Geometry and causality for efficient multiloop representations.

German F. R. SBORLINI

LTD team

G. Rodrigo, J. Aguilera Verdugo, F. Driencourt-Mangin, R. J. Hernández Pinto, J. Plenter, S. Ramírez Uribe, A. Rentería Olivo, W. J. Torres Bobadilla, L. Vale Silva

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

15th Radcor & LoopFest XIX

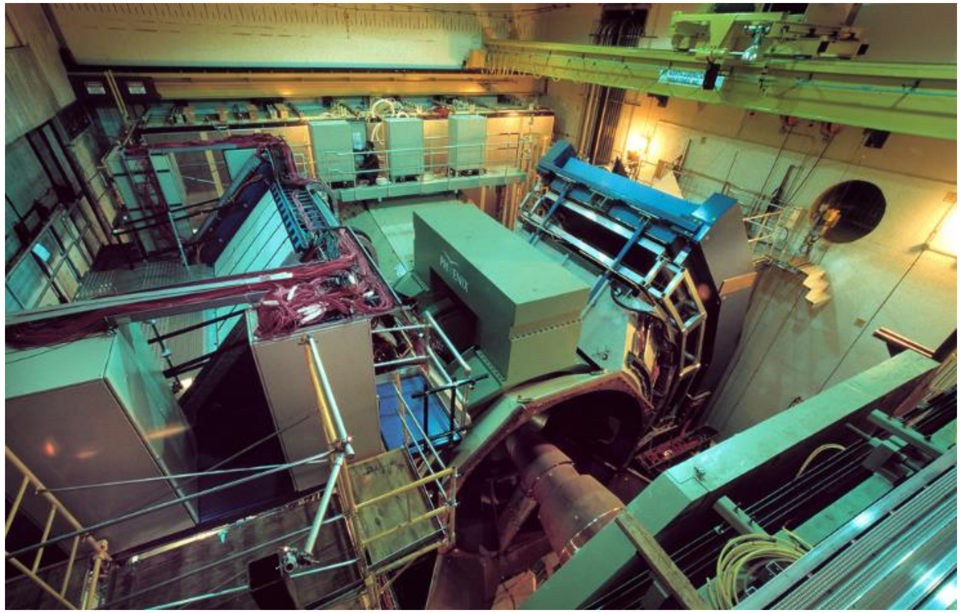
Tallahassee, FL (USA) – May 19th, 2021

DESY.

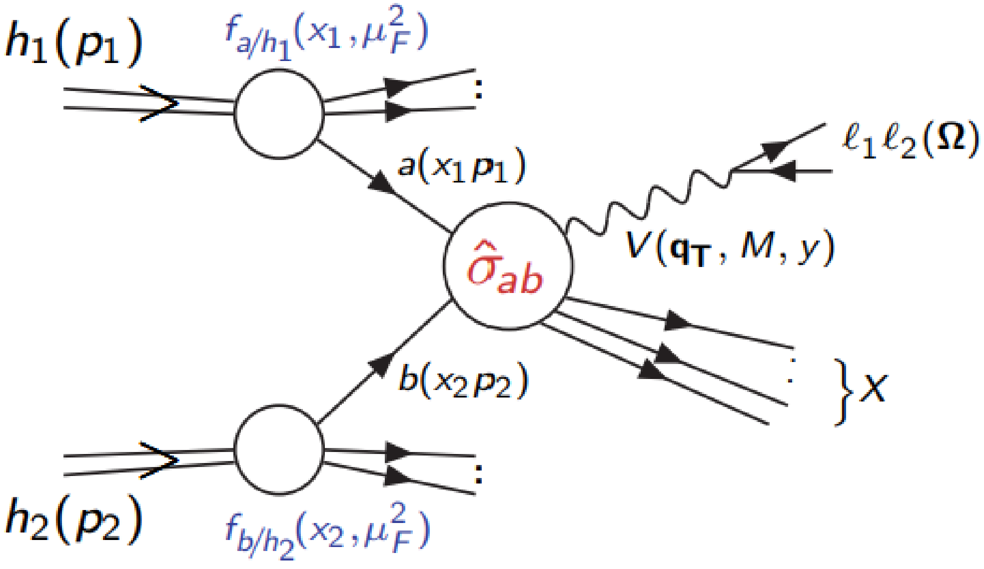
Index

1. Motivation
   a) Brief history of LTD-based methods
2. Nested residues
3. Manifestly Causal Representation
   a) Geometrical reconstruction NEW!!
   b) Quantum algorithms NEW!!
4. Conclusions and outlook

BASED ON: Phys. Rev. Lett. 124 (2020) 21, 211602
JHEP 12 (2019) 163; JHEP 01 (2021) 069
JHEP 02 (2021) 112; JHEP 04 (2021) 129
arXiv:2102.05062 [hep-ph]
arXiv:2105.08703 [hep-ph]



