



Causal representation and numerical evaluation of multi-loop Feynman integrals

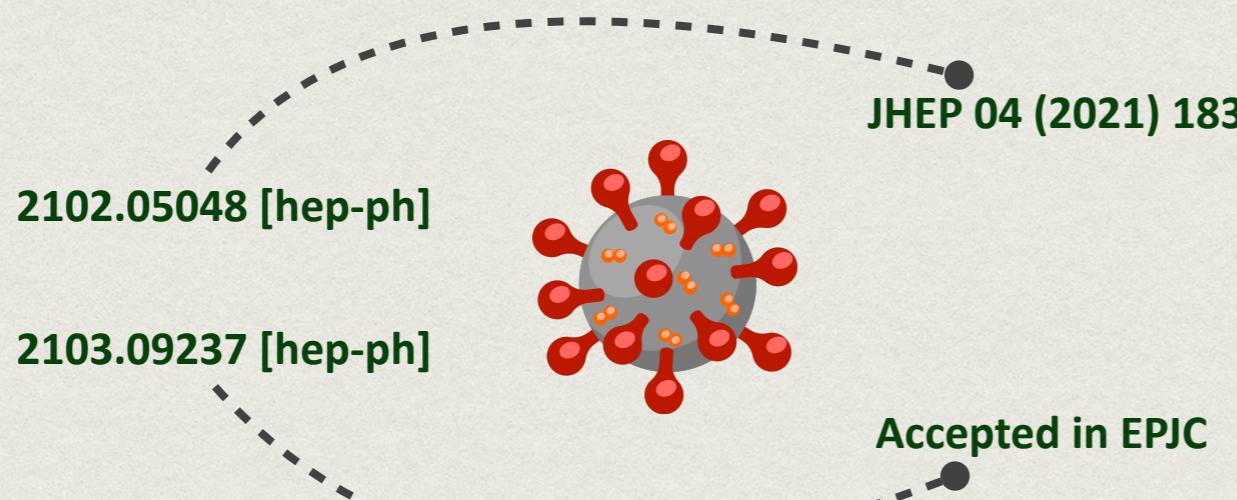
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<https://bitbucket.org/wjtorresb/lotty>

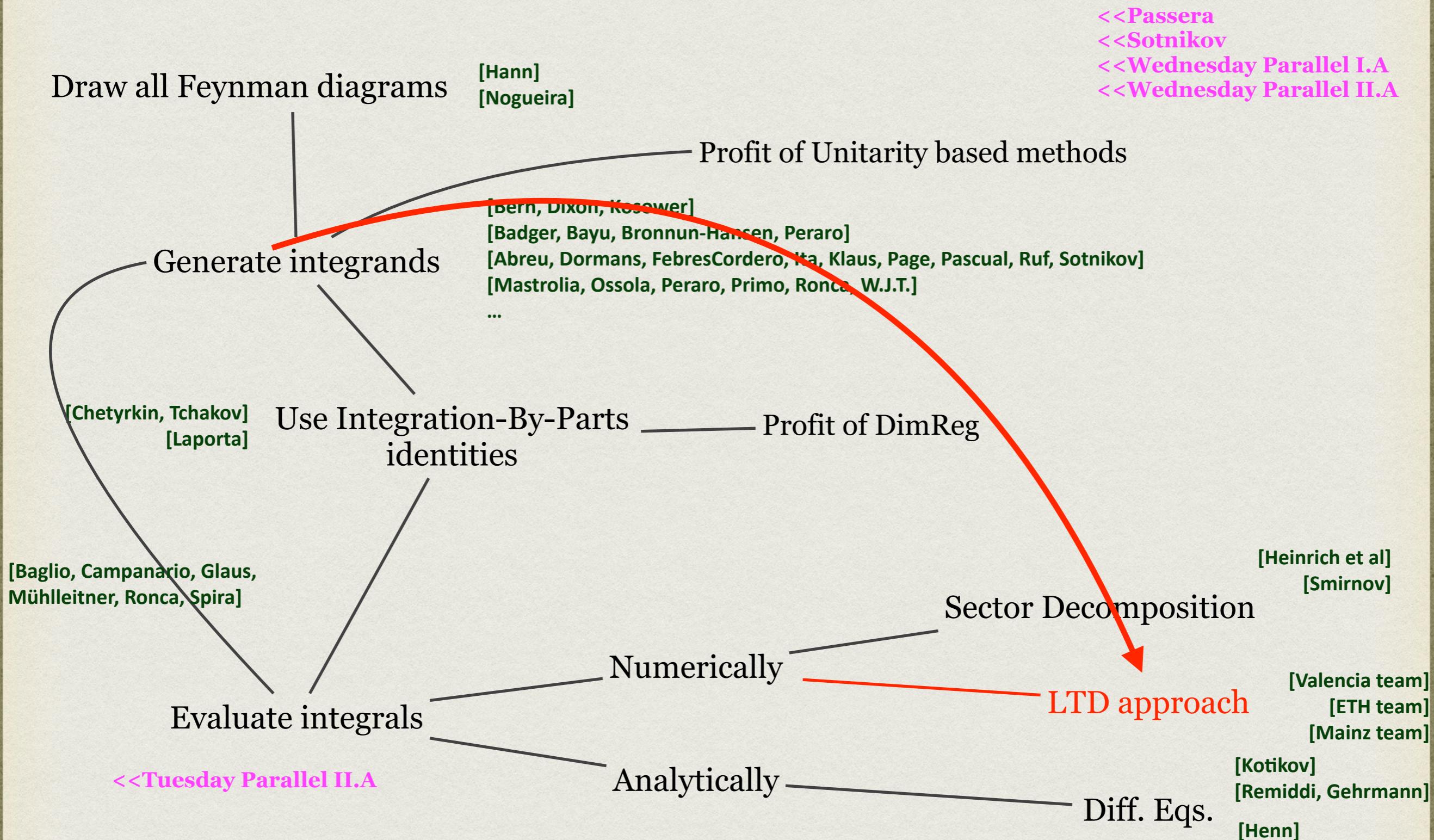


Outline

- Introduction
- Multi-loop LTD representation
- Causal Representation of Scattering Amplitudes
- Lotty – the loop-tree duality automation
- Conclusions & Outlook



Standard approach @multi-loop level



Standard approach @multi-loop level

Complete automation @ NNLO ?

Thresholds

UV

IR

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<https://doi.org/10.1140/epjc/s10052-021-08996-y>

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PHYSICAL JOURNAL C



Review

May the four be with you: novel IR-subtraction methods to tackle NNLO calculations

W. J. Torres Bobadilla^{1,2,a}, G. F. R. Sborlini³, P. Banerjee⁴, S. Catani⁵, A. L. Cherchiglia⁶, L. Cieri⁵, P. K. Dhani^{5,7}, F. Driencourt-Mangin², T. Engel^{4,8}, G. Ferrera⁹, C. Gnendiger⁴, R. J. Hernández-Pinto¹⁰, B. Hiller¹¹, G. Pelliccioli¹², J. Pires¹³, R. Pittau¹⁴, M. Rocco¹⁵, G. Rodrigo², M. Sampaio⁶, A. Signer^{4,8}, C. Signorile-Signorile^{16,17}, D. Stöckinger¹⁸, F. Tramontano¹⁹, Y. Ulrich^{4,8,20}

- ✿ FDH/FDR → transition rules both ren. schemes @NNLO
- ✿ FDU → preliminary mappings between VV & VR contributions
- ✿ IReg → Towards full renormalisation @ 2L
- ✿ Torino Scheme → general subtraction method for massless & final states QCD
- ✿ qt-subtraction → benefits from any existing calculation for “F+jet”
- ✿ Antenna subtraction → subtraction term at tree (RR) and one-loop (R) level
 - ✿ many more subs. schemes ...

*causal or
non-causal*



Causal representation → Display only physical **singularities**
→ Multi-loop LTD representation

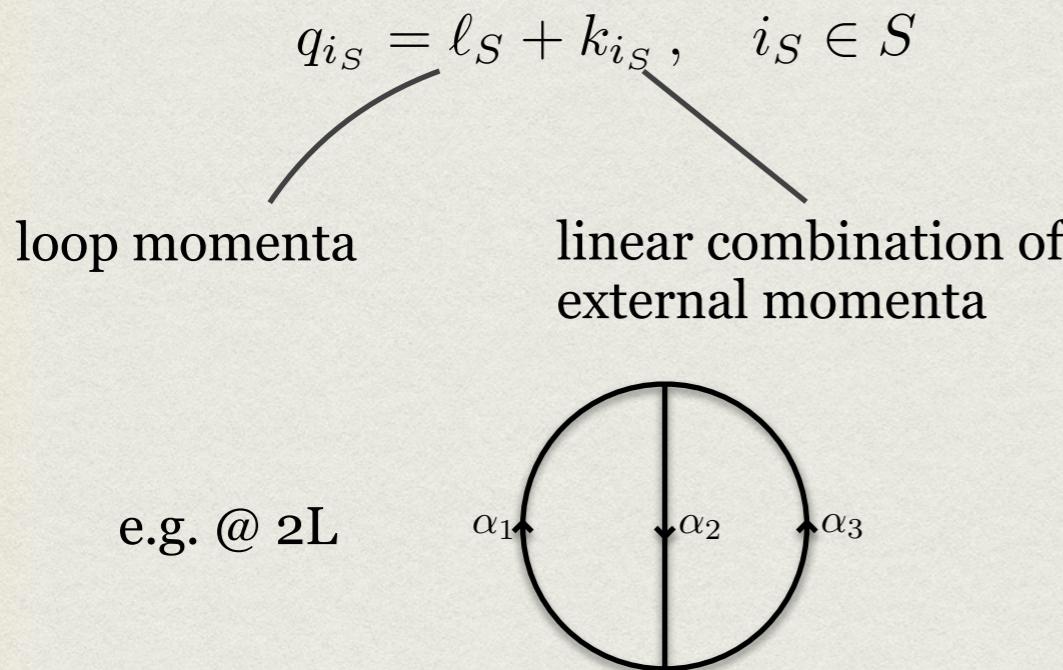
Multi-loop LTD representation

[Aguilera-Verdugo *et al* (2020)]

- Any multi-loop Feynman integral contains S sets of internal propagators

<< <https://indico.cern.ch/event/1021090/>

<< Sborlini



PHYSICAL REVIEW LETTERS **124**, 211602 (2020)

Open Loop Amplitudes and Causality to All Orders and Powers from the Loop-Tree Duality

J. Jesús Aguilera-Verdugo,^{1,*} Félix Driencourt-Mangin,^{1,†} Roger J. Hernández-Pinto^{1,‡} Judith Plenter^{1,§}, Selomit Ramírez-Uribe^{1,2,3,||} Andrés E. Rentería-Olivo^{1,¶} Germán Rodrigo^{1,**} Germán F. R. Sborlini^{1,††}, William J. Torres Bobadilla^{1,‡‡} and Szymon Tracz^{1,§§}

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- Feynman propagators

In terms of spatial components

$$G_F(q_{i_S}) = \frac{1}{q_{i_S}^2 - m_{i_S}^2 + i0} = \frac{1}{q_{i_S,0}^2 - (q_{i_S,0}^{(+)})^2}$$

$$q_{i_S,0}^{(+)} = +\sqrt{\mathbf{q}_{i_S}^2 + m_{i_S}^2 - i0}$$

Pull out full dependence of the energy components of loop momenta

usual Feynman **i0** prescription!

- Let's now apply the Cauchy residue thm for each “energy” integration

Multi-loop LTD representation

[Aguilera-Verdugo *et al* (2020)]

<< <https://indico.cern.ch/event/1021090/>

- LTD representation is written in terms of nested residues

<<**Sborlini**

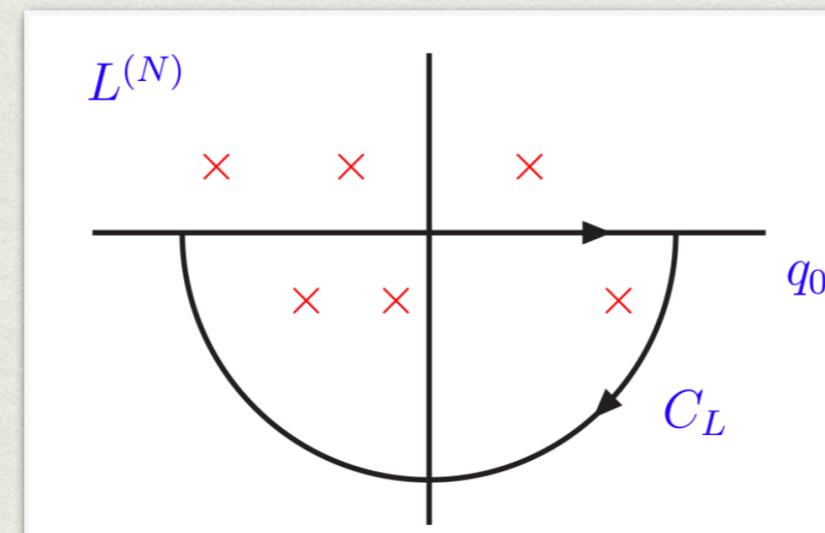
$$\mathcal{A}_D^{(L)} (1, \dots, r; r+1, \dots, n) \equiv -2\pi i \sum_{i_r \in r} \text{Res} \left(\mathcal{A}_D^{(L)} (1, \dots, r-1; r, \dots, n), \text{Im}(q_{i_r,0}) < 0 \right),$$

in terms of **on-shell** and **off-shell** propagators and

$$\mathcal{A}_D^{(L)} (1; 2, \dots, n) \equiv -2\pi \sum_{i_r \in r} \text{Res} \left(d\mathcal{A}_F^{(L)} (1, \dots, n), \text{Im}(q_{i_1,0}) < 0 \right),$$

$$\mathcal{A}_F^{(L)} (1, \dots, n) = \int_{\ell_1 \dots \ell_L} N \times G_F (1, \dots, n)$$

