Causal representation and numerical evaluation of multi-loop Feynman integrals

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https://bitbucket.org/wjtorresb/lotty

RadCor-LoopFest 2021
May 19th, 2021
Florida State University, Virtual Conference

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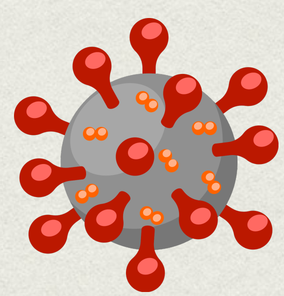
Outline

- Introduction
- Multi-loop LTD representation
- Causal Representation of Scattering Amplitudes
- Lotty — the loop-tree duality automation
- Conclusions & Outlook

JHEP 04 (2021) 183

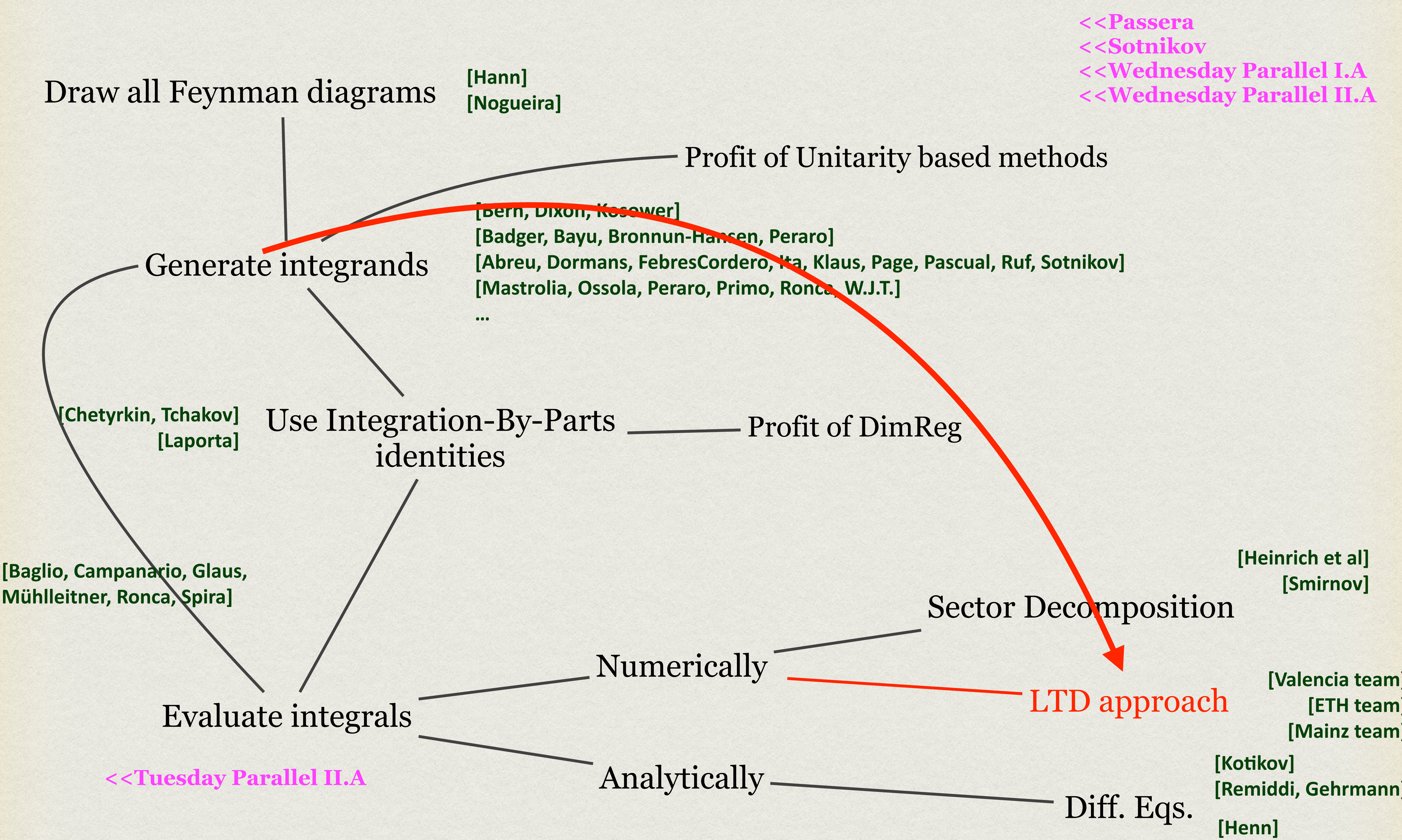
2102.05048 [hep-ph]



2103.09237 [hep-ph]

Accepted in EPJC

Standard approach @multi-loop level



Standard approach @multi-loop level

Complete automation @ NNLO ?

Thresholds

UV

IR

Eur. Phys. J. C (2021) 81:250
https://doi.org/10.1140/epjc/s10052-021-08996-y

THE EUROPEAN PHYSICAL JOURNAL C

Check for updates

Review

May the four be with you: novel IR-subtraction methods to tackle NNLO calculations

W. J. Torres Bobadilla[1,2,a], G. F. R. Sborlini[3], P. Banerjee[4], S. Catani[5], A. L. Cherchiglia[6], L. Cieri[5], P. K. Dhani[5,7], F. Driencourt-Mangin[2], T. Engel[4,8], G. Ferrera[9], C. Gnendiger[4], R. J. Hernández-Pinto[10], B. Hiller[11], G. Pelliccioli[12], J. Pires[13], R. Pittau[14], M. Rocco[15], G. Rodrigo[2], M. Sampaio[6], A. Signer[4,8], C. Signorile-Signorile[16,17], D. Stöckinger[18], F. Tramontano[19], Y. Ulrich[4,8,20]

- FDH/FDR —> transition rules both ren. schemes @NNLO
- FDU —> preliminary mappings between VV & VR contributions
- IReg —> Towards full renormalisation @ 2L
- Torino Scheme —> general subtraction method for massless & final states QCD
- qt-subtraction —> benefits from any existing calculation for “F+jet”
- Antenna subtraction —> subtraction term at tree (RR) and one-loop (R) level

- many more subs. schemes …

Causal representation —> *Display only physical* **singularities**

—> *Multi-loop LTD representation*

Multi-loop LTD representation

[Aguilera-Verdugo *et al* (2020)]

<< https://indico.cern.ch/event/1021090/

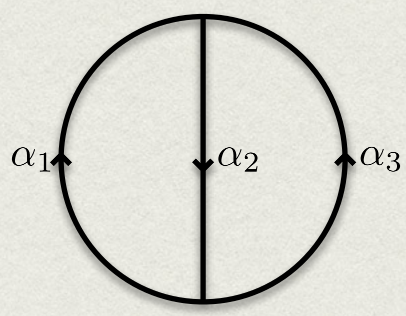
<<Sborlini

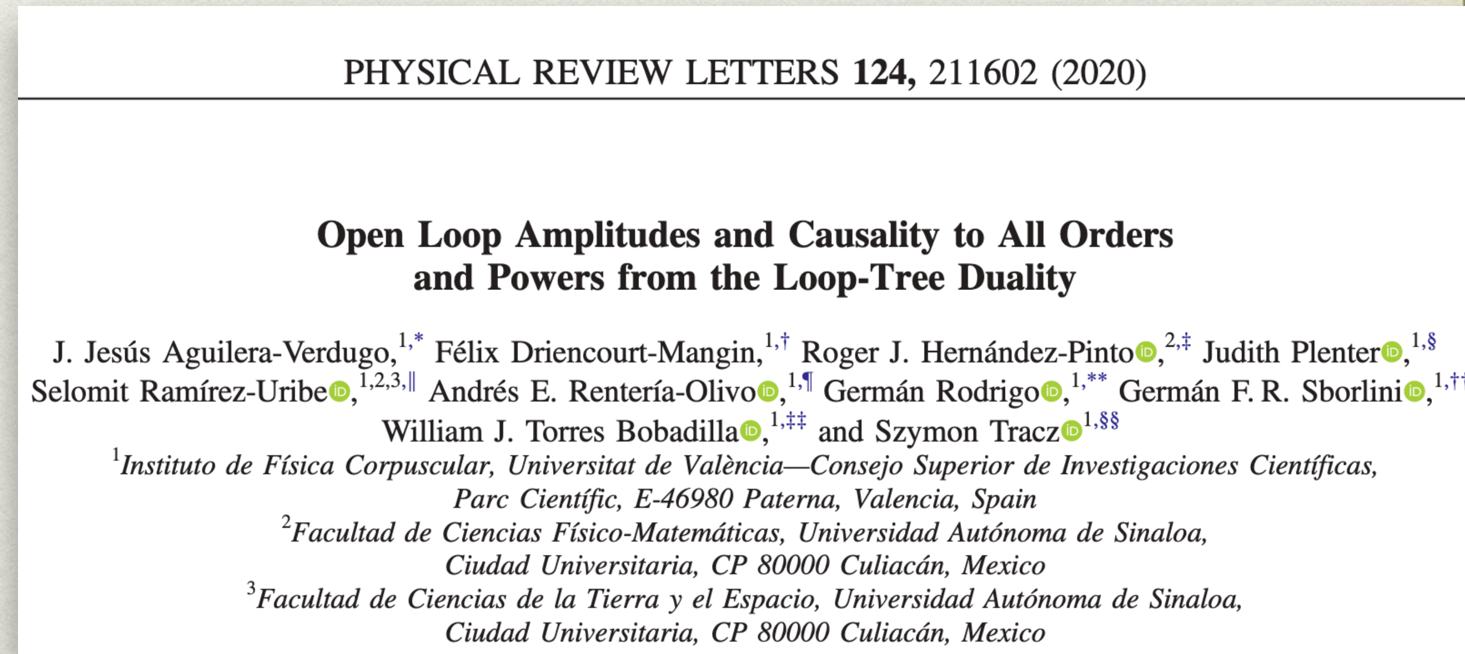
- Any multi-loop Feynman integral contains S sets of internal propagators

$$q_{i_S} = \ell_S + k_{i_S}\,, \quad i_S \in S$$

loop momenta — linear combination of external momenta

e.g. @ 2L





PHYSICAL REVIEW LETTERS **124,** 211602 (2020)

Open Loop Amplitudes and Causality to All Orders and Powers from the Loop-Tree Duality

J. Jesús Aguilera-Verdugo,[1,*] Félix Driencourt-Mangin,[1,†] Roger J. Hernández-Pinto,[2,‡] Judith Plenter,[1,§] Selomit Ramírez-Uribe,[1,2,3,∥] Andrés E. Rentería-Olivo,[1,¶] Germán Rodrigo,[1,**] Germán F. R. Sborlini,[1,††] William J. Torres Bobadilla,[1,‡‡] and Szymon Tracz[1,§§]
[1]*Instituto de Física Corpuscular, Universitat de València—Consejo Superior de Investigaciones Científicas, Parc Científic, E-46980 Paterna, Valencia, Spain*
[2]*Facultad de Ciencias Físico-Matemáticas, Universidad Autónoma de Sinaloa, Ciudad Universitaria, CP 80000 Culiacán, Mexico*
[3]*Facultad de Ciencias de la Tierra y el Espacio, Universidad Autónoma de Sinaloa, Ciudad Universitaria, CP 80000 Culiacán, Mexico*

- Feynman propagators

$$G_F(q_{i_S}) = \frac{1}{q_{i_S}^2 - m_{i_S}^2 + \imath 0} = \frac{1}{q_{i_S,0}^2 - (q_{i_S,0}^{(+)})^2}$$

In terms of spatial components

$$q_{i_S,0}^{(+)} = +\sqrt{\mathbf{q}_{i_S}^2 + m_{i_S}^2 - \imath 0}$$

Pull out full dependence of the energy components of loop momenta

usual Feynman ı0 prescription!

- *Let's now apply the Cauchy residue thm for each "energy" integration*

Multi-loop LTD representation

[Aguilera-Verdugo *et al* (2020)]

<< https://indico.cern.ch/event/1021090/

<<Sborlini

- LTD representation is written in terms of nested residues

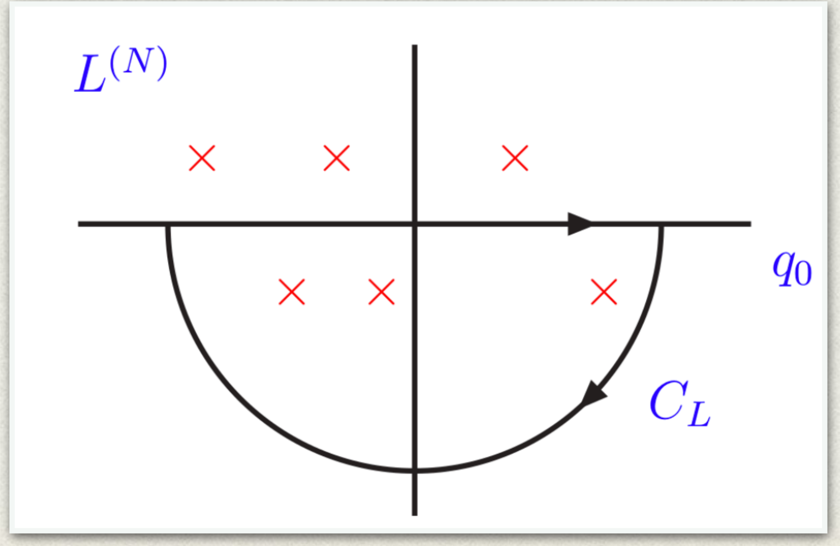
$$\mathcal{A}_D^{(L)}(1,\ldots,r;r+1,\ldots,n) \equiv -2\pi\imath \sum_{i_r\in r} \text{Res}\left(\mathcal{A}_D^{(L)}(1,\ldots,r-1;r,\ldots,n), \text{Im}(q_{i_r,0})<0\right),$$

in terms of on-shell and off-shell propagators and

$$\mathcal{A}_D^{(L)}(1;2,\ldots,n) \equiv -2\pi \sum_{i_r\in r} \text{Res}\left(d\mathcal{A}_F^{(L)}(1,\ldots,n), \text{Im}(q_{i_1,0})<0\right),$$

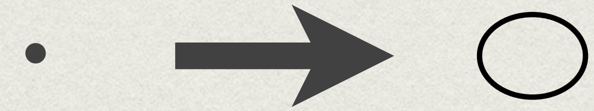
$$\mathcal{A}_F^{(L)}(1,\ldots,n) = \int_{\ell_1\cdots\ell_L} N \times G_F(1,\ldots,n)$$