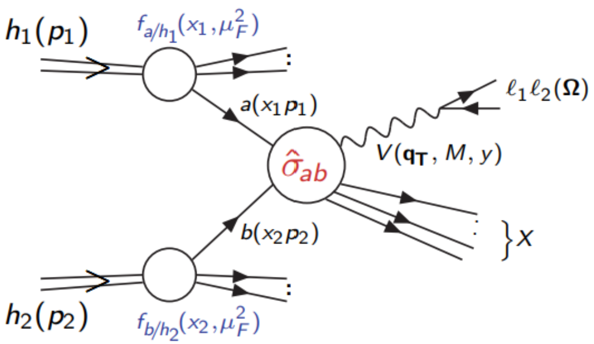
(COMPLICATED) EXPERIMENTS



THEORETICAL ABSTRACTION

Motivation

- **In the high-energy region, we can use the parton model**

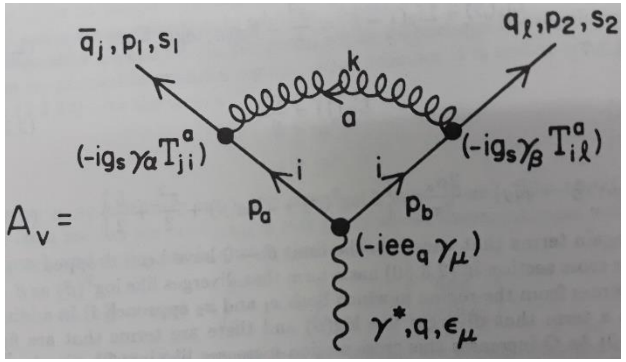


$$= \frac{d\sigma}{d^2\vec{q}_T\, dM^2 d\Omega dy} = \sum_{a,b} \int dx_1 dx_2\, f_a^{h_1}(x_1)\, f_b^{h_2}(x_2)\, \frac{d\hat{\sigma}_{ab\to V+X}}{d^2\vec{q}_T\, dM^2 d\Omega dy}$$

PDFs (non-perturbative)

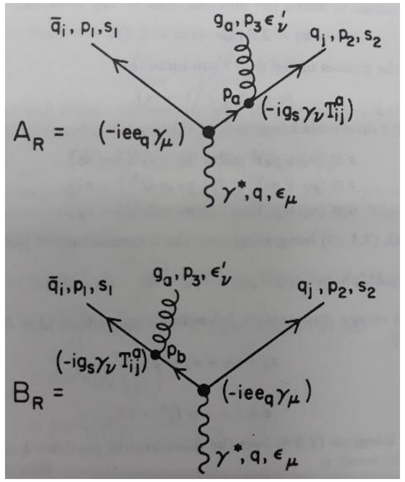
Partonic cross-section (perturbative)

- **Partonic cross sections are obtained from QFT (applying perturbative methods)**



Loop contributions (quantum fluctuations of vacuum)

+



Real corrections (additional particles)

+

*Appears **after** integration*

$$\frac{C_r}{\epsilon} \times d\sigma^{(0)}$$

Counter-terms (fix the problems of the other two)

=

FINITE NUMBER (compare to experiments)

CANCELLATION *AFTER* INTEGRATION

Motivation

- **Loop amplitudes are a bottleneck in current high-precision computations**
- Presence of **singularities and thresholds** prevents direct numerical implementations
- **Well-known theorems (KLN) guarantee the cancellation of singularities for physical observables**
- **Real-radiation** contributions are defined in **Euclidean space** (i.e. phase-space integrals)

