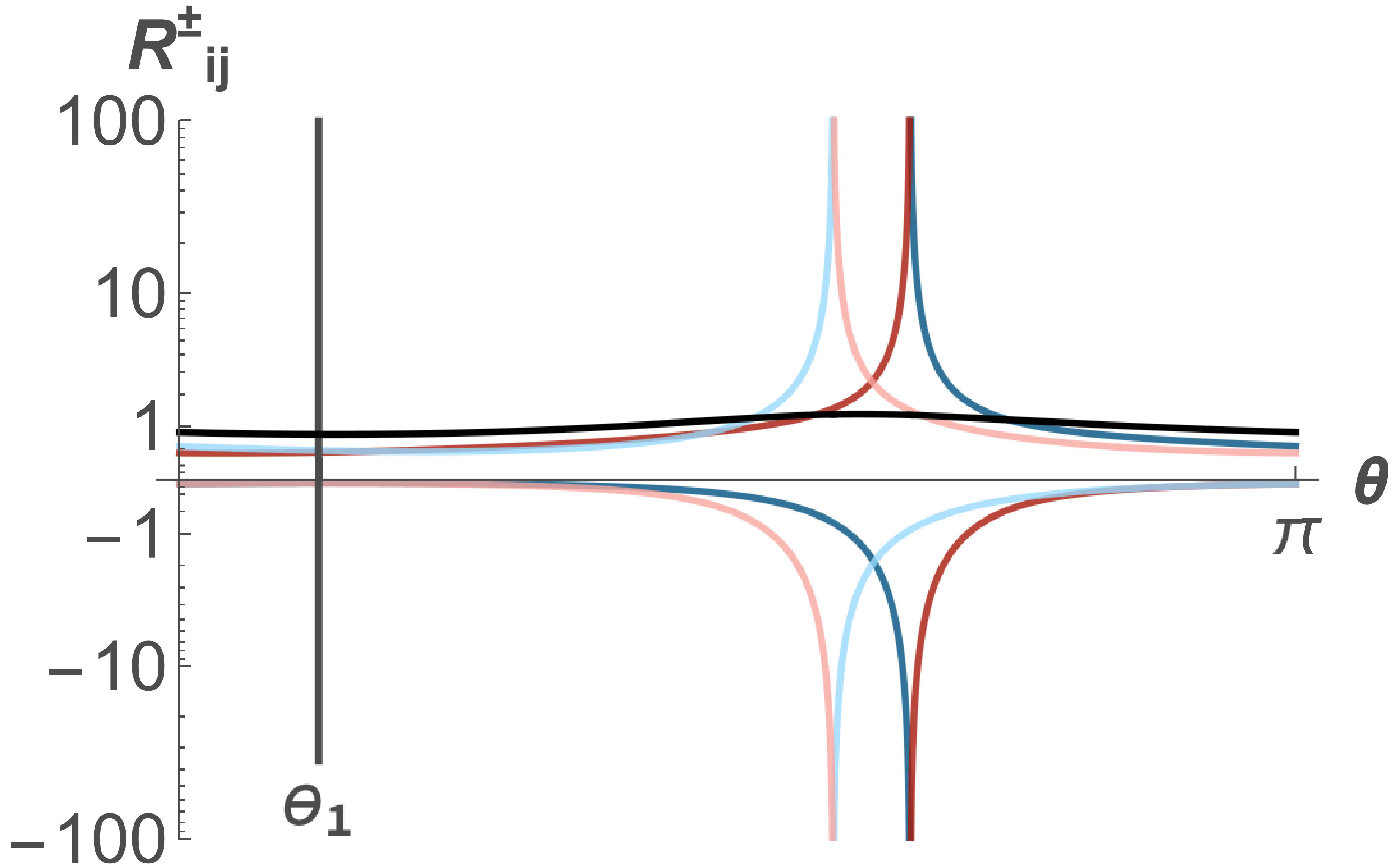
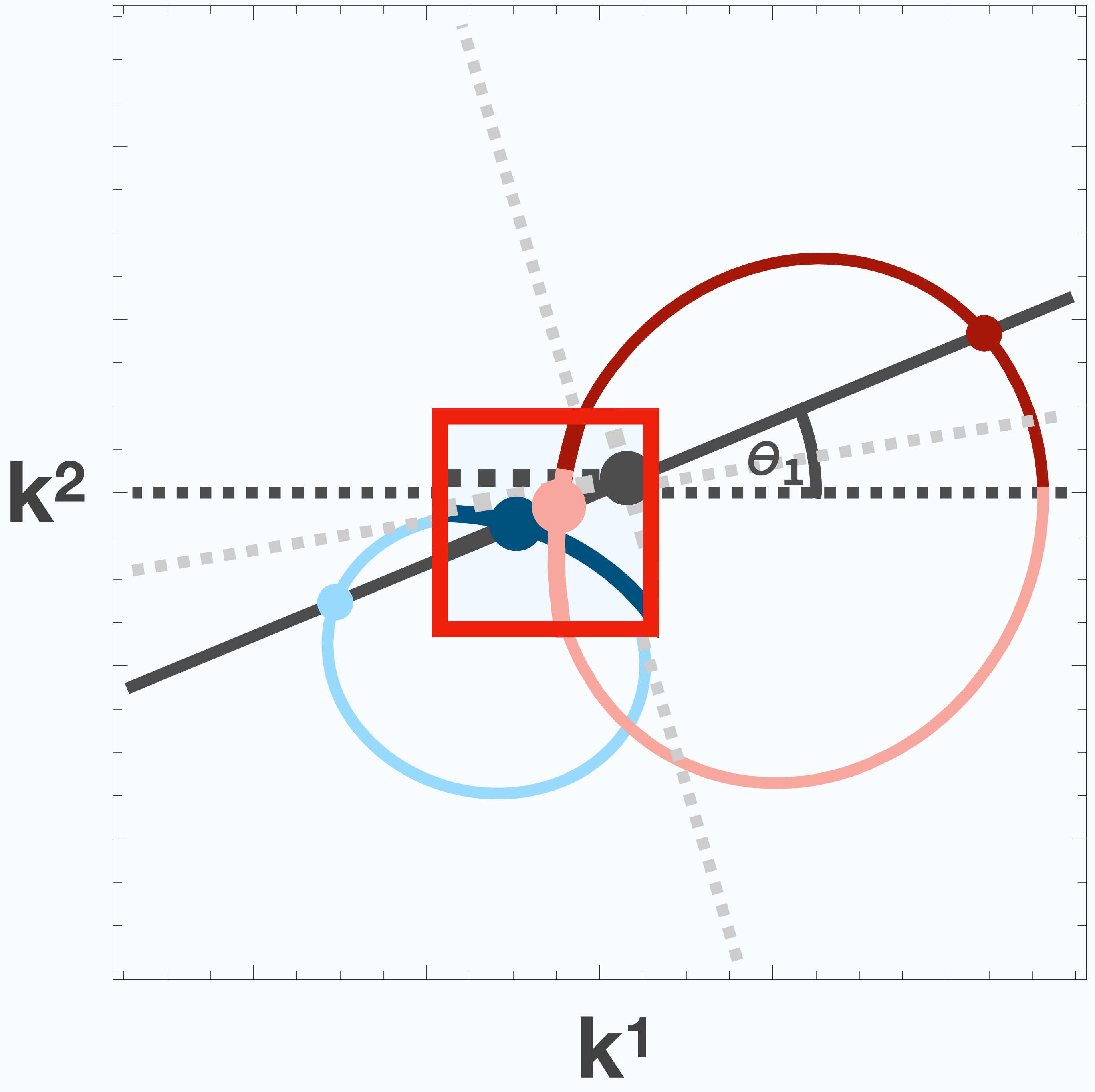
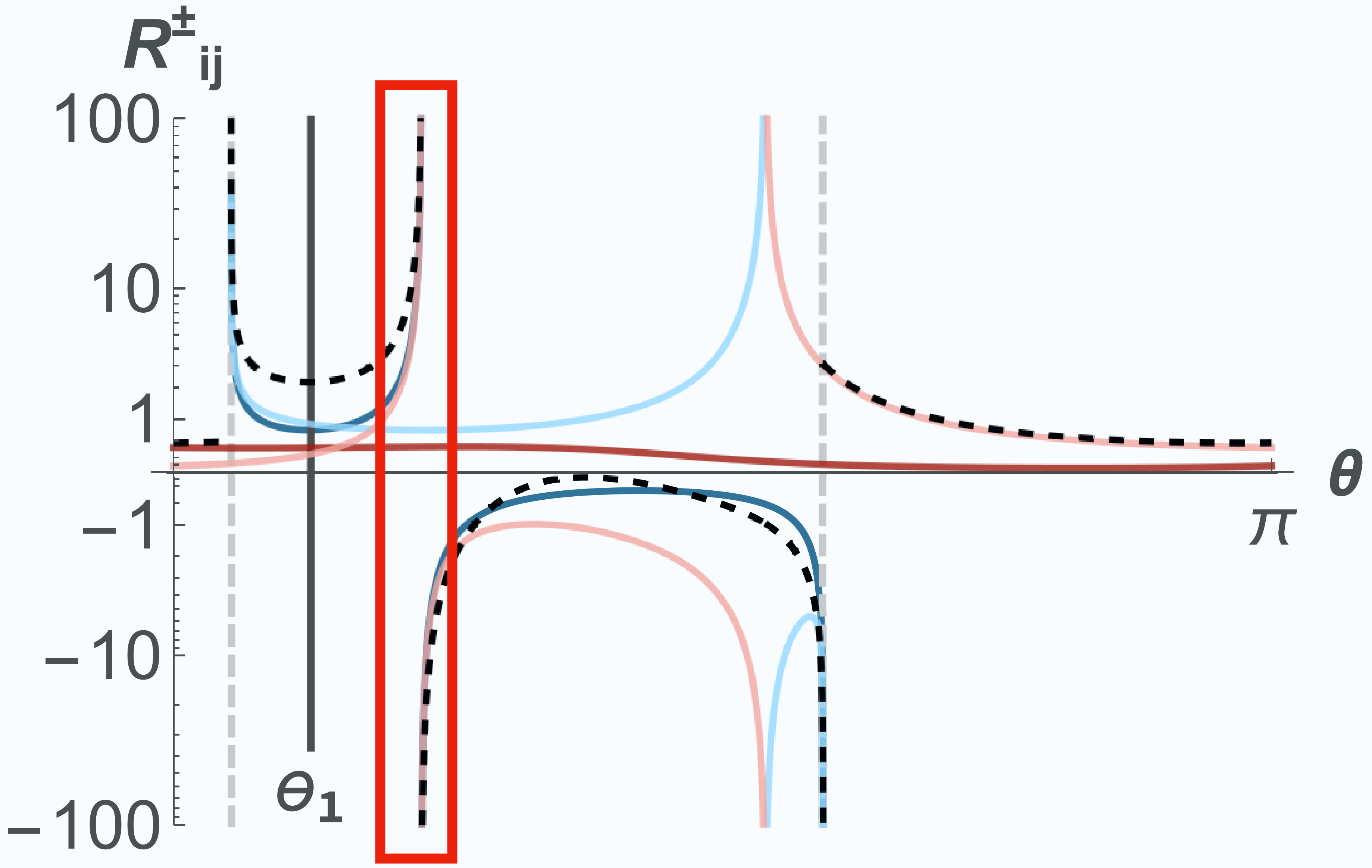
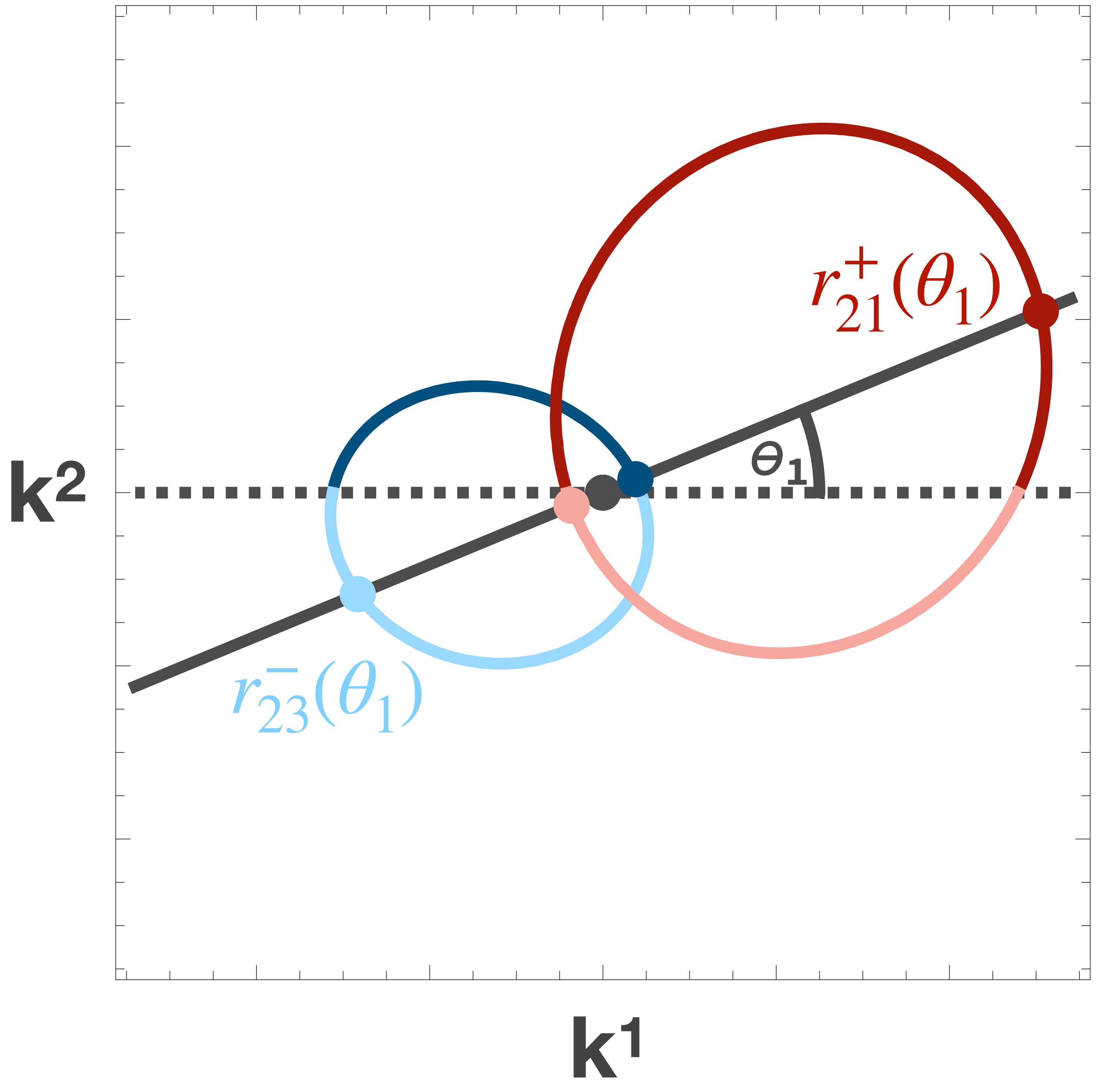