- Cauchy contour is always closed from below the real axis



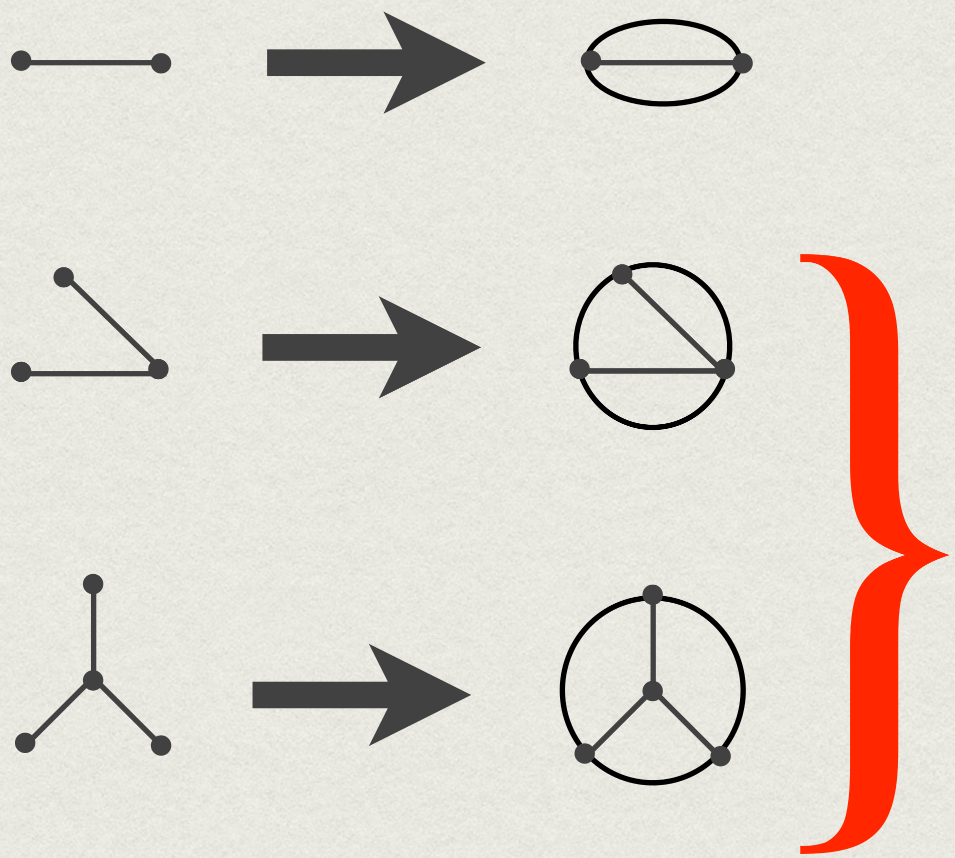
Let's recap

Everything started from 1 dot



All one-loop amplitudes under control

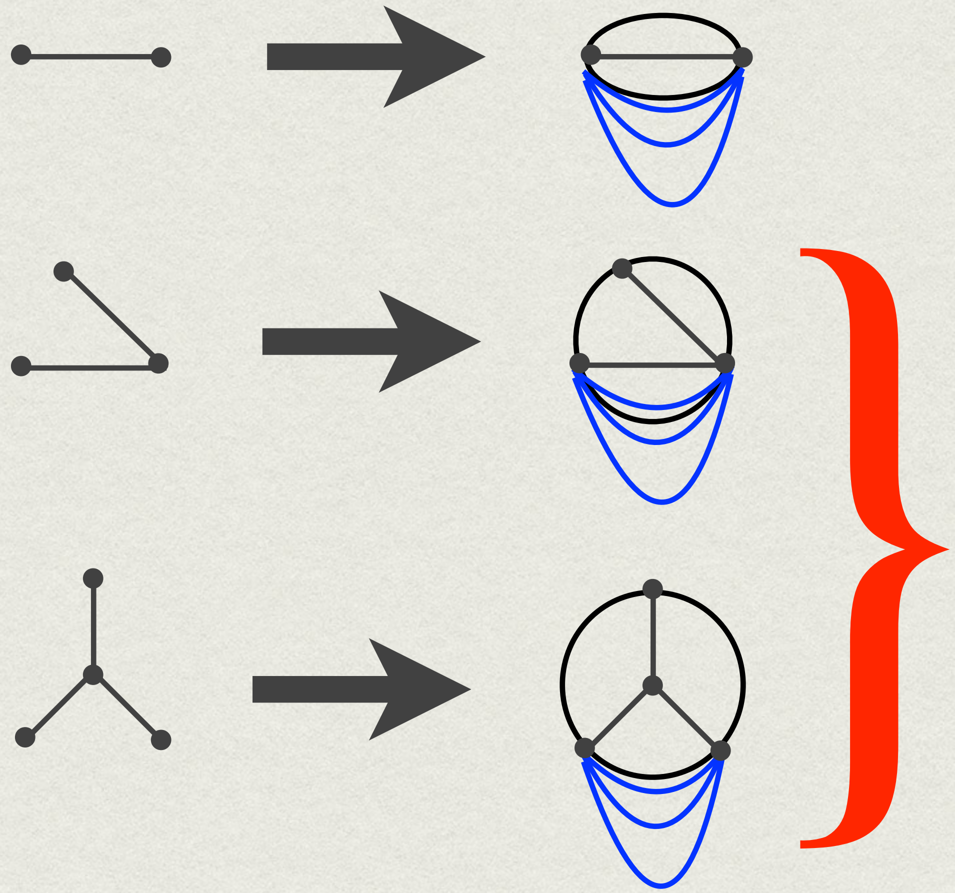
Let's recap



Two-loop amplitudes

Three-loop amplitudes

Let’s recap

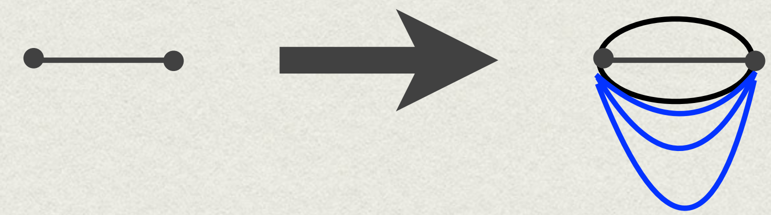


Two-loop amplitudes

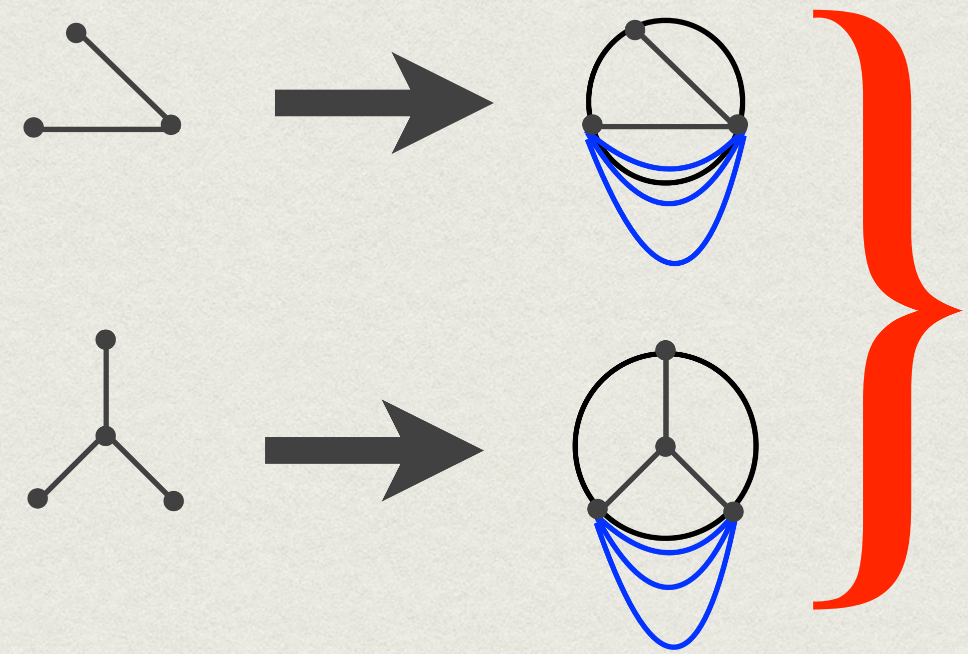
Three-loop amplitudes

Same functional structure @ L loops

Let's recap

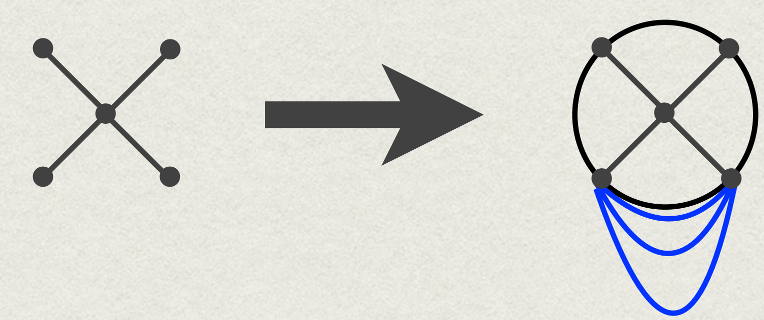


Two-loop amplitudes

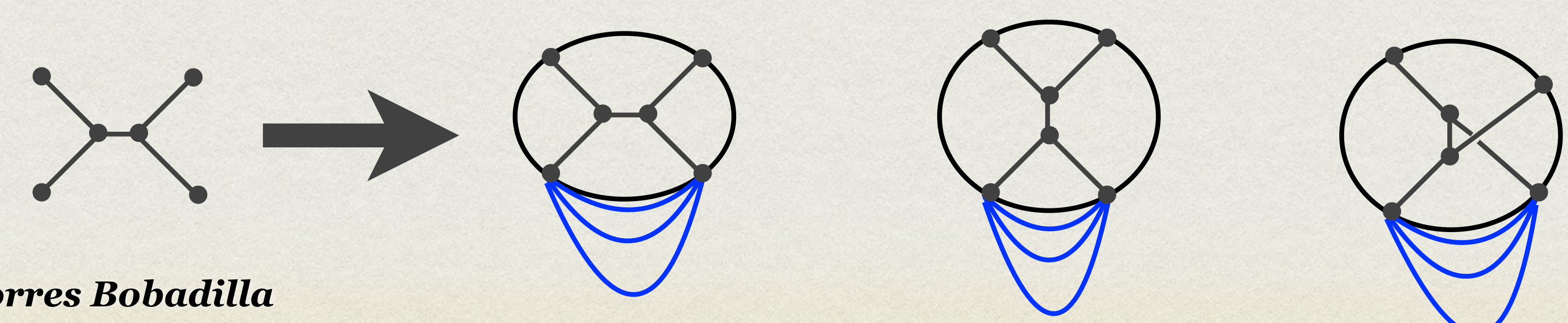


Three-loop amplitudes

Same functional structure @ L loops



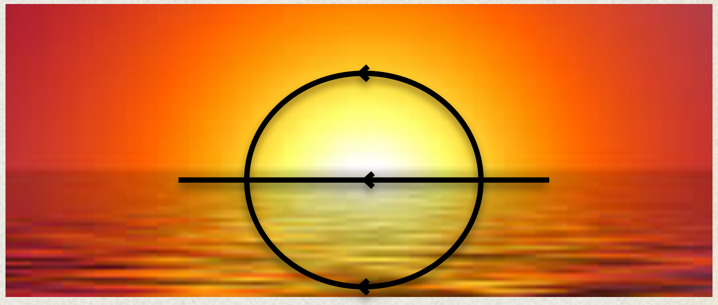
Follow the same approach @ 4 loops



Explicit applications

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

- So far —> relations among different kind of integral families
- The simplest application :: the two-loop sunrise diagram



$$\mathcal{A}_2^{(2)} = \int_{\ell_1 \ell_2} G_F(1, 2, 12) = \int_{\ell_1 \ell_2} \prod_{i=1,2,12} \frac{1}{\left(q_{i,0} - q_{i,0}^{(+)}\right)\left(q_{i,0} + q_{i,0}^{(+)}\right)} ,$$

$$q_i = \ell_i , \quad q_{12} = -\ell_1 - \ell_2 - p$$

applying the Cauchy residue thm in $\{\ell_1, \ell_2\}$

$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} \left[G_D(1, 2) + G_D\left(1, \overline{12}\right) + G_D\left(\overline{2}, \overline{12}\right)\right] ,$$

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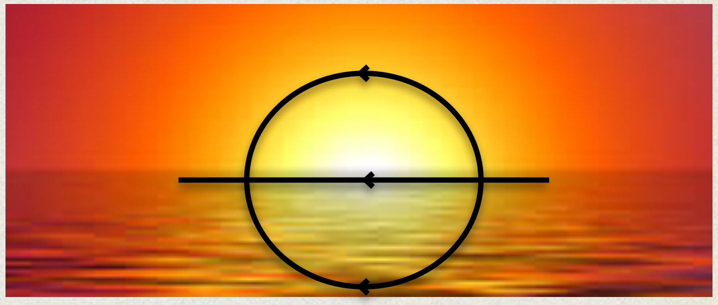

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$$G_D(1,2) = \frac{1}{4\, q_{1,0}^{(+)} q_{2,0}^{(+)} \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 + q_{12,0}^{(+)}\right)\left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 - q_{12,0}^{(+)}\right)}$$

Explicit applications

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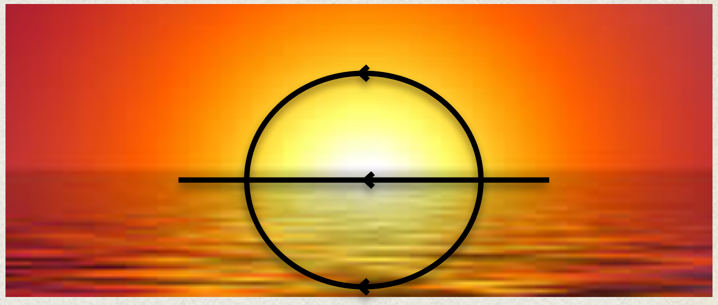

$$\mathcal{A}_2^{(2)} = \int_{\ell_1 \ell_2} G_F(1,2,12) = \int_{\ell_1 \ell_2} \prod_{i=1,2,12} \frac{1}{\left(q_{i,0} - q_{i,0}^{(+)}\right)\left(q_{i,0} + q_{i,0}^{(+)}\right)},$$

$$q_i = \ell_i\,, \quad q_{12} = -\ell_1 - \ell_2 - p$$

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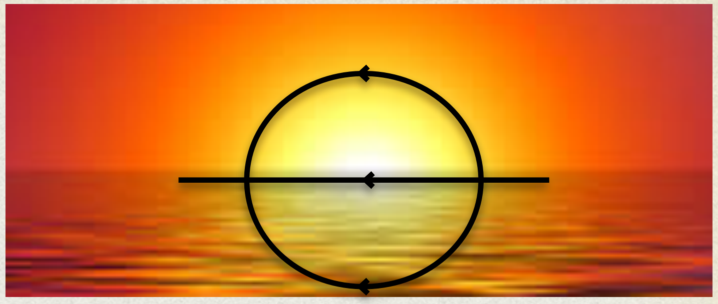
$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} \left[G_D(1,2) + G_D(1,\overline{12}) + G_D(\overline{2},\overline{12})\right],$$

$$G_D(i,j) \equiv \text{Res}\left(\text{Res}\left(G_F(1,2,12), \left\{q_{i,0} = q_{i,0}^{(+)}\right\}\right), \left\{q_{j,0} = q_{j,0}^{(+)}\right\}\right)$$

$$G_D(1,2) = \frac{1}{4\, q_{1,0}^{(+)} q_{2,0}^{(+)} \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 + q_{12,0}^{(+)}\right)\left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 - q_{12,0}^{(+)}\right)}$$

Explicit applications



- Some features with these individual residues

$$G_D(1,2) = \frac{1}{4\, q_{1,0}^{(+)} q_{2,0}^{(+)} \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} - p_0\right)\left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - q_{12,0}^{(+)} - p_0\right)}$$

Structure of $q_{i,0}^{(+)}$

$q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} = p_0$ → Causal threshold at p_0 $(p_0 > 0)$

$q_{1,0}^{(+)} + q_{2,0}^{(+)} - q_{12,0}^{(+)} = p_0$ → Non-Causal threshold
Introduces unphysical singularities

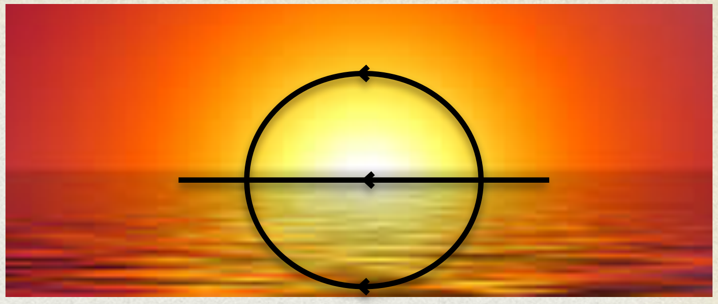
- Summing all contributions up

$$\mathcal{A}_2^{(2)} = -\int_{\vec{\ell}_1,\vec{\ell}_2} \frac{1}{8 q_{1,0}^{(+)} q_{2,0}^{(+)} q_{12,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} - p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} + p_0} \right)$$

Display causal structure only!

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

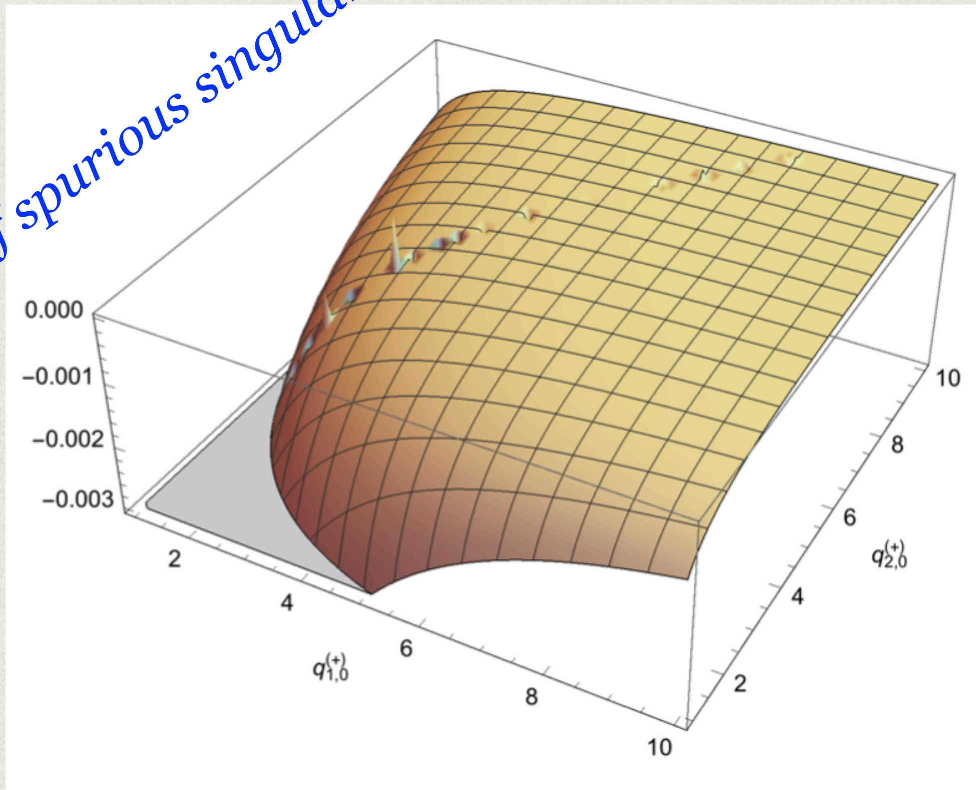
Explicit applications

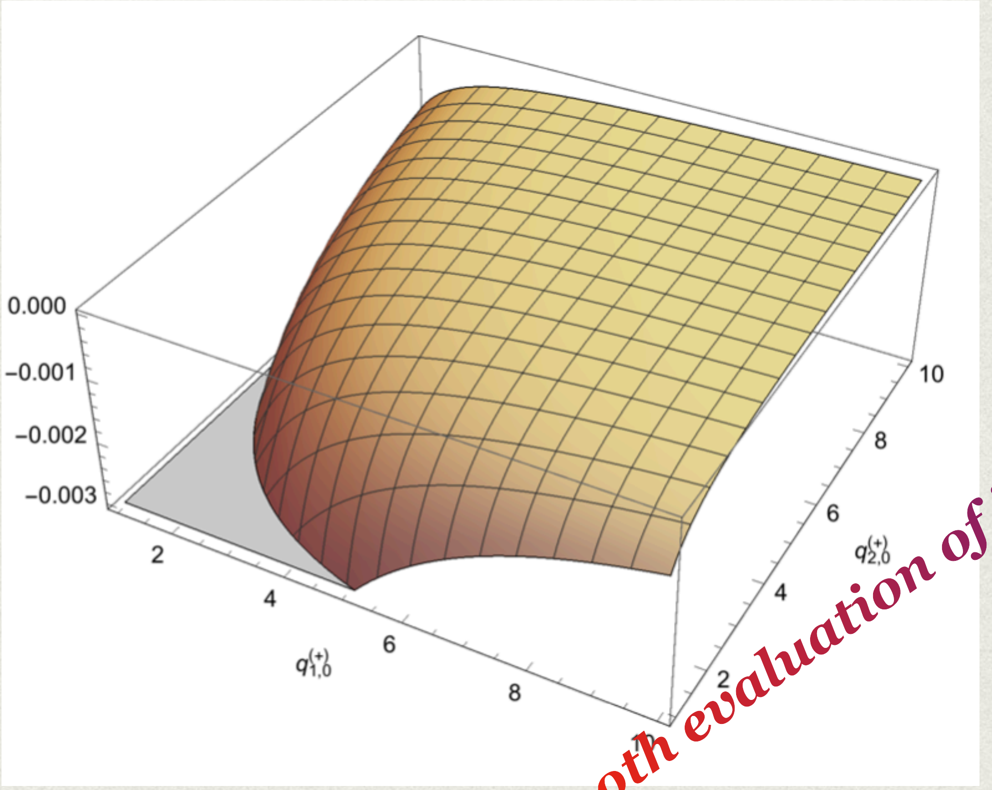


- Causal vs non-causal repr.