LOOP AMPLITUDES
- *Virtual internal momenta*
- *Defined in Minkowski space-time*

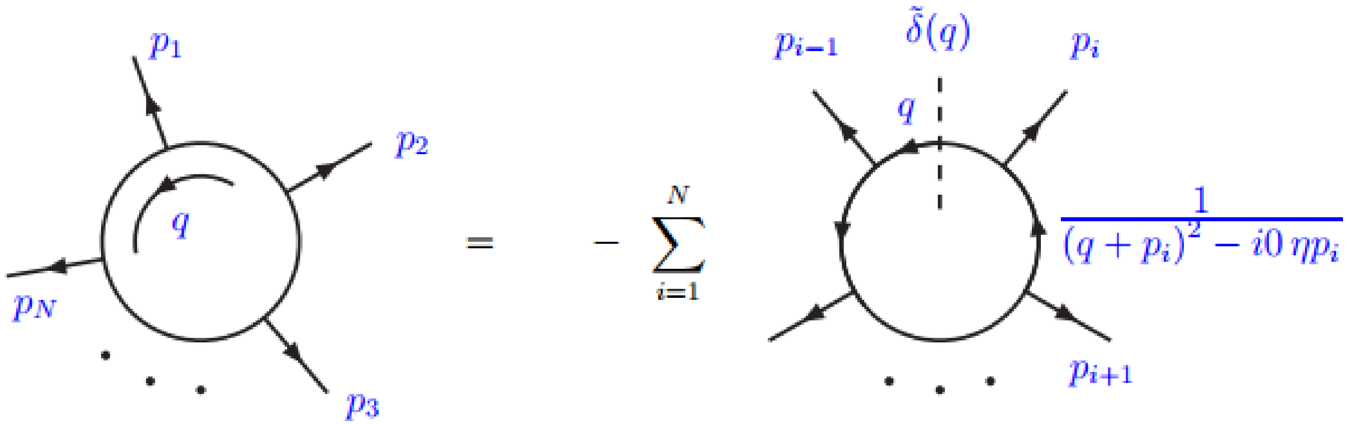
⇒ **Loop-Tree Duality** ⇒

DUAL AMPLITUDES
- *On-shell cut momenta*
- *Defined in Euclidean space-time*

To be combined +

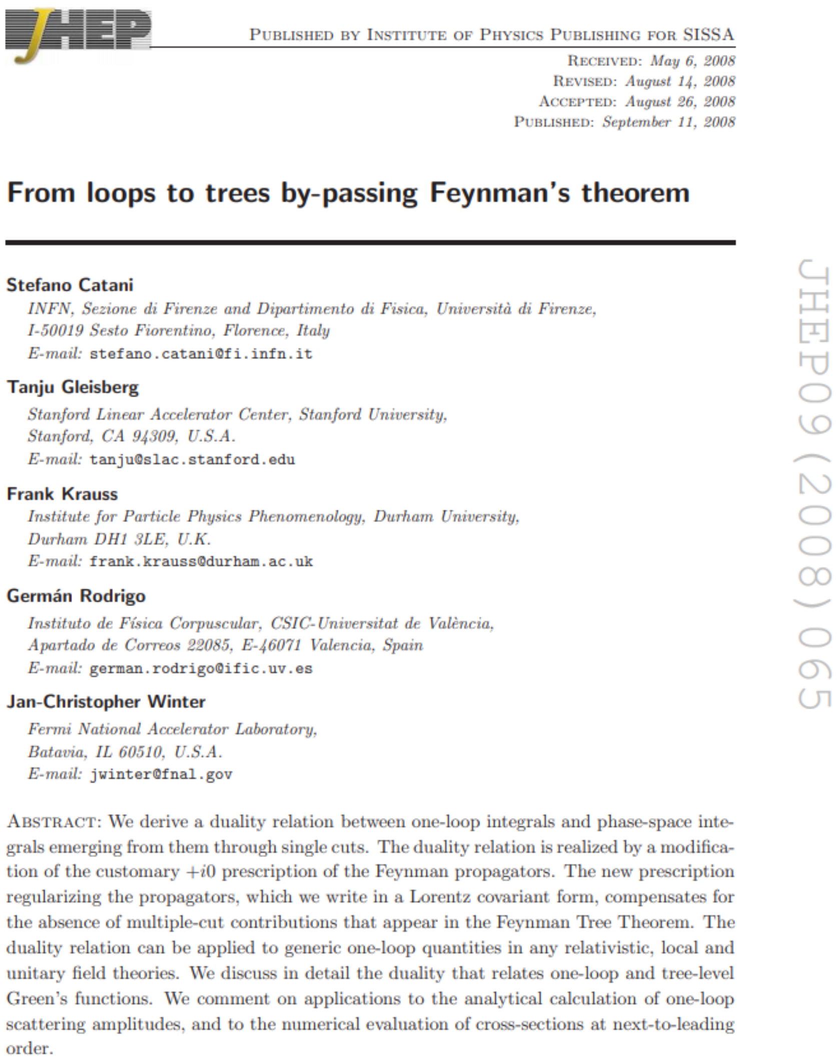
REAL CONTRIBUTIONS (AND ISR/UV COUNTER-TERMS)

Graphical representation of one-loop opening into trees *(original idea by Catani et al '08)*



Brief history of LTD-based methods

- **Foundational paper: a new way to decompose loop amplitudes**



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Accepted: *August 26, 2008*
Published: *September 11, 2008*

From loops to trees by-passing Feynman's theorem

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Abstract: We derive a duality relation between one-loop integrals and phase-space integrals emerging from them through single cuts. The duality relation is realized by a modification of the customary $+i0$ prescription of the Feynman propagators. The new prescription regularizing the propagators, which we write in a Lorentz covariant form, compensates for the absence of multiple-cut contributions that appear in the Feynman Tree Theorem. The duality relation can be applied to generic one-loop quantities in any relativistic, local and unitary field theories. We discuss in detail the duality that relates one-loop and tree-level Green's functions. We comment on applications to the analytical calculation of one-loop scattering amplitudes, and to the numerical evaluation of cross-sections at next-to-leading order.

JHEP09(2008)065

- Application of Cauchy theorem **taking care of Feynman prescription**: leads to a **new prescription!**

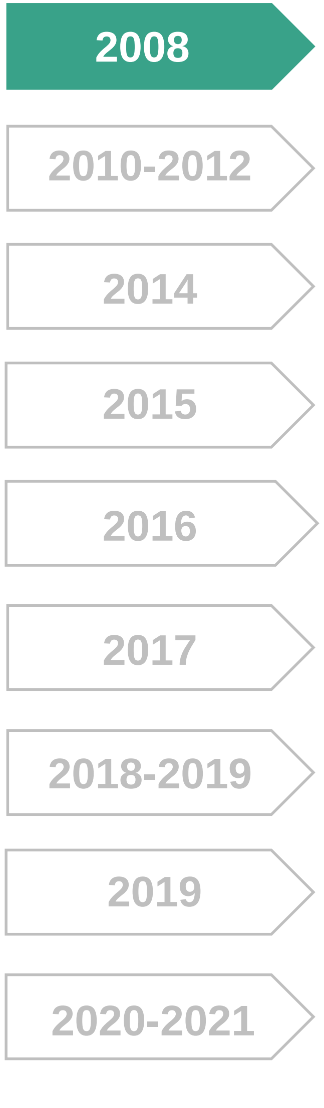
Feynman integral

$$L^{(1)}(p_1,\ldots,p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i) = \int_\ell \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0}$$

$$L^{(1)}(p_1,\ldots,p_N) = -\sum_{i=1}^{N} \int_\ell \tilde{\delta}(q_i) \prod_{j=1, j\neq i}^{N} G_D(q_i;q_j)$$

Dual integral

JHEP 09 (2008) 065



JHEP 02 (2016) 044

- Towards the computation of physical observables in four space-time dimensions
- **Tested on toy scalar model; local cancellation of IR divergences**

JHEP
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Received: September 2, 2015
Revised: December 6, 2015
Accepted: January 15, 2016
Published: February 5, 2016