- Cauchy contour is always closed from below the real axis



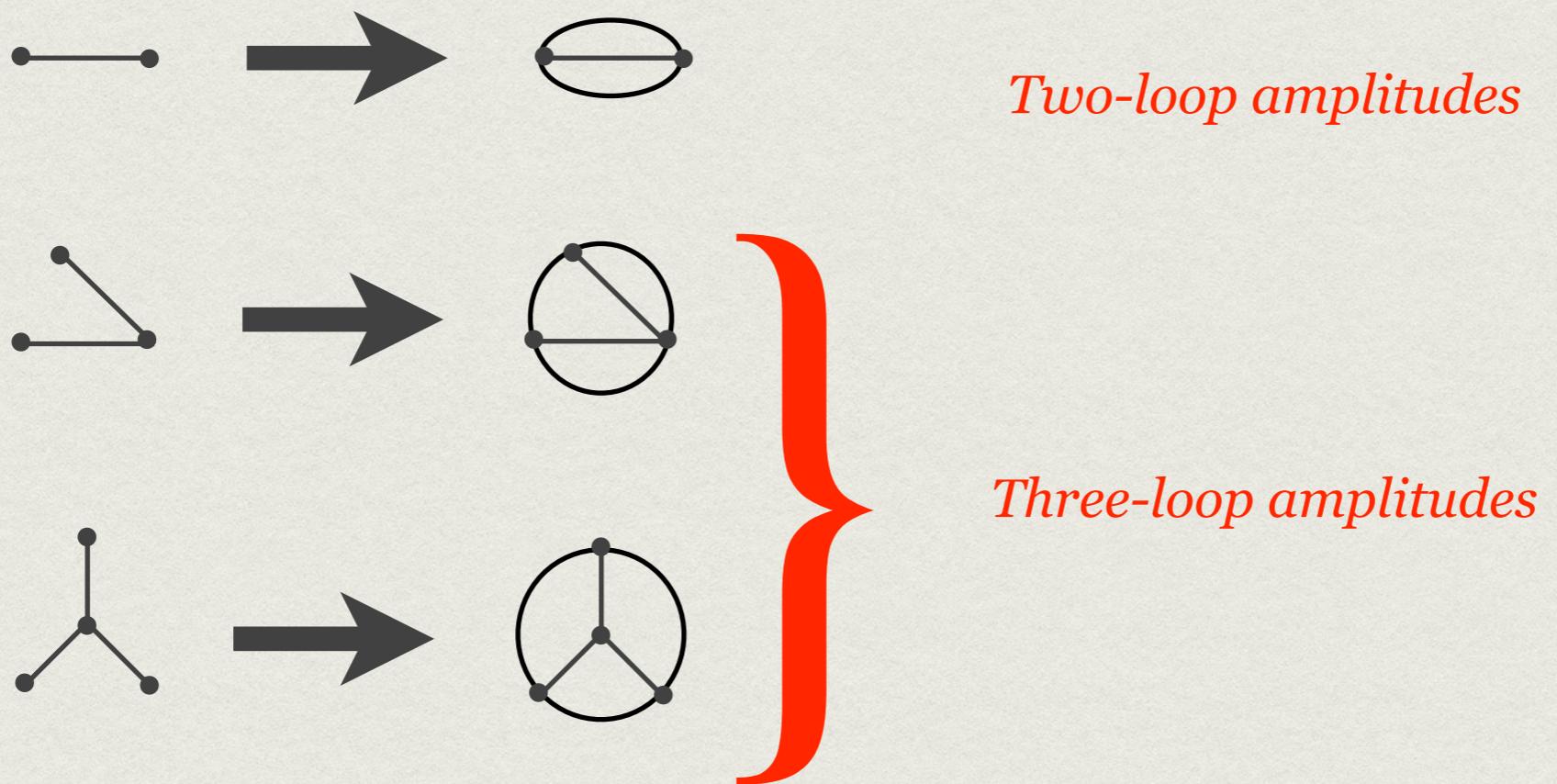
Let's recap

Everything started from 1 dot

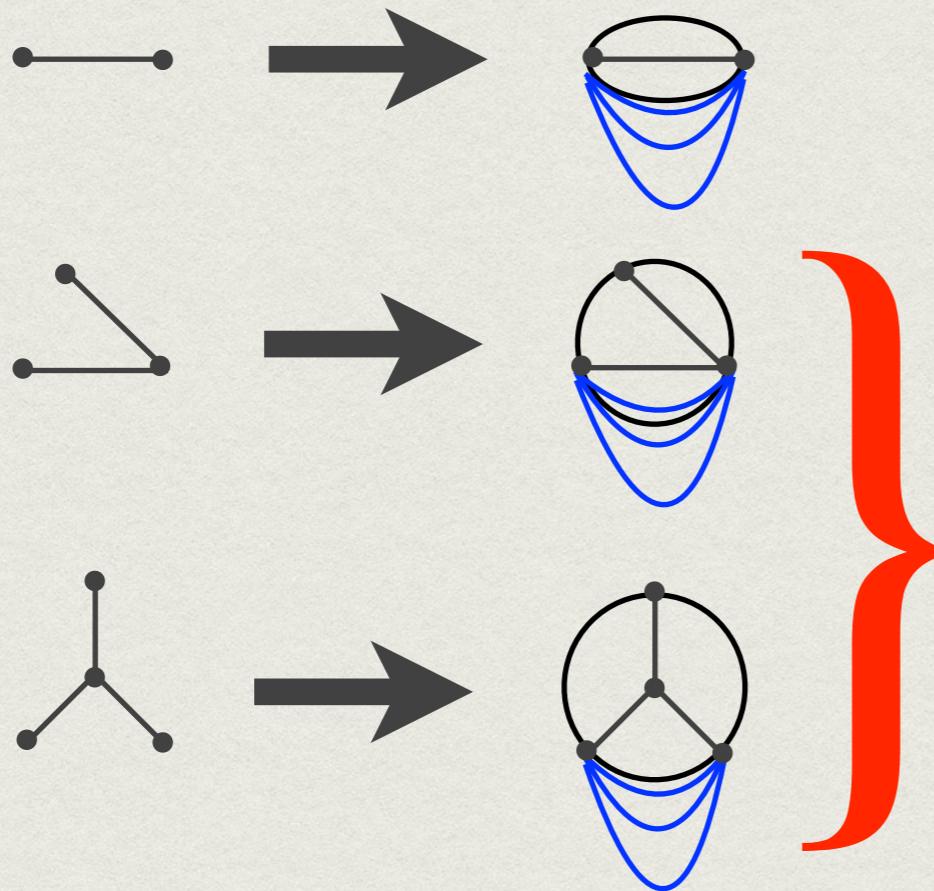


All one-loop amplitudes under control

Let's recap



Let's recap

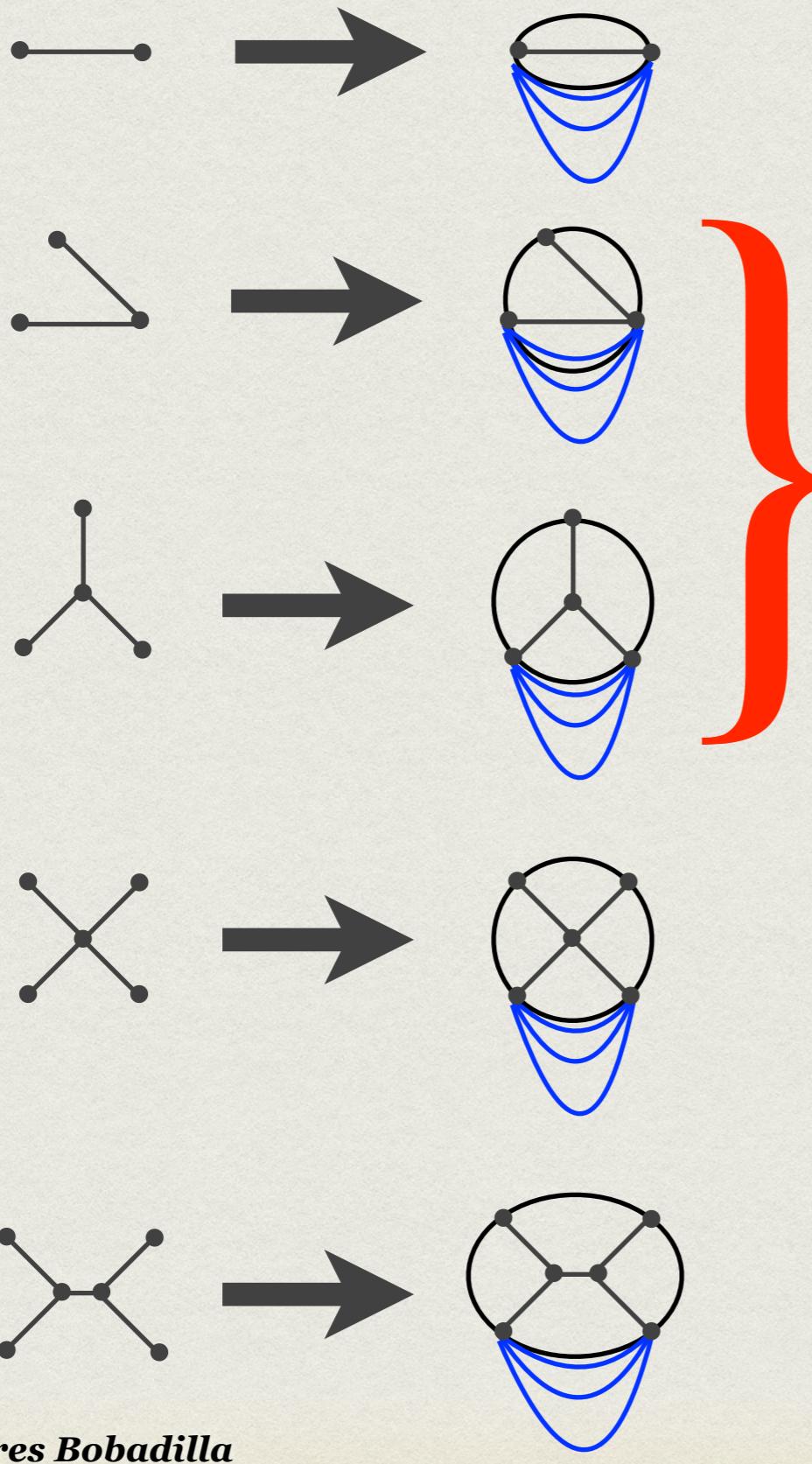


Two-loop amplitudes

Three-loop amplitudes

Same functional structure @ L loops

Let's recap



Two-loop amplitudes

Three-loop amplitudes

Same functional structure @ L loops

Follow the same approach @ 4 loops

Explicit applications

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

- So far \rightarrow relations among different kind of integral families
- The simplest application :: the two-loop sunrise diagram

$$\mathcal{A}_2^{(2)} = \int_{\ell_1 \ell_2} G_F(1, 2, 12) = \int_{\ell_1 \ell_2} \prod_{i=1,2,12} \frac{1}{(q_{i,0} - q_{i,0}^{(+)}) (q_{i,0} + q_{i,0}^{(+)})},$$
$$q_i = \ell_i, \quad q_{12} = -\ell_1 - \ell_2 - p$$



applying the Cauchy residue thm in $\{\ell_1, \ell_2\}$

$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} [G_D(1, 2) + G_D(1, \bar{1}\bar{2}) + G_D(\bar{2}, \bar{1}\bar{2})],$$

Explicit applications

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

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$$q_i = \ell_i, \quad q_{12} = -\ell_1 - \ell_2 - p$$



applying the Cauchy residue thm in $\{\ell_1, \ell_2\}$

$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} [G_D(1, 2) + G_D(1, \bar{1}\bar{2}) + G_D(\bar{2}, \bar{1}\bar{2})],$$

$$G_D(1, 2) = \frac{1}{4 q_{1,0}^{(+)} q_{2,0}^{(+)} \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 + q_{12,0}^{(+)}\right) \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 - q_{12,0}^{(+)}\right)}$$

Explicit applications

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

- So far \rightarrow relations among different kind of integral families
- The simplest application :: the two-loop sunrise diagram

$$\mathcal{A}_2^{(2)} = \int_{\ell_1 \ell_2} G_F(1, 2, 12) = \int_{\ell_1 \ell_2} \prod_{i=1,2,12} \frac{1}{(q_{i,0} - q_{i,0}^{(+)}) (q_{i,0} + q_{i,0}^{(+)})},$$

$$q_i = \ell_i, \quad q_{12} = -\ell_1 - \ell_2 - p$$



applying the Cauchy residue thm in $\{\ell_1, \ell_2\}$

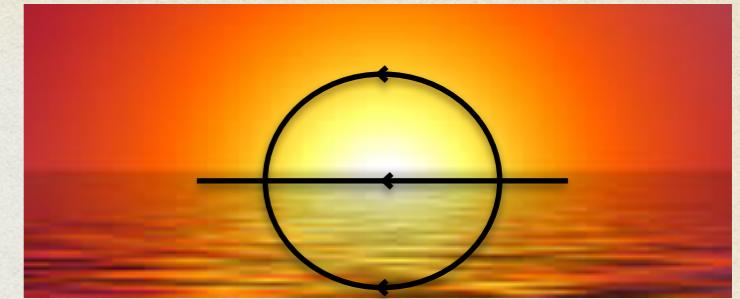
$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} [G_D(1, 2) + G_D(1, \bar{12}) + G_D(\bar{2}, \bar{12})],$$

$$G_D(i, j) \equiv \text{Res} \left(\text{Res} \left(G_F(1, 2, 12), \left\{ q_{i,0} = q_{i,0}^{(+)} \right\} \right), \left\{ q_{j,0} = q_{j,0}^{(+)} \right\} \right)$$

$$G_D(1, 2) = \frac{1}{4 q_{1,0}^{(+)} q_{2,0}^{(+)} \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 + q_{12,0}^{(+)} \right) \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0 - q_{12,0}^{(+)} \right)}$$

Explicit applications

- Some features with these individual residues



$$G_D(1, 2) = \frac{1}{4 q_{1,0}^{(+)} q_{2,0}^{(+)} \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} - p_0 \right) \left(q_{1,0}^{(+)} + q_{2,0}^{(+)} - q_{12,0}^{(+)} - p_0 \right)}$$

Structure of $q_{i,0}^{(+)}$

$q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} = p_0 \longrightarrow$ Causal threshold at p_0 ($p_0 > 0$)

$q_{1,0}^{(+)} + q_{2,0}^{(+)} - q_{12,0}^{(+)} = p_0 \longrightarrow$ Non-Causal threshold
Introduces unphysical singularities

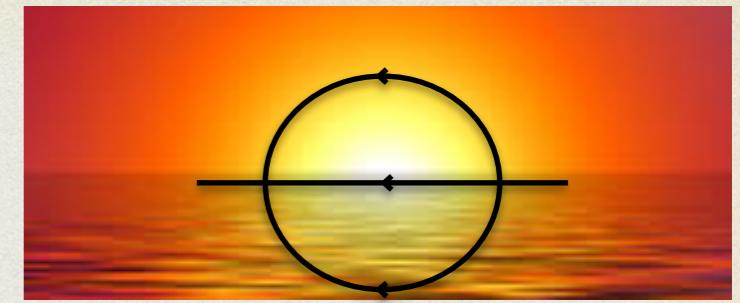
- Summing all contributions up

$$\mathcal{A}_2^{(2)} = - \int_{\vec{\ell}_1, \vec{\ell}_2} \frac{1}{8 q_{1,0}^{(+)} q_{2,0}^{(+)} q_{12,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} - p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} + p_0} \right)$$

Display causal structure only!

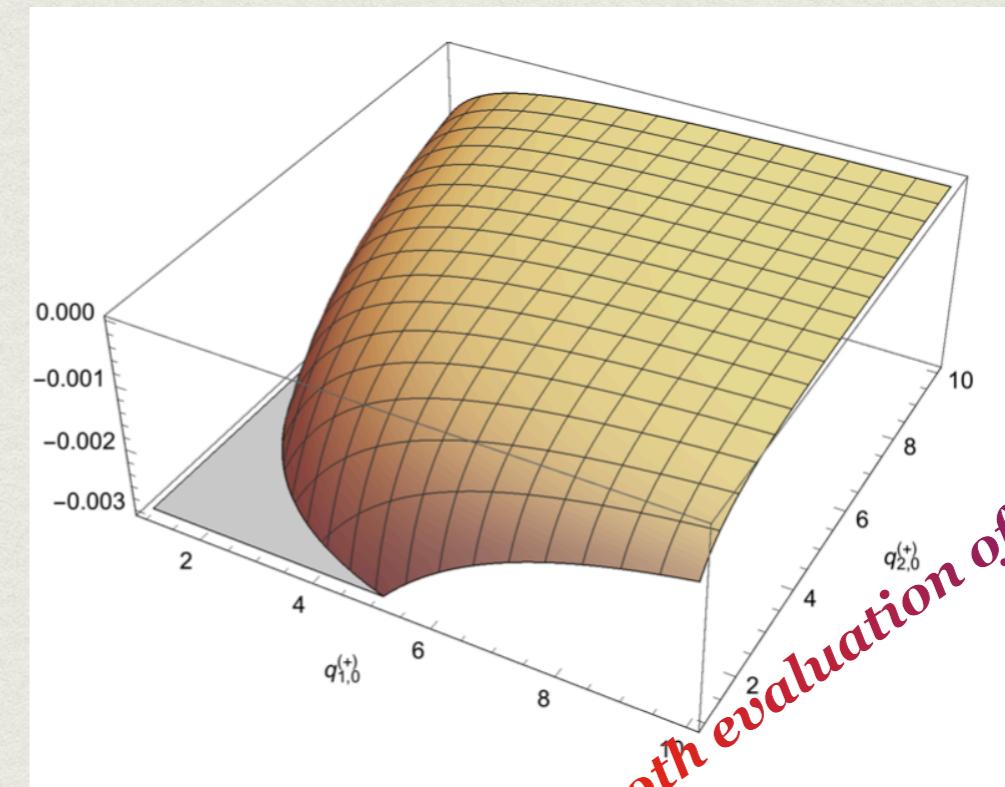
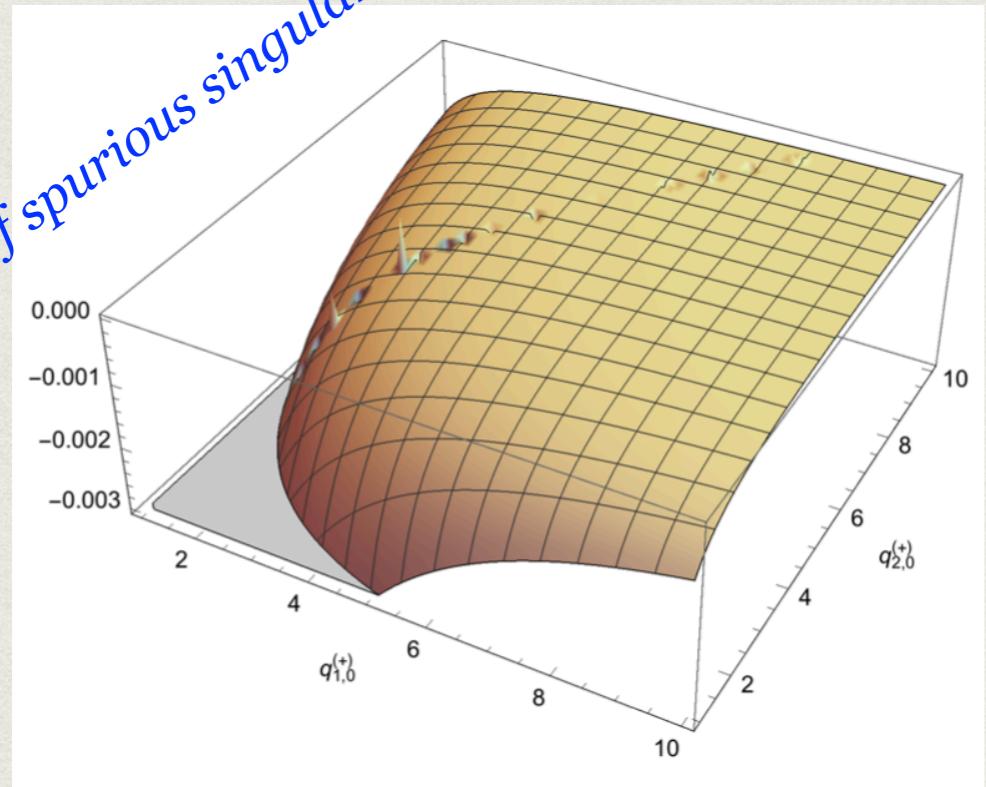
Explicit applications

- Causal vs non-causal repr.



$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} [G_D(1, 2) + G_D(1, \bar{12}) + G_D(\bar{2}, \bar{12})] ,$$

Presence of spurious singularities



Smooth evaluation of integrands

$$\mathcal{A}_2^{(2)} = - \int_{\vec{\ell}_1, \vec{\ell}_2} \frac{1}{8q_{1,0}^{(+)} q_{2,0}^{(+)} q_{12,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} - p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{12,0}^{(+)} + p_0} \right)$$

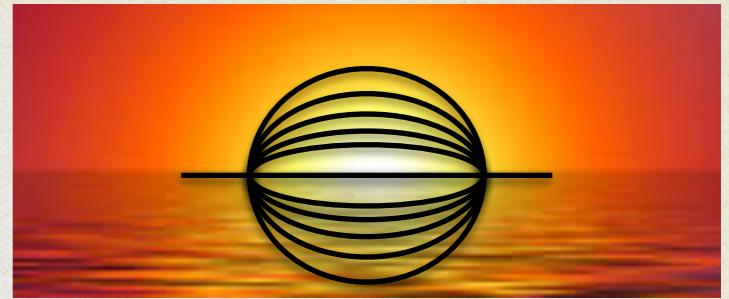
[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

MLT causal repr.

- Sunrise @ all orders

$$q_i = \ell_i, \quad \text{with } i \in \{1, \dots, L\},$$

$$q_{L+1} = - \sum_{i=1}^L \ell_i - p_1.$$



Compact causal repr.

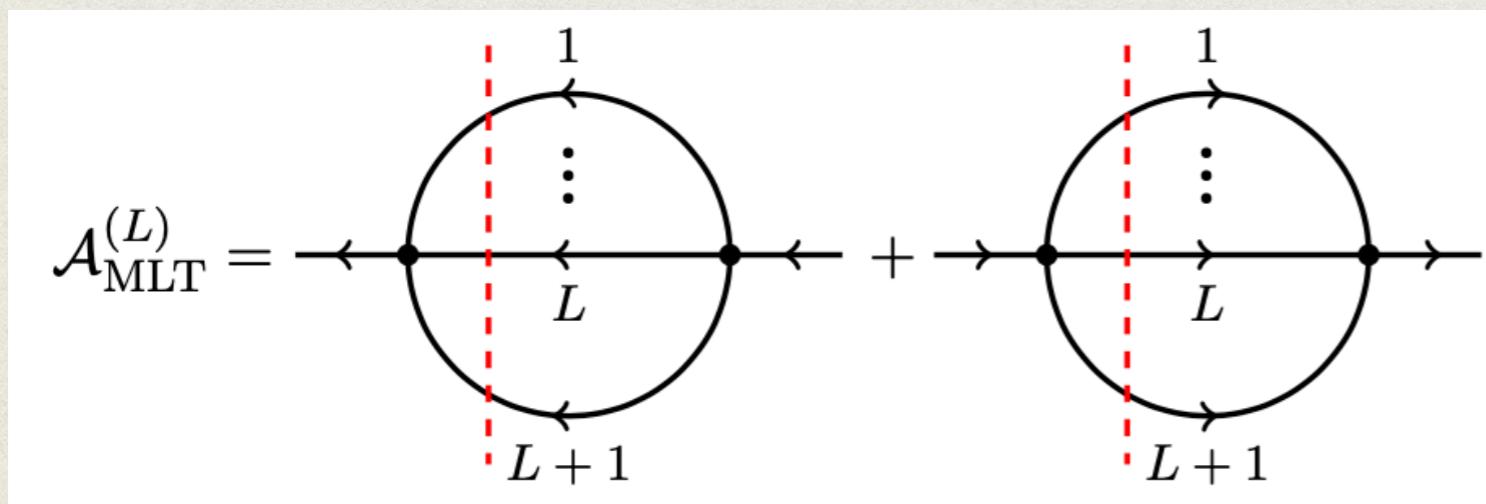
$$\mathcal{A}_{\text{MLT}}^{(L)} (1, 2, \dots, (L+1)_{-p_1}) = - \int_{\vec{\ell}_1, \dots, \vec{\ell}_L} \frac{1}{x_{L+1}} \left(\frac{1}{\lambda_1^-} + \frac{1}{\lambda_1^+} \right)$$

$$x_{L+k} = \prod_{i=1}^{L+k} 2q_{i,0}^{(+)}$$

$$\lambda_1^\pm = q_{(1,\dots,L+1),0}^{(+)} \pm p_{1,0}$$

sum over all int. lines

- Repr. in terms of causal thresholds



$\lambda_1^+ \rightarrow$ threshold if $p_{1,0} > 0$

$$\hookrightarrow q_{(1,\dots,L+1),0}^{(+)} = p_{1,0}$$

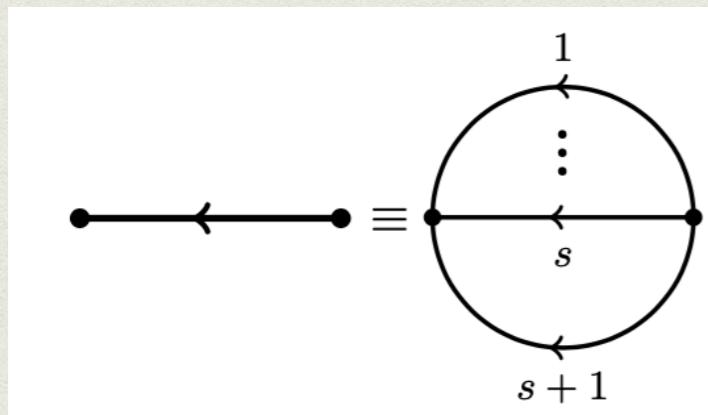
$\lambda_1^- \rightarrow$ threshold if $p_{1,0} < 0$

$$\hookrightarrow q_{(1,\dots,L+1),0}^{(+)} = -p_{1,0}$$

N^k MLT causal repr.