$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1\vec{\ell}_2} \left[G_D(1,2) + G_D\left(1,\overline{12}\right) + G_D\left(\bar{2},\overline{12}\right) \right] \,,$$

Presence of spurious singularities



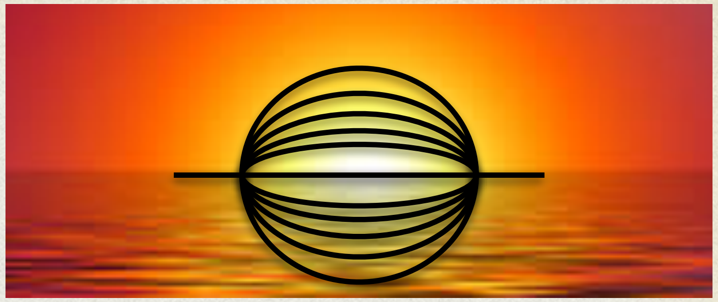


Smooth evaluation of integrands

$$\mathcal{A}_2^{(2)} = -\int_{\vec{\ell}_1,\vec{\ell}_2} \frac{1}{8q_{1,0}^{(+)}q_{2,0}^{(+)}q_{12,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)}+q_{2,0}^{(+)}+q_{12,0}^{(+)}-p_0} + \frac{1}{q_{1,0}^{(+)}+q_{2,0}^{(+)}+q_{12,0}^{(+)}+p_0} \right)$$

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

MLT causal repr.



- Sunrise @ all orders

$$q_i = \ell_i\,, \qquad \text{with } i \in \{1,\ldots,L\}\,, \qquad q_{L+1} = -\sum_{i=1}^{L} \ell_i - p_1\,.$$

Compact causal repr.

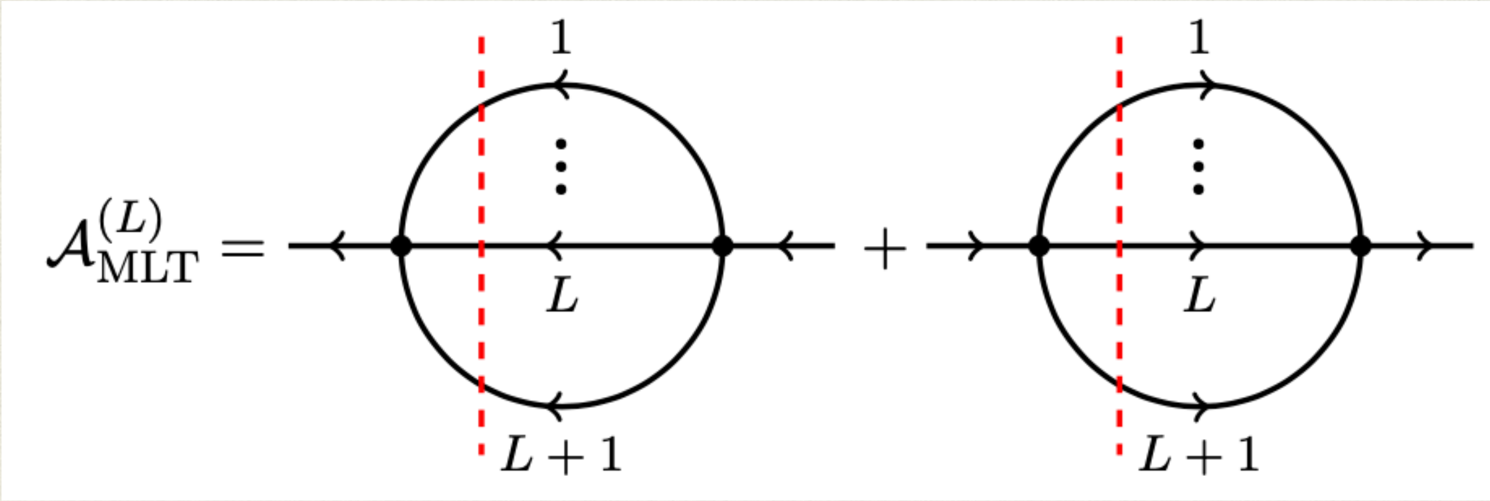
$$x_{L+k} = \prod_{i=1}^{L+k} 2q_{i,0}^{(+)}$$

$$\mathcal{A}_{\mathrm{MLT}}^{(L)}(1,2,\ldots,(L+1)_{-p_1}) = -\int_{\vec\ell_1,\cdots,\vec\ell_L} \frac{1}{x_{L+1}}\left(\frac{1}{\lambda_1^-}+\frac{1}{\lambda_1^+}\right)$$

$$\lambda_1^\pm = q_{(1,\ldots,L+1),0}^{(+)} \pm p_{1,0}$$

sum over all int. lines

- Repr. in terms of causal thresholds



$\lambda_1^+ \to$ threshold if $p_{1,0} > 0$

$\hookrightarrow q_{(1,\ldots,L+1),0}^{(+)} = p_{1,0}$

$\lambda_1^- \to$ threshold if $p_{1,0} < 0$

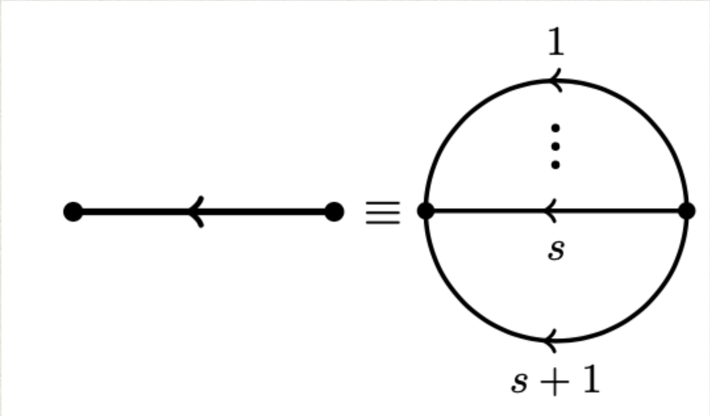
$\hookrightarrow q_{(1,\ldots,L+1),0}^{(+)} = -p_{1,0}$

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

N^kMLT causal repr.

[W.J.T. (2021)]

- Classify loop topologies



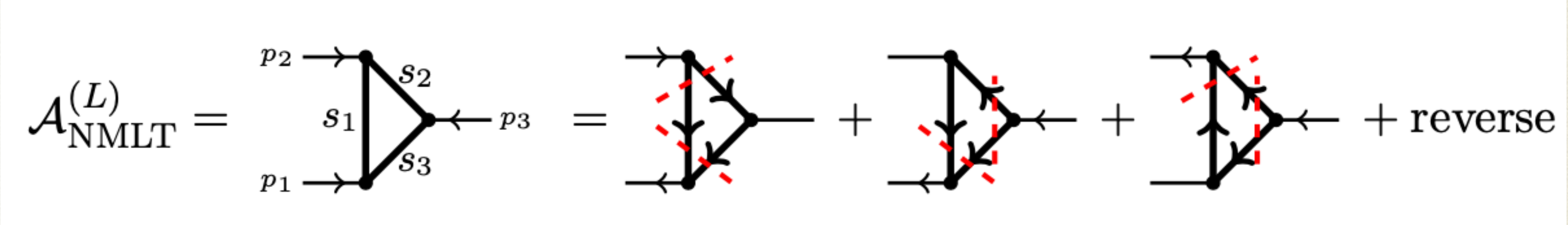
- ☆ ***Edges***
- ☆ ***Vertices***

- NMLT (3 vertices)

$$\mathcal{A}^{(L)}_{\rm NMLT} = \int_{\ell_1,\ldots,\ell_L} \frac{1}{x_{L+2}} \left(\frac{1}{\lambda_1^+ \lambda_2^-} + \frac{1}{\lambda_1^+ \lambda_3^-} + \frac{1}{\lambda_2^+ \lambda_3^-} + (\lambda_i^+ \leftrightarrow \lambda_i^-) \right)$$

$$\lambda_1^\pm = q^{(+)}_{(1,3),0} \pm p_{1,0}\,,$$
$$\lambda_2^\pm = q^{(+)}_{(1,2),0} \pm p_{2,0}\,,$$
$$\lambda_3^\pm = q^{(+)}_{(2,3),0} \pm p_{3,0}$$

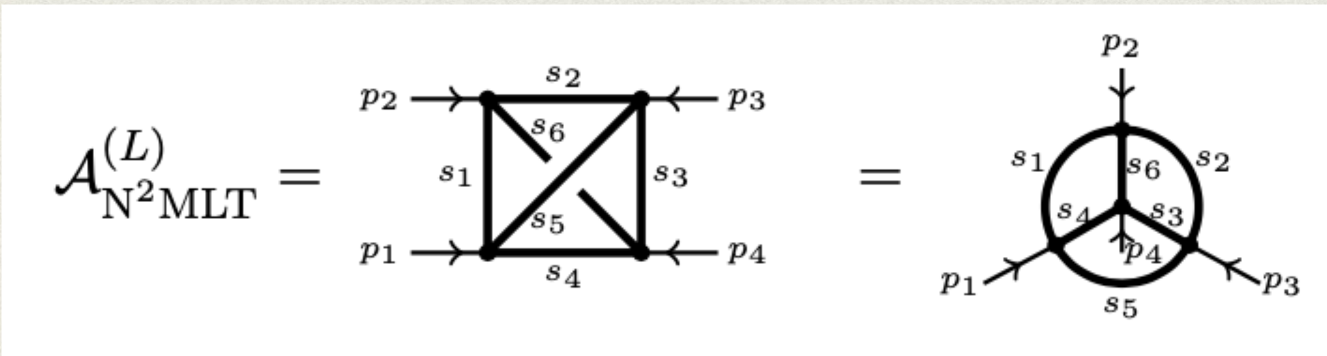
Two entangled causal thresholds



N^kMLT causal repr.

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]
[W.J.T. (2021)]

- N^{2}MLT (4 vertices)



all connections between vertices & edges!

[von Manteuffel, Schabinger (2015)]
[Peraro (2016, 2019)]
[Klappert, Klein, Lange (2020)]

*Reconstruct integrand from numerical evaluations over **finite fields***

$$\mathcal{A}^{(L)}_{\mathrm{N^2MLT}} = -\int_{\ell_1,\ldots,\ell_L} \frac{1}{x_{L+3}} \left[\frac{1}{\lambda_{12}^+}\left(\frac{1}{\lambda_1^+}+\frac{1}{\lambda_2^+}\right)\left(\frac{1}{\lambda_3^-}+\frac{1}{\lambda_4^-}\right) + \frac{1}{\lambda_{13}^+}\left(\frac{1}{\lambda_1^+}+\frac{1}{\lambda_3^+}\right)\left(\frac{1}{\lambda_2^-}+\frac{1}{\lambda_4^-}\right) + \frac{1}{\lambda_{23}^+}\left(\frac{1}{\lambda_2^+}+\frac{1}{\lambda_3^+}\right)\left(\frac{1}{\lambda_1^-}+\frac{1}{\lambda_4^-}\right) + (\lambda_i^+ \leftrightarrow \lambda_i^-) \right]$$

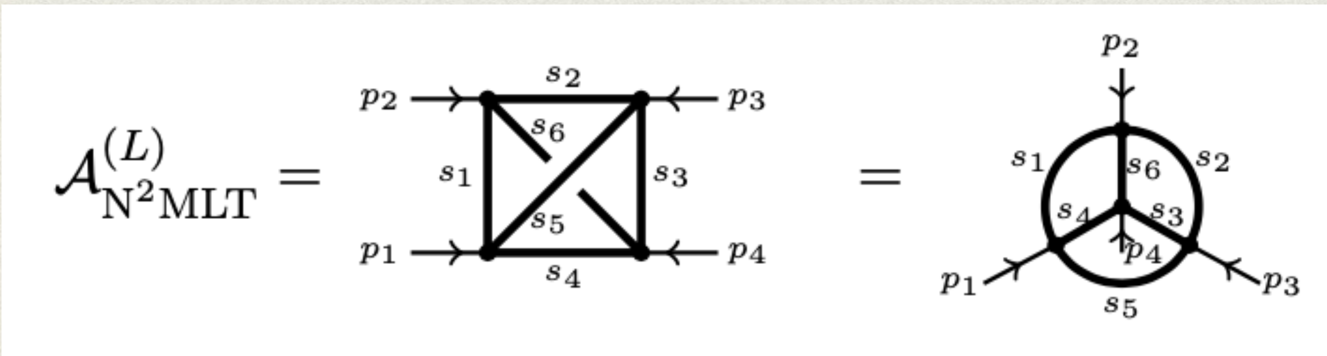
$$\lambda_1^\pm = q^{(+)}_{(1,4,5),0} \pm p_{1,0} \,,$$
$$\lambda_2^\pm = q^{(+)}_{(1,2,6),0} \pm p_{2,0} \,,$$
$$\lambda_3^\pm = q^{(+)}_{(2,3,5),0} \pm p_{3,0} \,,$$
$$\lambda_4^\pm = q^{(+)}_{(3,4,6),0} \pm p_{4,0} \,,$$
$$\lambda_{12}^\pm = q^{(+)}_{(2,4,5,6),0} \pm p_{12,0} \,,$$
$$\lambda_{13}^\pm = q^{(+)}_{(1,2,3,4),0} \pm p_{13,0} \,,$$
$$\lambda_{23}^\pm = q^{(+)}_{(1,3,5,6),0} \pm p_{23,0} \,,$$

Three entangled causal thresholds

N^kMLT causal repr.

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]
[W.J.T. (2021)]

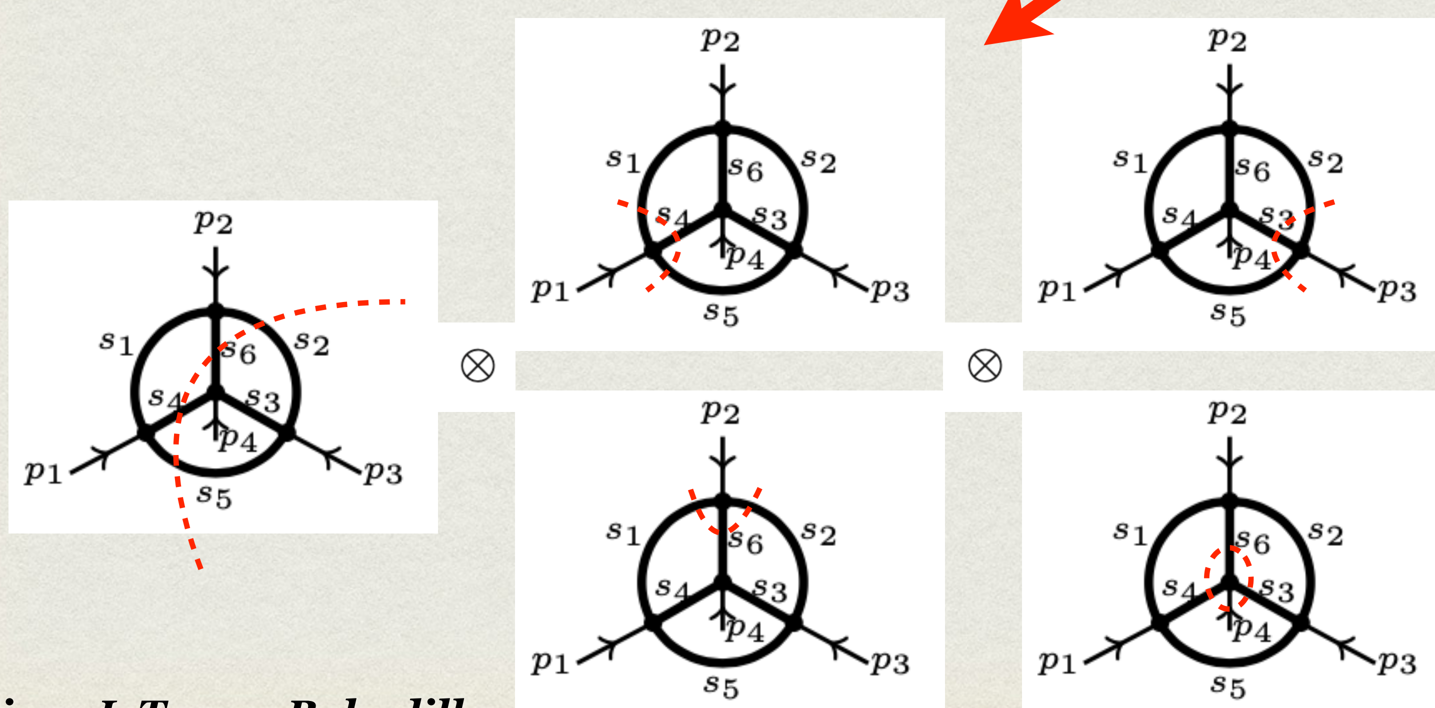
- N^2MLT (4 vertices)



all connections between vertices & edges!

$$\mathcal{A}^{(L)}_{\mathrm{N^2MLT}} = -\int_{\ell_1,\ldots,\ell_L} \frac{1}{x_{L+3}} \left[\frac{1}{\lambda_{12}^+}\left(\frac{1}{\lambda_1^+}+\frac{1}{\lambda_2^+}\right)\left(\frac{1}{\lambda_3^-}+\frac{1}{\lambda_4^-}\right) + \frac{1}{\lambda_{13}^+}\left(\frac{1}{\lambda_1^+}+\frac{1}{\lambda_3^+}\right)\left(\frac{1}{\lambda_2^-}+\frac{1}{\lambda_4^-}\right) + \frac{1}{\lambda_{23}^+}\left(\frac{1}{\lambda_2^+}+\frac{1}{\lambda_3^+}\right)\left(\frac{1}{\lambda_1^-}+\frac{1}{\lambda_4^-}\right) + (\lambda_i^+ \leftrightarrow \lambda_i^-) \right]$$

Three entangled causal thresholds



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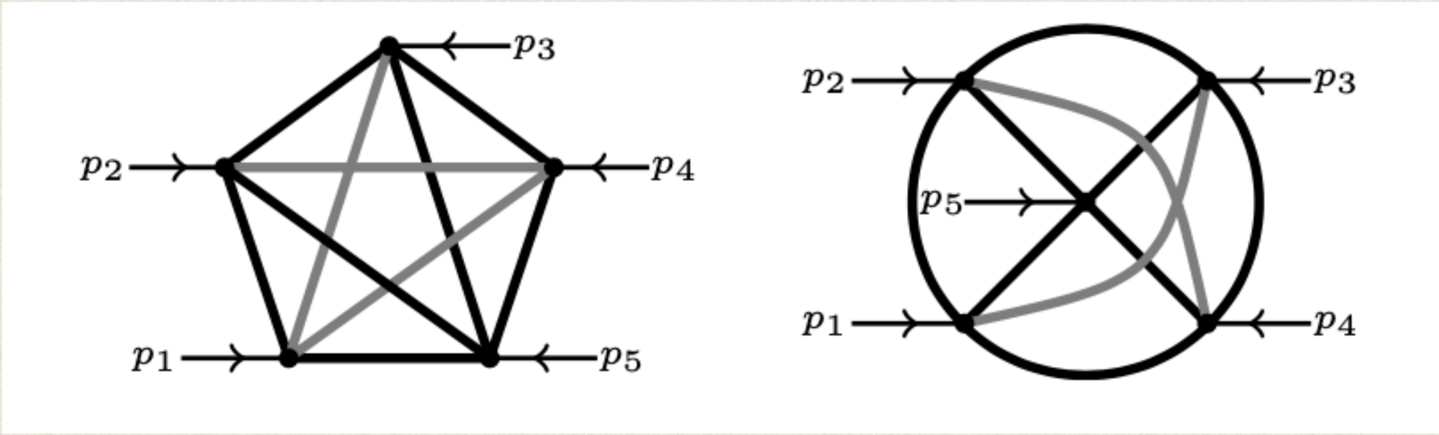
$$\lambda_{13}^\pm = q^{(+)}_{(1,2,3,4),0} \pm p_{13,0}\,,$$
$$\lambda_{23}^\pm = q^{(+)}_{(1,3,5,6),0} \pm p_{23,0}\,,$$

N^kMLT causal repr.