Towards gauge theories in four dimensions

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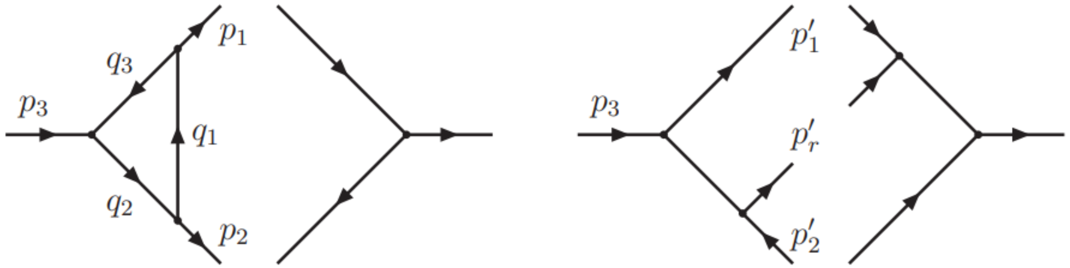
Abstract: The abundance of infrared singularities in gauge theories due to unresolved emission of massless particles (soft and collinear) represents the main difficulty in perturbative calculations. They are typically regularized in dimensional regularization, and their subtraction is usually achieved independently for virtual and real corrections. In this paper, we introduce a new method based on the loop-tree duality (LTD) theorem to accomplish the summation over degenerate infrared states directly at the integrand level such that the cancellation of the infrared divergences is achieved simultaneously, and apply it to reference examples as a proof of concept. Ultraviolet divergences, which are the consequence of the point-like nature of the theory, are also reinterpreted physically in this framework. The proposed method opens the intriguing possibility of carrying out purely four-dimensional implementations of higher-order perturbative calculations at next-to-leading order (NLO) and beyond free of soft and final-state collinear subtractions.

Keywords: NLO Computations

ArXiv ePrint: 1506.04617

JHEP02(2016)044



- **Introduction of real-dual mappings, to achieve a local cancellation of IR singularities!**

$$p_r'^{\mu} = q_1^{\mu}\,, \qquad p_1'^{\mu} = -q_3^{\mu} + \alpha_1\, p_2^{\mu} = p_1^{\mu} - q_1^{\mu} + \alpha_1\, p_2^{\mu}\,,$$

$$p_2'^{\mu} = (1-\alpha_1)\, p_2^{\mu}\,, \qquad \alpha_1 = \frac{q_3^2}{2 q_3\cdot p_2}\,,$$

- Purely four-dimensional representation of cross-sections
- **First study of dual UV *local* counter-terms:**

$$I_{\rm UV}^{\rm cnt} = \int_\ell \frac{1}{\left(q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0\right)^2}$$

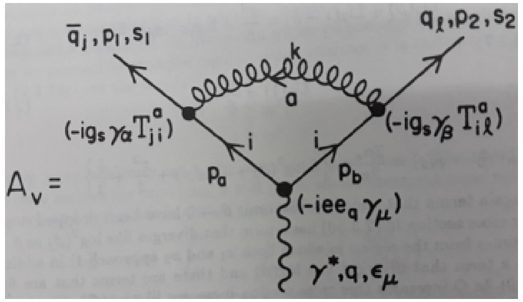
2008
2010-2012
2014
2015
2016
2017
2018-2019
2019
2020-2021

Brief history of LTD-based methods

JHEP 08 (2016) 160
JHEP 10 (2016) 162

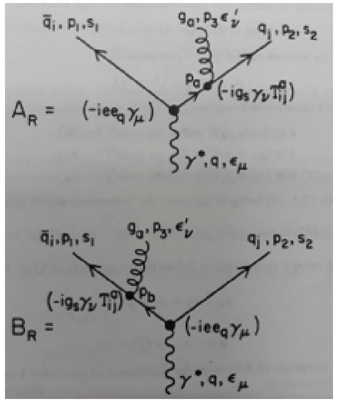
- Towards the computation of physical observables in four space-time dimensions
- **Tested on toy scalar model; local cancellation of IR divergences**

Partonic cross sections are obtained from QFT (applying perturbative methods)



Loop contributions (quantum fluctuations of vacuum)

+



Real corrections (additional particles)

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*Appears **after** integration*

$\frac{C_r}{\epsilon} \times d\sigma^{(0)}$

Counter-terms (fix the problems of the other two)

=

FINITE NUMBER (compare to experiments)

CANCELLATION *AFTER* INTEGRATION

2008
2010-2012
2014
2015
2016
2017
2018-2019
2019
2020-2021

- **Integrand-level** cancellation of IR and UV singularities!
- **No need of integrated counter-terms**
- Purely four-dimensional integration (**no DREG!**)

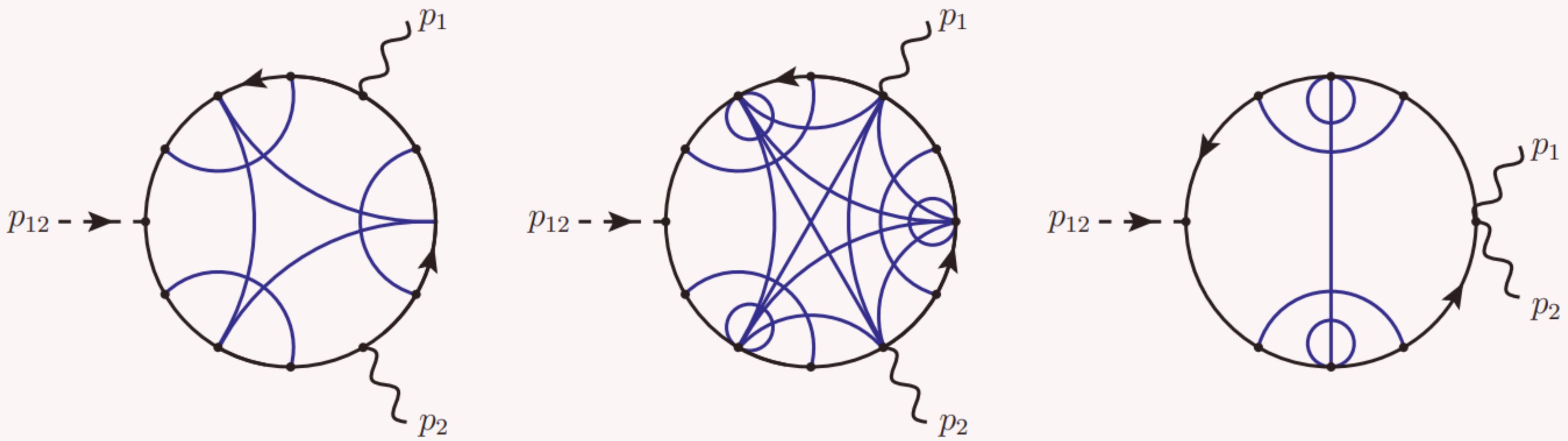
FIRST APPROACH TO LOCAL REPRESENTATIONS!!