[W.J.T. (2021)]

- Classify loop topologies



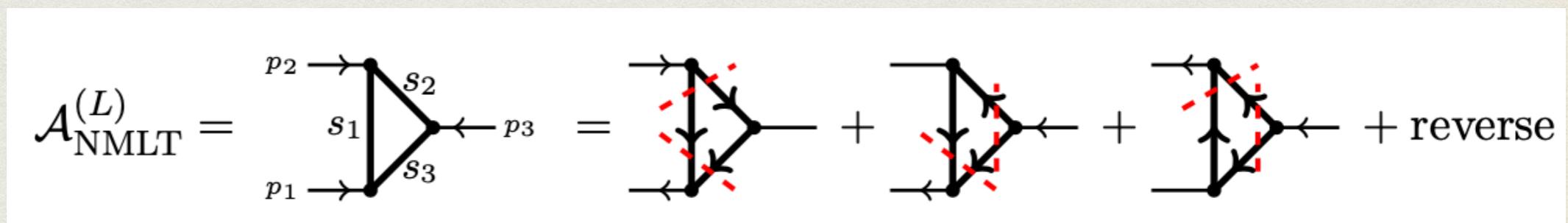
★ *Edges*
★ *Vertices*

- NMLT (3 vertices)

$$\mathcal{A}_{\text{NMLT}}^{(L)} = \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+2}} \left(\frac{1}{\lambda_1^+ \lambda_2^-} + \frac{1}{\lambda_1^+ \lambda_3^-} + \frac{1}{\lambda_2^+ \lambda_3^-} + (\lambda_i^+ \leftrightarrow \lambda_i^-) \right)$$

$$\begin{aligned}\lambda_1^\pm &= q_{(1,3),0}^{(+)} \pm p_{1,0}, \\ \lambda_2^\pm &= q_{(1,2),0}^{(+)} \pm p_{2,0}, \\ \lambda_3^\pm &= q_{(2,3),0}^{(+)} \pm p_{3,0}\end{aligned}$$

Two entangled causal thresholds

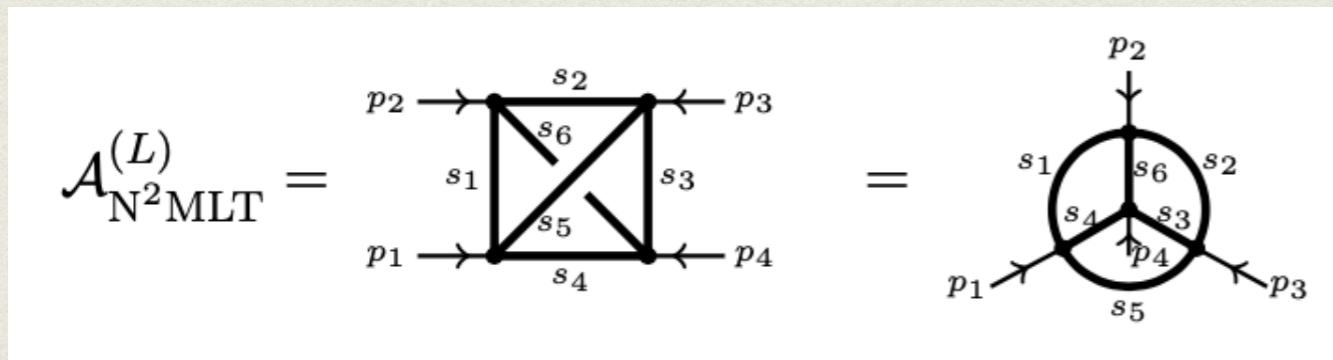


N^k MLT causal repr.

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

[W.J.T. (2021)]

- N²MLT (4 vertices)



all connections between vertices & edges!

[von Manteuffel, Schabinger (2015)]

[Peraro (2016, 2019)]

Reconstruct integrand from numerical evaluations over finite fields

[Klappert, Klein, Lange (2020)]

$$\begin{aligned} \mathcal{A}_{N^2\text{MLT}}^{(L)} = - \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+3}} & \left[\frac{1}{\lambda_{12}^+} \left(\frac{1}{\lambda_1^+} + \frac{1}{\lambda_2^+} \right) \left(\frac{1}{\lambda_3^-} + \frac{1}{\lambda_4^-} \right) + \frac{1}{\lambda_{13}^+} \left(\frac{1}{\lambda_1^+} + \frac{1}{\lambda_3^+} \right) \left(\frac{1}{\lambda_2^-} + \frac{1}{\lambda_4^-} \right) \right. \\ & \left. + \frac{1}{\lambda_{23}^+} \left(\frac{1}{\lambda_2^+} + \frac{1}{\lambda_3^+} \right) \left(\frac{1}{\lambda_1^-} + \frac{1}{\lambda_4^-} \right) + (\lambda_i^+ \leftrightarrow \lambda_i^-) \right] \end{aligned}$$

$$\lambda_1^\pm = q_{(1,4,5),0}^{(+)} \pm p_{1,0},$$

$$\lambda_2^\pm = q_{(1,2,6),0}^{(+)} \pm p_{2,0},$$

$$\lambda_3^\pm = q_{(2,3,5),0}^{(+)} \pm p_{3,0},$$

$$\lambda_4^\pm = q_{(3,4,6),0}^{(+)} \pm p_{4,0},$$

$$\lambda_{12}^\pm = q_{(2,4,5,6),0}^{(+)} \pm p_{12,0},$$

$$\lambda_{13}^\pm = q_{(1,2,3,4),0}^{(+)} \pm p_{13,0},$$

$$\lambda_{23}^\pm = q_{(1,3,5,6),0}^{(+)} \pm p_{23,0},$$

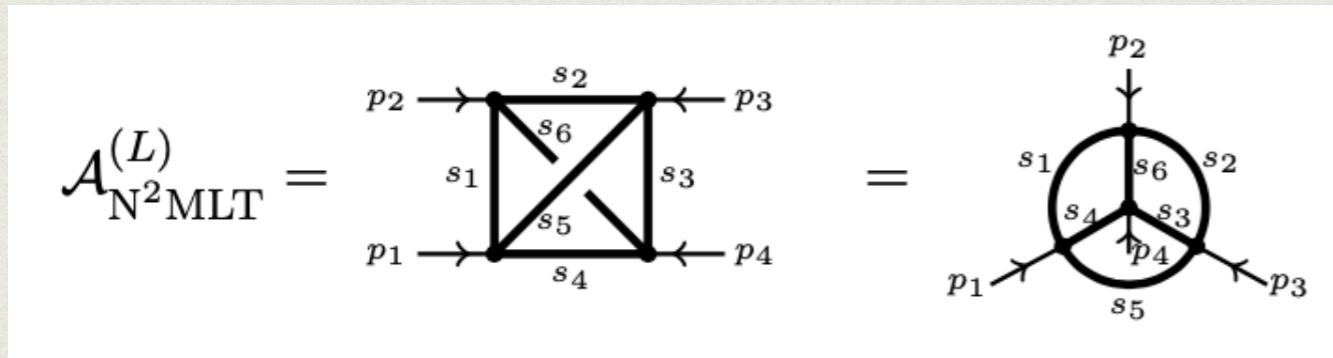
Three entangled causal thresholds

N^k MLT causal repr.

[Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, W.J.T. (2020)]

[W.J.T. (2021)]

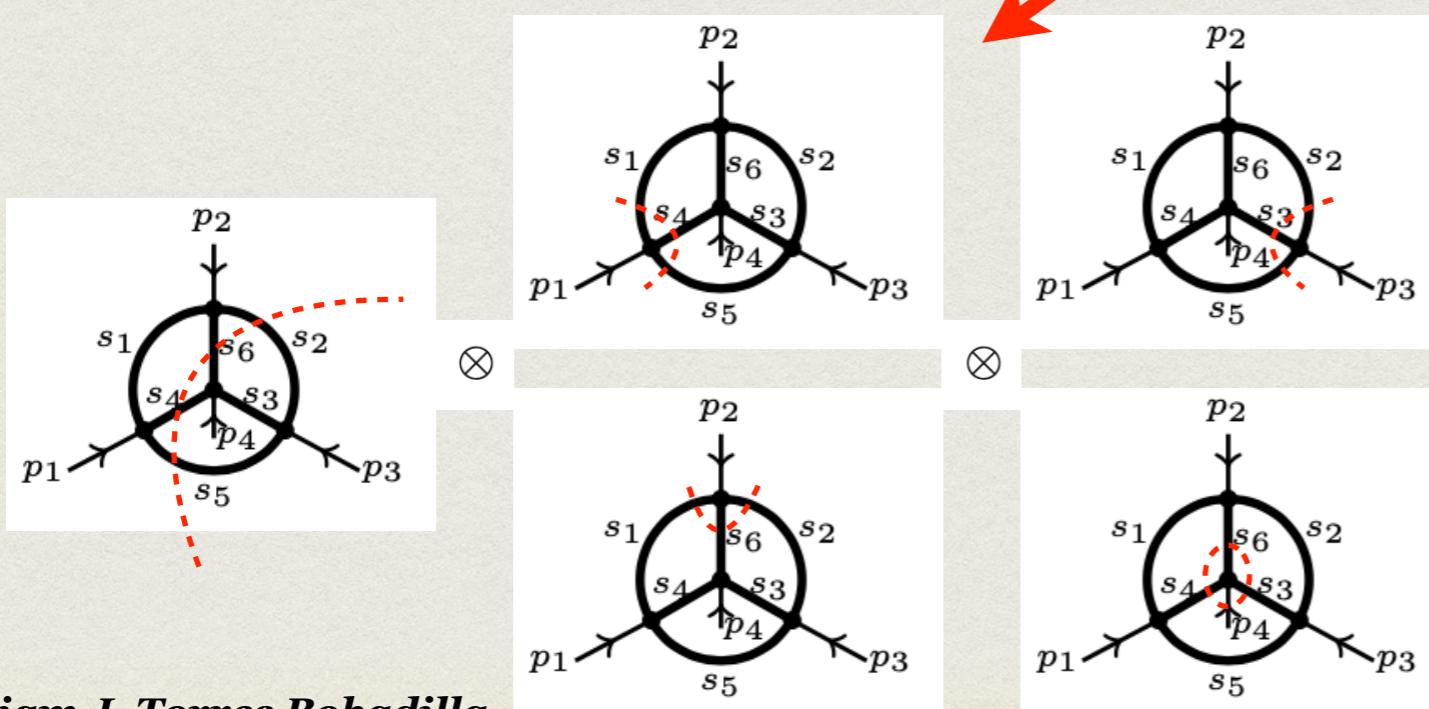
- N²MLT (4 vertices)



all connections between vertices & edges!

$$\mathcal{A}_{N^2\text{MLT}}^{(L)} = - \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+3}} \left[\frac{1}{\lambda_{12}^+} \left(\frac{1}{\lambda_1^+} + \frac{1}{\lambda_2^+} \right) \left(\frac{1}{\lambda_3^-} + \frac{1}{\lambda_4^-} \right) + \frac{1}{\lambda_{13}^+} \left(\frac{1}{\lambda_1^+} + \frac{1}{\lambda_3^+} \right) \left(\frac{1}{\lambda_2^-} + \frac{1}{\lambda_4^-} \right) \right. \\ \left. + \frac{1}{\lambda_{23}^+} \left(\frac{1}{\lambda_2^+} + \frac{1}{\lambda_3^+} \right) \left(\frac{1}{\lambda_1^-} + \frac{1}{\lambda_4^-} \right) + (\lambda_i^+ \leftrightarrow \lambda_i^-) \right]$$

Three entangled causal thresholds



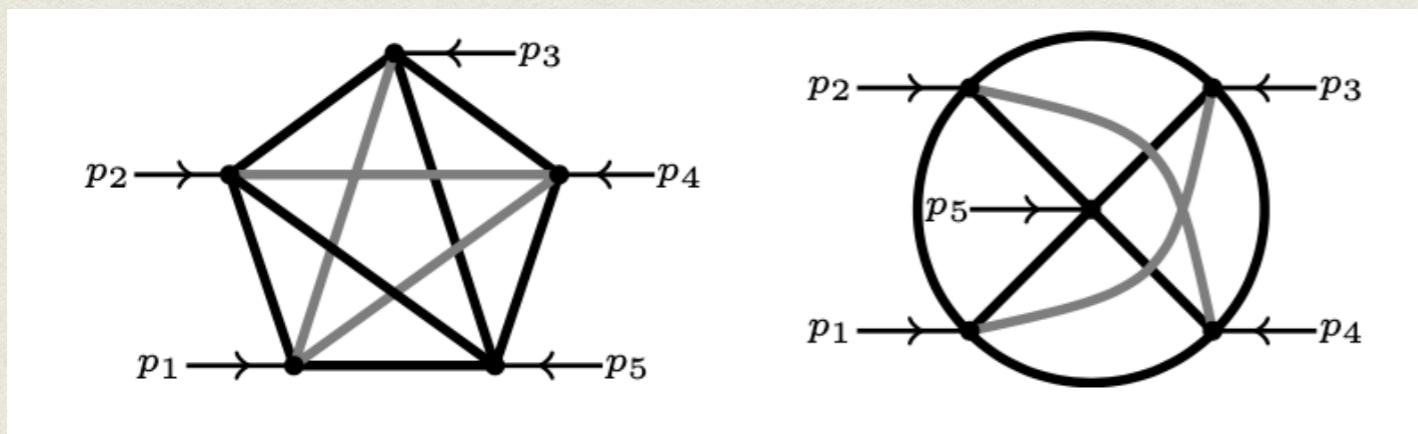
$$\begin{aligned} \lambda_1^\pm &= q_{(1,4,5),0}^{(+)} \pm p_{1,0}, \\ \lambda_2^\pm &= q_{(1,2,6),0}^{(+)} \pm p_{2,0}, \\ \lambda_3^\pm &= q_{(2,3,5),0}^{(+)} \pm p_{3,0}, \\ \lambda_4^\pm &= q_{(3,4,6),0}^{(+)} \pm p_{4,0}, \\ \lambda_{12}^\pm &= q_{(2,4,5,6),0}^{(+)} \pm p_{12,0}, \\ \lambda_{13}^\pm &= q_{(1,2,3,4),0}^{(+)} \pm p_{13,0}, \\ \lambda_{23}^\pm &= q_{(1,3,5,6),0}^{(+)} \pm p_{23,0}, \end{aligned}$$

N^k MLT causal repr.

[W.J.T. (2021)]

Four entangled causal thresholds

- N^3 MLT (5 vertices)



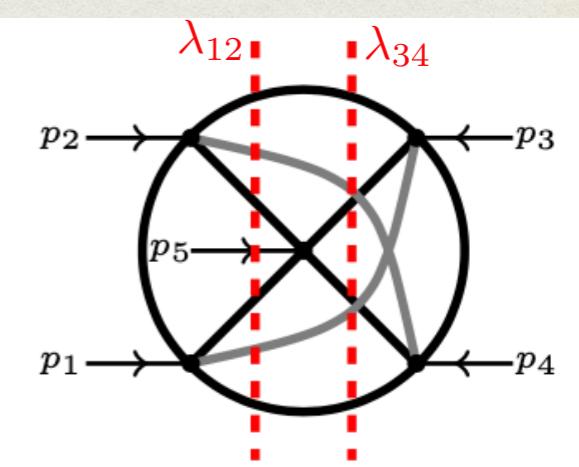
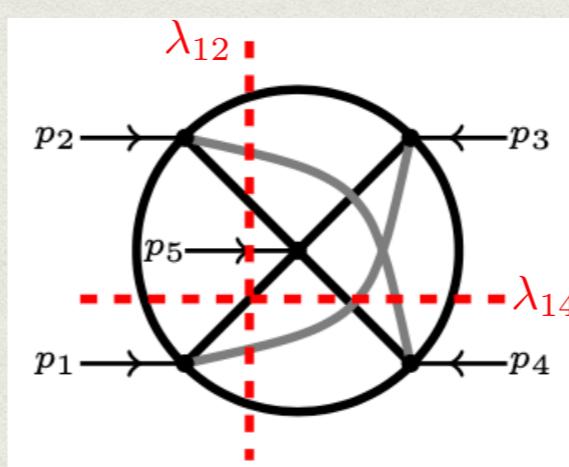
Direct application of LTD

$$\begin{aligned} \lambda_1^\pm &= q_{(1,4,5,7),0}^{(+)} \pm p_{1,0}, & \lambda_{12}^\pm &= q_{(2,4,5,6,7,8),0}^{(+)} \pm p_{12,0}, & \lambda_{24}^\pm &= q_{(1,2,3,4,8,10),0}^{(+)} \pm p_{24,0}, \\ \lambda_2^\pm &= q_{(1,2,6,8),0}^{(+)} \pm p_{2,0}, & \lambda_{13}^\pm &= q_{(1,2,3,4,7,9),0}^{(+)} \pm p_{13,0}, & \lambda_{35}^\pm &= q_{(2,3,5,7,8,10),0}^{(+)} \pm p_{35,0}, \\ \lambda_3^\pm &= q_{(2,3,5,9),0}^{(+)} \pm p_{3,0}, & \lambda_{23}^\pm &= q_{(1,3,5,6,8,9),0}^{(+)} \pm p_{23,0}, & \lambda_{34}^\pm &= q_{(2,4,5,6,9,10),0}^{(+)} \pm p_{34,0}, \\ \lambda_4^\pm &= q_{(3,4,6,10),0}^{(+)} \pm p_{4,0}, & \lambda_{45}^\pm &= q_{(3,4,6,7,8,9),0}^{(+)} \pm p_{45,0}, & \lambda_{25}^\pm &= q_{(1,2,6,7,9,10),0}^{(+)} \pm p_{25,0}, \\ \lambda_5^\pm &= q_{(7,8,9,10),0}^{(+)} \pm p_{5,0}, & \lambda_{14}^\pm &= q_{(1,3,5,6,7,10),0}^{(+)} \pm p_{14,0}, & \lambda_{15}^\pm &= q_{(1,4,5,8,9,10),0}^{(+)} \pm p_{15,0}. \end{aligned}$$

Reconstruct integrand
→ finite fields

$$\mathcal{A}_{N^3\text{MLT}}^{(L)} = \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1 \\ j=i+1}}^5 \sum_{\substack{k=1 \\ l=k+1 \\ k,l \neq i,j}}^5 L_{ij}^+ L_{kl}^-.$$

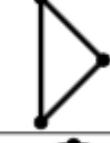
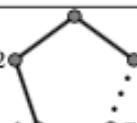
$$L_{ij}^\pm = \frac{1}{\lambda_{ij}^\pm} \left(\frac{1}{\lambda_i^\pm} + \frac{1}{\lambda_j^\pm} \right).$$



N^kMLT causal repr.

[W.J.T. (2021)]

- In general

	Vertices	Edges	Loops	$\dim \lambda_i^\pm$
	2	1	0	1
	3	3	1	3
	4	6	3	7
	5	10	6	15
\vdots	\vdots	\vdots	\vdots	\vdots
	n	$\binom{n}{2}$	$\frac{(n-1)(n-2)}{2}$	$2^{n-1} - 1$

★ *Edges*

★ *Vertices*

All subsets of up-to $[n/2]$ elements

$$\vec{\lambda}_i^\pm = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \\ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$$

intersection of subsets $\rightarrow \emptyset$

$$\{\{\{1, 2\}, \{3, 4\}\}, \{\{1, 2\}, \{3, 5\}\}, \{\{1, 2\}, \{4, 5\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 3\}, \{2, 5\}\}, \\ \{\{1, 3\}, \{4, 5\}\}, \{\{1, 4\}, \{2, 3\}\}, \{\{1, 4\}, \{2, 5\}\}, \{\{1, 4\}, \{3, 5\}\}, \{\{1, 5\}, \{2, 3\}\}, \\ \{\{1, 5\}, \{2, 4\}\}, \{\{1, 5\}, \{3, 4\}\}, \{\{2, 3\}, \{4, 5\}\}, \{\{2, 4\}, \{3, 5\}\}, \{\{2, 5\}, \{3, 4\}\}\},$$

$$\hookrightarrow \{\{i, j\}, \{k, l\}\} \rightarrow L_{ij}^+ L_{kl}^- + L_{ij}^- L_{kl}^+$$

Same causal repr. w/out LTD !

summing over all contributions

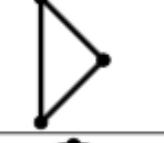
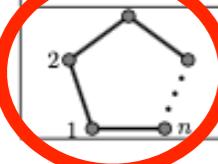
$$d\mathcal{A}_{N^3MLT}^{(L)} \sim \sum_{\substack{i=1 \\ j=i+1}}^5 \sum_{\substack{k=1 \\ l=k+1 \\ k,l \neq i,j}}^5 L_{ij}^+ L_{kl}^-.$$

N^kMLT causal repr.