[W.J.T. (2021)]

Four entangled causal thresholds

- N^{3}MLT (5 vertices)



$$q_i = \begin{cases} q_i = \ell_i, & i = 1, \ldots, 6 \\ \ell_1 - \ell_4 - \ell_5 - p_1, & i = 7 \\ \ell_2 - \ell_1 - \ell_6 - p_2, & i = 8 \\ \ell_3 - \ell_2 + \ell_5 - p_3, & i = 9 \\ \ell_4 - \ell_3 + \ell_6 - p_4 & i = 10 \end{cases}$$

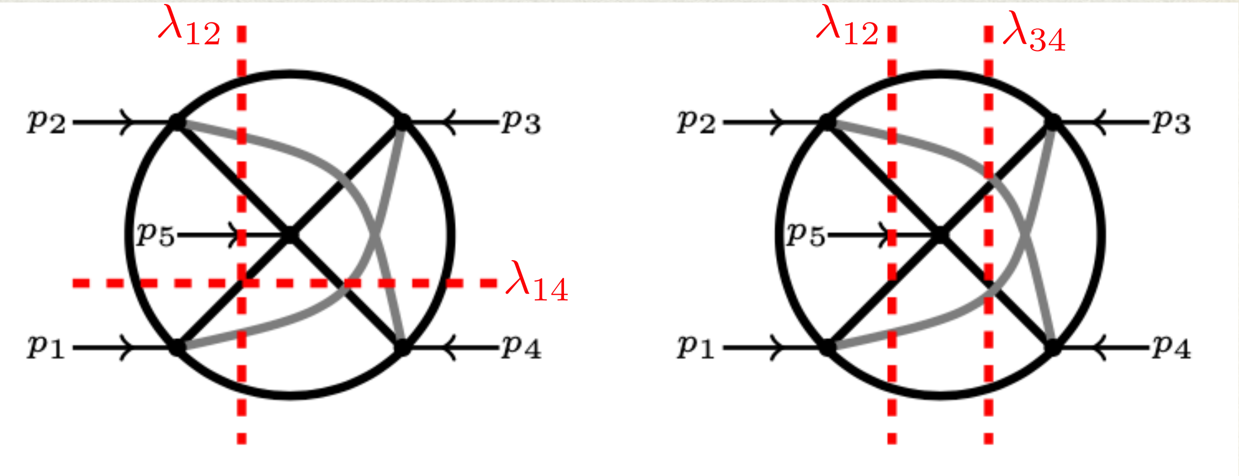
Direct application of LTD

$$\begin{aligned}
\lambda_1^\pm &= q^{(+)}_{(1,4,5,7),0} \pm p_{1,0}\,, & \lambda_{12}^\pm &= q^{(+)}_{(2,4,5,6,7,8),0} \pm p_{12,0}\,, & \lambda_{24}^\pm &= q^{(+)}_{(1,2,3,4,8,10),0} \pm p_{24,0}\,, \\
\lambda_2^\pm &= q^{(+)}_{(1,2,6,8),0} \pm p_{2,0}\,, & \lambda_{13}^\pm &= q^{(+)}_{(1,2,3,4,7,9),0} \pm p_{13,0}\,, & \lambda_{35}^\pm &= q^{(+)}_{(2,3,5,7,8,10),0} \pm p_{35,0}\,, \\
\lambda_3^\pm &= q^{(+)}_{(2,3,5,9),0} \pm p_{3,0}\,, & \lambda_{23}^\pm &= q^{(+)}_{(1,3,5,6,8,9),0} \pm p_{23,0}\,, & \lambda_{34}^\pm &= q^{(+)}_{(2,4,5,6,9,10),0} \pm p_{34,0}\,, \\
\lambda_4^\pm &= q^{(+)}_{(3,4,6,10),0} \pm p_{4,0}\,, & \lambda_{45}^\pm &= q^{(+)}_{(3,4,6,7,8,9),0} \pm p_{45,0}\,, & \lambda_{25}^\pm &= q^{(+)}_{(1,2,6,7,9,10),0} \pm p_{25,0}\,, \\
\lambda_5^\pm &= q^{(+)}_{(7,8,9,10),0} \pm p_{5,0}\,, & \lambda_{14}^\pm &= q^{(+)}_{(1,3,5,6,7,10),0} \pm p_{14,0}\,, & \lambda_{15}^\pm &= q^{(+)}_{(1,4,5,8,9,10),0} \pm p_{15,0}\,.
\end{aligned}$$

Reconstruct integrand
—> finite fields

$$\mathcal{A}^{(L)}_{\mathrm{N^3MLT}} = \int_{\ell_1,\ldots,\ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1 \\ j=i+1}}^{5} \sum_{\substack{k=1 \\ l=k+1 \\ k,l\neq i,j}}^{5} L^+_{ij}\, L^-_{kl}\,.$$

$$L^\pm_{ij} = \frac{1}{\lambda^\pm_{ij}} \left(\frac{1}{\lambda^\pm_i} + \frac{1}{\lambda^\pm_j} \right).$$

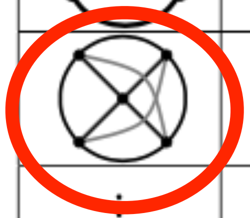
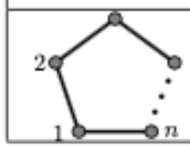


Overlapped *vs* *Entangled*

N^kMLT causal repr.

[W.J.T. (2021)]

- In general

	Vertices	Edges	Loops	dim $\lambda_i^\pm$
	2	1	0	1
	3	3	1	3
	4	6	3	7
	5	10	6	15
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	n	$\binom{n}{2}$	$\frac{(n-1)(n-2)}{2}$	$2^{n-1}-1$

☆ ***Edges***

☆ ***Vertices***

All subsets of up-to [n/2] elements

$$\vec{\lambda}_i^\pm = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

intersection of subsets —> $\emptyset$

$$\{\{\{1,2\},\{3,4\}\},\{\{1,2\},\{3,5\}\},\{\{1,2\},\{4,5\}\},\{\{1,3\},\{2,4\}\},\{\{1,3\},\{2,5\}\},\{\{1,3\},\{4,5\}\},\{\{1,4\},\{2,3\}\},\{\{1,4\},\{2,5\}\},\{\{1,4\},\{3,5\}\},\{\{1,5\},\{2,3\}\},\{\{1,5\},\{2,4\}\},\{\{1,5\},\{3,4\}\},\{\{2,3\},\{4,5\}\},\{\{2,4\},\{3,5\}\},\{\{2,5\},\{3,4\}\}\},$$

$$\hookrightarrow \{\{i,j\},\{k,l\}\} \to L_{ij}^+ L_{kl}^- + L_{ij}^- L_{kl}^+$$

Same causal repr. w/out LTD !

summing over all contributions

$$d\mathcal{A}^{(L)}_{\mathrm{N^3MLT}} \sim \sum_{\substack{i=1\\ j=i+1}}^{5} \sum_{\substack{k=1\\ l=k+1\\ k,l\neq i,j}}^{5} L_{ij}^+ L_{kl}^- \,.$$

N^kMLT causal repr.

[W.J.T. (2021)]

- In general

loop topology made of ***k+2*** *vertices —> N^kMLT*

	Vertices	Edges	Loops	dim $\lambda_i^{\pm}$
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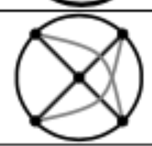
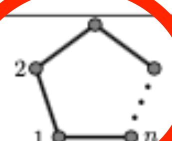
	5	10	6	15
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	n	$\binom{n}{2}$	$\frac{(n-1)(n-2)}{2}$	$2^{n-1}-1$

- ***Edges***
- ***Vertices***

All-loop causal representation

$$d\mathcal{A}_{L+k+1}^{(L)} = \frac{(-1)^{k+1}}{x_{L+k+1}} \mathcal{F}_{L+k+1}\left(\lambda_i^{\pm}\right) \, .$$

N^kMLT causal repr.

[W.J.T. (2021)]

- In general

*loop topology made of **k+2** vertices —> N^kMLT*

	Vertices	Edges	Loops	dim $\lambda_i^{\pm}$
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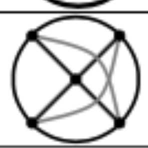
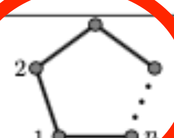
	5	10	6	15
⋮	⋮	⋮	⋮	⋮
	n	$\binom{n}{2}$	$\frac{(n-1)(n-2)}{2}$	$2^{n-1}-1$

- ***Edges***
- ***Vertices***

universal structure $x_{L+k} = \prod_{i=1}^{L+k} 2q_{i,0}^{(+)}$

depends on # lines

All-loop causal representation

$$d\mathcal{A}_{L+k+1}^{(L)} = \frac{(-1)^{k+1}}{x_{L+k+1}} \mathcal{F}_{L+k+1}\left(\lambda_i^{\pm}\right) .$$

N^kMLT causal repr.

[W.J.T. (2021)]

- In general

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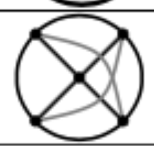	5	10	6	15
⋮	⋮	⋮	⋮	⋮
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- ☆ ***Edges***
- ☆ ***Vertices***

universal structure $x_{L+k} = \prod_{i=1}^{L+k} 2q_{i,0}^{(+)}$

depends on # lines

All-loop causal representation

$$dA_{L+k+1}^{(L)} = \frac{(-1)^{k+1}}{x_{L+k+1}} \mathcal{F}_{L+k+1}\left(\lambda_i^{\pm}\right).$$

$$\mathcal{F}_{L+k} = \sum_{\substack{i_1 \ll i_{N_i} \\ j_1 \ll j_{N_j}}}^{k+2} \Omega_{\vec{i}}^{\vec{j}} \, L^+_{i_1 i_2 \dots i_{N_i}} L^-_{j_1 j_2 \dots j_{N_j}} ,$$

$$\Omega_{\vec{i}}^{\vec{j}} = \begin{cases} 1 & \text{If } \vec{i} \cap \vec{j} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

causal thresholds cannot overlap

$$L^{\pm}_{i_1 i_2 \dots i_N} = \frac{1}{\lambda^{\pm}_{i_1 i_2 \dots i_N}} \sum_{\substack{j_1 \ll j_{N-1} \\ \vec{j} \subset \vec{i}}} L^{\pm}_{j_1 j_2 \dots j_{N-1}} , \qquad L^{\pm}_{i_1} = \frac{1}{\lambda^{\pm}_{i_1}} .$$

All-loop order & multiplicity

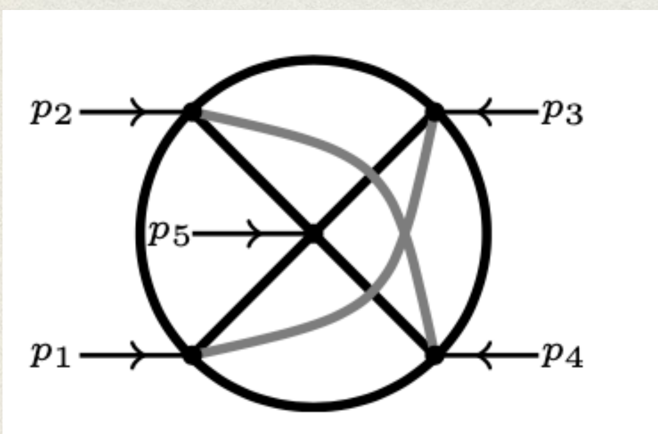
Removing procedure

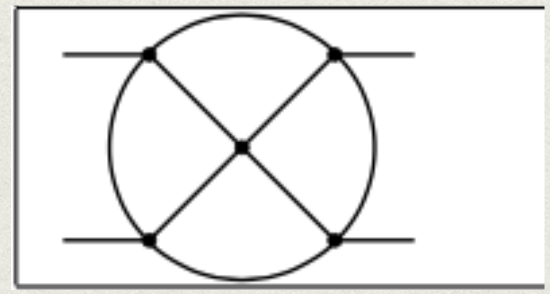
[W.J.T. (2021)]

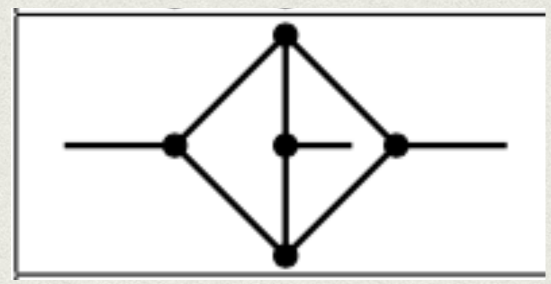
$$\mathcal{A}^{(L)}_{\mathrm{N^3MLT}} = \int_{\ell_1,\ldots,\ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1\\ j=i+1}}^{5} \sum_{\substack{k=1\\ l=k+1\\ k,l\neq i,j}}^{5} L^+_{ij}\, L^-_{kl} \,.$$

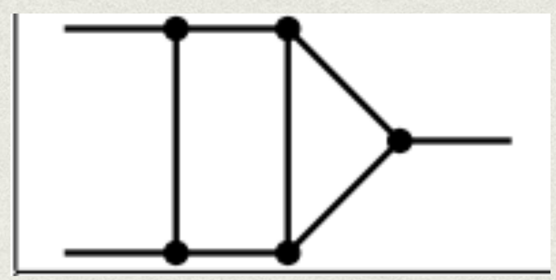
$$L^\pm_{ij} = \frac{1}{\lambda^\pm_{ij}} \left(\frac{1}{\lambda^\pm_i} + \frac{1}{\lambda^\pm_j} \right) .$$

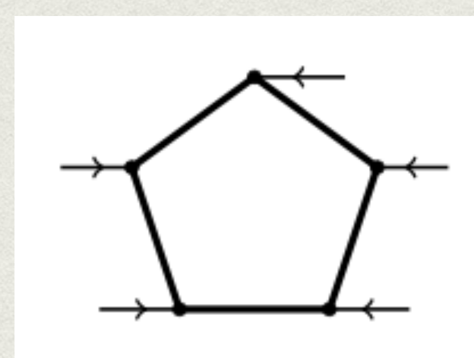
Compute it once and for all











- ☆ ***Equal # Vertices***
- ☆ ***Less # Edges***

$$\lambda^\pm_1 = q^{(+)}_{(1,4,5,7),0} \pm p_{1,0} \,,$$

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$$\lambda^\pm_3 = q^{(+)}_{(2,3,5,9),0} \pm p_{3,0} \,,$$

$$\lambda^\pm_4 = q^{(+)}_{(3,4,6,10),0} \pm p_{4,0} \,,$$

$$\lambda^\pm_5 = q^{(+)}_{(7,8,9,10),0} \pm p_{5,0} \,,$$

Get causal repr. for free!