JHEP 02 (2019) 143

JHEP 12 (2019) 163

- Full analysis of Higgs decays at two-loop (inclusion of EW effects)
- **First realization of local UV counter-terms at two-loop level**

Locality explored at two-loops… what's next?



- **New singular structures arise beyond one-loop**
- More diagrams, more variables… starts to be cumbersome!
- **Explore novel representations of the integrands**
- Point towards fully local cancellations of IR/UV singularities

UNDERSTANDING SINGULARITIES IS CRUCIAL!! EXPLORE THEM!!

2008

2010-2012

2014

2015

2016

2017

2018-2019

2019

2020-2021

Brief history of LTD-based methods

PHYSICAL REVIEW LETTERS 124, 211602 (2020)

Open Loop Amplitudes and Causality to All Orders and Powers from the Loop-Tree Duality

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Selomit Ramírez-Uribe,[1,2,3,||] Andrés E. Rentería-Olivo,[1,¶] Germán Rodrigo,[1,**] ...
William J. Torres Bobadilla,[1,††] and Szymon T...

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Ciudad Universitari...
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Ciud...

(Received 16 January 2020...

Mult...

NEW TECHNOLOGY FOR MULTILOOP COMPUTATIONS!!

arXiv:2006.112...

Causal repres...

Authors: J. Jesus Ag... Torres Bobadilla

Abstract: The numeric... requires to deal with bo... offers a powerful framew... analytically the underling e...

Submitted 19 June, 2020; origina...

Comments: 24 pages, 8 figures

Report number: IFIC/20-27

Jun. '...

...heir application to multi-loop scattering

...rnandez-Pinto, German Rodrigo, German F. R. Sborlini, William J.

... multi-loop multi-leg scattering amplitudes plays a key role to improve the precision of ...s for particle physics at high-energy colliders. In this work, we focus on the mathematical ... the novel integrand-level representation of Feynman integrals, which is based on the Loop-Tree Duality ... We explore the behaviour of the multi-loop iterated res... ▽ More

Submitted 24 October, 2020; originally announced October 2020.

Comments: 29 pages + appendices, 11 figures

Report number: IFIC/20-30; DESY 20-172; MPP-2020-184

Oct. '20

...ering amplitudes to trees

...ger J. Hernandez-Pinto, German Rodrigo, German F. R. Sborlini, William J. Torres

...rbative approach to quantum field theories has made it possible to obtain incredibly accurate ... predictions in high-energy physics. Although various techniques have been developed to boost the ...ciency of these calculations, some ingredients remain specially challenging. This is the case of multiloop scattering amplitudes that constitute a hard bottleneck to solve. In this Let... ▽ More

Submitted 24 June, 2020; originally announced June 2020.

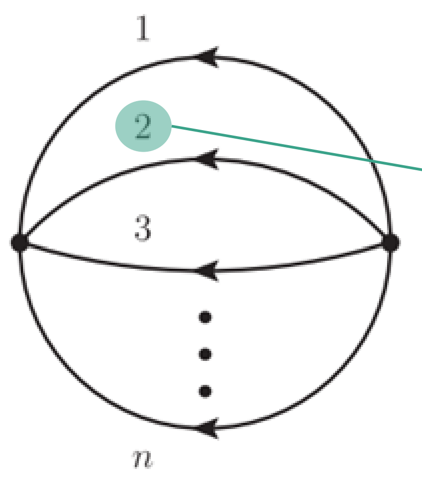
Comments: 7 pages, 4 figures

Report number: IFIC/20-29

Jun. '20

- 2008
- 2010-2012
- 2014
- 2015
- 2016
- 2017
- 2018-2019
- 2019
- **2020-2021**

- *Starting point:* multiloop Feynman integrals and scattering amplitudes
- ***Iterated** application of the Cauchy residue theorem to remove one DOF for each loop momenta*



Multiloop diagram

Notation setup

Sets of momenta $i_s \in s$ → $q_{i_s} = \ell_s + k_{i_s}$

Combination of external momenta

Loop momentum (integration)

Loop-energy component

$$G_F(q_{i_s}) = \frac{1}{q_{i_s,0}^2 - (q_{i_s,0}^{(+)})^2}$$

Standard propagator

$$q_{i_s,0}^{(+)} = \sqrt{\mathbf{q}_{i_s}^2 + m_{i_s}^2 - \imath 0}$$

On-shell energy (*complex number!*)

Loop space-vector (*Euclidean*)