[W.J.T. (2021)]

- In general

loop topology made of $k+2$ vertices $\rightarrow N^kMLT$

	Vertices	Edges	Loops	$\dim \lambda_i^\pm$
	2	1	0	1
	3	3	1	3
	4	6	3	7
	5	10	6	15
\vdots	\vdots	\vdots	\vdots	\vdots
	n	$\binom{n}{2}$	$\frac{(n-1)(n-2)}{2}$	$2^{n-1} - 1$

★ *Edges*

★ *Vertices*

All-loop causal representation

$$d\mathcal{A}_{L+k+1}^{(L)} = \frac{(-1)^{k+1}}{x_{L+k+1}} \mathcal{F}_{L+k+1}(\lambda_i^\pm) .$$

$N^k M L T$ causal repr.

[W.J.T. (2021)]

- In general

loop topology made of $k+2$ vertices $\rightarrow N^k M L T$

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★ *Edges*

★ *Vertices*

universal structure

$$x_{L+k} = \prod_{i=1}^{L+k} 2q_{i,0}^{(+)}$$

depends on # lines

All-loop causal representation

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★ Edges

★ Vertices

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All-loop causal representation

$$d\mathcal{A}_{L+k+1}^{(L)} = \frac{(-1)^{k+1}}{x_{L+k+1}} \mathcal{F}_{L+k+1}(\lambda_i^\pm)$$

$$\mathcal{F}_{L+k} = \sum_{\substack{i_1 \ll i_{N_i} \\ j_1 \ll j_{N_j}}}^{k+2} \Omega_{\vec{i}}^{\vec{j}} L_{i_1 i_2 \dots i_{N_i}}^+ L_{j_1 j_2 \dots j_{N_j}}^- ,$$

$\Omega_{\vec{i}}^{\vec{j}} = \begin{cases} 1 & \text{If } \vec{i} \cap \vec{j} = \emptyset \\ 0 & \text{otherwise} \end{cases}$

causal thresholds cannot overlap

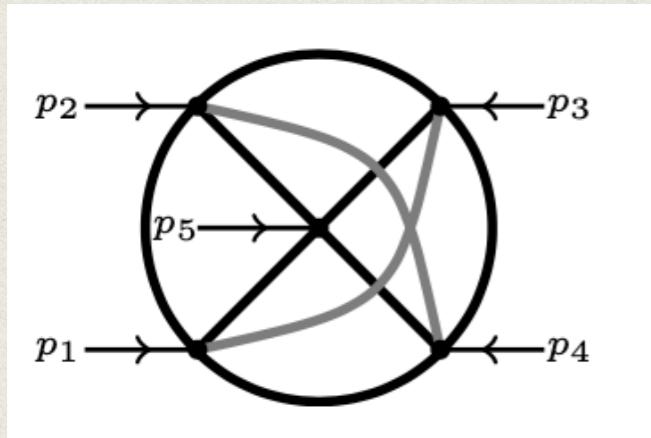
$$L_{i_1 i_2 \dots i_N}^\pm = \frac{1}{\lambda_{i_1 i_2 \dots i_N}^\pm} \sum_{\substack{j_1 \ll j_{N-1} \\ \vec{j} \subset \vec{i}}} L_{j_1 j_2 \dots j_{N-1}}^\pm , \quad L_{i_1}^\pm = \frac{1}{\lambda_{i_1}^\pm} .$$

All-loop order & multiplicity

$$\mathcal{A}_{\text{N}^3\text{MLT}}^{(L)} = \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1 \\ j=i+1}}^5 \sum_{\substack{k=1 \\ l=k+1 \\ k,l \neq i,j}}^5 L_{ij}^+ L_{kl}^-.$$

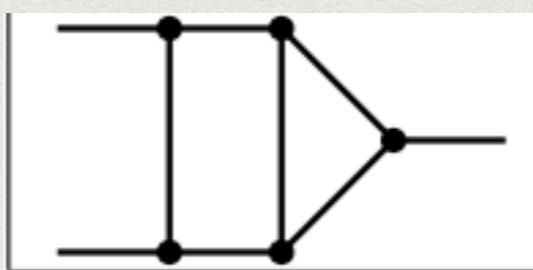
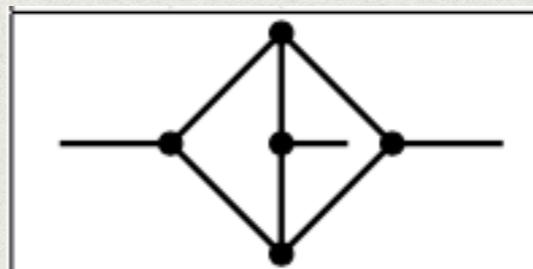
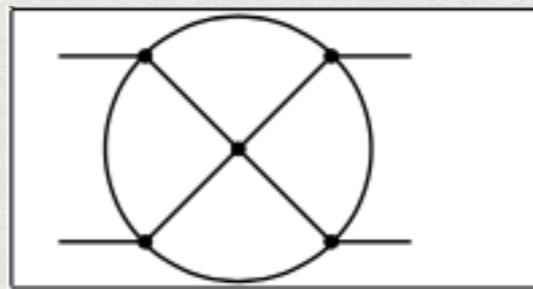
$$L_{ij}^\pm = \frac{1}{\lambda_{ij}^\pm} \left(\frac{1}{\lambda_i^\pm} + \frac{1}{\lambda_j^\pm} \right).$$

Compute it once and for all



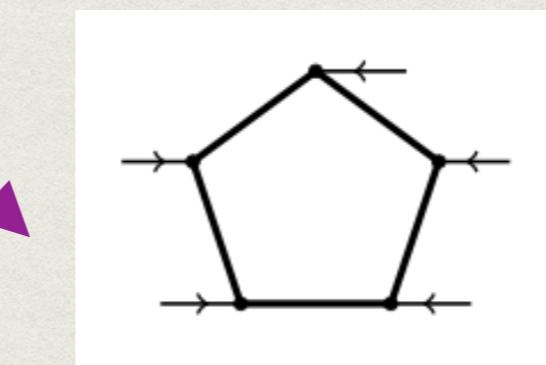
Removing procedure

[W.J.T. (2021)]



★ Equal # Vertices

★ Less # Edges



Get causal repr. for free!

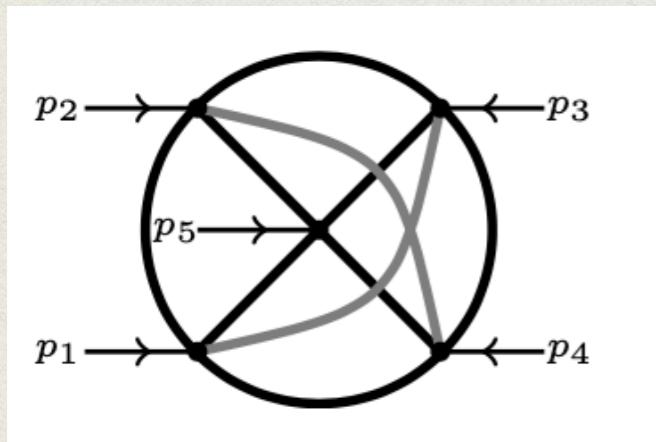
$$\begin{aligned} \lambda_1^\pm &= q_{(1,4,5,7),0}^{(+)} \pm p_{1,0}, \\ \lambda_2^\pm &= q_{(1,2,6,8),0}^{(+)} \pm p_{2,0}, \\ \lambda_3^\pm &= q_{(2,3,5,9),0}^{(+)} \pm p_{3,0}, \\ \lambda_4^\pm &= q_{(3,4,6,10),0}^{(+)} \pm p_{4,0}, \\ \lambda_5^\pm &= q_{(7,8,9,10),0}^{(+)} \pm p_{5,0}, \end{aligned}$$

All-loop order & multiplicity

$$\mathcal{A}_{N^3MLT}^{(L)} = \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1 \\ j=i+1}}^5 \sum_{\substack{k=1 \\ l=k+1 \\ k,l \neq i,j}}^5 L_{ij}^+ L_{kl}^- .$$

$$L_{ij}^\pm = \frac{1}{\lambda_{ij}^\pm} \left(\frac{1}{\lambda_i^\pm} + \frac{1}{\lambda_j^\pm} \right) .$$

Compute it once and for all



$$\lambda_1^\pm = q_{(1,4,5,7),0}^{(+)} \pm p_{1,0} ,$$

$$\lambda_2^\pm = q_{(1,2,6,8),0}^{(+)} \pm p_{2,0} ,$$

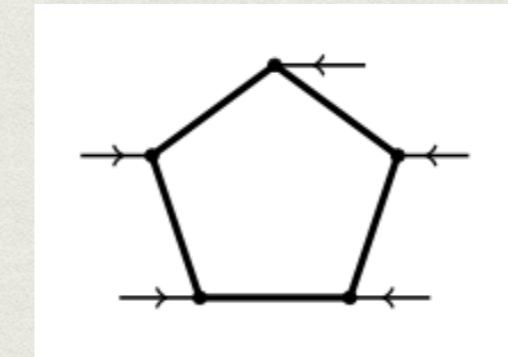
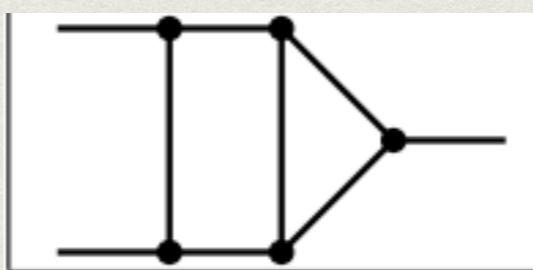
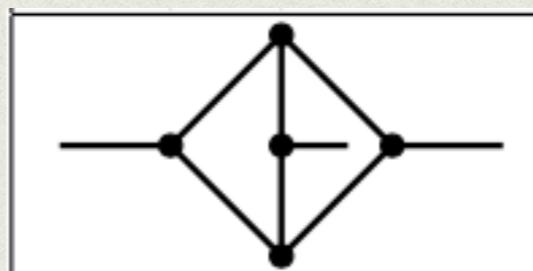
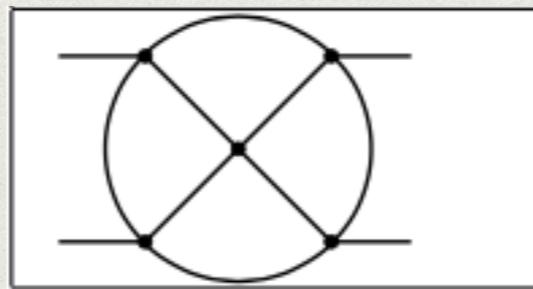
$$\lambda_3^\pm = q_{(2,3,5,9),0}^{(+)} \pm p_{3,0} ,$$

$$\lambda_4^\pm = q_{(3,4,6,10),0}^{(+)} \pm p_{4,0} ,$$

$$\lambda_5^\pm = q_{(7,8,9,10),0}^{(+)} \pm p_{5,0} ,$$

Removing procedure

[W.J.T. (2021)]



★ Equal # Vertices

★ Less # Edges

$$d\mathcal{A}_{N^kMLT}^{(L-|s_i|)} = \lim_{q_{s_i}^{(+)} \rightarrow 0} 2q_{s_i}^{(+)} d\mathcal{A}_{N^kMLT}^{(L)},$$

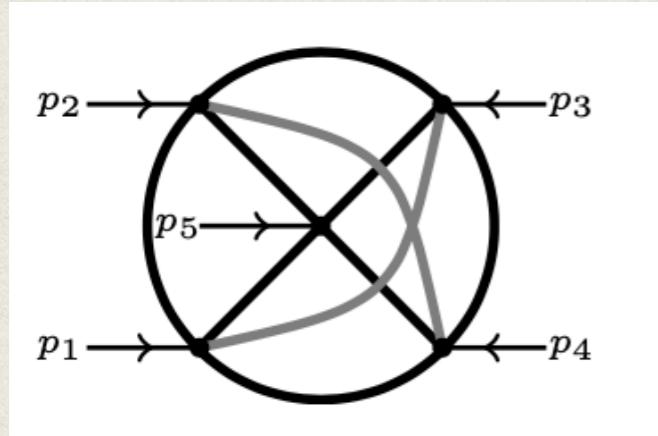
Get causal repr. for free!

All-loop order & multiplicity

$$\mathcal{A}_{\text{N}^3\text{MLT}}^{(L)} = \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1 \\ j=i+1}}^5 \sum_{\substack{k=1 \\ l=k+1 \\ k,l \neq i,j}}^5 L_{ij}^+ L_{kl}^- .$$

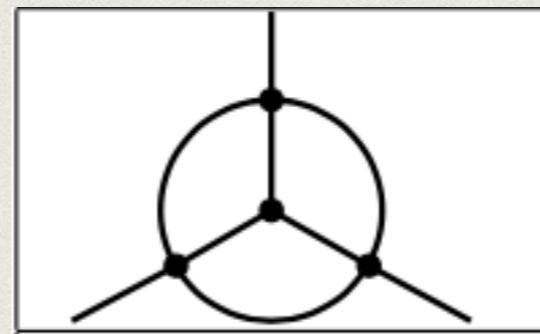
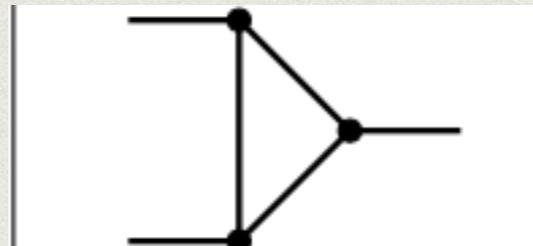
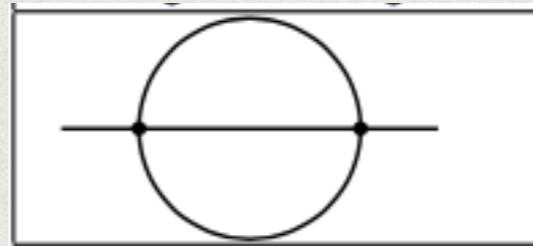
$$L_{ij}^\pm = \frac{1}{\lambda_{ij}^\pm} \left(\frac{1}{\lambda_i^\pm} + \frac{1}{\lambda_j^\pm} \right) .$$

Compute it once and for all



Collapsing procedure

[W.J.T. (2021)]



- ★ *Less # Vertices*
- ★ *Less # Edges*

Get causal repr. for free!

$$\lambda_1^\pm = q_{(1,4,5,7),0}^{(+)} \pm p_{1,0} ,$$

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$$\lambda_4^\pm = q_{(3,4,6,10),0}^{(+)} \pm p_{4,0} ,$$

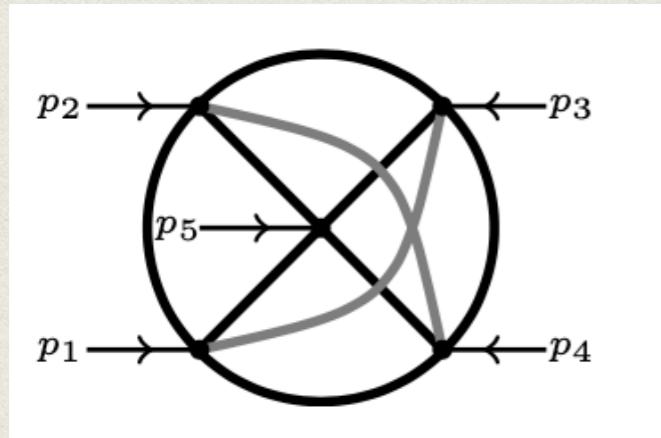
$$\lambda_5^\pm = q_{(7,8,9,10),0}^{(+)} \pm p_{5,0} ,$$

All-loop order & multiplicity

$$\mathcal{A}_{N^3MLT}^{(L)} = \int_{\ell_1, \dots, \ell_L} \frac{1}{x_{L+4}} \sum_{\substack{i=1 \\ j=i+1}}^5 \sum_{\substack{k=1 \\ l=k+1 \\ k,l \neq i,j}}^5 L_{ij}^+ L_{kl}^-.$$

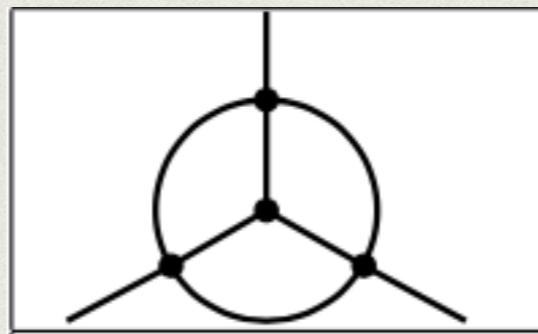
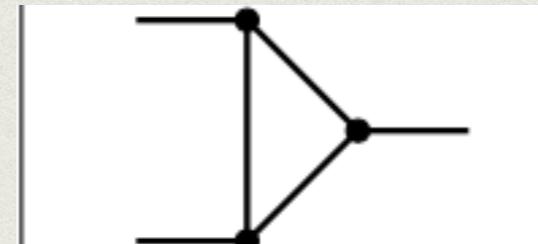
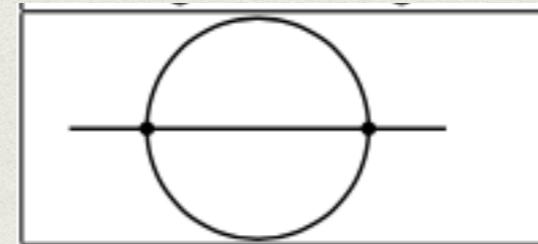
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Compute it once and for all



Collapsing procedure

[W.J.T. (2021)]



- ★ *Less # Vertices*
- ★ *Less # Edges*

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$$\begin{aligned} \lambda_1^\pm &= q_{(1,4,5,7),0}^{(+)} \pm p_{1,0}, \\ \lambda_2^\pm &= q_{(1,2,6,8),0}^{(+)} \pm p_{2,0}, \\ \lambda_3^\pm &= q_{(2,3,5,9),0}^{(+)} \pm p_{3,0}, \\ \lambda_4^\pm &= q_{(3,4,6,10),0}^{(+)} \pm p_{4,0}, \\ \lambda_5^\pm &= q_{(7,8,9,10),0}^{(+)} \pm p_{5,0}, \end{aligned}$$

$$d\mathcal{A}_{N^2MLT} = -\frac{1}{2} \lim_{s_7, s_8, s_9, s_{10} \rightarrow 0} \prod_{i \in s_7 \cup s_8 \cup s_9 \cup s_{10}} 2q_{i,0}^{(+)} \lim_{\lambda_5^\pm \rightarrow 0} \lambda_5^\pm d\mathcal{A}_{N^3MLT}.$$

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

What about numerators?

“pull out” energies

$$\mathcal{A}_F^{(L)}(1, \dots, n) = \int_{\ell_1 \dots \ell_L} N \times G_F(1, \dots, n)$$

make use of poly. div. \rightarrow scalar integrands

$$q_i \cdot q_j = q_i^{(+)} \cdot q_j^{(+)} - q_{i,0}^{(+)} q_{j,0}^{(+)} + q_{i,0} q_{j,0},$$

$$\hookrightarrow N = \sum_{i=1}^R c_i \prod_{j=1}^L \ell_{j,0}^{\alpha_{ij}},$$

Application: four gluons @2L

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

What about numerators?