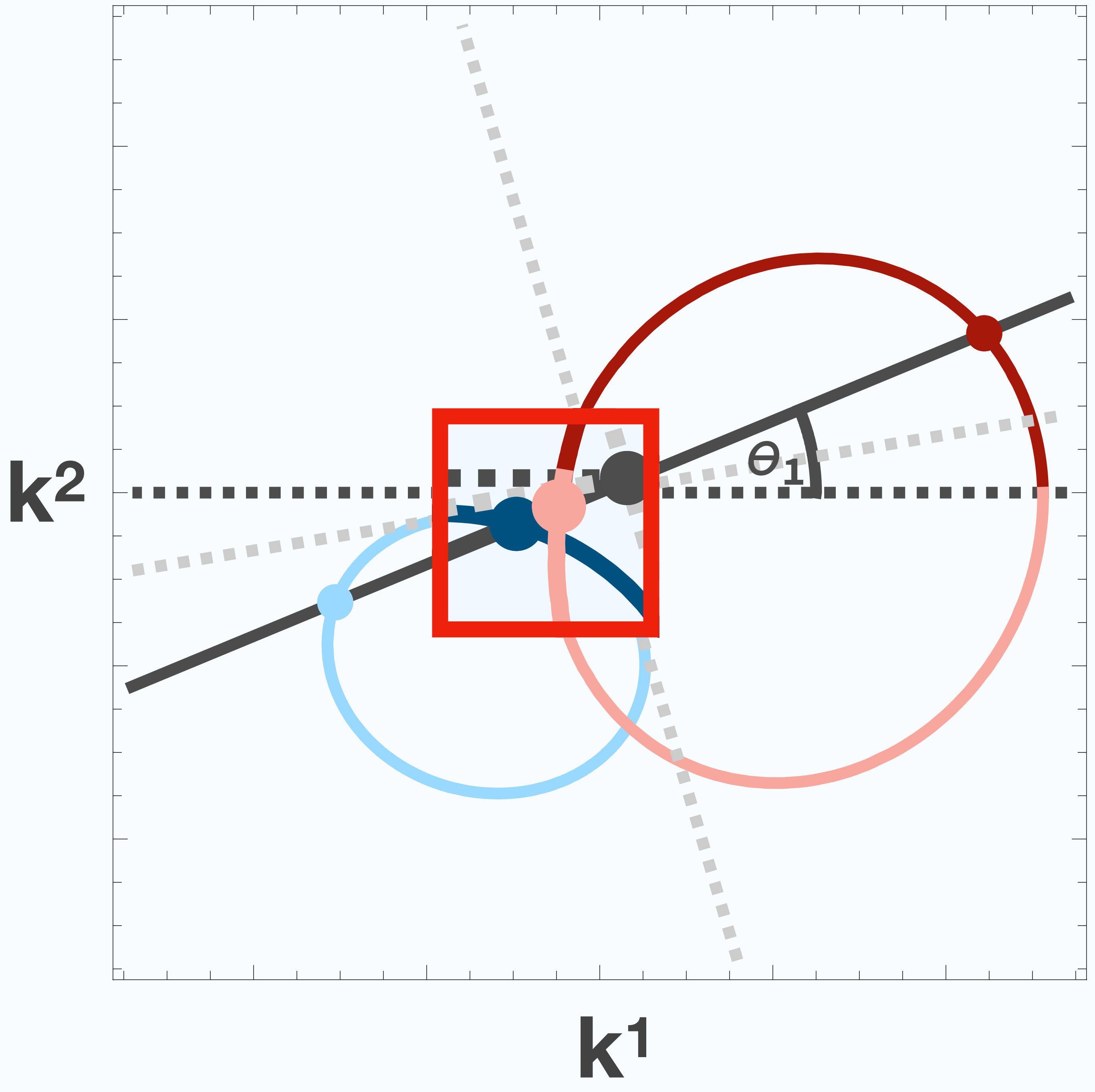




Origin 2 (pinched pole)







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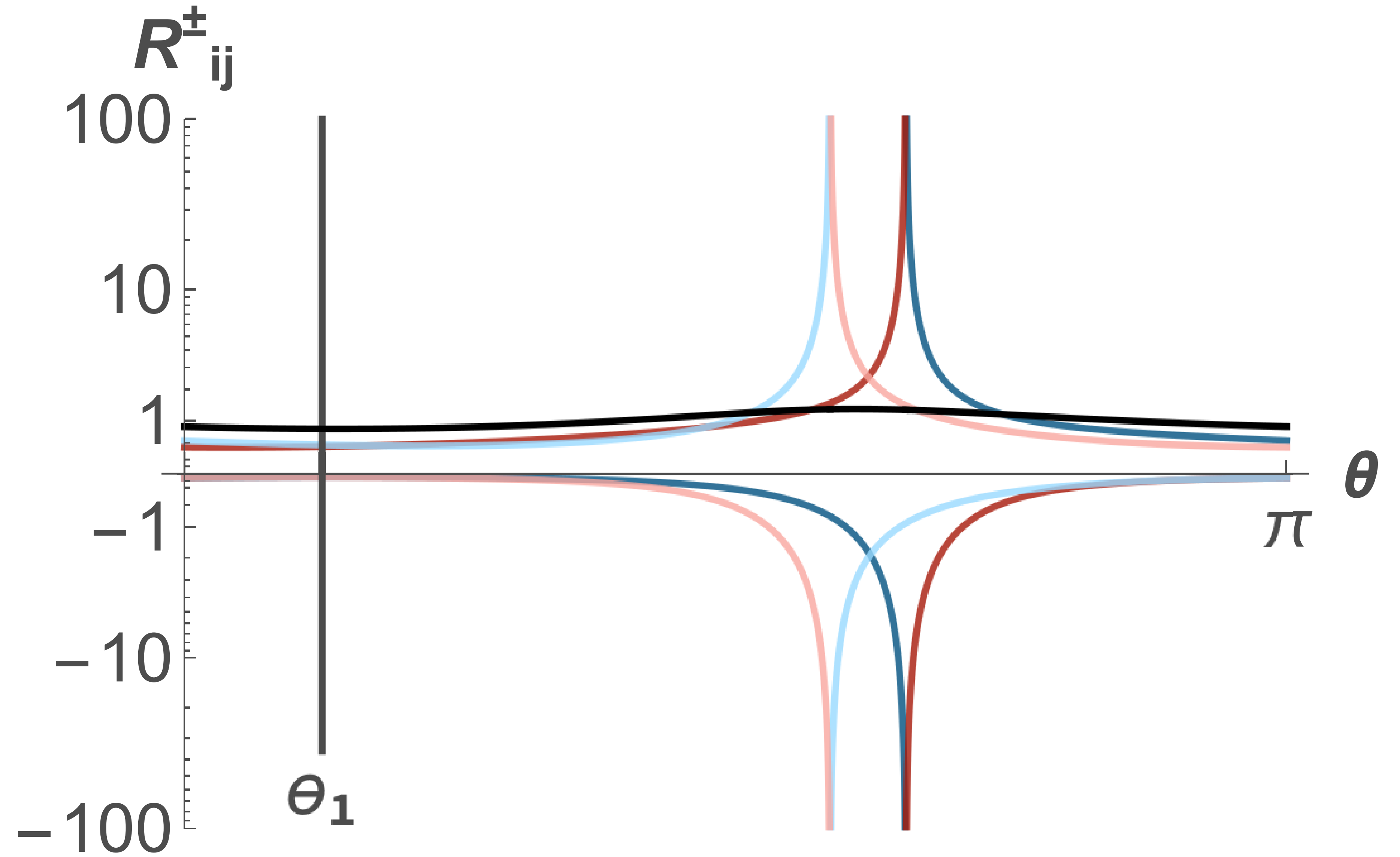