Feynman i0 prescription

- Using this notation, we write *any* L-loop N-particle scattering amplitude:

$$\mathcal{A}_N^{(L)}(1,\dots,n) = \int_{\ell_1,\dots,\ell_L} \mathcal{N}(\{\ell_i\}_L, \{p_j\}_N) G_F(1,\dots,n) \quad \text{with} \quad G_F(1,\dots,n) = \prod_{i \in 1 \cup \dots \cup n} (G_F(q_i))^{a_i}$$

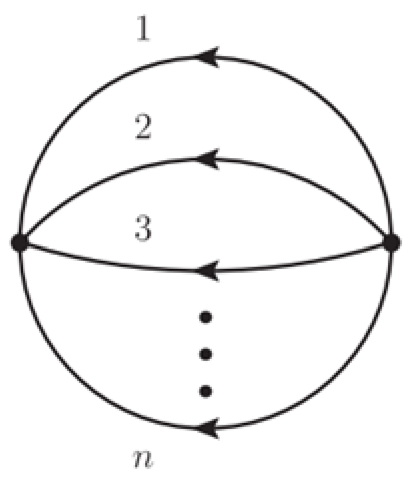
D-dimensional loop momenta (*Minkowski*)

Sets of momenta

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

Nested residues: Details

- *Starting point:* multiloop Feynman integrals and scattering amplitudes
- ***Iterated** application of the Cauchy residue theorem to remove one DOF for each loop momenta*



Multiloop diagram

Application of Cauchy's theorem

$$G_F(1,\ldots,n) = \prod_{i\in 1\cup\cdots\cup n} (G_F(q_i))^{a_i} \;\Rightarrow\; G_D(s;t) = -2\pi\imath \sum_{i_s\in s} \mathrm{Res}(G_F(s,t), \mathrm{Im}(\eta\cdot q_{i_s})<0)$$

Dual function (1st step)

Select one set "s" and compute the residue

Other sets (no residue computation)

Sum over all the elements of the set

Pole selection criteria! **(IMPORTANT)**

- **Observation 1:** For single powers and $\eta = (1,\mathbf{0})$ we get the well-know one-loop LTD formula:

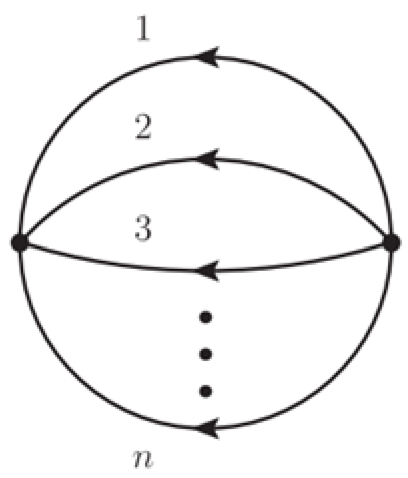
$$G_D(s) = -\sum_{i_s\in s} \tilde{\delta}(q_{i_s}) \prod_{\substack{j_s\neq i_s \\ j_s\in s}} \frac{1}{(q^{(+)}_{i_s,0} + k_{j_s i_s,0})^2 - (q^{(+)}_{j_s,0})^2}$$

- **Observation 2:** The equivalence with previous LTD representation is encoded in $\mathrm{Im}(\eta\cdot q_{i_s})<0$ for the integration contour selection (*"dual prescription"*)

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

Nested residues: Details

- *Starting point:* multiloop Feynman integrals and scattering amplitudes
- ***Iterated** application of the Cauchy residue theorem to remove one DOF for each loop momenta*



Multiloop diagram

Iterated application of Cauchy's theorem

Remaining sets (no residue evaluation)

$$G_D(1,\ldots,r;n) = -2\pi\imath \sum_{i_r\in r} \mathrm{Res}(G_D(1,\ldots,r-1;r,n), \mathrm{Im}(\eta\cdot q_{i_r})<0)$$

r^{th} residue evaluation

Sum over all the elements of the r^{th} set

$(r-1)^{th}$ dual function

Depends on integration variables (q_i)

Poles could be in-or-out depending on specific momenta…

- Dual representation for L-loop amplitudes is obtained after the L^{th} residue evaluation
- *Equivalent to: **"Number of cuts equal number of loops"***
- ***Sum over all possible poles is implicit: some contributions vanish inside each iteration***

Iterated residues (all the poles) → cancellations, **"Displaced poles"** → **Nested residues *(only physical ones)***

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

Nested residues: Displaced poles

- *Theorem*: Given a generic* rational function $F(x_i, x_j) = \frac{P(x_i, x_j)}{((x_i - a_i)^2 - y_i^2)^{\gamma_i}((x_i + x_j - a_{ij})^2 - y_k^2)^{\gamma_k}}$

  then: $$\text{Res}(\text{Res}(F(x_i,x_j), \{x_i, y_i + a_i\}), \{x_j, y_k - y_i + a_{ij} - a_i\})$$
  $$= -\text{Res}\left(\text{Res}\left(F(x_i, x_j), \{x_i, y_k - x_j + a_{ij}\}\right), \{x_j, y_k - y_i + a_{ij} - a_i\}\right)$$

- **Physical consequences:**

  1. **Displaced poles** are associated to **un-physical** contributions:

     "they can not be mapped into cuts"