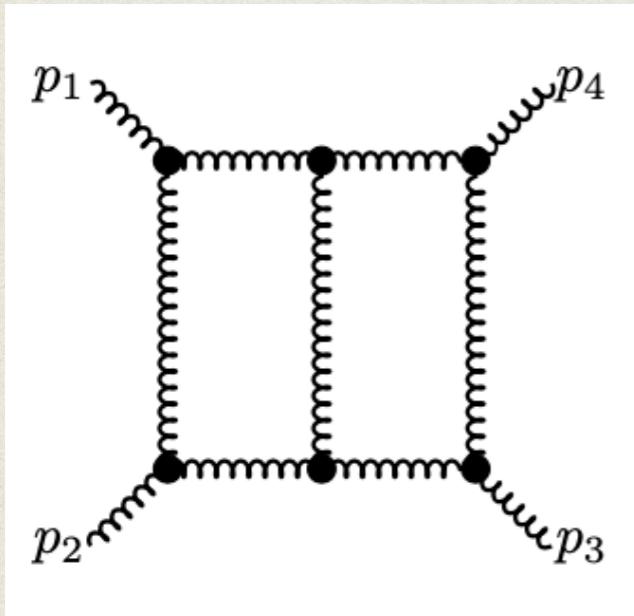
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Application: four gluons @2L



$$N = \sum_{r,s}^{\text{rank}} c_{rs} \ell_{1,0}^r \ell_{2,0}^s, \quad \text{with } r, s \leq 4 \wedge r + s \leq 6.$$

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

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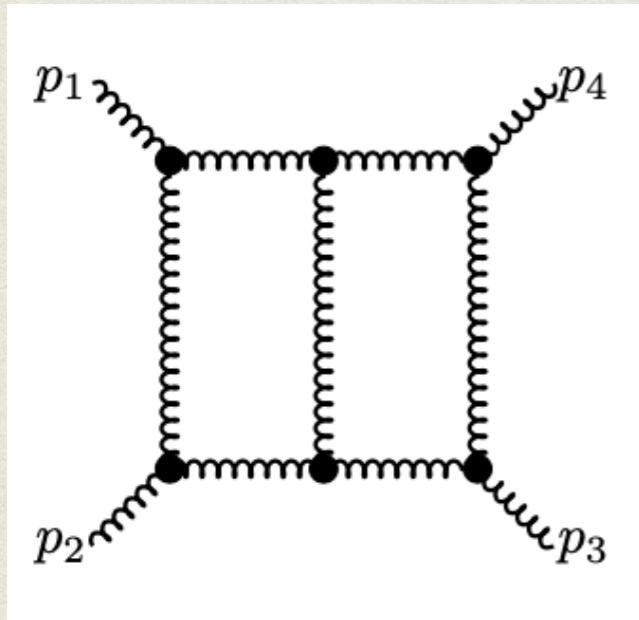
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$$\begin{aligned}
 &= \tilde{c}_8 J_{\{0,0,1,0,0,1,1\}} + \tilde{c}_7 J_{\{0,0,1,0,1,0,1\}} + \tilde{c}_{27} J_{\{0,0,1,0,1,1,1\}} + \tilde{c}_9 J_{\{0,0,1,1,0,1,1\}} \\
 &\quad + \tilde{c}_{22} J_{\{0,0,1,1,1,1,1\}} + \tilde{c}_3 J_{\{0,1,0,0,0,1,1\}} + \tilde{c}_4 J_{\{0,1,0,0,1,0,1\}} + \tilde{c}_{25} J_{\{0,1,0,0,1,1,1\}} \\
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 &\quad + \tilde{c}_6 J_{\{1,0,0,0,0,1,1\}} + \tilde{c}_5 J_{\{1,0,0,0,1,0,1\}} + \tilde{c}_{26} J_{\{1,0,0,0,1,1,1\}} + \tilde{c}_{16} J_{\{1,0,1,0,0,1,1\}} \\
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 \end{aligned}$$

All-loop order & multiplicity

[W.J.T. (w.i.p.)]

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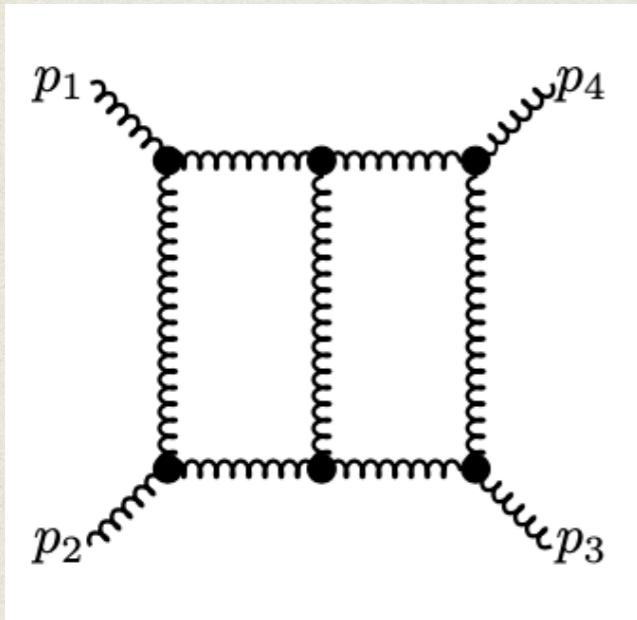
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$$\frac{c_{1,4} \left((p_{1,0} + p_{2,0}) (p_{1,0} + p_{2,0} + 2p_{3,0}) - \left(q_{1,0}^{(+)}\right)^2 + \left(q_{3,0}^{(+)}\right)^2 \right)}{4p_{1,0} (p_{1,0} + p_{2,0}) p_{3,0}}$$

Application: four gluons @2L



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with $r, s \leq 4 \wedge r + s \leq 6$.

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 \end{aligned}$$

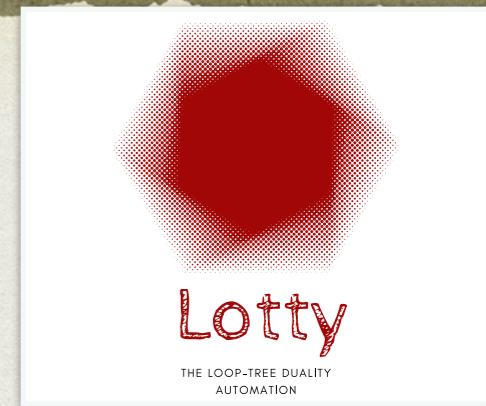
Lotty – The loop-tree duality automation

In[1]:= << Lotty`

```
-----  
Lotty -- the LOop-Tree dualITY automation  
by William J. Torres Bobadilla (MPP)  
Version 1.0 (March 14th of 2021)  
All functions are stored in the variable $LottyFunctions  
-----
```

Download Lotty from

<https://bitbucket.org/wjtorresb/lotty>



[W.J.T. (2021)]

- ➊ Dual repr. of multi-loop Feynman integrands
- ➋ All-loop causal repr. of scalar Feynman integrands
- ➌ Singular structure of any multi-loop topology
- ➍ Direct translation to spherical coordinates

Lotty – The loop-tree duality automation

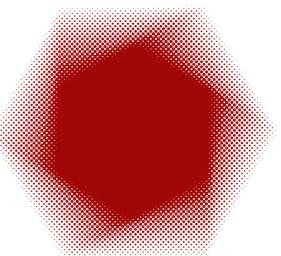
In[1]:= << Lotty`

Lotty -- the LLoop-Tree dualITY automation

by William J. Torres Bobadilla (MPP)

Version 1.0 (March 14th of 2021)

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Lotty

THE LOOP-TREE DUALITY
AUTOMATION

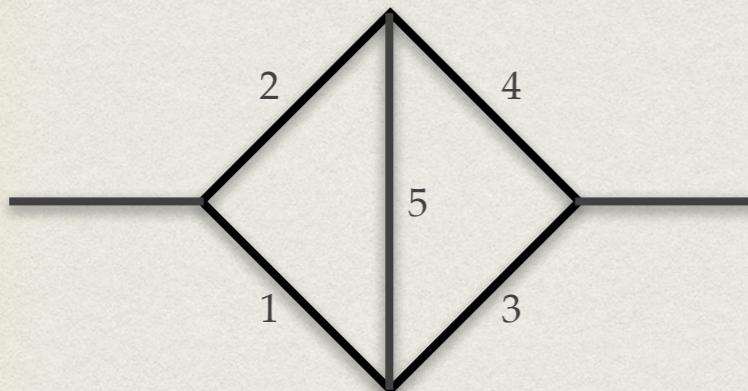
[W.J.T. (2021)]

Download Lotty from

<https://bitbucket.org/wjtorresb/lotty>

- ➊ Dual repr. of multi-loop Feynman integrands
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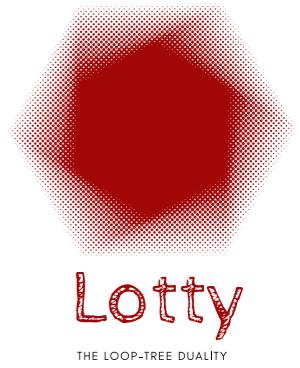
○ Two-loop kite diagrams



All two-loop Scattering amplitudes

$$\mathcal{A}_2^{(2)} = \int_{\vec{\ell}_1 \vec{\ell}_2} [G_D(1, 2) + G_D(1, \bar{1}\bar{2}) + G_D(\bar{2}, \bar{1}\bar{2})] ,$$
$$\alpha_1 = \{1, 2\}, \alpha_2 = \{3, 4\}, \alpha_3 = \{5\} .$$

Lotty – The loop-tree duality automation



In[1]:= << Lotty`

Lotty -- the LLoop-Tree dualITY automation

by William J. Torres Bobadilla (MPP)

Version 1.0 (March 14th of 2021)

All functions are stored in the variable \$LottyFunctions

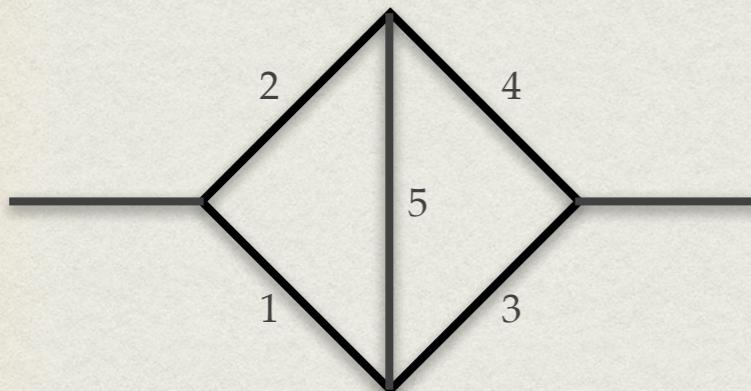
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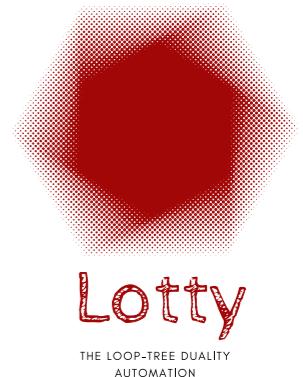
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Lotty :: Computes all residues

{GD[1, 3], GD[2, 3], GD[3, 5], GD[1, 4], GD[2, 4], GD[4, 5], GD[-2, 5], GD[-1, 5]}

Lotty – The loop-tree duality automation



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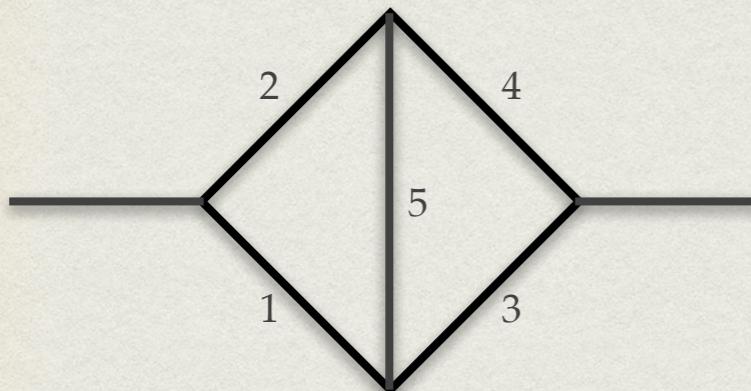
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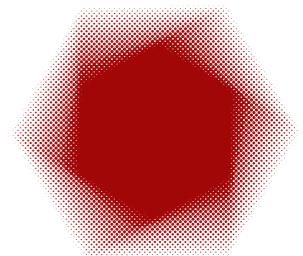
{GD[1, 3], GD[2, 3], GD[3, 5], GD[1, 4], GD[2, 4], GD[4, 5], GD[-2, 5], GD[-1, 5]}

Causal representation

William J. Torres Bobadilla

$$\begin{aligned}
 & -\frac{1}{2\lambda_3\lambda_1^+\lambda_2^-} - \frac{1}{2\lambda_6\lambda_1^+\lambda_2^-} - \frac{1}{2\lambda_3\lambda_1^-\lambda_2^+} - \frac{1}{2\lambda_6\lambda_1^-\lambda_2^+} - \frac{1}{2\lambda_3\lambda_6\lambda_4^-} - \frac{1}{2\lambda_6\lambda_1^-\lambda_4^-} - \\
 & \frac{1}{2\lambda_3\lambda_2^-\lambda_4^-} - \frac{1}{2\lambda_1^-\lambda_2^-\lambda_4^-} - \frac{1}{2\lambda_3\lambda_6\lambda_4^+} - \frac{1}{2\lambda_6\lambda_1^+\lambda_4^+} - \frac{1}{2\lambda_3\lambda_2^+\lambda_4^+} - \frac{1}{2\lambda_1^+\lambda_2^+\lambda_4^+} - \frac{1}{2\lambda_3\lambda_6\lambda_5^-} - \\
 & \frac{1}{2\lambda_3\lambda_1^-\lambda_5^-} - \frac{1}{2\lambda_6\lambda_2^-\lambda_5^-} - \frac{1}{2\lambda_1^-\lambda_2^-\lambda_5^-} - \frac{1}{2\lambda_3\lambda_6\lambda_5^+} - \frac{1}{2\lambda_3\lambda_1^+\lambda_5^+} - \frac{1}{2\lambda_6\lambda_2^+\lambda_5^+} - \frac{1}{2\lambda_1^+\lambda_2^+\lambda_5^+}
 \end{aligned}$$

Lotty – The loop-tree duality automation



Lotty
THE LOOP-TREE DUALITY AUTOMATION

In[1]:= << Lotty`

Lotty -- the LOop-Tree dualITY automation

by William J. Torres Bobadilla (MPP)

Version 1.0 (March 14th of 2021)

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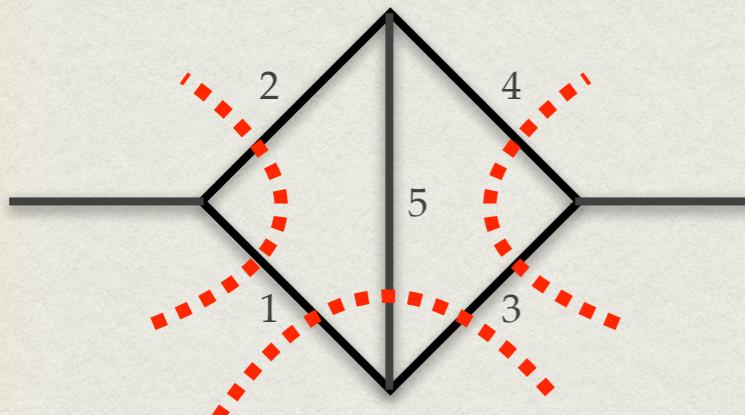
[W.J.T. (2021)]

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Lotty :: Computes all residues

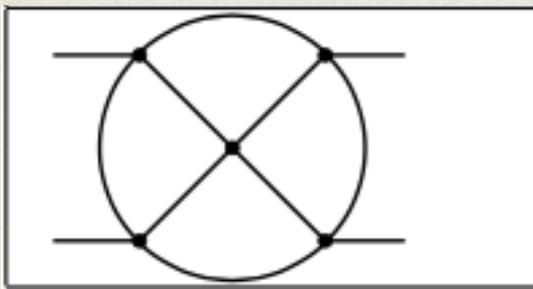
{GD[1, 3], GD[2, 3], GD[3, 5], GD[1, 4], GD[2, 4], GD[4, 5], GD[-2, 5], GD[-1, 5]}

Causal representation

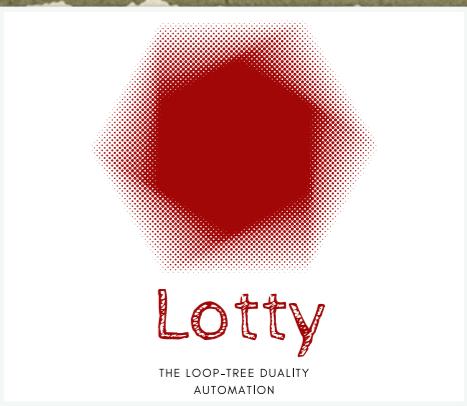
William J. Torres Bobadilla

$$\begin{aligned}
 & -\frac{1}{2\lambda_3\lambda_1^+\lambda_2^-} - \frac{1}{2\lambda_6\lambda_1^+\lambda_2^-} - \frac{1}{2\lambda_3\lambda_1^-\lambda_2^+} - \frac{1}{2\lambda_6\lambda_1^-\lambda_2^+} - \frac{1}{2\lambda_3\lambda_6\lambda_4^-} - \frac{1}{2\lambda_6\lambda_1^-\lambda_4^-} - \\
 & \quad \circled{1} \quad \frac{1}{2\lambda_3\lambda_2^-\lambda_4^-} - \frac{1}{2\lambda_1^-\lambda_2^-\lambda_4^-} - \frac{1}{2\lambda_3\lambda_6\lambda_4^+} - \frac{1}{2\lambda_6\lambda_1^+\lambda_4^+} - \frac{1}{2\lambda_3\lambda_2^+\lambda_4^+} - \frac{1}{2\lambda_1^+\lambda_2^+\lambda_4^+} - \frac{1}{2\lambda_3\lambda_6\lambda_5^-} - \\
 & \quad \frac{1}{2\lambda_3\lambda_1^-\lambda_5^-} - \frac{1}{2\lambda_6\lambda_2^-\lambda_5^-} - \frac{1}{2\lambda_1^-\lambda_2^-\lambda_5^-} - \frac{1}{2\lambda_3\lambda_6\lambda_5^+} - \frac{1}{2\lambda_3\lambda_1^+\lambda_5^+} - \frac{1}{2\lambda_6\lambda_2^+\lambda_5^+} - \frac{1}{2\lambda_1^+\lambda_2^+\lambda_5^+}
 \end{aligned}$$

Lotty – *The loop-tree duality automation*

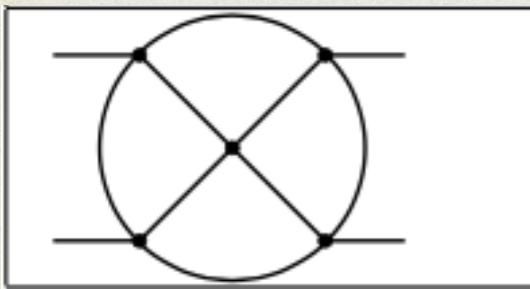


in a nutshell :: Dual & causal representation

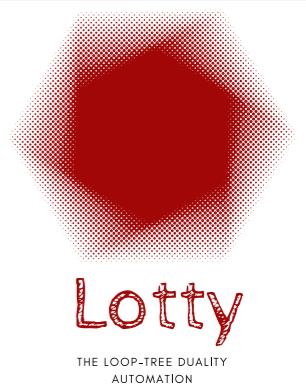


[W.J.T. (2021)]

Lotty – The loop-tree duality automation



in a nutshell :: Dual & causal representation



[W.J.T. (2021)]