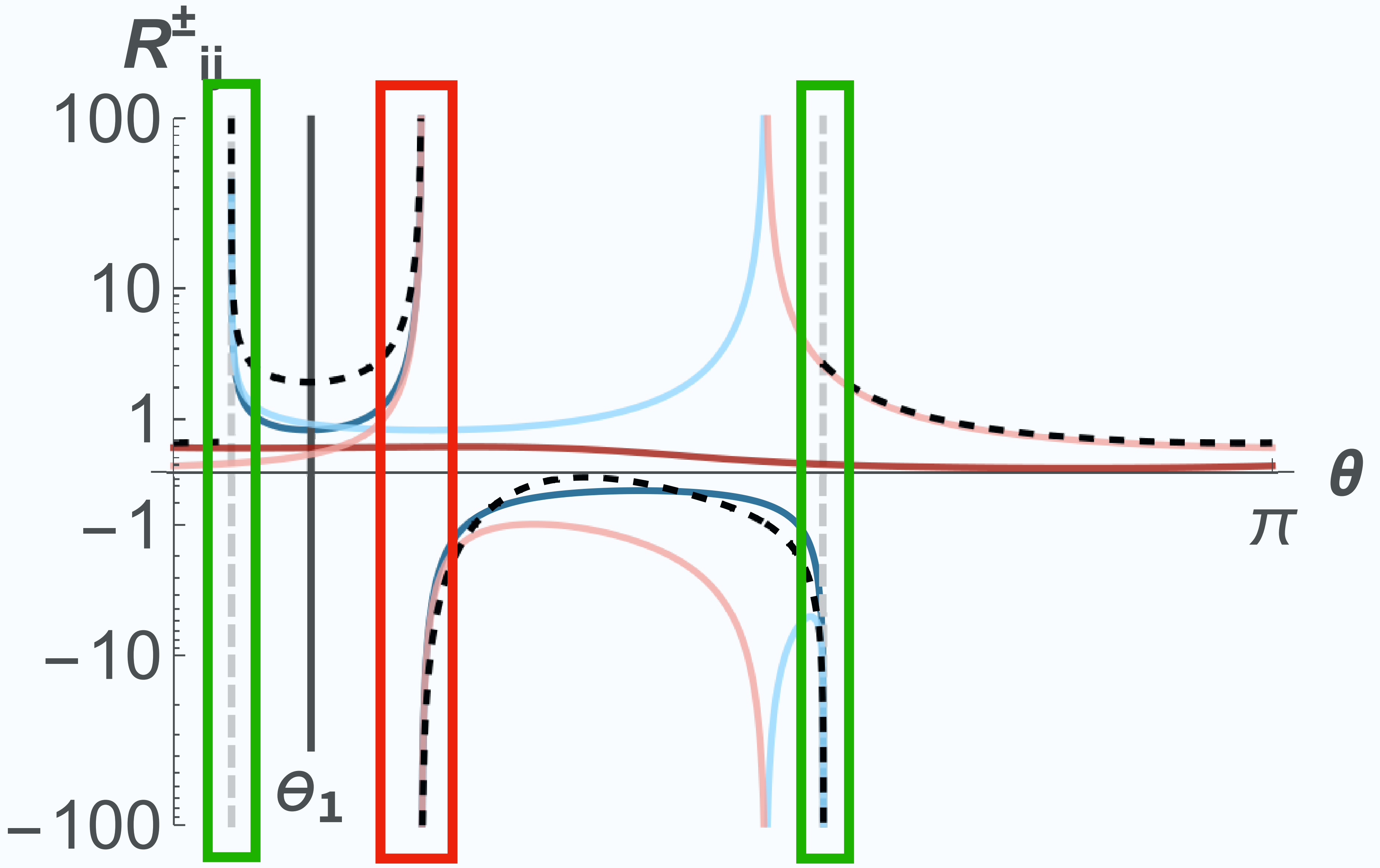
Origin 2 (pinched pole)





Exterior origin introduces integrable singularities

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