All-loop order & multiplicity

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[W.J.T. (2021)]

$$\mathcal{A}^{(L)}_{\mathrm{N^3MLT}} = \int_{\ell_1,\ldots,\ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1\\ j=i+1}}^{5} \sum_{\substack{k=1\\ l=k+1\\ k,l\neq i,j}}^{5} L^+_{ij}\, L^-_{kl} \,.$$

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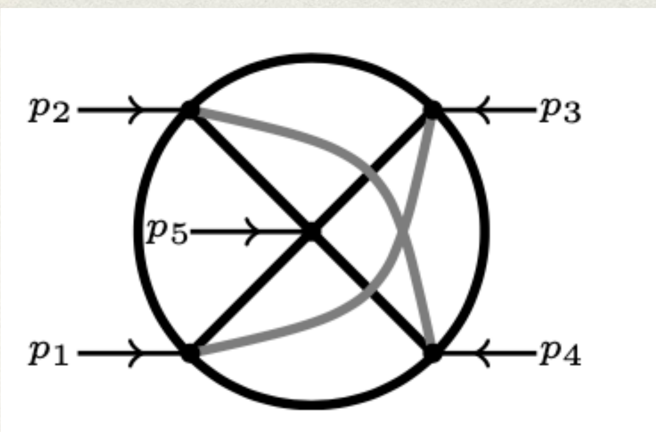

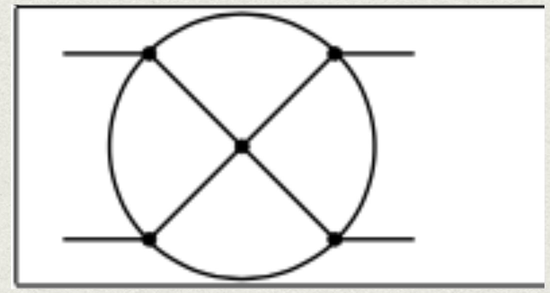

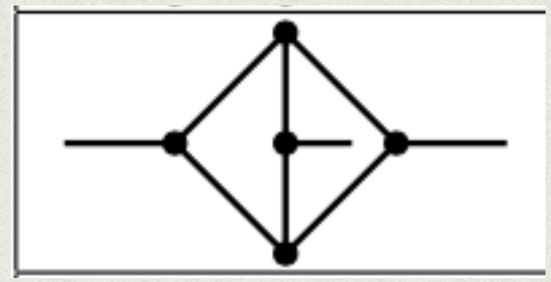

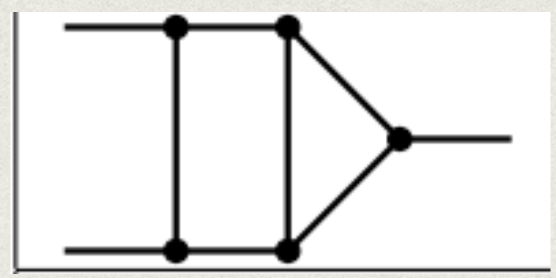

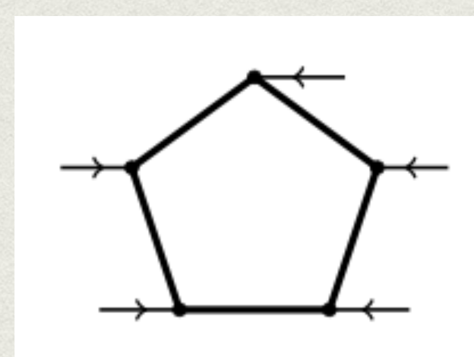

- ☆ ***Equal # Vertices***
- ☆ ***Less # Edges***

$$d\mathcal{A}^{(L-|s_i|)}_{\mathrm{N^kMLT}} = \lim_{q^{(+)}_{s_i}\to 0} 2q^{(+)}_{s_i}\, d\mathcal{A}^{(L)}_{\mathrm{N^kMLT}} \,,$$

$$\lambda^{\pm}_1 = q^{(+)}_{(1,4,5,7),0} \pm p_{1,0} \,,$$

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All-loop order & multiplicity

Collapsing procedure

[W.J.T. (2021)]

$$\mathcal{A}^{(L)}_{\mathrm{N^3MLT}} = \int_{\ell_1,\dots,\ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1\\ j=i+1}}^{5} \sum_{\substack{k=1\\ l=k+1\\ k,l\neq i,j}}^{5} L^+_{ij}\, L^-_{kl}\,.$$

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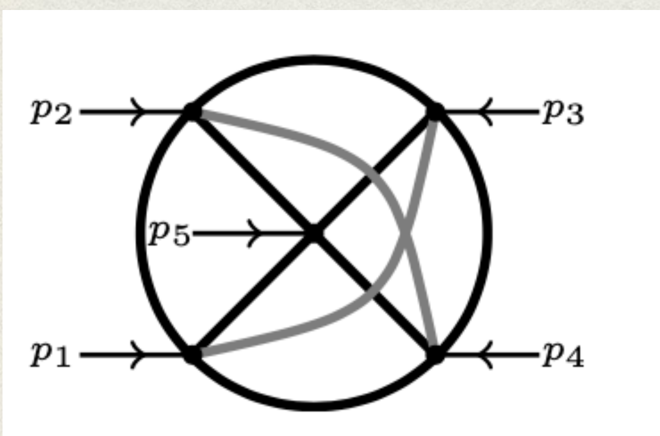
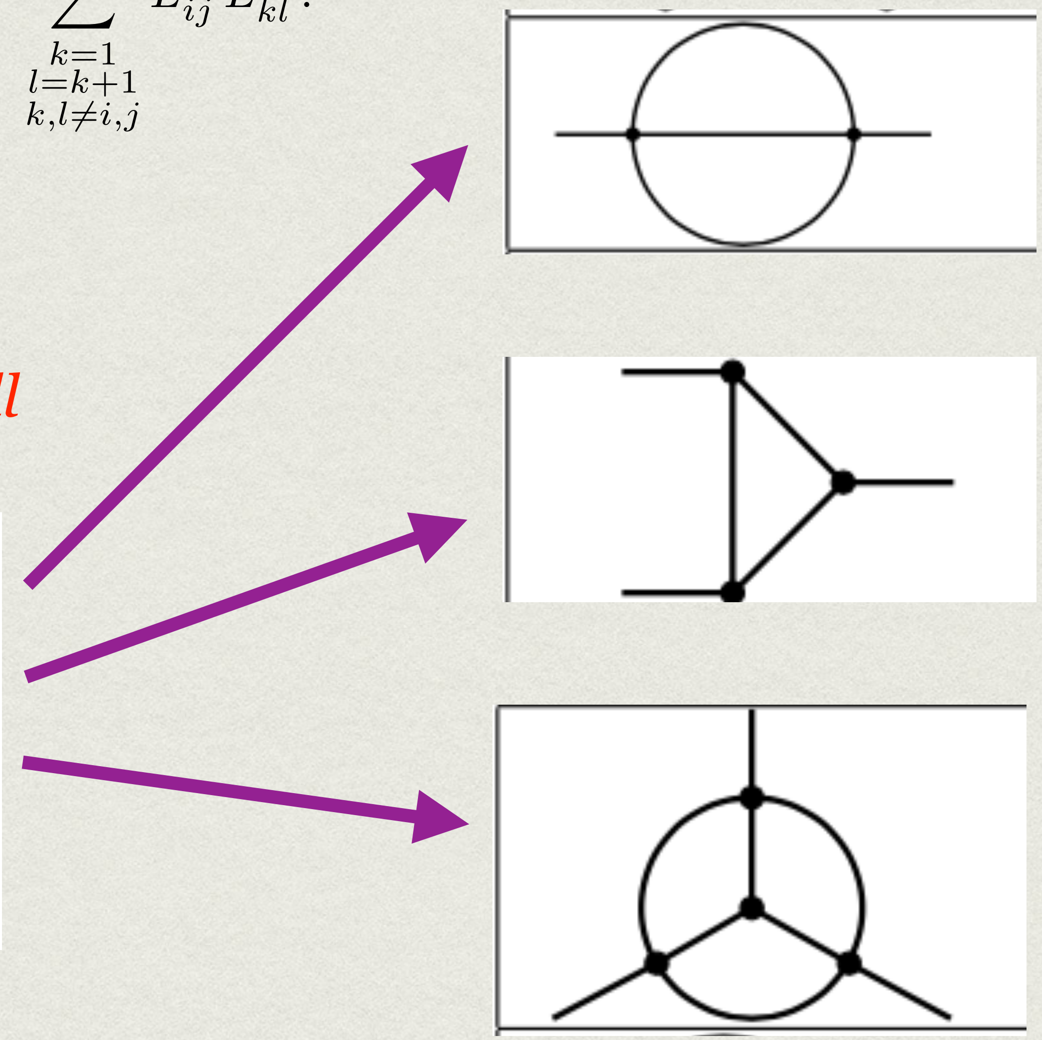
Compute it once and for all





- ***Less # Vertices***
- ***Less # Edges***

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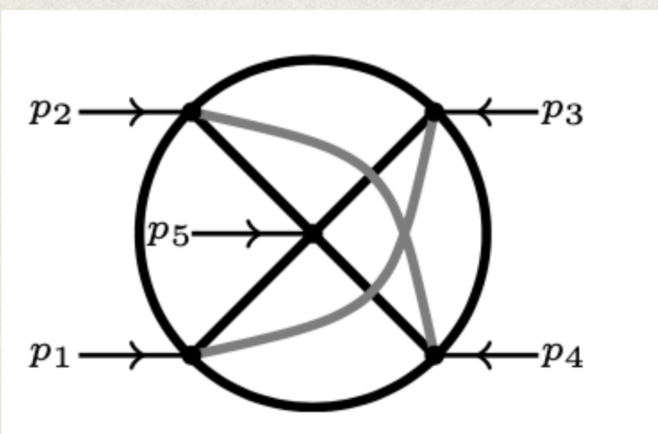

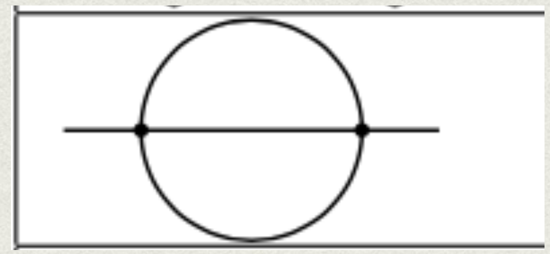

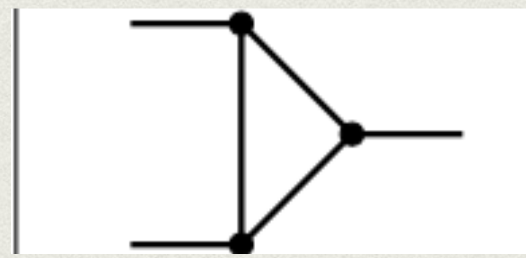

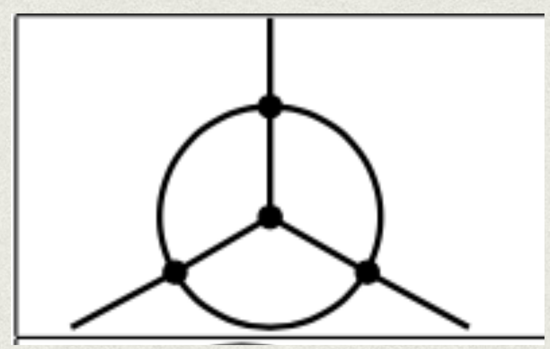

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$$\lambda^{\pm}_4 = q^{(+)}_{(3,4,6,10),0} \pm p_{4,0} \,,$$
$$\lambda^{\pm}_5 = q^{(+)}_{(7,8,9,10),0} \pm p_{5,0} \,,$$

Get causal repr. for free!

$$d\mathcal{A}_{\mathrm{N^2MLT}} = -\frac{1}{2} \lim_{s_7,s_8,s_9,s_{10}\to 0} \prod_{i\in s_7\cup s_8\cup s_9\cup s_{10}} 2q^{(+)}_{i,0} \lim_{\lambda^{\pm}_5\to 0} \lambda^{\pm}_5 \, d\mathcal{A}_{\mathrm{N^3MLT}} \,.$$

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

What about numerators?

$$\mathcal{A}_F^{(L)}(1,\ldots,n) = \int_{\ell_1\cdots\ell_L} N \times G_F(1,\ldots,n)$$

make use of poly. div. —> scalar integrands

"pull out" energies

$$q_i \cdot q_j = q_i^{(+)} \cdot q_j^{(+)} - q_{i,0}^{(+)} q_{j,0}^{(+)} + q_{i,0} q_{j,0}\,,$$

$$\hookrightarrow N = \sum_{i=1}^{R} c_i \prod_{j=1}^{L} \ell_{j,0}^{\alpha_{ij}}\,,$$

☑ *Application: four gluons @2L*

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

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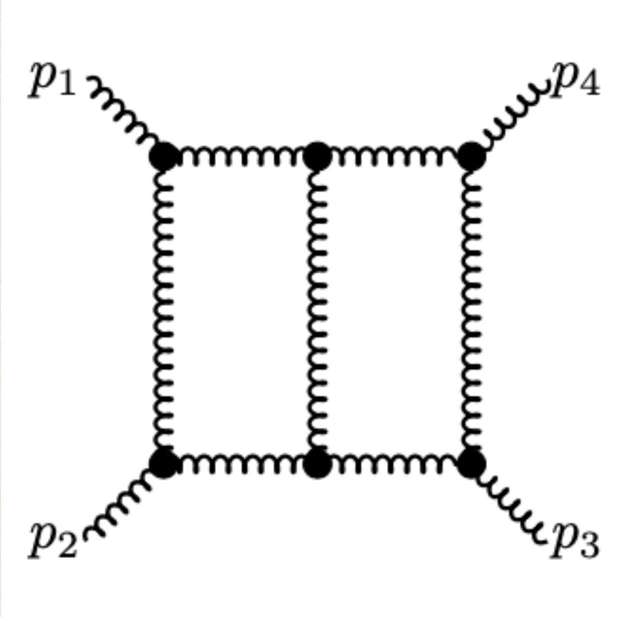

$$N = \sum_{r,s}^{\text{rank}} c_{rs}\,\ell_{1,0}^r \ell_{2,0}^s\,, \qquad \text{with } r,s\le 4 \wedge r+s\le 6\,.$$

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

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$$\mathcal{A}_F^{(L)}(1,\ldots,n) = \int_{\ell_1\cdots\ell_L} N \times G_F(1,\ldots,n)$$

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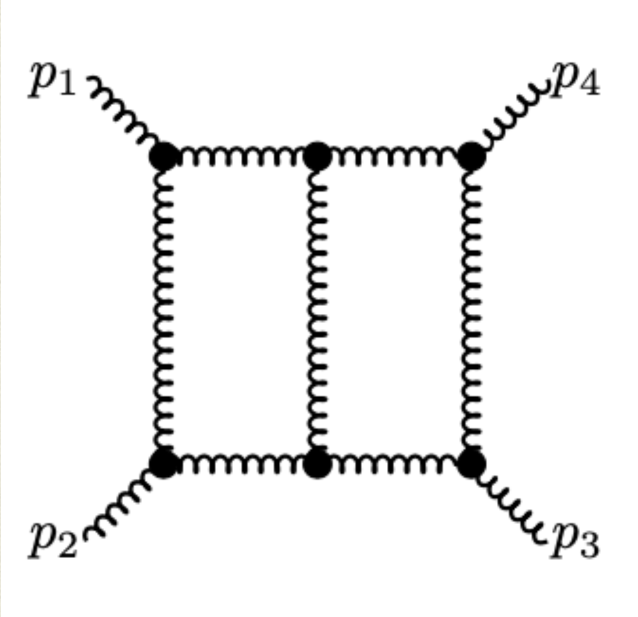


$$N = \sum_{r,s}^{\text{rank}} c_{rs}\, \ell_{1,0}^r \ell_{2,0}^s\,, \qquad \text{with } r,s \le 4 \wedge r+s \le 6\,.$$

$$\begin{aligned}
&= \tilde{c}_8 J_{\{0,0,1,0,0,1,1\}} + \tilde{c}_7 J_{\{0,0,1,0,1,0,1\}} + \tilde{c}_{27} J_{\{0,0,1,0,1,1,1\}} + \tilde{c}_9 J_{\{0,0,1,1,0,1,1\}} \\
&\quad + \tilde{c}_{22} J_{\{0,0,1,1,1,1,1\}} + \tilde{c}_3 J_{\{0,1,0,0,0,1,1\}} + \tilde{c}_4 J_{\{0,1,0,0,1,0,1\}} + \tilde{c}_{25} J_{\{0,1,0,0,1,1,1\}} \\
&\quad + \tilde{c}_{18} J_{\{0,1,0,1,1,1,1\}} + \tilde{c}_{13} J_{\{0,1,1,0,0,1,1\}} + \tilde{c}_{14} J_{\{0,1,1,0,1,0,1\}} + \tilde{c}_2 J_{\{0,1,1,0,1,1,0\}} \\
&\quad + \tilde{c}_{29} J_{\{0,1,1,0,1,1,1\}} + \tilde{c}_{19} J_{\{0,1,1,1,0,1,1\}} + \tilde{c}_{17} J_{\{0,1,1,1,1,1,0\}} + \tilde{c}_{31} J_{\{0,1,1,1,1,1,1\}} \\
&\quad + \tilde{c}_6 J_{\{1,0,0,0,0,1,1\}} + \tilde{c}_5 J_{\{1,0,0,0,1,0,1\}} + \tilde{c}_{26} J_{\{1,0,0,0,1,1,1\}} + \tilde{c}_{16} J_{\{1,0,1,0,0,1,1\}} \\
&\quad + \tilde{c}_{15} J_{\{1,0,1,0,1,0,1\}} + \tilde{c}_1 J_{\{1,0,1,0,1,1,0\}} + \tilde{c}_{28} J_{\{1,0,1,0,1,1,1\}} + \tilde{c}_{20} J_{\{1,0,1,1,0,1,1\}} \\
&\quad + \tilde{c}_{23} J_{\{1,0,1,1,1,1,1\}} + \tilde{c}_{12} J_{\{1,1,1,0,0,1,1\}} + \tilde{c}_{11} J_{\{1,1,1,0,1,0,1\}} + \tilde{c}_{10} J_{\{1,1,1,0,1,1,0\}} \\
&\quad + \tilde{c}_{30} J_{\{1,1,1,0,1,1,1\}} + \tilde{c}_{21} J_{\{1,1,1,1,0,1,1\}} + \tilde{c}_{24} J_{\{1,1,1,1,1,1,1\}}
\end{aligned}$$

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

What about numerators?

$$\mathcal{A}_F^{(L)}(1,\ldots,n) = \int_{\ell_1\cdots\ell_L} N \times G_F(1,\ldots,n)$$

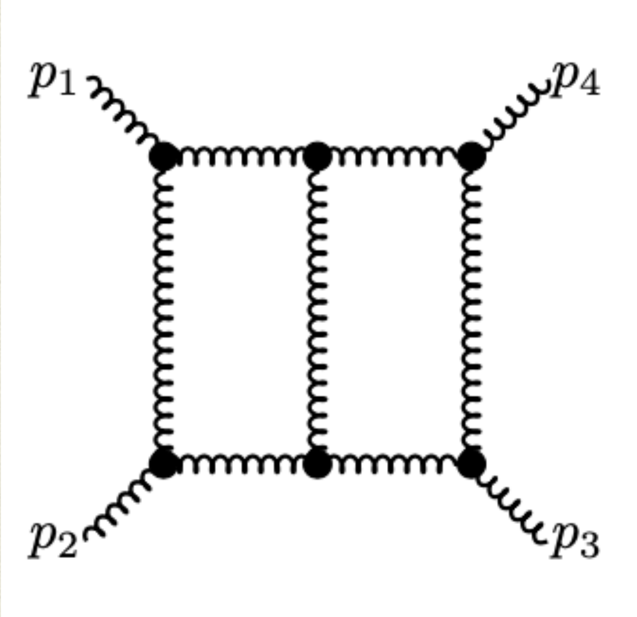
make use of poly. div. —> scalar integrands

"pull out" energies

$$q_i \cdot q_j = q_i^{(+)} \cdot q_j^{(+)} - q_{i,0}^{(+)} q_{j,0}^{(+)} + q_{i,0} q_{j,0}\,,$$

$$\hookrightarrow N = \sum_{i=1}^{R} c_i \prod_{j=1}^{L} \ell_{j,0}^{\alpha_{ij}}\,,$$