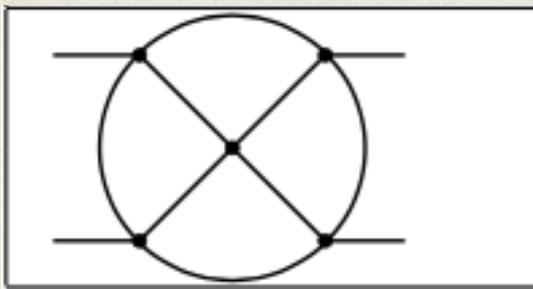
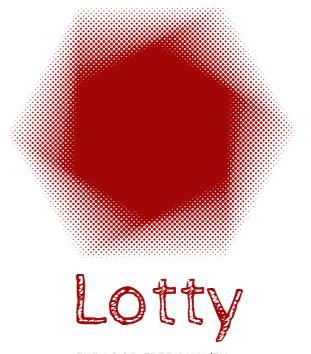
```
In[5]:= num = 1;
LoopMom = {l1, l2, l3, l4};
propagators = {l1, l2, l3, l4, -l1 + l2 - p[1], -l2 + l3 - p[2], -l3 + l4 - p[3], -l4 + l1 - p[4]};
assumptions = (Im[Subscript[p[#], 0]] == 0) & /@ Range[4];
\[Downarrow] | ⓘ
In[9]:= tmp = GetDual[num, propagators, LoopMom, "Assumptions" \[Rule] assumptions]; // AbsoluteTiming
tmp2 = tmp * GetCausalProps[tmp, propagators, "GetPref" \[Rule] True] // Total;
Out[9]= {1.6389, Null}

In[11]:= tmp1 = RefineDual[tmp, num, propagators, LoopMom]; // AbsoluteTiming
tmp1
Out[11]= {4.73747, Null}

Out[12]= {GD[{-1, -2, -3, -4}], GD[{2, 3, 4, -5}], GD[{-2, -3, -4, -8}], GD[{-1, -3, -4, -5}], GD[{1, 3, 4, -6}],
GD[{3, 4, -5, -6}], GD[{-3, -4, 6, -8}], GD[{-3, -4, -5, -8}], GD[{-1, -2, -4, -6}],
GD[{-2, -4, 5, -6}], GD[{-2, -4, -6, -8}], GD[{-1, -4, -5, -6}], GD[{1, 2, 4, -7}],
GD[{2, 4, -5, -7}], GD[{-2, -4, 7, -8}], GD[{1, 4, 5, -7}], GD[{1, 4, -6, -7}], GD[{4, -5, -6, -7}],
GD[{-4, 6, 7, -8}], GD[{-4, -5, 7, -8}], GD[{-4, -5, -6, -8}], GD[{-1, -2, -3, -7}],
GD[{-2, -3, 5, -7}], GD[{-2, -3, -7, -8}], GD[{-1, -3, -5, -7}], GD[{-1, -3, 6, -7}],
GD[{-3, 5, 6, -7}], GD[{-3, 6, -7, -8}], GD[{-3, -5, -7, -8}], GD[{-1, -2, -6, -7}],
GD[{-2, 5, -6, -7}], GD[{-2, -6, -7, -8}], GD[{-1, -5, -6, -7}], GD[{1, 2, 3, -8}], GD[{1, 3, 5, -8}],
GD[{1, 3, -6, -8}], GD[{1, 2, 6, -8}], GD[{1, 5, 6, -8}], GD[{1, 2, -7, -8}], GD[{1, 5, -7, -8}],
GD[{1, -6, -7, -8}], GD[{2, 3, -5, -8}], GD[{2, -5, 6, -8}], GD[{2, -5, -7, -8}], GD[{3, -5, -6, -8}]}

In[13]:= GetCausalProps[tmp, propagators]
ALL λ[i] ARE STORED IN THE FUNCTION λ2qi0
Out[13]= {p[1]₀ - λ[1], p[1]₀ + λ[1], p[2]₀ - λ[2], p[2]₀ + λ[2], p[3]₀ - λ[3], p[3]₀ + λ[3], p[4]₀ - λ[4], p[4]₀ + λ[4],
p[1]₀ + p[2]₀ - λ[5], p[1]₀ + p[2]₀ + λ[5], p[2]₀ + p[3]₀ - λ[6], p[2]₀ + p[3]₀ + λ[6], p[1]₀ + p[4]₀ - λ[7],
p[1]₀ + p[4]₀ + λ[7], p[3]₀ + p[4]₀ - λ[8], p[3]₀ + p[4]₀ + λ[8], p[1]₀ + p[2]₀ + p[3]₀ + p[4]₀ - λ[9],
p[1]₀ + p[2]₀ + p[3]₀ + p[4]₀ + λ[9], p[1]₀ + p[2]₀ + p[3]₀ - λ[10], p[1]₀ + p[2]₀ + p[3]₀ + λ[10],
p[1]₀ + p[2]₀ + p[4]₀ - λ[11], p[1]₀ + p[2]₀ + p[4]₀ + λ[11], p[1]₀ + p[3]₀ + p[4]₀ - λ[12],
p[1]₀ + p[3]₀ + p[4]₀ + λ[12], p[2]₀ + p[3]₀ + p[4]₀ - λ[13], p[2]₀ + p[3]₀ + p[4]₀ + λ[13]}
```

Lotty – The loop-tree duality automation



in a nutshell :: Dual & causal representation

[W.J.T. (2021)]

```
In[5]:= num = 1;
LoopMom = {l1, l2, l3, l4};
propagators = {l1, l2, l3, l4, -l1 + l2 - p[1], -l2 + l3 - p[2], -l3 + l4 - p[3], -l4 + l1 - p[4]};
assumptions = (Im[Subscript[p[#], 0]] == 0) & /@ Range[4];
\[Downarrow] | \[Information]

In[9]:= tmp = GetDual[num, propagators, LoopMom, "Assumptions" \[Rule] assumptions]; // AbsoluteTiming
tmp2 = tmp * GetCausalProps[tmp, propagators, "GetPref" \[Rule] True] // Total;

Out[9]= {1.6389, Null}

In[11]:= tmp1 = RefineDual[tmp, num, propagators, LoopMom]; // AbsoluteTiming
tmp1

Out[11]= {4.73747, Null}

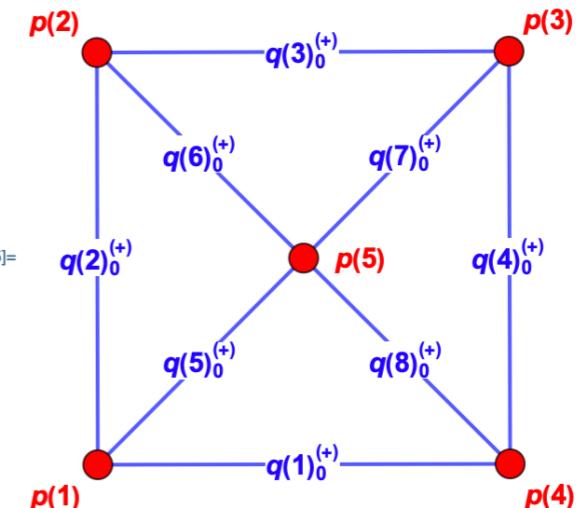
Out[12]= {GD[{-1, -2, -3, -4}], GD[{2, 3, 4, -5}], GD[{-2, -3, -4, -8}], GD[{-1, -3, -4, -5}], GD[{1, 3, 4, -6}],
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GD[{1, -6, -7, -8}], GD[{2, 3, -5, -8}], GD[{2, -5, 6, -8}], GD[{2, -5, -7, -8}], GD[{3, -5, -6, -8}]}

In[13]:= GetCausalProps[tmp, propagators]
ALL  $\lambda[i]$  ARE STORED IN THE FUNCTION  $\lambda2q[i]$ 

Out[13]= {p[1]_0 - \lambda[1], p[1]_0 + \lambda[1], p[2]_0 - \lambda[2], p[2]_0 + \lambda[2], p[3]_0 - \lambda[3], p[3]_0 + \lambda[3], p[4]_0 - \lambda[4], p[4]_0 + \lambda[4],
p[1]_0 + p[2]_0 - \lambda[5], p[1]_0 + p[2]_0 + \lambda[5], p[2]_0 + p[3]_0 - \lambda[6], p[2]_0 + p[3]_0 + \lambda[6], p[1]_0 + p[4]_0 - \lambda[7],
p[1]_0 + p[4]_0 + \lambda[7], p[3]_0 + p[4]_0 - \lambda[8], p[3]_0 + p[4]_0 + \lambda[8], p[1]_0 + p[2]_0 + p[3]_0 + p[4]_0 - \lambda[9],
p[1]_0 + p[2]_0 + p[3]_0 + p[4]_0 + \lambda[9], p[1]_0 + p[2]_0 + p[3]_0 - \lambda[10], p[1]_0 + p[2]_0 + p[3]_0 + \lambda[10],
p[1]_0 + p[2]_0 + p[4]_0 - \lambda[11], p[1]_0 + p[2]_0 + p[4]_0 + \lambda[11], p[1]_0 + p[3]_0 + p[4]_0 - \lambda[12],
p[1]_0 + p[3]_0 + p[4]_0 + \lambda[12], p[2]_0 + p[3]_0 + p[4]_0 - \lambda[13], p[2]_0 + p[3]_0 + p[4]_0 + \lambda[13]}
```

```
In[14]:= SetSingLam = {
q[{1}] \[Rule] {1, 2, 5},
q[{2}] \[Rule] {2, 3, 6},
q[{3}] \[Rule] {3, 4, 7},
q[{4}] \[Rule] {4, 1, 8},
q[{5}] \[Rule] {5, 6, 7, 8}
};

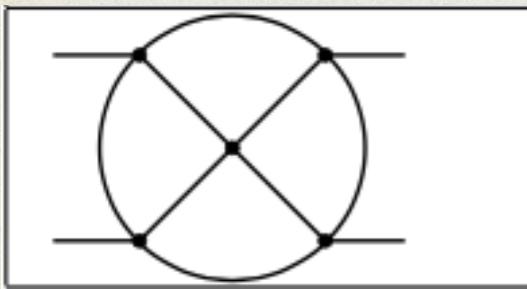
In[15]:= PlotTop[SetSingLam]
```



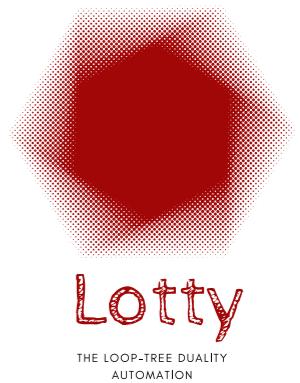
```
In[46]:= tmp3 = AllCausal[5]
tmp3 // Expand // Length

Out[46]= (L[-][{3, 4}] + L[-][{3, 5}] + L[-][{4, 5}]) L[+][{1, 2}] +
(L[-][{2, 4}] + L[-][{2, 5}] + L[-][{4, 5}]) L[+][{1, 3}] +
(L[-][{2, 3}] + L[-][{2, 5}] + L[-][{3, 5}]) L[+][{1, 4}] +
(L[-][{2, 3}] + L[-][{2, 4}] + L[-][{3, 4}]) L[+][{1, 5}] +
(L[-][{1, 4}] + L[-][{1, 5}] + L[-][{4, 5}]) L[+][{2, 3}] +
(L[-][{1, 3}] + L[-][{1, 5}] + L[-][{3, 5}]) L[+][{2, 4}] +
(L[-][{1, 3}] + L[-][{1, 4}] + L[-][{3, 4}]) L[+][{2, 5}] +
(L[-][{1, 2}] + L[-][{1, 5}] + L[-][{2, 5}]) L[+][{3, 4}] +
(L[-][{1, 2}] + L[-][{1, 4}] + L[-][{2, 4}]) L[+][{3, 5}] +
(L[-][{1, 2}] + L[-][{1, 3}] + L[-][{2, 3}]) L[+][{4, 5}]
```