$$R_{ij}^{\pm} = \Theta\left(r_{ij}^{\pm} \in \mathbb{R}\right) \frac{1}{-4E_iE_j} \frac{r^2}{\left(\frac{\vec{q}_i}{E_i} + \frac{\vec{q}_j}{E_j}\right) \cdot \hat{k}} \Bigg|_{r=r_{ij}^{\pm}} \frac{N}{\prod_{l \neq i,j} D_l} \Bigg|_{ij}$$

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outward normal of ellipsoid

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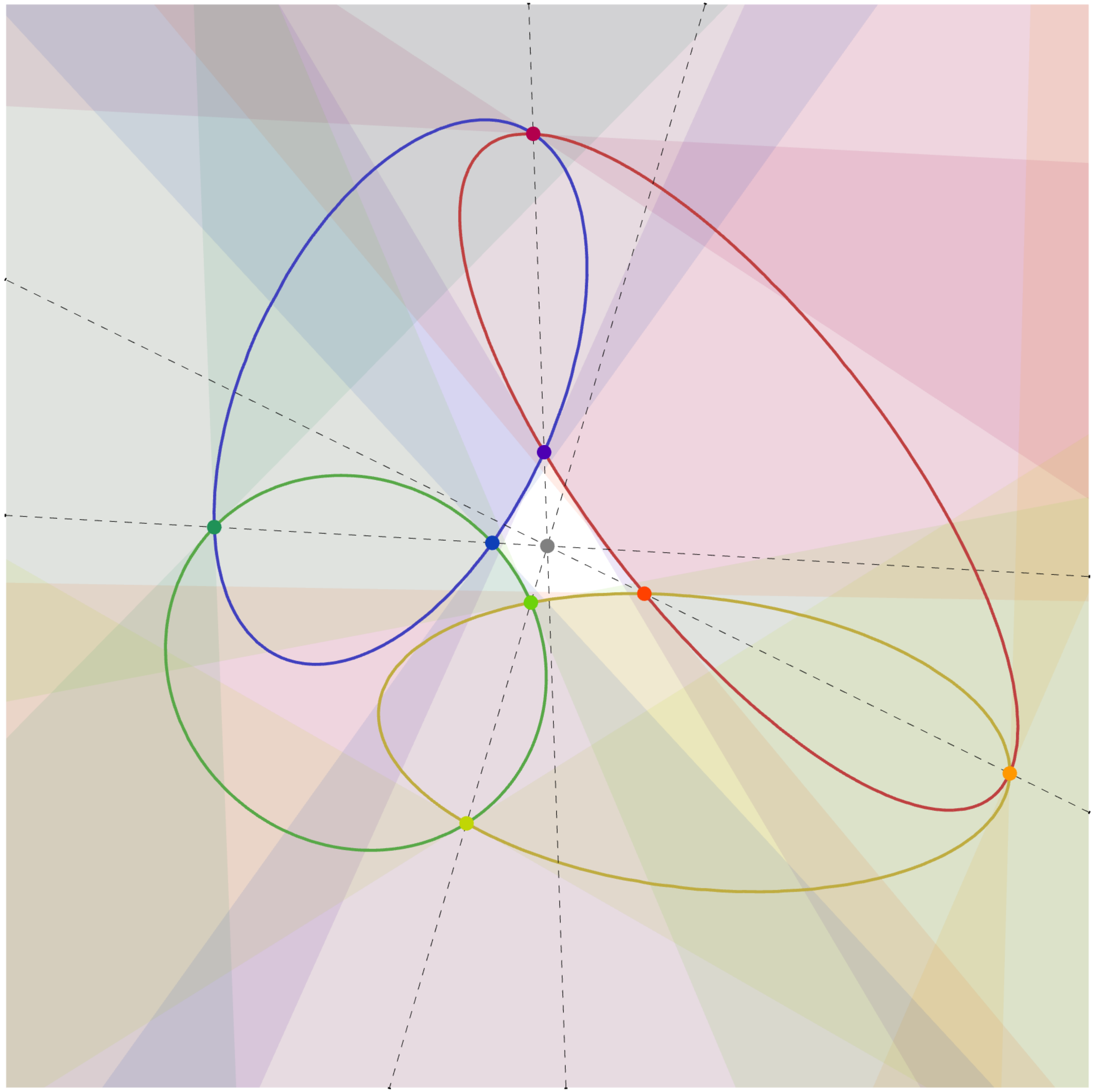
outward normal of ellipsoid

No pinched poles:



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