Application: four gluons @2L



$$-\frac{c_{1,4}\left((p_{1,0}+p_{2,0})(p_{1,0}+p_{2,0}+2p_{3,0}) - \left(q_{1,0}^{(+)}\right)^2 + \left(q_{3,0}^{(+)}\right)^2\right)}{4p_{1,0}(p_{1,0}+p_{2,0})\,p_{3,0}}$$

$$N = \sum_{r,s}^{\text{rank}} c_{rs}\, \ell_{1,0}^r \ell_{2,0}^s\,, \quad \text{with } r,s \le 4 \wedge r+s \le 6\,.$$

$$\begin{aligned}
&= \tilde{c}_8 J_{\{0,0,1,0,0,1,1\}} + \tilde{c}_7 J_{\{0,0,1,0,1,0,1\}} + \tilde{c}_{27} J_{\{0,0,1,0,1,1,1\}} + \tilde{c}_9 J_{\{0,0,1,1,0,1,1\}} \\
&\quad + \tilde{c}_{22} J_{\{0,0,1,1,1,1,1\}} + \tilde{c}_3 J_{\{0,1,0,0,0,1,1\}} + \tilde{c}_4 J_{\{0,1,0,0,1,0,1\}} + \tilde{c}_{25} J_{\{0,1,0,0,1,1,1\}} \\
&\quad + \tilde{c}_{18} J_{\{0,1,0,1,1,1,1\}} + \tilde{c}_{13} J_{\{0,1,1,0,0,1,1\}} + \tilde{c}_{14} J_{\{0,1,1,0,1,0,1\}} + \tilde{c}_2 J_{\{0,1,1,0,1,1,0\}} \\
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\end{aligned}$$

`Lotty` — *The loop-tree duality automation*

```
In[1]:= << Lotty`

    ------------------------------------------------------------
    Lotty -- the LOop-Tree dualiTY automation
        by William J. Torres Bobadilla (MPP)
            Version 1.0 (March 14th of 2021)
    All functions are stored in the variable $LottyFunctions
    ------------------------------------------------------------
```

Lotty
THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

Download `Lotty` *from*

https://bitbucket.org/wjtorresb/lotty

- *Dual repr. of multi-loop Feynman integrands*
- *All-loop causal repr. of scalar Feynman integrands*
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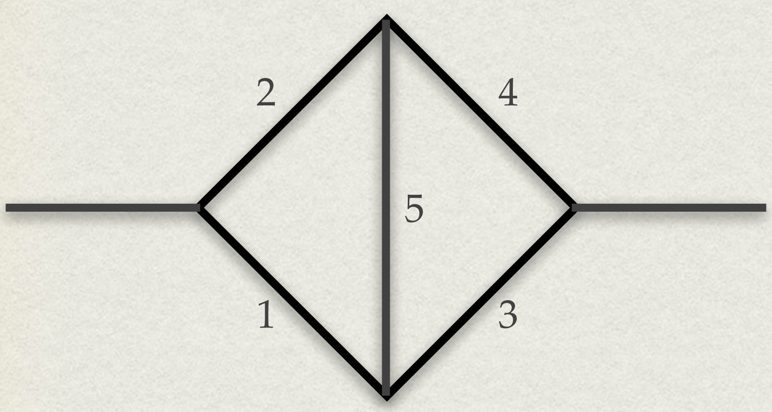
- *Two-loop kite diagrams*



All two-loop Scattering amplitudes

$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} \left[G_D\left(1,2\right) + G_D\left(1,\overline{12}\right) + G_D\left(\bar{2},\overline{12}\right) \right] ,$$

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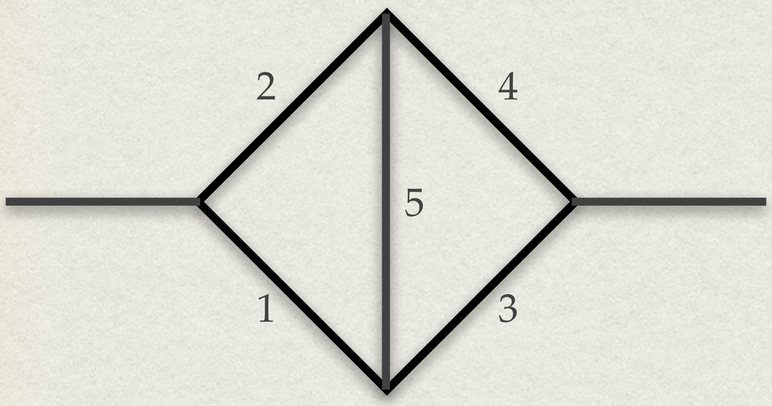

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`Lotty ::` *Computes all residues*

```
{GD[1, 3], GD[2, 3], GD[3, 5], GD[1, 4], GD[2, 4], GD[4, 5], GD[-2, 5], GD[-1, 5]}
```

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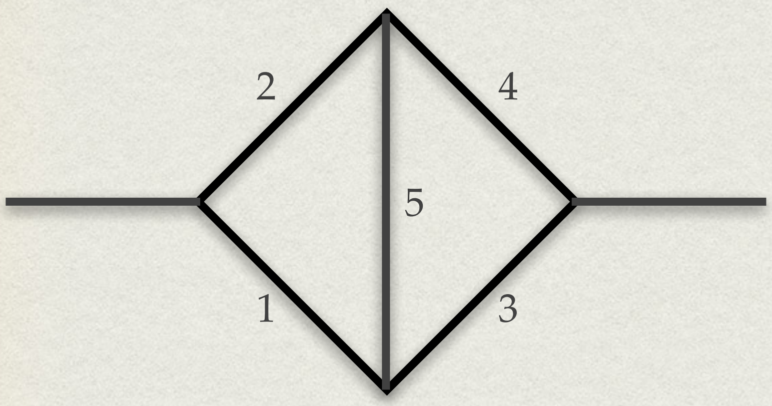

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```

Causal representation

$$-\frac{1}{2\lambda_3\lambda_1^+\lambda_2^-}-\frac{1}{2\lambda_6\lambda_1^+\lambda_2^-}-\frac{1}{2\lambda_3\lambda_1^-\lambda_2^+}-\frac{1}{2\lambda_6\lambda_1^-\lambda_2^+}-\frac{1}{2\lambda_3\lambda_6\lambda_4^-}-\frac{1}{2\lambda_6\lambda_1^-\lambda_4^-}-$$
$$\frac{1}{2\lambda_3\lambda_2^-\lambda_4^-}-\frac{1}{2\lambda_1^-\lambda_2^-\lambda_4^-}-\frac{1}{2\lambda_3\lambda_6\lambda_4^+}-\frac{1}{2\lambda_6\lambda_1^+\lambda_4^+}-\frac{1}{2\lambda_3\lambda_2^+\lambda_4^+}-\frac{1}{2\lambda_1^+\lambda_2^+\lambda_4^+}-\frac{1}{2\lambda_3\lambda_6\lambda_5^-}-$$
$$\frac{1}{2\lambda_3\lambda_1^-\lambda_5^-}-\frac{1}{2\lambda_6\lambda_2^-\lambda_5^-}-\frac{1}{2\lambda_1^-\lambda_2^-\lambda_5^-}-\frac{1}{2\lambda_3\lambda_6\lambda_5^+}-\frac{1}{2\lambda_3\lambda_1^+\lambda_5^+}-\frac{1}{2\lambda_6\lambda_2^+\lambda_5^+}-\frac{1}{2\lambda_1^+\lambda_2^+\lambda_5^+}$$

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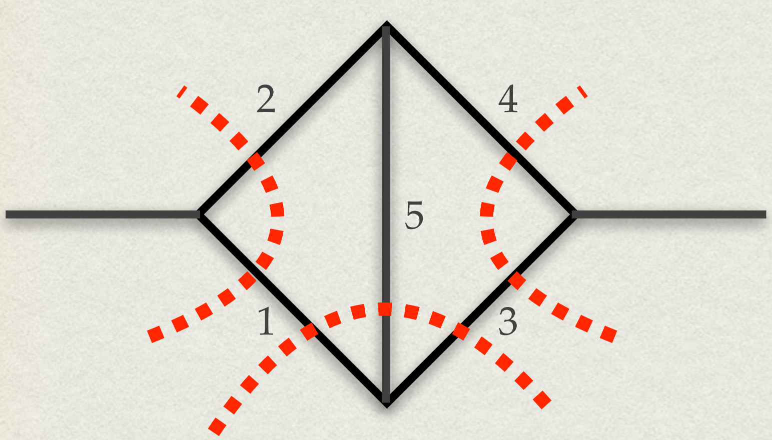

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$$\frac{1}{2\lambda_3\lambda_2^-\lambda_4^-} - \frac{1}{2\lambda_1^-\lambda_2^-\lambda_4^-} - \frac{1}{2\lambda_3\lambda_6\lambda_4^+} - \frac{1}{2\lambda_6\lambda_1^+\lambda_4^+} - \frac{1}{2\lambda_3\lambda_2^+\lambda_4^+} - \frac{1}{2\lambda_1^+\lambda_2^+\lambda_4^+} - \frac{1}{2\lambda_3\lambda_6\lambda_5^-} -$$
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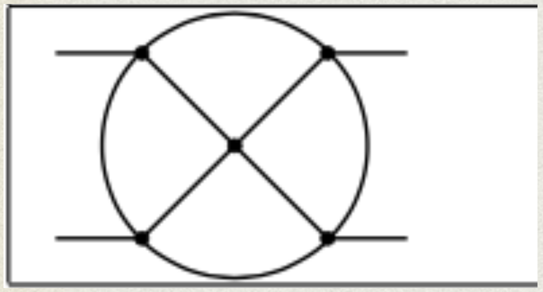
Lotty — *The loop-tree duality automation*



in a nutshell :: Dual & causal representation

[W.J.T. (2021)]

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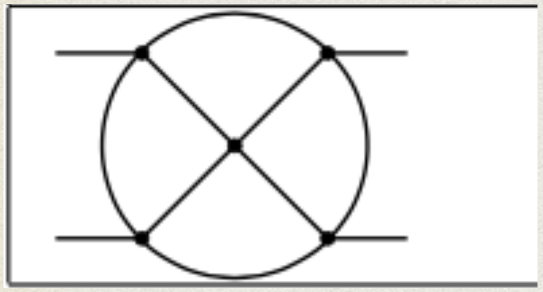

in a nutshell :: Dual & causal representation

[W.J.T. (2021)]

```
In[5]:= num = 1;
        LoopMom = {l1, l2, l3, l4};
        propagators = {l1, l2, l3, l4, -l1 + l2 - p[1], -l2 + l3 - p[2], -l3 + l4 - p[3], -l4 + l1 - p[4]};
        assumptions = (Im[Subscript[p[#], 0]] == 0) & /@ Range[4];

In[9]:= tmp = GetDual[num, propagators, LoopMom, "Assumptions" → assumptions]; // AbsoluteTiming
        tmp2 = tmp * GetCausalProps[tmp, propagators, "GetPref" → True] // Total;

Out[9]= {1.6389, Null}

In[11]:= tmp1 = RefineDual[tmp, num, propagators, LoopMom]; // AbsoluteTiming
         tmp1

Out[11]= {4.73747, Null}

Out[12]= {GD[{-1, -2, -3, -4}], GD[{2, 3, 4, -5}], GD[{-2, -3, -4, -8}], GD[{-1, -3, -4, -5}], GD[{1, 3, 4, -6}],
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  GD[{2, 4, -5, -7}], GD[{-2, -4, 7, -8}], GD[{1, 4, 5, -7}], GD[{1, 4, -6, -7}], GD[{4, -5, -6, -7}],
  GD[{-4, 6, 7, -8}], GD[{-4, -5, 7, -8}], GD[{-4, -5, -6, -8}], GD[{-1, -2, -3, -7}],
  GD[{-2, -3, 5, -7}], GD[{-2, -3, -7, -8}], GD[{-1, -3, -5, -7}], GD[{-1, -3, 6, -7}],
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In[13]:= GetCausalProps[tmp, propagators]

         ALL λ[i] ARE STORED IN THE FUNCTION λ2qi0

Out[13]= {p[1]_0 - λ[1], p[1]_0 + λ[1], p[2]_0 - λ[2], p[2]_0 + λ[2], p[3]_0 - λ[3], p[3]_0 + λ[3], p[4]_0 - λ[4], p[4]_0 + λ[4],
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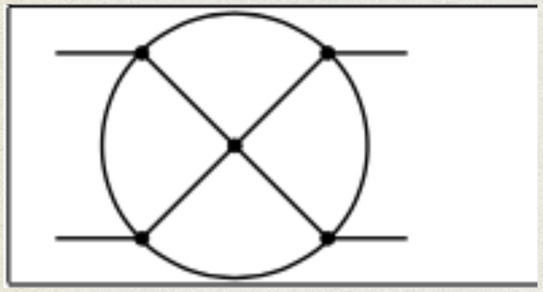
  p[1]_0 + p[2]_0 + p[4]_0 - λ[11], p[1]_0 + p[2]_0 + p[4]_0 + λ[11], p[1]_0 + p[3]_0 + p[4]_0 - λ[12],
  p[1]_0 + p[3]_0 + p[4]_0 + λ[12], p[2]_0 + p[3]_0 + p[4]_0 - λ[13], p[2]_0 + p[3]_0 + p[4]_0 + λ[13]}
```

Lotty — *The loop-tree duality automation*



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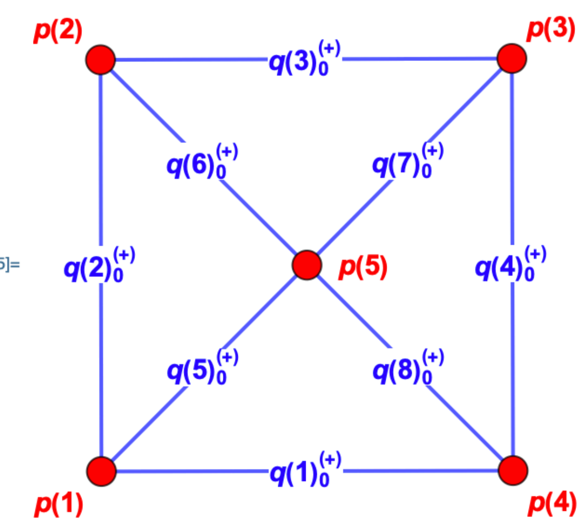
```

```
In[14]:= SetSingLam = {
           q[{1}] → {1, 2, 5},
           q[{2}] → {2, 3, 6},
           q[{3}] → {3, 4, 7},
           q[{4}] → {4, 1, 8},
           q[{5}] → {5, 6, 7, 8}
         };

In[15]:= PlotTop[SetSingLam]

Out[15]=
```

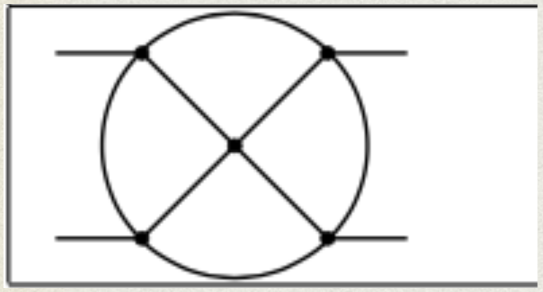



```
In[46]:= tmp3 = AllCausal[5]
         tmp3 // Expand // Length

Out[46]= (L[-][{3, 4}] + L[-][{3, 5}] + L[-][{4, 5}]) L[+][{1, 2}] +
  (L[-][{2, 4}] + L[-][{2, 5}] + L[-][{4, 5}]) L[+][{1, 3}] +
  (L[-][{2, 3}] + L[-][{2, 5}] + L[-][{3, 5}]) L[+][{1, 4}] +
  (L[-][{2, 3}] + L[-][{2, 4}] + L[-][{3, 4}]) L[+][{1, 5}] +
  (L[-][{1, 4}] + L[-][{1, 5}] + L[-][{4, 5}]) L[+][{2, 3}] +
  (L[-][{1, 3}] + L[-][{1, 5}] + L[-][{3, 5}]) L[+][{2, 4}] +
  (L[-][{1, 3}] + L[-][{1, 4}] + L[-][{3, 4}]) L[+][{2, 5}] +
  (L[-][{1, 2}] + L[-][{1, 5}] + L[-][{2, 5}]) L[+][{3, 4}] +
  (L[-][{1, 2}] + L[-][{1, 4}] + L[-][{2, 4}]) L[+][{3, 5}] +
  (L[-][{1, 2}] + L[-][{1, 3}] + L[-][{2, 3}]) L[+][{4, 5}]
```

William J. Torres Bobadilla

Lotty — *The loop-tree duality automation*



in a nutshell :: Dual & causal representation

[W.J.T. (2021)]

```
In[5]:= num = 1;
        LoopMom = {l1, l2, l3, l4};
        propagators = {l1, l2, l3, l4, -l1 + l2 - p[1], -l2 + l3 - p[2], -l3 + l4 - p[3], -l4 + l1 - p[4]};
        assumptions = (Im[Subscript[p[#], 0]] == 0) & /@ Range[4];

In[9]:= tmp = GetDual[num, propagators, LoopMom, "Assumptions" → assumptions]; // AbsoluteTiming
        tmp2 = tmp * GetCausalProps[tmp, propagators, "GetPref" → True] // Total;

Out[9]= {1.6389, Null}

In[11]:= tmp1 = RefineDual[tmp, num, propagators, LoopMom]; // AbsoluteTiming
         tmp1

Out[11]= {4.73747, Null}

Out[12]= {GD[{-1, -2, -3, -4}], GD[{2, 3, 4, -5}], GD[{-2, -3, -4, -8}], GD[{-1, -3, -4, -5}], GD[{1, 3, 4, -6}],
  GD[{3, 4, -5, -6}], GD[{-3, -4, 6, -8}], GD[{-3, -4, -5, -8}], GD[{-1, -2, -4, -6}],
  GD[{-2, -4, 5, -6}], GD[{-2, -4, -6, -8}], GD[{-1, -4, -5, -6}], GD[{1, 2, 4, -7}],
  GD[{2, 4, -5, -7}], GD[{-2, -4, 7, -8}], GD[{1, 4, 5, -7}], GD[{1, 4, -6, -7}], GD[{4, -5, -6, -7}],
  GD[{-4, 6, 7, -8}], GD[{-4, -5, 7, -8}], GD[{-4, -5, -6, -8}], GD[{-1, -2, -3, -7}],
  GD[{-2, -3, 5, -7}], GD[{-2, -3, -7, -8}], GD[{-1, -3, -5, -7}], GD[{-1, -3, 6, -7}],
  GD[{-3, 5, 6, -7}], GD[{-3, 6, -7, -8}], GD[{
  GD[{-2, 5, -6, -7}], GD[{-2, -6, -7, -8}], GD
  GD[{1, 3, -6, -8}], GD[{1, 2, 6, -8}], GD[{1,
  GD[{1, -6, -7, -8}], GD[{2, 3, -5, -8}], GD[{