Lotty – The loop-tree duality automation



in a nutshell :: Dual & causal representation



[W.J.T. (2021)]

```
In[5]:= num = 1;
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∞ | ⓘ

In[9]:= tmp = GetDual[num, propagators, LoopMom, "Assumptions" → assumptions]; // AbsoluteTiming
tmp2 = tmp * GetCausalProps[tmp, propagators, "GetPref" → True] // Total;

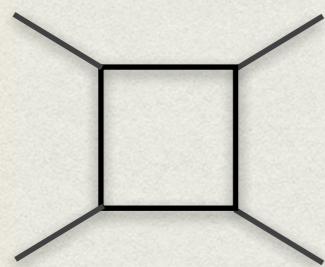
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In[11]:= tmp1 = RefineDual[tmp, num, propagators, LoopMom]; // AbsoluteTiming
tmp1

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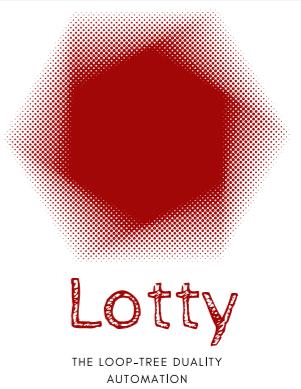
Out[12]= {GD[{-1, -2, -3, -4}], GD[{2, 3, 4, -5}], GD[{-2, -3, -4, -8}], GD[{-1, -3, -4, -5}], GD[{1, 3, 4, -6}],
GD[{3, 4, -5, -6}], GD[{-3, -4, 6, -8}], GD[{-3, -4, -5, -8}], GD[{-1, -2, -4, -6}],
GD[{-2, -4, 5, -6}], GD[{-2, -4, -6, -8}], GD[{-1, -4, -5, -6}], GD[{1, 2, 4, -7}],
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GD[{-4, 6, 7, -8}], GD[{-4, -5, 7, -8}], GD[{-4, -5, -6, -8}], GD[{-1, -2, -3, -7}],
GD[{-2, -3, 5, -7}], GD[{-2, -3, -7, -8}], GD[{-1, -3, -5, -7}], GD[{-1, -3, 6, -7}],
GD[{-3, 5, 6, -7}], GD[{-3, 6, -7, -8}], GD[{1, 3, -6, -8}], GD[{1, 2, 6, -8}], GD[{1, 6, -7, -8}],
GD[{2, 3, -5, -8}], GD[{2, 3, 4, -5}], GD[{2, 3, 5, -4}], GD[{2, 3, 6, -3}], GD[{2, 3, 7, -2}], GD[{2, 3, 8, -1}], GD[{2, 3, 9, 0}], GD[{2, 3, 10, 1}], GD[{2, 3, 11, 2}], GD[{2, 3, 12, 3}], GD[{2, 3, 13, 4}], GD[{2, 3, 14, 5}], GD[{2, 3, 15, 6}], GD[{2, 3, 16, 7}], GD[{2, 3, 17, 8}], GD[{2, 3, 18, 9}], GD[{2, 3, 19, 10}], GD[{2, 3, 20, 11}], GD[{2, 3, 21, 12}], GD[{2, 3, 22, 13}], GD[{2, 3, 23, 14}], GD[{2, 3, 24, 15}], GD[{2, 3, 25, 16}], GD[{2, 3, 26, 17}], GD[{2, 3, 27, 18}], GD[{2, 3, 28, 19}], GD[{2, 3, 29, 20}], GD[{2, 3, 30, 21}], GD[{2, 3, 31, 22}], GD[{2, 3, 32, 23}], GD[{2, 3, 33, 24}], GD[{2, 3, 34, 25}], GD[{2, 3, 35, 26}], GD[{2, 3, 36, 27}], GD[{2, 3, 37, 28}], GD[{2, 3, 38, 29}], GD[{2, 3, 39, 30}], GD[{2, 3, 40, 31}], GD[{2, 3, 41, 32}], GD[{2, 3, 42, 33}], GD[{2, 3, 43, 34}], GD[{2, 3, 44, 35}], GD[{2, 3, 45, 36}], GD[{2, 3, 46, 37}], GD[{2, 3, 47, 38}], GD[{2, 3, 48, 39}], GD[{2, 3, 49, 40}], GD[{2, 3, 50, 41}], GD[{2, 3, 51, 42}], GD[{2, 3, 52, 43}], GD[{2, 3, 53, 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GD[{2, 3, 285, 276}], GD[{2, 3, 286, 277}], GD[{2, 3, 287, 278}], GD[{2, 3, 288, 279}], GD[{2, 3, 289, 280}], GD[{2, 3, 290, 281}], GD[{2, 3, 291, 282}], GD[{2, 3, 292, 283}], GD[{2, 3, 293, 284}], GD[{2, 3, 294, 285}], GD[{2, 3, 295, 286}], GD[{2, 3, 296, 287}], GD[{2, 3, 297, 288}], GD[{2, 3, 298, 289}], GD[{2, 3, 299, 290}], GD[{2, 3, 300, 291}], GD[{2, 3, 301, 292}], GD[{2, 3, 302, 293}], GD[{2, 3, 303, 294}], GD[{2, 3, 304, 295}], GD[{2, 3, 305, 296}], GD[{2, 3, 306, 297}], GD[{2, 3, 307, 298}], GD[{2, 3, 308, 299}], GD[{2, 3, 309, 300}], GD[{2, 3, 310, 301}], GD[{2, 3, 311, 302}], GD[{2, 3, 312, 303}], GD[{2, 3, 313, 304}], GD[{2, 3, 314, 305}], GD[{2, 3, 315, 306}], GD[{2, 3, 316, 307}], GD[{2, 3, 317, 308}], GD[{2, 3, 318, 309}], GD[{2, 3, 319, 310}], GD[{2, 3, 320, 311}], GD[{2, 3, 321, 312}], GD[{2, 3, 322, 313}], GD[{2, 3, 323, 314}], GD[{2, 3, 324, 315}], GD[{2, 3, 325, 316}], GD[{2, 3, 326, 317}], GD[{2, 3, 327, 318}], GD[{2, 3, 328, 319}], GD[{2, 3, 329, 320}], GD[{2, 3, 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```

Numerical evaluations



$$= -\imath \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1,2,3,4) , \quad q_i^2 = (\ell_1 - k_i)^2 + \imath 0$$

$$k_i = p_1 + \dots + p_{i-1}$$

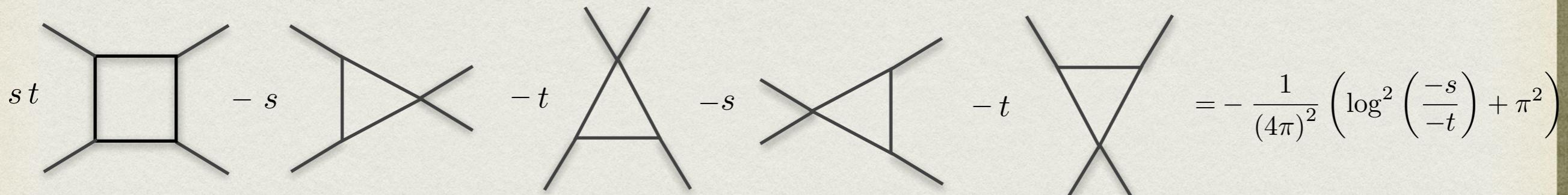


Lotty

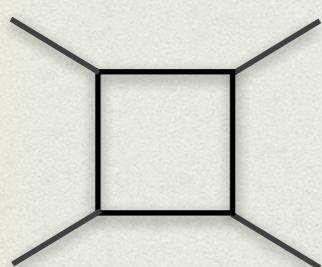
THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

IR safe combination of Feynman integrals in $d=4$

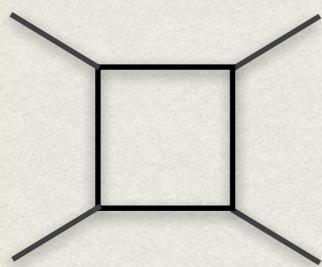


IR finite integral in $d=6$



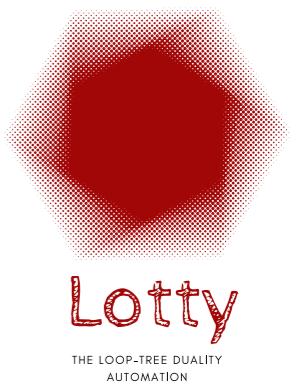
$$= \frac{1}{(4\pi)^3} \frac{1}{2(s+t)} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right)$$

Numerical evaluations



$$= -i \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1, 2, 3, 4), \quad q_i^2 = (\ell_1 - k_i)^2 + i0$$

$$k_i = p_1 + \dots + p_{i-1}$$

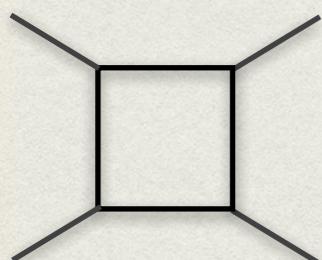


[W.J.T. (2021)]

IR safe combination of Feynman integrals in $d=4$

$$s t \begin{array}{c} \text{square loop} \\ \text{---} \\ \text{---} \end{array} - s \begin{array}{c} \text{square loop} \\ \diagdown \diagup \\ \text{---} \end{array} - t \begin{array}{c} \text{square loop} \\ \diagup \diagdown \\ \text{---} \end{array} - s \begin{array}{c} \text{square loop} \\ \diagup \diagdown \\ \text{---} \end{array} - t \begin{array}{c} \text{square loop} \\ \diagup \diagdown \\ \text{---} \end{array} = - \frac{1}{(4\pi)^2} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right)$$

IR finite integral in $d=6$



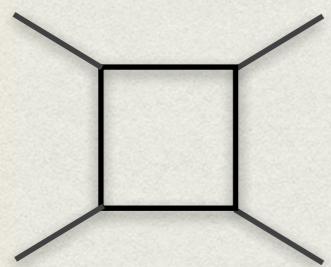
$$= \frac{1}{(4\pi)^3} \frac{1}{2(s+t)} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right)$$

$$p_1^\alpha = \frac{\sqrt{s}}{2} \left\{ 1, 1, \vec{0}_{d-2} \right\}, \quad p_3^\alpha = \frac{\sqrt{s}}{2} \left\{ -1, \sin \theta, \cos \theta, \vec{0}_{d-1} \right\},$$

$$p_2^\alpha = \frac{\sqrt{s}}{2} \left\{ 1, -1, \vec{0}_{d-2} \right\}, \quad p_4^\alpha = \frac{\sqrt{s}}{2} \left\{ -1, -\sin \theta, -\cos \theta, \vec{0}_{d-1} \right\}.$$

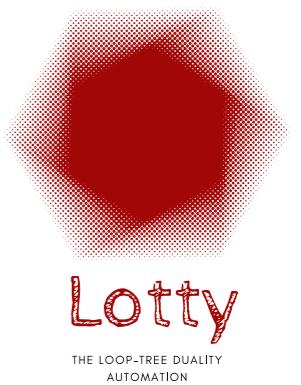
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Numerical evaluations



$$= -i \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1, 2, 3, 4), \quad q_i^2 = (\ell_1 - k_i)^2 + i0$$

$$k_i = p_1 + \dots + p_{i-1}$$

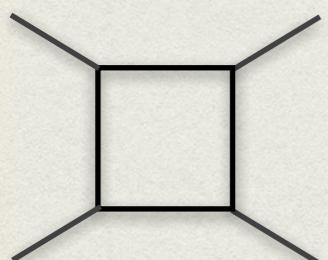


[W.J.T. (2021)]

IR safe combination of Feynman integrals in $d=4$

$$\begin{aligned} s t & \quad - s \quad - t \quad - s \quad - t \\ \text{Feynman diagram} & - \text{triangle} - \text{triangle} - \text{triangle} = - \frac{1}{(4\pi)^2} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right) \end{aligned}$$

IR finite integral in $d=6$



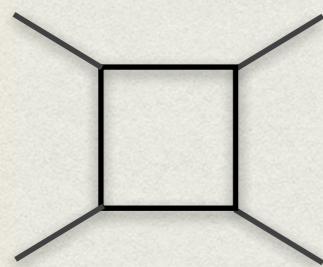
$$= \frac{1}{(4\pi)^3} \frac{1}{2(s+t)} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right)$$

Straightforward contour deformation @1L $\xrightarrow{\hspace{1cm}}$ Richardson extrapolation
[Ronca, W.J.T. (w.i.p.)]

$$\boxed{\begin{aligned} p_1^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, 1, \vec{0}_{d-2} \right\}, & p_3^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, \sin \theta, \cos \theta, \vec{0}_{d-1} \right\}, \\ p_2^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, -1, \vec{0}_{d-2} \right\}, & p_4^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, -\sin \theta, -\cos \theta, \vec{0}_{d-1} \right\}. \end{aligned}}$$

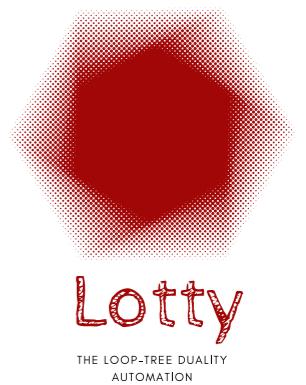
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Numerical evaluations



$$= -i \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1, 2, 3, 4), \quad q_i^2 = (\ell_1 - k_i)^2 + i0$$

$$k_i = p_1 + \dots + p_{i-1}$$

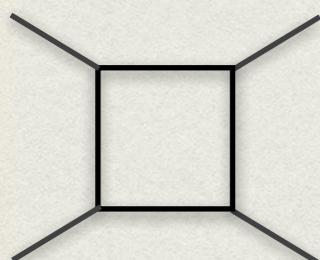


[W.J.T. (2021)]

IR safe combination of Feynman integrals in $d=4$

$$\begin{aligned} s t & \quad - s \quad - t \quad - s \quad - t \\ \text{Feynman diagram} & - \text{Feynman diagram} - \text{Feynman diagram} - \text{Feynman diagram} = - \frac{1}{(4\pi)^2} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right) \\ \{s = 3 + i0, \theta = \pi/6\} \\ \{s = 5 + i0, \theta = \pi/6\} & \quad \text{LTD} = (-1.2170(4) - i5.5158(4)) * 10^{-2} \end{aligned}$$

IR finite integral in $d=6$



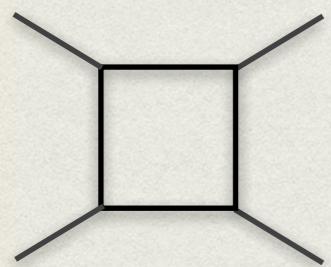
$$= \frac{1}{(4\pi)^3} \frac{1}{2(s+t)} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right)$$

Straightforward contour deformation @1L $\xrightarrow{\hspace{1cm}}$ Richardson extrapolation
[Ronca, W.J.T. (w.i.p.)]

$$\boxed{\begin{aligned} p_1^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, 1, \vec{0}_{d-2} \right\}, & p_3^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, \sin \theta, \cos \theta, \vec{0}_{d-1} \right\}, \\ p_2^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, -1, \vec{0}_{d-2} \right\}, & p_4^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, -\sin \theta, -\cos \theta, \vec{0}_{d-1} \right\}. \end{aligned}}$$

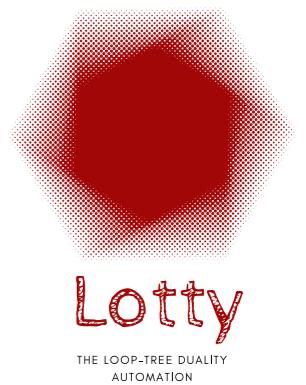
COM

Numerical evaluations



$$= -i \int \frac{d^d \ell_1}{(2\pi)^d} G_F(1, 2, 3, 4), \quad q_i^2 = (\ell_1 - k_i)^2 + i0$$

$$k_i = p_1 + \dots + p_{i-1}$$

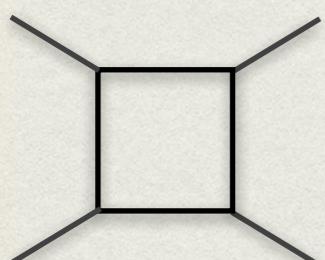


[W.J.T. (2021)]

IR safe combination of Feynman integrals in $d=4$

$$\begin{aligned} s t & \quad - s \quad - t \quad - s \quad - t \\ \text{Feynman diagram} & - \text{triangle} - \text{triangle} - \text{triangle} = - \frac{1}{(4\pi)^2} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right) \\ \{s = 3 + i0, \theta = \pi/6\} \\ \{s = 5 + i0, \theta = \pi/6\} & \quad \text{LTD} = (-1.2170(4) - i5.5158(4)) * 10^{-2} \end{aligned}$$

IR finite integral in $d=6$



$$= \frac{1}{(4\pi)^3} \frac{1}{2(s+t)} \left(\log^2 \left(\frac{-s}{-t} \right) + \pi^2 \right)$$

LTD

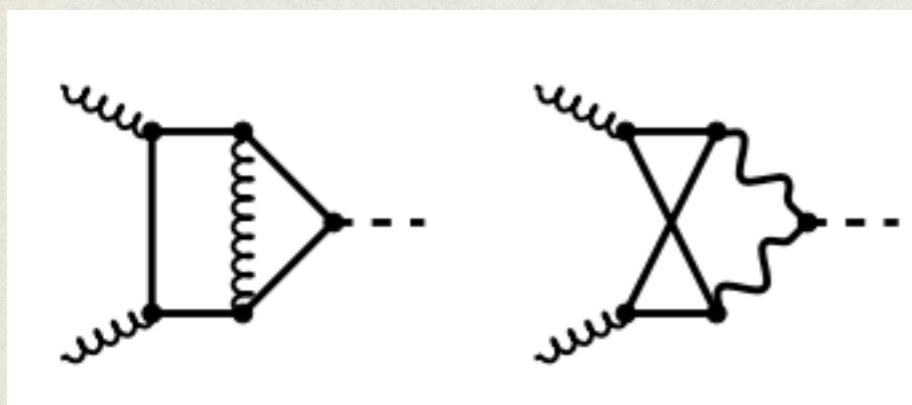
$$\begin{aligned} \{s = 3 + i0, \theta = \pi/6\} & \rightarrow (1.2901(2) - i5.8528(2)) * 10^{-4} \\ \{s = 5 + i0, \theta = \pi/6\} & \rightarrow (2.1489(4) - i9.7549(4)) * 10^{-4} \end{aligned}$$

Straightforward contour deformation @1L → Richardson extrapolation
[Ronca, W.J.T. (w.i.p.)]

$$\begin{aligned} p_1^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, 1, \vec{0}_{d-2} \right\}, & p_3^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, \sin \theta, \cos \theta, \vec{0}_{d-1} \right\}, \\ p_2^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, -1, \vec{0}_{d-2} \right\}, & p_4^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, -\sin \theta, -\cos \theta, \vec{0}_{d-1} \right\}. \end{aligned}$$

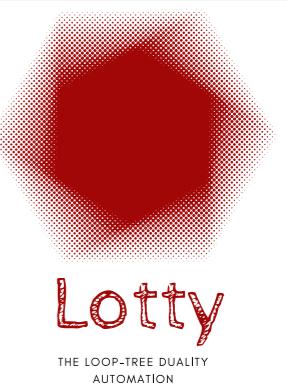
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Numerical evaluations



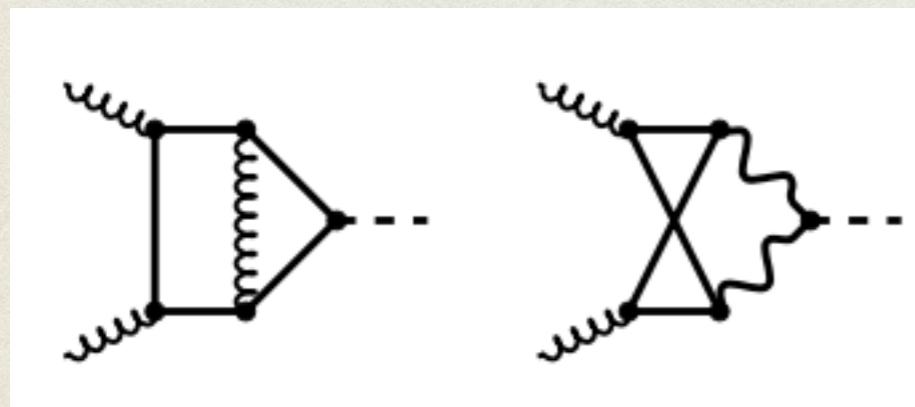
QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]



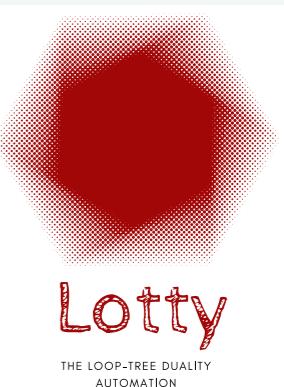
[W.J.T. (2021)]

Numerical evaluations



QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]



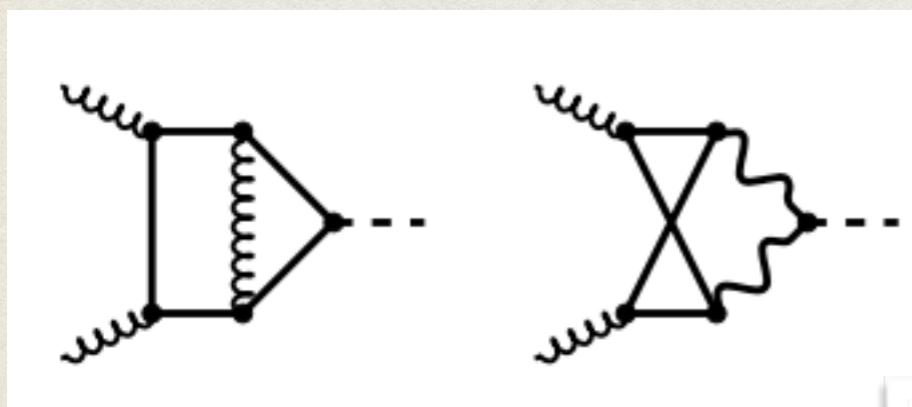
[W.J.T. (2021)]

On-shell energies

```
In[54]:= myqi0 = Subsuperscript[q[#], 0, "(+)" ]&/@Range[Length@propagators];
value = myqi0/.Subsuperscript[q[ii_], __]:>Sqrt[sp@propagators[[ii]]+m[ii]^2];
value = value/.LoopToSC[LoopMom,dim]/.spatial;
myrepl = Thread[myqi0->value]//FullSimplify

Out[54]= {q[1]_0^{(+)}\rightarrow \sqrt{r1^2 + m[1]^2},
          q[2]_0^{(+)}\rightarrow \frac{1}{2} \sqrt{Ecm^2 + 4 Ecm r1 \text{Cos}[\theta 11] + 4 (r1^2 + m[2]^2)},
          q[3]_0^{(+)}\rightarrow \sqrt{r1^2 + m[3]^2},
          q[4]_0^{(+)}\rightarrow \sqrt{r2^2 + m[4]^2},
          q[5]_0^{(+)}\rightarrow \sqrt{r2^2 + m[5]^2},
          q[6]_0^{(+)}\rightarrow \sqrt{r1^2+r2^2+m[6]^2+2r1r2 (\text{Cos}[\theta 11]\text{Cos}[\theta 21] + \text{Cos}[\theta 12-\theta 22]\text{Sin}[\theta 11]\text{Sin}[\theta 21])} }
```

Numerical evaluations



QED & QCD/EW

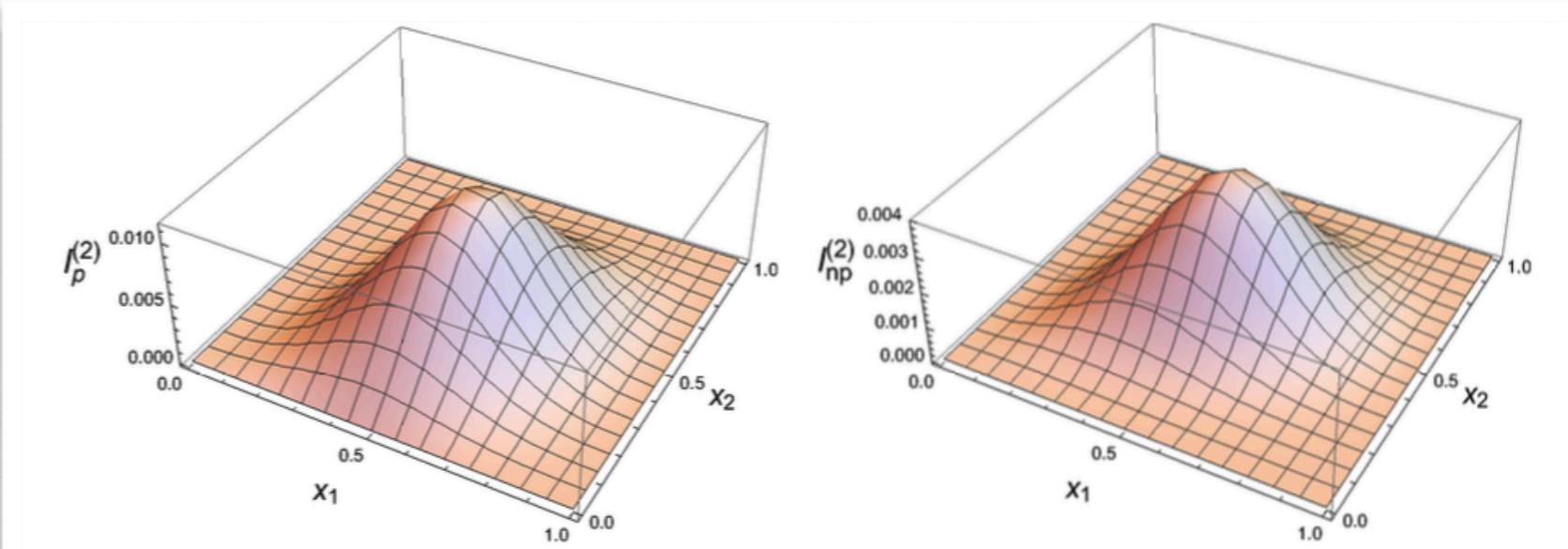
[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]

Lotty

THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

On-shell energies



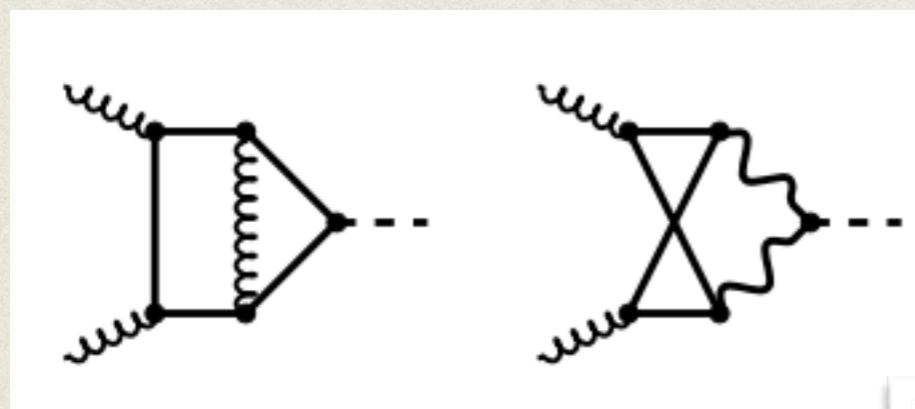
```
In[54]:= myqi0 = Subsuperscript[q[#], 0, "(+)" ]&/@Range[Length@propagators];
value = myqi0/.Subsuperscript[q[ii_], __]:>Sqrt[sp@propagators[[ii]]+m[ii]^2];
value = value/.LoopToSC[LoopMom,dim]/.spatial;
myrepl = Thread[myqi0->value]//FullSimplify
```