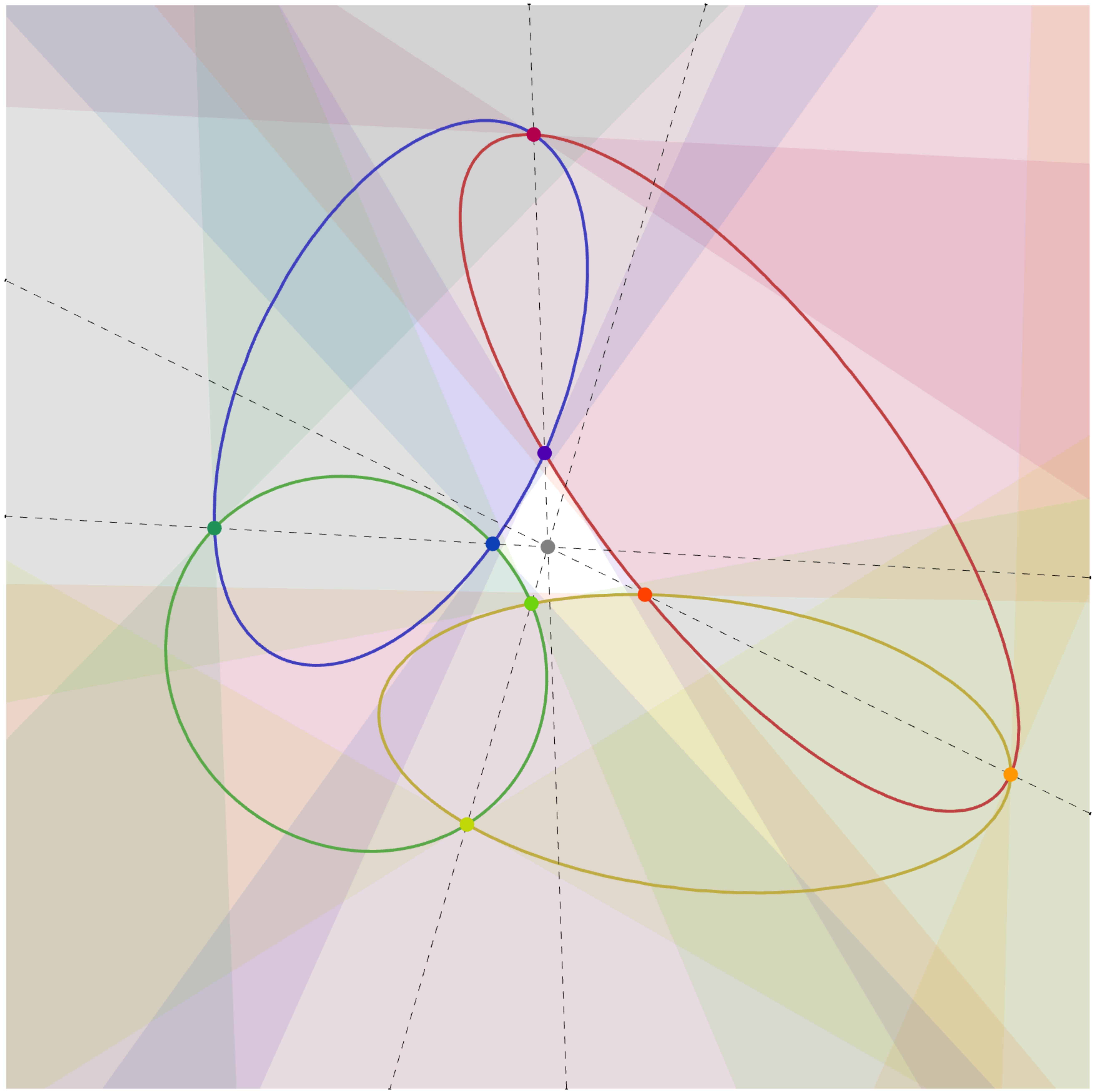

Topology	Kin.	N_E	N_G	N_G^{max}	N_p	Phase	Exp.	Reference	Numerical	$\Delta\ [\sigma]$	$\Delta\ [\%]$	$\Delta\ [\%] \lvert\cdot\rvert$
Box	Box_4E	4	1	1	10^9	Re	-08	-7.437071	-7.430810 +/- 0.017054	0.367	0.084	0.101
					10^9	Im		6.578304	6.570476 +/- 0.005288	1.480	0.119	