In[13]:= GetCausalProps[tmp, propagators]

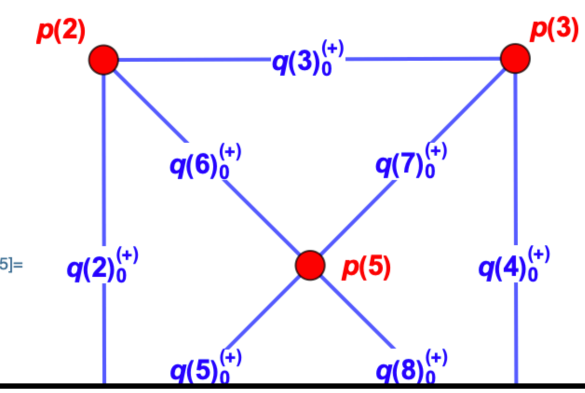
   ALL λ[i] ARE STORED IN THE FUNCTION λ2qi0

Out[13]= {p[1]0 - λ[1], p[1]0 + λ[1], p[2]0 - λ[2], p[2]0 +
  p[1]0 + p[2]0 - λ[5], p[1]0 + p[2]0 + λ[5], p[2]0
  p[1]0 + p[4]0 + λ[7], p[3]0 + p[4]0 - λ[8], p[3]0
  p[1]0 + p[2]0 + p[3]0 + p[4]0 + λ[9], p[1]0 + p[2]
  p[1]0 + p[2]0 + p[4]0 - λ[11], p[1]0 + p[2]0 + p[4
  p[1]0 + p[3]0 + p[4]0 + λ[12], p[2]0 + p[3]0 + p[4
```

```
In[14]:= SetSingLam = {
           q[{1}] → {1, 2, 5},
           q[{2}] → {2, 3, 6},
           q[{3}] → {3, 4, 7},
           q[{4}] → {4, 1, 8},
           q[{5}] → {5, 6, 7, 8}
         };

In[15]:= PlotTop[SetSingLam]

Out[15]=
```




```
In[48]:= tmp3 = AllCausal[5, "ExpandToLambda" → True];
         momcon = {p[4] → -p[1] - p[2] - p[3]};
         toLambda = Lamb2qij[tmp3, SetSingLam];
         tmp4 = tmp3 //. toLambda /. momcon /. Subscript[p[5 | 6], __] :→ 0;

In[52]:= (* numerical check *)

         repl1 = (Subsuperscript[q[#], 0, "(+)"] → RandomInteger[{10^3, 10^4}] / RandomInteger[{10^4, 10^5}]) & /@ Range[Length@propagators];

         repl1 = Join[repl1, (Subscript[p[#], 0] → RandomInteger[{10^3, 10^4}] / RandomInteger[{10^4, 10^5}] I) & /@ Range[3]]

         tmp2 - tmp4 /. momcon /. repl1

Out[53]= {q[1]0(+) → 1596/15493, q[2]0(+) → 4537/36068, q[3]0(+) → 747/7679, q[4]0(+) → 7975/82068, q[5]0(+) → 8834/93397,
  q[6]0(+) → 6580/67731, q[7]0(+) → 1939/3987, q[8]0(+) → 9740/37749, p[1]0 → 7741 i/55425, p[2]0 → 949 i/12276, p[3]0 → 8599 i/59525}

Out[54]= 0
```

```
 L[+][{1, 2}] +
) L[+][{1, 3}] +
) L[+][{1, 4}] +
) L[+][{1, 5}] +
) L[+][{2, 3}] +
) L[+][{2, 4}] +
) L[+][{2, 5}] +
) L[+][{3, 4}] +
) L[+][{3, 5}] +
) L[+][{4, 5}]
```

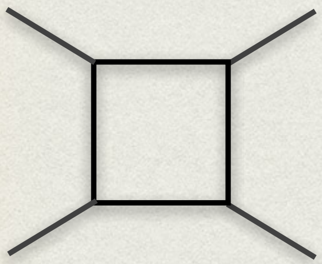
William J. Torres Bobadilla

Numerical evaluations

Lotty

THE LOOP-TREE DUALITY AUTOMATION

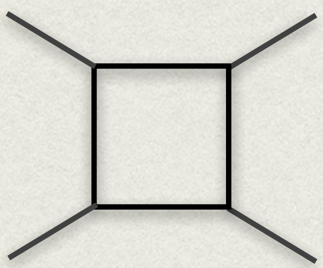
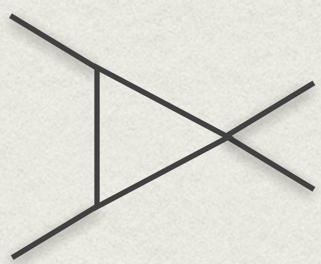
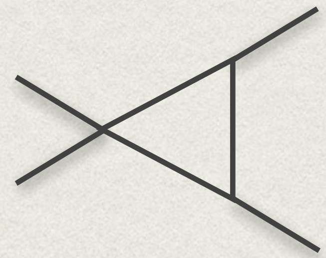
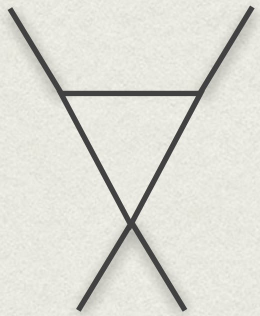
[W.J.T. (2021)]



$$= -\imath \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1,2,3,4) \ , \qquad q_i^2 = (\ell_1 - k_i)^2 + \imath 0$$

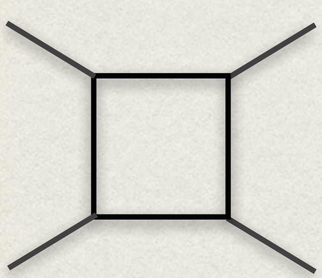
$$k_i = p_1 + \ldots + p_{i-1}$$

IR safe combination of Feynman integrals in ***d=4***

$s\,t$  $-\ s$  $-\ t$  $-s$  $-\ t$ 

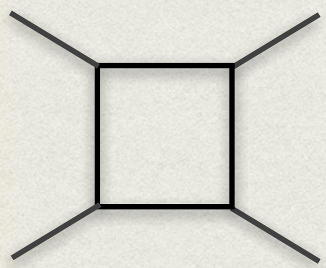
$$= -\frac{1}{(4\pi)^2}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$$

IR finite integral in ***d=6***



$$= \frac{1}{(4\pi)^3}\frac{1}{2\,(s+t)}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$$

Numerical evaluations



$$= -\imath \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1,2,3,4) \,, \qquad q_i^2 = (\ell_1 - k_i)^2 + \imath 0$$

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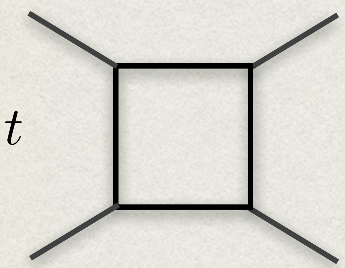
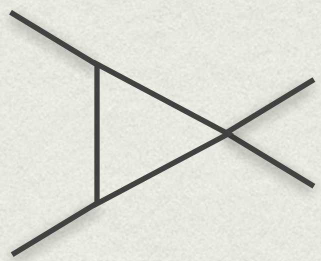
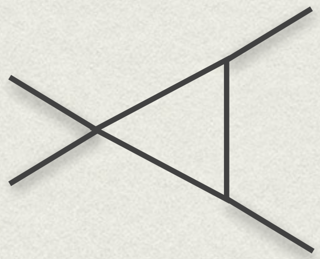
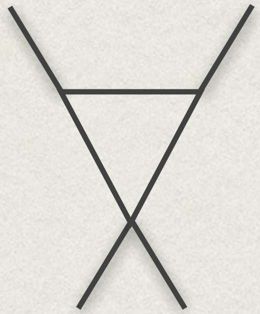
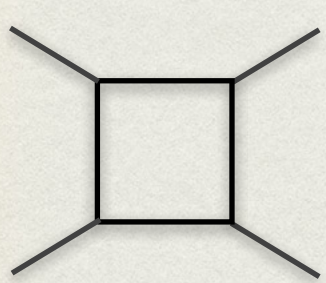
Lotty — THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

IR safe combination of Feynman integrals in ***d=4***

$s\,t$  $-\,s$  $-\,t$  $-s$  $-\,t$  $= -\frac{1}{(4\pi)^2}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$

IR finite integral in ***d=6***

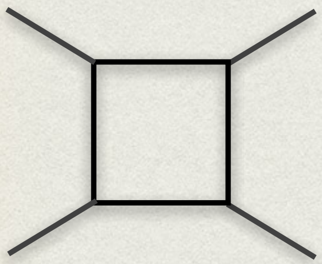


$$= \frac{1}{(4\pi)^3} \frac{1}{2\,(s+t)} \left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$$

$$p_1^\alpha = \frac{\sqrt{s}}{2}\left\{1, 1, \vec{0}_{d-2}\right\}, \qquad p_3^\alpha = \frac{\sqrt{s}}{2}\left\{-1, \sin\theta, \cos\theta, \vec{0}_{d-1}\right\},$$
$$p_2^\alpha = \frac{\sqrt{s}}{2}\left\{1, -1, \vec{0}_{d-2}\right\}, \qquad p_4^\alpha = \frac{\sqrt{s}}{2}\left\{-1, -\sin\theta, -\cos\theta, \vec{0}_{d-1}\right\}.$$

COM

Numerical evaluations



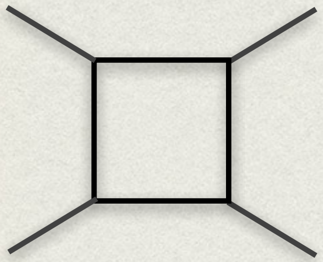
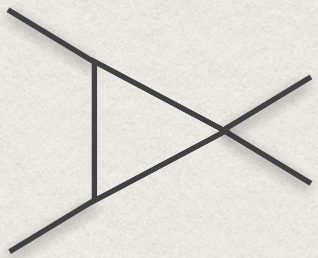
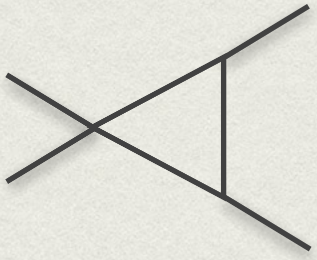
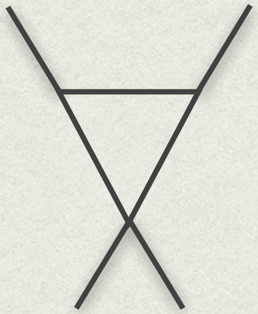
$$= -\imath \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1,2,3,4) \,, \qquad q_i^2 = (\ell_1 - k_i)^2 + \imath 0$$

$$k_i = p_1 + \ldots + p_{i-1}$$

Lotty
THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

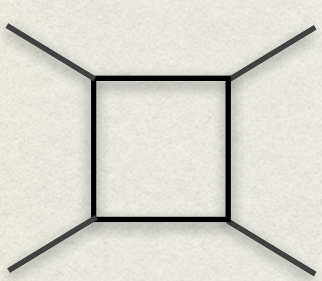
IR safe combination of Feynman integrals in ***d=4***

$s\,t$  $-\,s$  $-\,t$  $-s$  $-\,t$  $= -\frac{1}{(4\pi)^2}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$

IR finite integral in ***d=6***

Straightforward contour deformation @1L → Richardson extrapolation

[Ronca, W.J.T. (w.i.p.)]



$$= \frac{1}{(4\pi)^3}\frac{1}{2(s+t)}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$$

$$p_1^\alpha = \frac{\sqrt{s}}{2}\left\{1, 1, \vec{0}_{d-2}\right\}, \qquad p_3^\alpha = \frac{\sqrt{s}}{2}\left\{-1, \sin\theta, \cos\theta, \vec{0}_{d-1}\right\},$$
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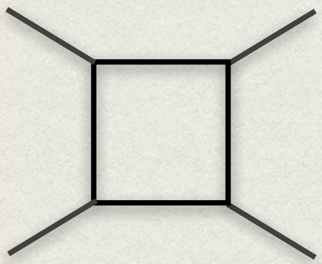
COM

Numerical evaluations



$$= -\imath \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1,2,3,4) \,, \qquad q_i^2 = (\ell_1 - k_i)^2 + \imath 0$$

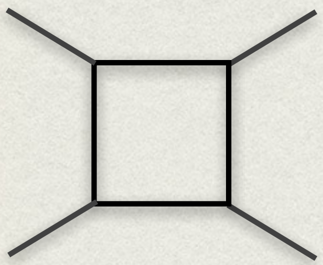
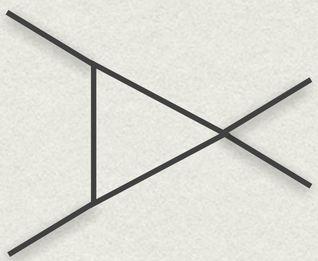
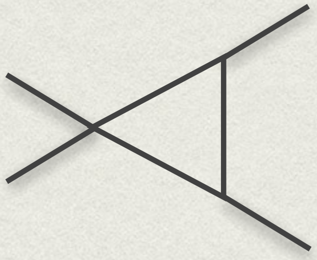
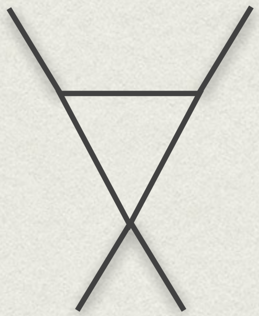
$$k_i = p_1 + \ldots + p_{i-1}$$

Lotty

THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

IR safe combination of Feynman integrals in ***d=4***

$s\,t$  $-\,s$  $-\,t$  $-s$  $-\,t$ 

$$= -\frac{1}{(4\pi)^2}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$$

$$\{s = 3 + \imath 0, \theta = \pi/6\}$$

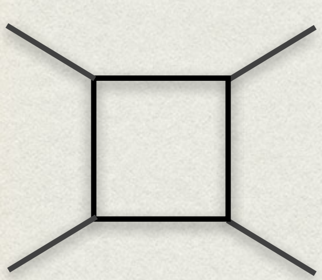
$$\{s = 5 + \imath 0, \theta = \pi/6\}$$

LTD $= \ (-1.2170(4) - \imath\, 5.5158(4)) * 10^{-2}$

IR finite integral in ***d=6***

Straightforward contour deformation @1L → Richardson extrapolation

[Ronca, W.J.T. (w.i.p.)]



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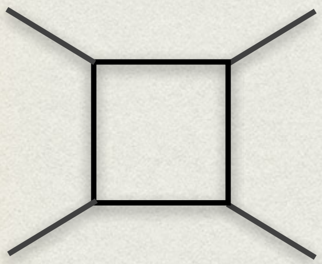
COM

Numerical evaluations



$$= -\imath \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1,2,3,4) \,, \qquad q_i^2 = (\ell_1 - k_i)^2 + \imath 0$$

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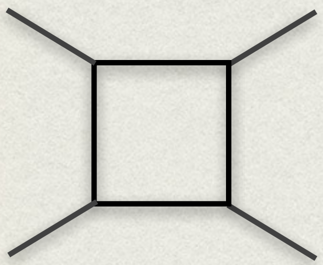
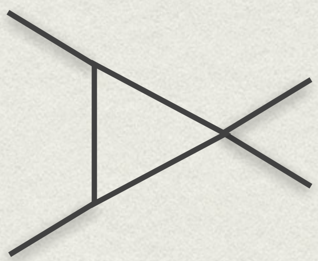
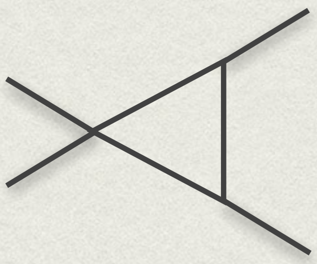
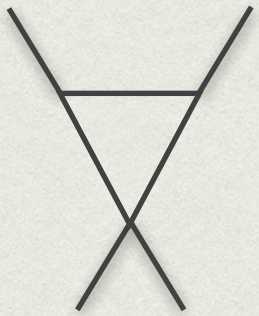
Lotty

THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

IR safe combination of Feynman integrals in ***d=4***

$s\,t$  $-\,s$  $-\,t$  $-s$  $-\,t$  $= -\frac{1}{(4\pi)^2}\left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$

$\{s = 3 + \imath 0, \theta = \pi/6\}$

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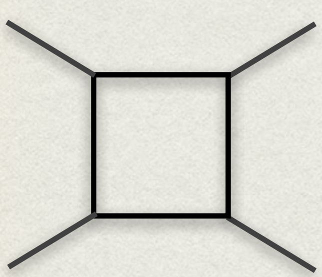
LTD $= \ (-1.2170(4) - \imath\, 5.5158(4)) * 10^{-2}$

Straightforward contour deformation @1L → Richardson extrapolation

[Ronca, W.J.T. (w.i.p.)]

IR finite integral in ***d=6***



$$= \frac{1}{(4\pi)^3} \frac{1}{2(s+t)} \left(\log^2\left(\frac{-s}{-t}\right) + \pi^2\right)$$

$$p_1^\alpha = \frac{\sqrt{s}}{2}\left\{1, 1, \vec{0}_{d-2}\right\}, \qquad p_3^\alpha = \frac{\sqrt{s}}{2}\left\{-1, \sin\theta, \cos\theta, \vec{0}_{d-1}\right\},$$

$$p_2^\alpha = \frac{\sqrt{s}}{2}\left\{1, -1, \vec{0}_{d-2}\right\}, \qquad p_4^\alpha = \frac{\sqrt{s}}{2}\left\{-1, -\sin\theta, -\cos\theta, \vec{0}_{d-1}\right\}.$$

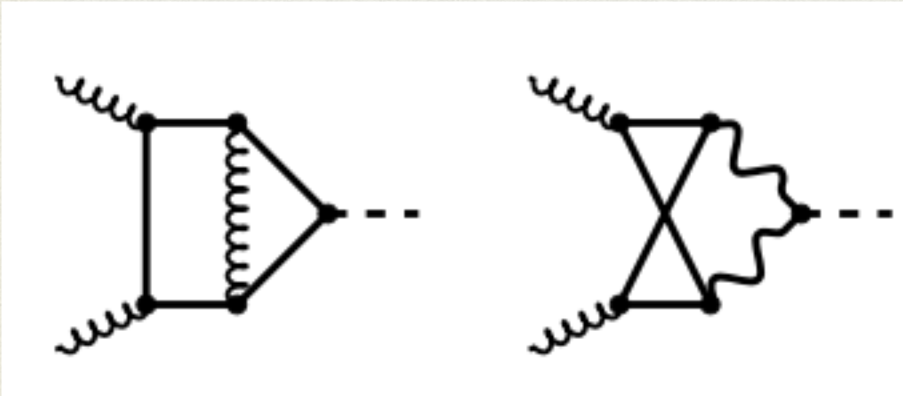
COM

LTD

$\{s = 3 + \imath 0, \theta = \pi/6\} \to (1.2901(2) - \imath\, 5.8528(2)) * 10^{-4}$

$\{s = 5 + \imath 0, \theta = \pi/6\} \to (2.1489(4) - \imath\, 9.7549(4)) * 10^{-4}$

Numerical evaluations

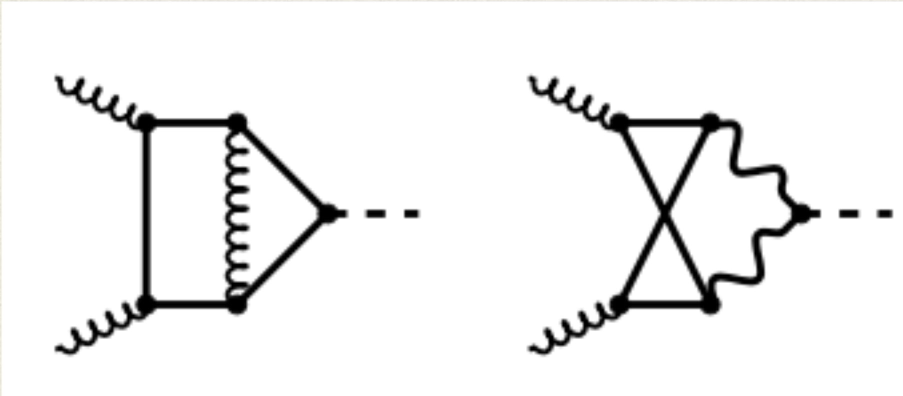


QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]

[W.J.T. (2021)]

Numerical evaluations



QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]

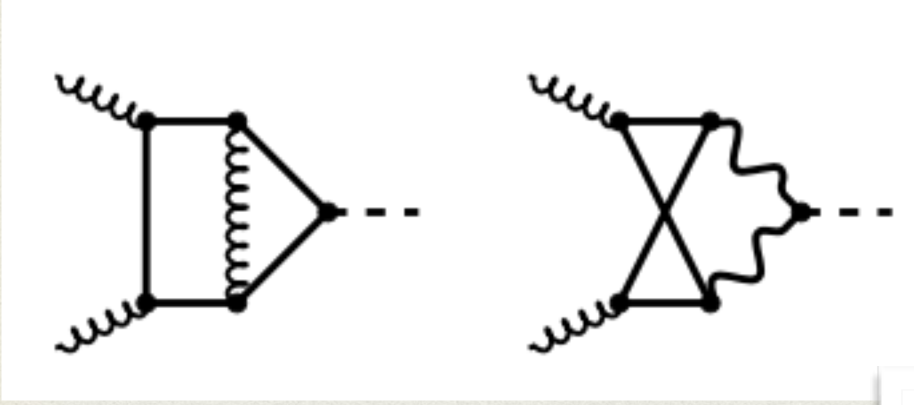
[W.J.T. (2021)]

On-shell energies