```
Out[54]= {q[1]_0^(<+>) -> Sqrt[r1^2 + m[1]^2],
q[2]_0^(<+>) -> 1/2 Sqrt[Ecm^2 + 4 Ecm r1 Cos[\theta11] + 4 (r1^2 + m[2]^2)],
q[3]_0^(<+>) -> Sqrt[r1^2 + m[3]^2],
q[4]_0^(<+>) -> Sqrt[r2^2 + m[4]^2],
q[5]_0^(<+>) -> Sqrt[r2^2 + m[5]^2],
q[6]_0^(<+>) -> Sqrt[r1^2+r2^2+m[6]^2+2r1r2 (Cos[\theta11]Cos[\theta21] + Cos[\theta12-\theta22]Sin[\theta11]Sin[\theta21]) } }
```

Compactify variables

$$r_i \rightarrow \frac{1 - x_i}{x_i}$$

Numerical evaluations



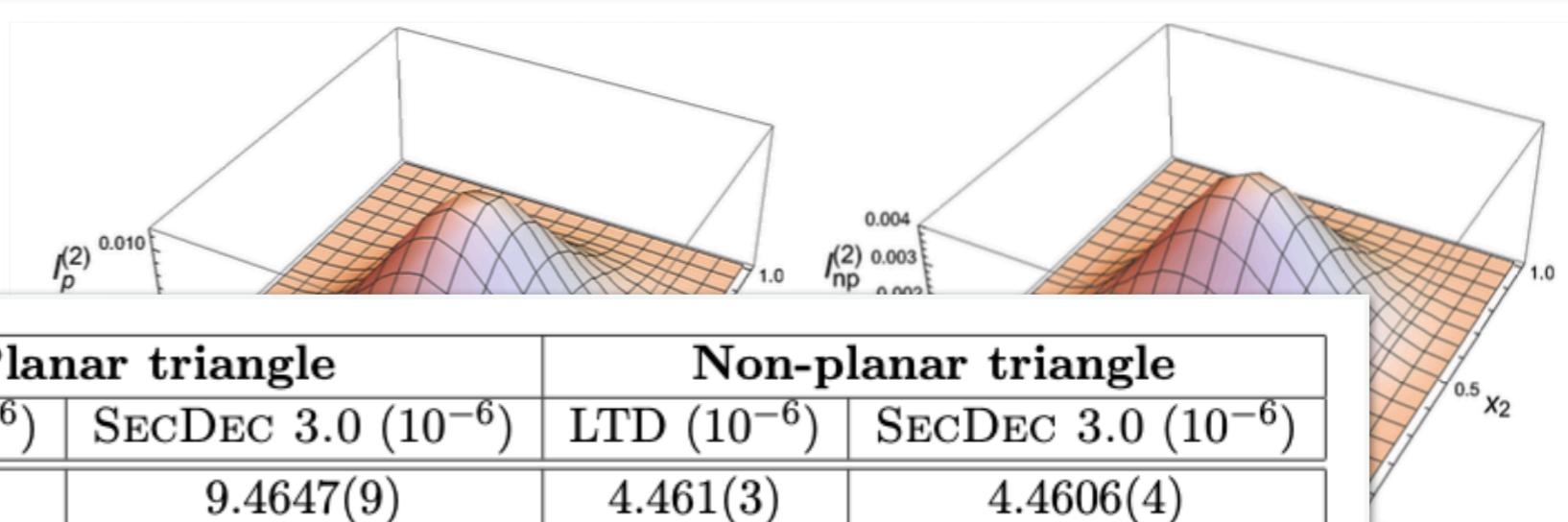
QED & QCD/EW

[Driencourt-Mangin, Rodrigo, Sborlini, W.J.T. (2019) x 2]

Lotty

THE LOOP-TREE DUALITY AUTOMATION

[W.J.T. (2021)]

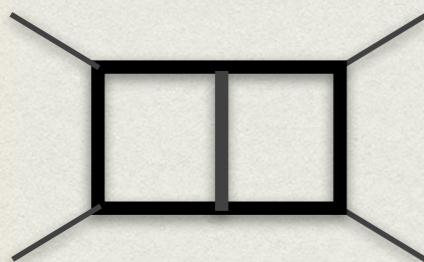


	Planar triangle		Non-planar triangle	
$\frac{s}{m^2}$	LTD (10^{-6})	SECDEC 3.0 (10^{-6})	LTD (10^{-6})	SECDEC 3.0 (10^{-6})
$-\frac{1}{4}$	9.48(5)	9.4647(9)	4.461(3)	4.4606(4)
-1	8.10(5)	8.0885(8)	4.101(3)	4.1012(4)
$-\frac{9}{4}$	6.49(3)	6.4760(6)	3.627(5)	3.6276(3)
-4	5.02(2)	5.0188(5)	3.15(5)	3.1334(3)
$+\frac{1}{4}$	10.68(6)	10.651(1)	4.743(3)	4.7436(4)
1	13.11(8)	13.070(1)	5.259(3)	5.2590(5)
$+\frac{9}{4}$	20.81(1)	20.748(2)	6.533(3)	6.5331(6)
$+\frac{25}{16}$	15.74(9)	15.700(1)	5.748(3)	5.7474(6)

On-shell energies

```
In[54]:= myqio = Subsuper;
value = myqio/.{q1,q2,q3,q4,q5,q6,q7,q8,q9,q10,q11,q12,q13,q14,q15,q16,q17,q18,q19,q20,q21,q22,q23,q24,q25,q26,q27,q28,q29,q30,q31,q32,q33,q34,q35,q36,q37,q38,q39,q40,q41,q42,q43,q44,q45,q46,q47,q48,q49,q50,q51,q52,q53,q54,q55,q56,q57,q58,q59,q60,q61,q62,q63,q64,q65,q66,q67,q68,q69,q70,q71,q72,q73,q74,q75,q76,q77,q78,q79,q80,q81,q82,q83,q84,q85,q86,q87,q88,q89,q90,q91,q92,q93,q94,q95,q96,q97,q98,q99,q100,q101,q102,q103,q104,q105,q106,q107,q108,q109,q110,q111,q112,q113,q114,q115,q116,q117,q118,q119,q120,q121,q122,q123,q124,q125,q126,q127,q128,q129,q130,q131,q132,q133,q134,q135,q136,q137,q138,q139,q140,q141,q142,q143,q144,q145,q146,q147,q148,q149,q150,q151,q152,q153,q154,q155,q156,q157,q158,q159,q160,q161,q162,q163,q164,q165,q166,q167,q168,q169,q170,q171,q172,q173,q174,q175,q176,q177,q178,q179,q180,q181,q182,q183,q184,q185,q186,q187,q188,q189,q190,q191,q192,q193,q194,q195,q196,q197,q198,q199,q200,q201,q202,q203,q204,q205,q206,q207,q208,q209,q210,q211,q212,q213,q214,q215,q216,q217,q218,q219,q220,q221,q222,q223,q224,q225,q226,q227,q228,q229,q230,q231,q232,q233,q234,q235,q236,q237,q238,q239,q240,q241,q242,q243,q244,q245,q246,q247,q248,q249,q250,q251,q252,q253,q254,q255,q256,q257,q258,q259,q260,q261,q262,q263,q264,q265,q266,q267,q268,q269,q270,q271,q272,q273,q274,q275,q276,q277,q278,q279,q280,q281,q282,q283,q284,q285,q286,q287,q288,q289,q290,q291,q292,q293,q294,q295,q296,q297,q298,q299,q300,q301,q302,q303,q304,q305,q306,q307,q308,q309,q310,q311,q312,q313,q314,q315,q316,q317,q318,q319,q320,q321,q322,q323,q324,q325,q326,q327,q328,q329,q330,q331,q332,q333,q334,q335,q336,q337,q338,q339,q340,q341,q342,q343,q344,q345,q346,q347,q348,q349,q350,q351,q352,q353,q354,q355,q356,q357,q358,q359,q360,q361,q362,q363,q364,q365,q366,q367,q368,q369,q370,q371,q372,q373,q374,q375,q376,q377,q378,q379,q380,q381,q382,q383,q384,q385,q386,q387,q388,q389,q390,q391,q392,q393,q394,q395,q396,q397,q398,q399,q400,q401,q402,q403,q404,q405,q406,q407,q408,q409,q410,q411,q412,q413,q414,q415,q416,q417,q418,q419,q420,q421,q422,q423,q424,q425,q426,q427,q428,q429,q430,q431,q432,q433,q434,q435,q436,q437,q438,q439,q440,q441,q442,q443,q444,q445,q446,q447,q448,q449,q450,q451,q452,q453,q454,q455,q456,q457,q458,q459,q460,q461,q462,q463,q464,q465,q466,q467,q468,q469,q470,q471,q472,q473,q474,q475,q476,q477,q478,q479,q480,q481,q482,q483,q484,q485,q486,q487,q488,q489,q490,q491,q492,q493,q494,q495,q496,q497,q498,q499,q500,q501,q502,q503,q504,q505,q506,q507,q508,q509,q510,q511,q512,q513,q514,q515,q516,q517,q518,q519,q520,q521,q522,q523,q524,q525,q526,q527,q528,q529,q530,q531,q532,q533,q534,q535,q536,q537,q538,q539,q540,q541,q542,q543,q544,q545,q546,q547,q548,q549,q550,q551,q552,q553,q554,q555,q556,q557,q558,q559,q560,q561,q562,q563,q564,q565,q566,q567,q568,q569,q570,q571,q572,q573,q574,q575,q576,q577,q578,q579,q580,q581,q582,q583,q584,q585,q586,q587,q588,q589,q589,q590,q591,q592,q593,q594,q595,q596,q597,q598,q599,q599,q600,q601,q602,q603,q604,q605,q606,q607,q608,q609,q609,q610,q611,q612,q613,q614,q615,q616,q617,q618,q619,q619,q620,q621,q622,q623,q624,q625,q626,q627,q628,q629,q629,q630,q631,q632,q633,q634,q635,q636,q637,q638,q639,q639,q640,q641,q642,q643,q644,q645,q646,q647,q648,q649,q649,q650,q651,q652,q653,q654,q655,q656,q657,q658,q659,q659,q660,q661,q662,q663,q664,q665,q666,q667,q668,q669,q669,q670,q671,q672,q673,q674,q675,q676,q677,q678,q679,q679,q680,q681,q682,q683,q684,q685,q686,q687,q688,q689,q689,q690,q691,q692,q693,q694,q695,q696,q697,q698,q698,q699,q699,q700,q701,q702,q703,q704,q705,q706,q707,q708,q709,q709,q710,q711,q712,q713,q714,q715,q716,q717,q718,q719,q719,q720,q721,q722,q723,q724,q725,q726,q727,q728,q729,q729,q730,q731,q732,q733,q734,q735,q736,q737,q738,q739,q739,q740,q741,q742,q743,q744,q745,q746,q747,q748,q749,q749,q750,q751,q752,q753,q754,q755,q756,q757,q758,q759,q759,q760,q761,q762,q763,q764,q765,q766,q767,q768,q769,q769,q770,q771,q772,q773,q774,q775,q776,q777,q778,q779,q779,q780,q781,q782,q783,q784,q785,q786,q787,q788,q789,q789,q790,q791,q792,q793,q794,q795,q796,q797,q798,q798,q799,q799,q800,q799,q801,q802,q803,q804,q805,q806,q807,q808,q809,q809,q810,q811,q812,q813,q814,q815,q816,q817,q818,q819,q819,q820,q821,q822,q823,q824,q825,q826,q827,q828,q829,q829,q830,q831,q832,q833,q834,q835,q836,q837,q838,q839,q839,q840,q841,q842,q843,q844,q845,q846,q847,q848,q849,q849,q850,q851,q852,q853,q854,q855,q856,q857,q858,q859,q859,q860,q861,q862,q863,q864,q865,q866,q867,q868,q869,q869,q870,q871,q872,q873,q874,q875,q876,q877,q878,q879,q879,q880,q881,q882,q883,q884,q885,q886,q887,q888,q889,q889,q890,q891,q892,q893,q894,q895,q896,q897,q898,q898,q899,q899,q900,q899,q901,q902,q903,q904,q905,q906,q907,q908,q909,q909,q910,q911,q912,q913,q914,q915,q916,q917,q918,q919,q919,q920,q921,q922,q923,q924,q925,q926,q927,q928,q929,q929,q930,q931,q932,q933,q934,q935,q936,q937,q938,q939,q939,q940,q941,q942,q943,q944,q945,q946,q947,q948,q949,q949,q950,q951,q952,q953,q954,q955,q956,q957,q958,q959,q959,q960,q961,q962,q963,q964,q965,q966,q967,q968,q969,q969,q970,q971,q972,q973,q974,q975,q976,q977,q978,q979,q979,q980,q981,q982,q983,q984,q985,q986,q987,q988,q988,q989,q989,q990,q989,q991,q992,q993,q994,q995,q996,q997,q997,q998,q998,q999,q999,q1000,q999,q1000,q1001,q1002,q1003,q1004,q1005,q1006,q1007,q1008,q1009,q1009,q1010,q1011,q1012,q1013,q1014,q1015,q1016,q1017,q1018,q1019,q1019,q1020,q1021,q1022,q1023,q1024,q1025,q1026,q1027,q1028,q1029,q1029,q1030,q1031,q1032,q1033,q1034,q1035,q1036,q1037,q1038,q1039,q1039,q1040,q1041,q1042,q1043,q1044,q1045,q1046,q1047,q1048,q1049,q1049,q1050,q1051,q1052,q1053,q1054,q1055,q1056,q1057,q1058,q1059,q1059,q1060,q1061,q1062,q1063,q1064,q1065,q1066,q1067,q1068,q1069,q1069,q1070,q1071,q1072,q1073,q1074,q1075,q1076,q1077,q1078,q1079,q1079,q1080,q1081,q1082,q1083,q1084,q1085,q1086,q1087,q1088,q1089,q1089,q1090,q1091,q1092,q1093,q1094,q1095,q1096,q1097,q1097,q1098,q1099,q1099,q1100,q1101,q1102,q1103,q1104,q1105,q1106,q1107,q1108,q1109,q1109,q1110,q1111,q1112,q1113,q1114,q1115,q1116,q1117,q1118,q1119,q1119,q1120,q1121,q1122,q1123,q1124,q1125,q1126,q1127,q1128,q1129,q1129,q1130,q1131,q1132,q1133,q1134,q1135,q1136,q1137,q1138,q1139,q1139,q1140,q1141,q1142,q1143,q1144,q1145,q1146,q1147,q1148,q1149,q1149,q1150,q1151,q1152,q1153,q1154,q1155,q1156,q1157,q1158,q1159,q1159,q1160,q1161,q1162,q1163,q1164,q1165,q1166,q1167,q1168,q1169,q1169,q1170,q1171,q1172,q1173,q1174,q1175,q1176,q1177,q1178,q1179,q1179,q1180,q1181,q1182,q1183,q1184,q1185,q1186,q1187,q1188,q1189,q1189,q1190,q1191,q1192,q1193,q1194,q1195,q1196,q1197,q1197,q1198,q1199,q1199,q1200,q1199,q1200,q1201,q1202,q1203,q1204,q1205,q1206,q1207,q1208,q1209,q1209,q1210,q1211,q1212,q1213,q1214,q1215,q1216,q1217,q1218,q1219,q1219,q1220,q1221,q1222,q1223,q1224,q1225,q1226,q1227,q1228,q1229,q1229,q1230,q1231,q1232,q1233,q1234,q1235,q1236,q1237,q1238,q1239,q1239,q1240,q1241,q1242,q1243,q1244,q1245,q1246,q1247,q1248,q1249,q1249,q1250,q1251,q1252,q1253,q1254,q1255,q1256,q1257,q1258,q1259,q1259,q1260,q1261,q1262,q1263,q1264,q1265,q1266,q1267,q1268,q1269,q1269,q1270,q1271,q1272,q1273,q1274,q1275,q1276,q1277,q1278,q1279,q1279,q1280,q1281,q1282,q1283,q1284,q1285,q1286,q1287,q1288,q1289,q1289,q1290,q1291,q1292,q1293,q1294,q1295,q1296,q1297,q1297,q1298,q1299,q1299,q1300,q1299,q1300,q1301,q1302,q1303,q1304,q1305,q1306,q1307,q1308,q1309,q1309,q1310,q1311,q1312,q1313,q1314,q1315,q1316,q1317,q1318,q1319,q1319,q1320,q1321,q1322,q1323,q1324,q1325,q1326,q1327,q1328,q1329,q1329,q1330,q1331,q1332,q1333,q1334,q1335,q1336,q1337,q1338,q1339,q1339,q1340,q1341,q1342,q1343,q1344,q1345,q1346,q1347,q1348,q1349,q1349,q1350,q1351,q1352,q1353,q1354,q1355,q1356,q1357,q1358,q1359,q1359,q1360,q1361,q1362,q1363,q1364,q1365,q1366,q1367,q1368,q1369,q1369,q1370,q1371,q1372,q1373,q1374,q1375,q1376,q1377,q1378,q1379,q1379,q1380,q1381,q1382,q1383,q1384,q1385,q1386,q1387,q1388,q1389,q1389,q1390,q1391,q1392,q1393,q1394,q1395,q1396,q1397,q1397,q1398,q1399,q1399,q1400,q1399,q1400,q1401,q1402,q1403,q1404,q1405,q1406,q1407,q1408,q1409,q1409,q1410,q1411,q1412,q1413,q1414,q1415,q1416,q1417,q1418,q1419,q1419,q1420,q1421,q1422,q1423,q1424,q1425,q1426,q1427,q1428,q1429,q1429,q1430,q1431,q1432,q1433,q1434,q1435,q1436,q1437,q1438,q1439,q1439,q1440,q1441,q1442,q1443,q1444,q1445,q1446,q1447,q1448,q1449,q1449,q1450,q1451,q1452,q1453,q1454,q1455,q1456,q1457,q1458,q1459,q1459,q1460,q1461,q1462,q1463,q1464,q1465,q1466,q1467,q1468,q1469,q1469,q1470,q1471,q1472,q1473,q1474,q1475,q1476,q1477,q1478,q1479,q1479,q1480,q1481,q1482,q1483,q1484,q1485,q1486,q1487,q1488,q1489,q1489,q1490,q1491,q1492,q1493,q1494,q1495,q1496,q1497,q1497,q1498,q1499,q1499,q1500,q1499,q1500,q1501,q1502,q1503,q1504,q1505,q1506,q1507,q1508,q1509,q1509,q1510,q1511,q1512,q1513,q1514,q1515,q1516,q1517,q1518,q1519,q1519,q15
```

Numerical evaluations



$$= - \int \frac{d^d \ell_1}{(2\pi)^d} \frac{d^d \ell_2}{(2\pi)^d} G_F(1, 2, 3, \dots, 7), \quad G_F(i) = q_i^2 - m^2 + i0$$

d=4

$$\begin{aligned} \{s = -3/5, \theta = \pi/6\} &\rightarrow -1.5912(2) * 10^{-6} \\ \{s = -2/7, \theta = \pi/6\} &\rightarrow -1.6991(1) * 10^{-6} \end{aligned}$$

d=6

$$\begin{aligned} \{s = -3/5, \theta = \pi/6\} &\rightarrow -2.73(7) * 10^{-8} \\ \{s = -2/7, \theta = \pi/6\} &\rightarrow -2.79(7) * 10^{-8} \end{aligned}$$

O(10") per point

Lotty
THE LOOP-TREE DUALITY
AUTOMATION

[W.J.T. (2021)]

$$\begin{aligned} p_1^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, 1, \vec{0}_{d-2} \right\}, & p_3^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, \sin \theta, \cos \theta, \vec{0}_{d-1} \right\}, \\ p_2^\alpha &= \frac{\sqrt{s}}{2} \left\{ 1, -1, \vec{0}_{d-2} \right\}, & p_4^\alpha &= \frac{\sqrt{s}}{2} \left\{ -1, -\sin \theta, -\cos \theta, \vec{0}_{d-1} \right\}. \end{aligned}$$

COM

Interesting set of integrals to look at in **d=6**

[Ronca, W.J.T. (w.i.p.)]

graph	$I^{[d]}-\text{integral}$	$I^{[d=6-2e]}(s = -\frac{1}{7}, t = -\frac{1}{3}, m^2 = 1)$
	$I^{[d]}(1, 1, 1, 0, 1, 0, 1, 1, 1)$	$-1.219372 - i 0.294408$
	$I^{[d]}(1, 2, 1, 0, 1, 0, 1, 1, 1)$	$0.98317 + i 1.00335$
	$I^{[d]}(1, 1, 1, 0, 1, 0, 1, 2, 1)$	$12.039969 + i 6.660946$
	$I^{[d]}(2, 1, 1, 0, 0, 0, 1, 1, 1)$	$-0.554605 - i 0.06984485$
	$I^{[d]}(1, 1, 1, 0, 0, 0, 1, 2, 1)$	$-1.91103 + i 0.241649$
	$I^{[d]}(3, 1, 1, 0, 0, 0, 1, 1, 1)$	$0.525679 + i 0.248668$

Multi-loop LTD causal representation

- In summary

$$\mathcal{A}^{(L)}(1, 2, \dots, n) = \int_{\vec{\ell}_1 \dots \vec{\ell}_L} \mathcal{I}\left(q_{i,0}^{(+)}\right)$$

Integration in the spatial components

$$q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$$

$$\int \prod_{i=1}^n dq_{i,0}^{(+)} \mathcal{I}\left(q_{i,0}^{(+)}\right)$$

- ➊ Rational function in $q_{i,0}^{(+)}$
- ➋ Dual repr. of Feynman integrands
- ➌ All-loop causal repr.
- ➍ Described by loop topologies' features

- ★ Edges
- ★ Vertices

- Straightforward numerical evaluation

Set dimension and evaluate
(finite integrand)

- ➊ Find local UV/IR counter-terms
- ➋ Find a repr. for real corrections
- ➌ Provide a complete implementation
in $d=4$ at NNLO.

[Rodrigo, Ronca, Sborlini, W.J.T., Tramontano (w.i.p)]

- Independently of the structure of numerator

- ➊ Make of use poly. div \rightarrow Groebner basis

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