  2. After applying C.R.T. to all the loop momenta and **summing over the physical poles**:

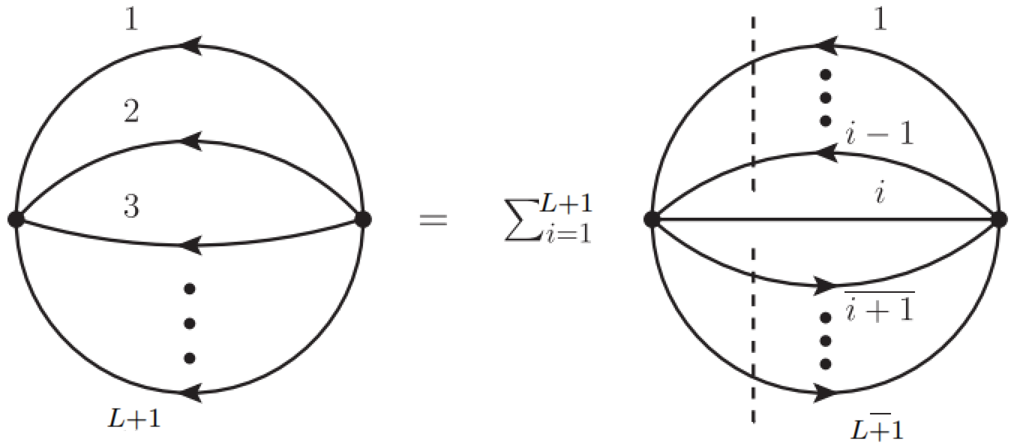
     "only same-sign combinations of $q_{k,0}^{(+)}$ *remain"*

Cancellation of displaced poles → **"Aligned contributions"** → **Causal propagators**

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph] & JHEP 02 (2021) 112

- **Cancellation of displaced poles leads to very compact formulae for the dual representation:**

Maximal Loop Topology (2 vertices, L+1 lines)



REMARK: External particles can be attached to each momenta set

On-shell lines with reversed momenta

Lines = sets of propagators

$$\mathcal{A}^{(L)}_{\mathrm{MLT}}(1,2,\dots,L+1) = \int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \sum_{i=1}^{L+1} \mathcal{A}_D(1,\dots,i-1,\overline{i+1},\dots,\overline{L+1};i)$$

1 off-shell line

Defined in Minkowski space

Defined in Euclidean space

On-shell lines

- We define the Maximal Loop Topology (MLT) as a building block to describe multi-loop amplitudes
- **Important:** "*Any one and two-loop amplitude can be described by MLT topologies*"

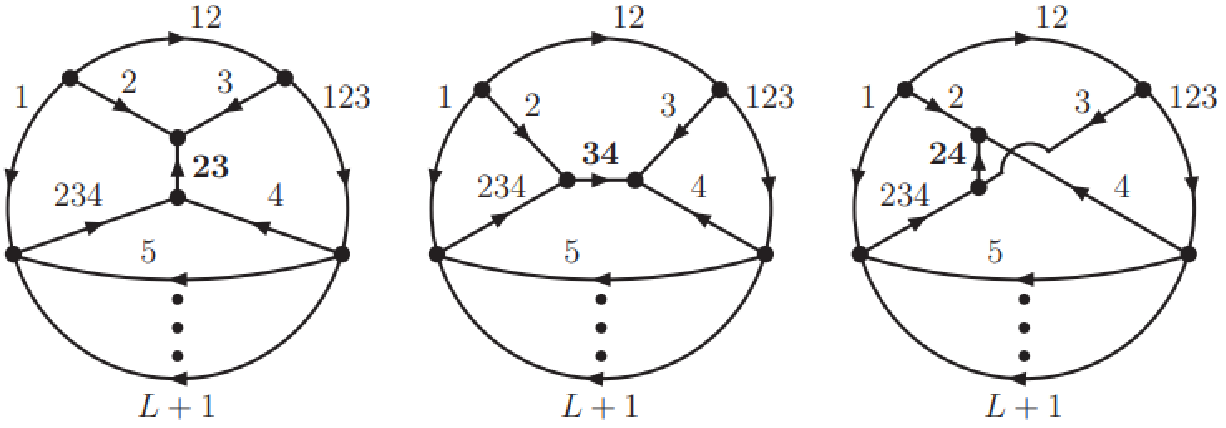
Inductive proofs of these formulae to all-loop orders available in JHEP 02 (2021) 112

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602 & JHEP 01 (2021) 069

Nested residues: Compact representations

- It works also for (much) more complicated topologies!!!

NNNN Maximal Loop Topologies (6 vertices, L+5 lines)

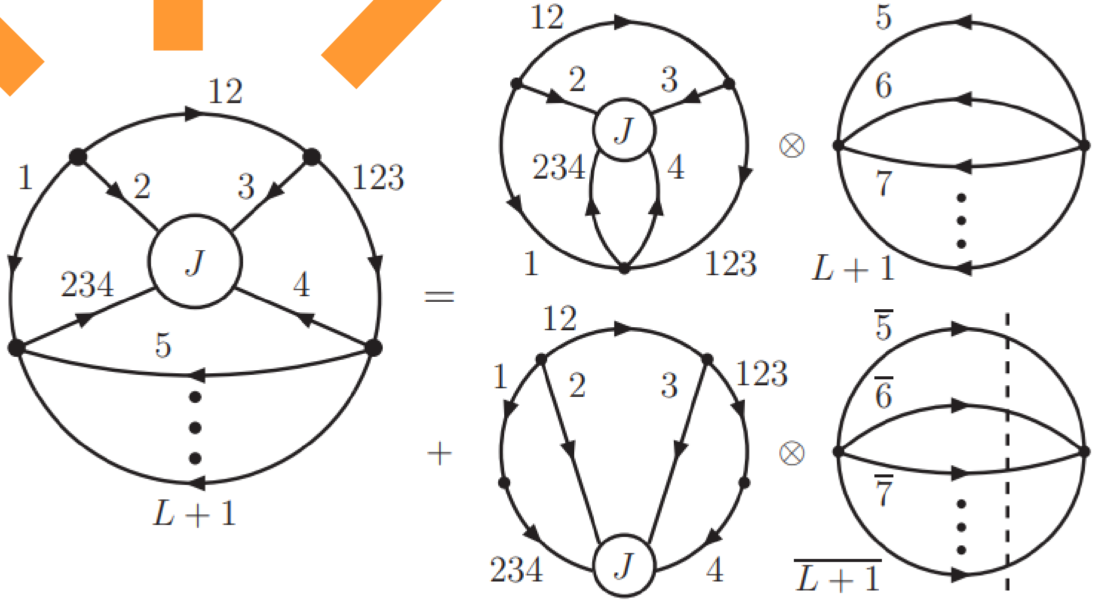


Thanks to factorization properties, the singular and **causal** structure is given in terms of simpler objects