```
In[54]:= myqi0 = Subsuperscript[q[#],0,"(+)"]&/@Range[Length@propagators];
         value = myqi0/.Subsuperscript[q[ii_],__]:>Sqrt[sp@propagators[[ii]]+m[ii]^2];
         value = value/.LoopToSC[LoopMom,dim]/.spatial;
         myrepl = Thread[myqi0->value]//FullSimplify
```

$$\text{Out[54]=}\ \{q[1]_0^{(+)} \to \sqrt{r1^2 + m[1]^2},$$
$$q[2]_0^{(+)} \to \frac{1}{2}\sqrt{Ecm^2 + 4\ Ecm\ r1\ Cos[\theta 11] + 4\ (r1^2 + m[2]^2)},$$
$$q[3]_0^{(+)} \to \sqrt{r1^2 + m[3]^2},$$
$$q[4]_0^{(+)} \to \sqrt{r2^2 + m[4]^2},$$
$$q[5]_0^{(+)} \to \sqrt{r2^2 + m[5]^2},$$
$$q[6]_0^{(+)} \to \sqrt{r1^2+r2^2+m[6]^2+2r1r2\ (Cos[\theta 11]Cos[\theta 21] + Cos[\theta 12-\theta 22]Sin[\theta 11]Sin[\theta 21])}\ \}$$

Numerical evaluations



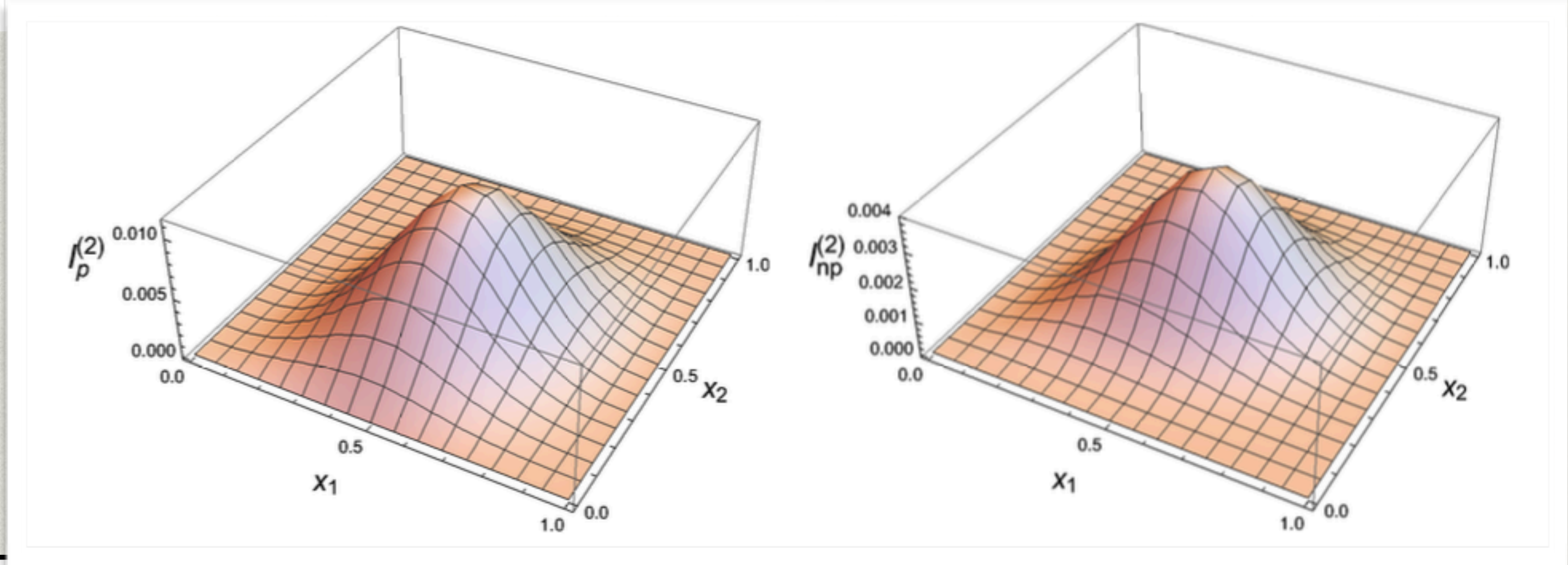
QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]

Lotty

THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]



On-shell energies

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         value = myqi0/.Subsuperscript[q[ii_],__]:>Sqrt[sp@propagators[[ii]]+m[ii]^2];
         value = value/.LoopToSC[LoopMom,dim]/.spatial;
         myrepl = Thread[myqi0->value]//FullSimplify
```

Out[54]=

$$\{q[1]_0^{(+)} \to \sqrt{r1^2 + m[1]^2},$$
$$q[2]_0^{(+)} \to \frac{1}{2}\sqrt{Ecm^2 + 4\ Ecm\ r1\ Cos[\theta 11] + 4\ (r1^2 + m[2]^2)},$$
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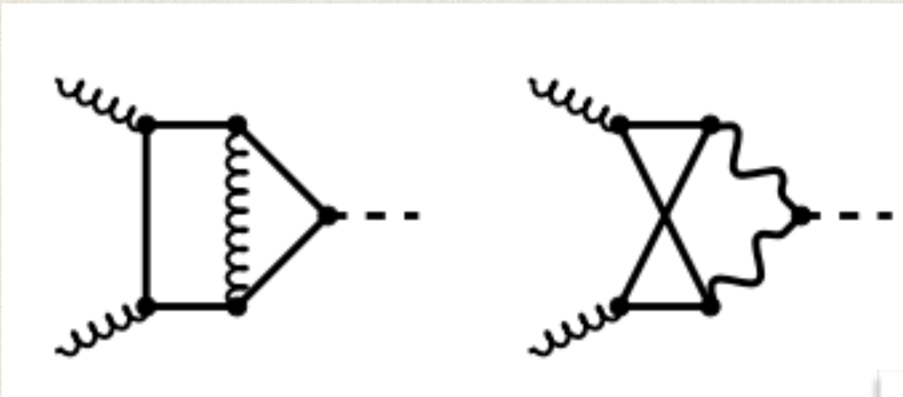
$$q[6]_0^{(+)} \to \sqrt{r1^2+r2^2+m[6]^2+2r1r2\ (Cos[\theta 11]Cos[\theta 21] + Cos[\theta 12-\theta 22]Sin[\theta 11]Sin[\theta 21])}\ \}$$

Compactify variables

$$r_i \to \frac{1-x_i}{x_i}$$

Numerical evaluations



QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]

Lotty
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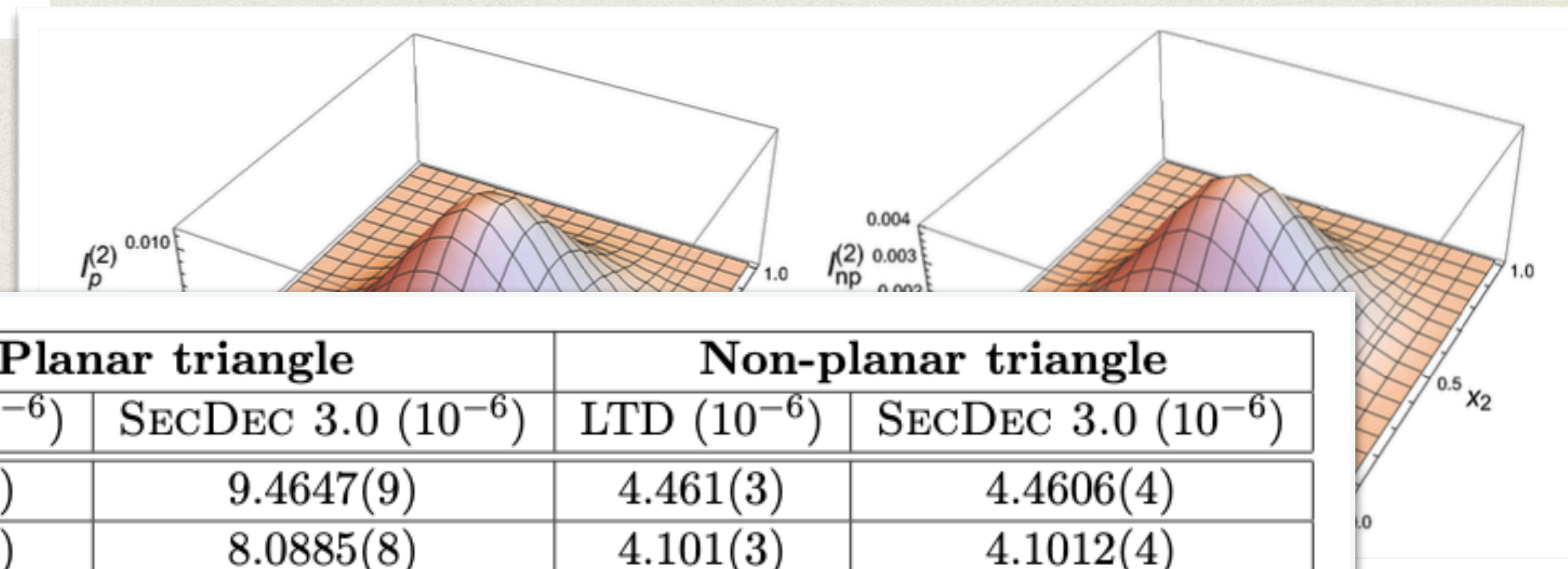
[W.J.T. (2021)]



	Planar triangle		Non-planar triangle	
$\frac{s}{m^2}$	LTD (10^{-6})	SecDec 3.0 (10^{-6})	LTD (10^{-6})	SecDec 3.0 (10^{-6})
$-\frac{1}{4}$	9.48(5)	9.4647(9)	4.461(3)	4.4606(4)
-1	8.10(5)	8.0885(8)	4.101(3)	4.1012(4)
$-\frac{9}{4}$	6.49(3)	6.4760(6)	3.627(5)	3.6276(3)
-4	5.02(2)	5.0188(5)	3.15(5)	3.1334(3)
$+\frac{1}{4}$	10.68(6)	10.651(1)	4.743(3)	4.7436(4)
1	13.11(8)	13.070(1)	5.259(3)	5.2590(5)
$+\frac{9}{4}$	20.81(1)	20.748(2)	6.533(3)	6.5331(6)
$+\frac{25}{16}$	15.74(9)	15.700(1)	5.748(3)	5.7474(6)

On-shell energies

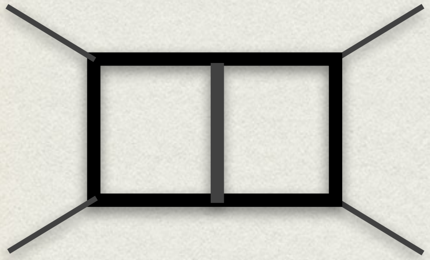
```
In[54]:= myqi0 = Subsupe
         value = myqi0/.
         value = value/.
         myrepl = Thread[

Out[54]= {q[1]_0^(+) -> √r
          q[2]_0^(+) -> 1/2 √E
          q[3]_0^(+) -> √r
          q[4]_0^(+) -> √(r2^2 + m[4]^2),
          q[5]_0^(+) -> √(r2^2 + m[5]^2),
          q[6]_0^(+) -> √(r1^2+r2^2+m[6]^2+2r1r2 (Cos[θ11]Cos[θ21] + Cos[θ12-θ22]Sin[θ11]Sin[θ21]) }
```

riables

Check with SecDec 3.0

Numerical evaluations



$$= -\int \frac{d^d\ell_1}{(2\pi)^d}\frac{d^d\ell_2}{(2\pi)^d} G_F(1,2,3,\ldots,7)\,, \qquad G_F(i) = q_i^2 - m^2 + \imath 0$$

Lotty

THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

d=4

$$\{s=-3/5, \theta=\pi/6\} \to -1.5912(2) * 10^{-6}$$
$$\{s=-2/7, \theta=\pi/6\} \to -1.6991(1) * 10^{-6}$$

d=6

$$\{s=-3/5, \theta=\pi/6\} \to -2.73(7) * 10^{-8}$$
$$\{s=-2/7, \theta=\pi/6\} \to -2.79(7) * 10^{-8}$$

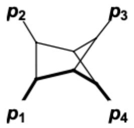
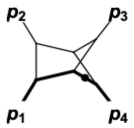
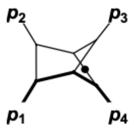
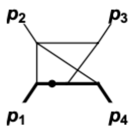
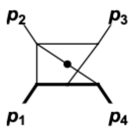
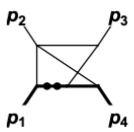
O(10") per point

$$p_1^\alpha = \frac{\sqrt{s}}{2}\left\{1,1,\vec{0}_{d-2}\right\}, \qquad p_3^\alpha = \frac{\sqrt{s}}{2}\left\{-1,\sin\theta,\cos\theta,\vec{0}_{d-1}\right\},$$
$$p_2^\alpha = \frac{\sqrt{s}}{2}\left\{1,-1,\vec{0}_{d-2}\right\}, \qquad p_4^\alpha = \frac{\sqrt{s}}{2}\left\{-1,-\sin\theta,-\cos\theta,\vec{0}_{d-1}\right\}.$$

COM

Interesting set of integrals to look at in ***d=6***

[Ronca, W.J.T. (w.i.p.)]

graph	$I^{[d]}$–integral	$I^{[d=6-2\epsilon]}(s=-\frac{1}{7}, t=-\frac{1}{3}, m^2=1)$
	$I^{[d]}(1,1,1,0,1,0,1,1,1)$	$-1.219372 - \imath\,0.294408$
	$I^{[d]}(1,2,1,0,1,0,1,1,1)$	$0.98317 + \imath\,1.00335$
	$I^{[d]}(1,1,1,0,1,0,1,2,1)$	$12.039969 + \imath\,6.660946$
	$I^{[d]}(2,1,1,0,0,0,1,1,1)$	$-0.554605 - \imath\,0.06984485$
	$I^{[d]}(1,1,1,0,0,0,1,2,1)$	$-1.91103 + \imath\,0.241649$
	$I^{[d]}(3,1,1,0,0,0,1,1,1)$	$0.525679 + \imath\,0.248668$

[Divita, Laporta, Mastrolia, Primo, Schubert (2018)]

Multi-loop LTD causal representation

- In summary

$$\mathcal{A}^{(L)}(1,2,\ldots,n) = \int_{\vec{\ell}_1\cdots\vec{\ell}_L} \mathcal{I}\left(q_{i,0}^{(+)}\right)$$

Integration in the spatial components

$$q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - \imath 0}$$

$$\int \prod_{i=1}^{n} dq_{i,0}^{(+)}\, \mathcal{I}\left(q_{i,0}^{(+)}\right)$$

- Rational function in $q_{i,0}^{(+)}$
- Dual repr. of Feynman integrands
- All-loop causal repr.
- Described by loop topologies' features
  - ☆ **Edges**
  - ☆ **Vertices**

- Straightforward numerical evaluation

Set dimension and evaluate
(finite integrand)

- Find local UV/IR counter-terms
- Find a repr. for real corrections
- Provide a complete implementation in d=4 at NNLO.

[Rodrigo, Ronca, Sborlini, W.J.T., Tramontano (w.i.p)]

- Independently of the structure of numerator

- Make of use poly. div —> Groebner basis

- Completely automated code —> `Lotty`

- Pedestrian appl. of all the theory developed for LTD

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Thanks