Lines = sets of propagators

$$\mathcal{A}^{(L)}_{\mathrm{N^4MLT}}(1,\ldots,L+1,12,123,234,J)$$
$$= \mathcal{A}^{(4)}_{\mathrm{N^4MLT}}(1,2,3,4,12,123,234,J)$$
$$\otimes\, \mathcal{A}^{(L-4)}_{\mathrm{MLT}}(5,\ldots,L+1)$$
$$+\, \mathcal{A}^{(3)}_{\mathrm{N^2MLT}}(1\cup 234,2,3,4\cup 123,12,J)$$
$$\otimes\, \mathcal{A}^{(L-3)}_{\mathrm{MLT}}(\bar{5},\ldots,\overline{L+1})$$

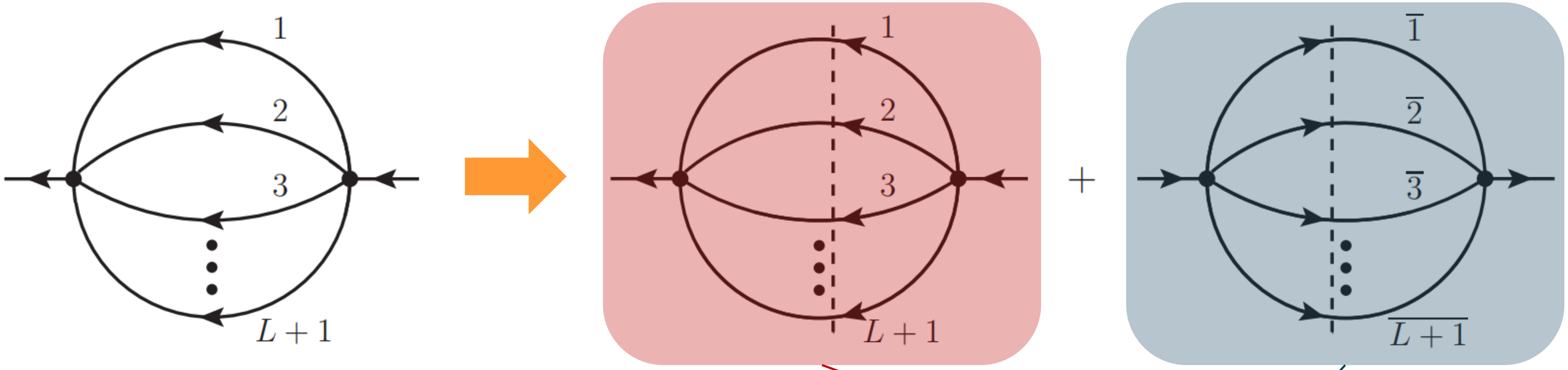
$\mathrm{N^4MLT}$ universal topology



Ramírez Uribe et al (2020) arXiv:2006.13818 [hep-ph] & JHEP 04 (2021) 129

Manifestly Causal Representation

- The cancellation of displaced poles implies un-physical terms vanish in the final representation
- Moreover, there is a strict connection between **aligned contributions** and **causal terms!!!**
- *MLT example*: If we **sum over all the possible cuts**, we get this **extremely compact** result:



$$\mathcal{A}^{(L)}_{\mathrm{MLT}}(1,2,\ldots,(L+1)_{-p_1}) = -\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{1}{x_{L+1}} \left(\frac{1}{\lambda_1^-} + \frac{1}{\lambda_1^+} \right)$$

with $\lambda_1^{\pm} = \sum_{i=1}^{L+1} q_{i,0}^{(+)} \pm p_{1,0}$ and $x_{L+k} = 2^{L+k} \prod_{i=1}^{L+k} q_{i,0}^{(+)}$

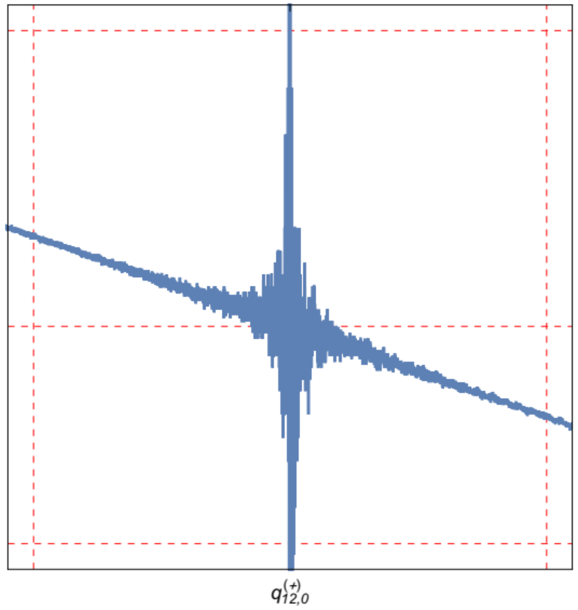
CAUSAL PROPAGATORS

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph] & JHEP 01 (2021) 069

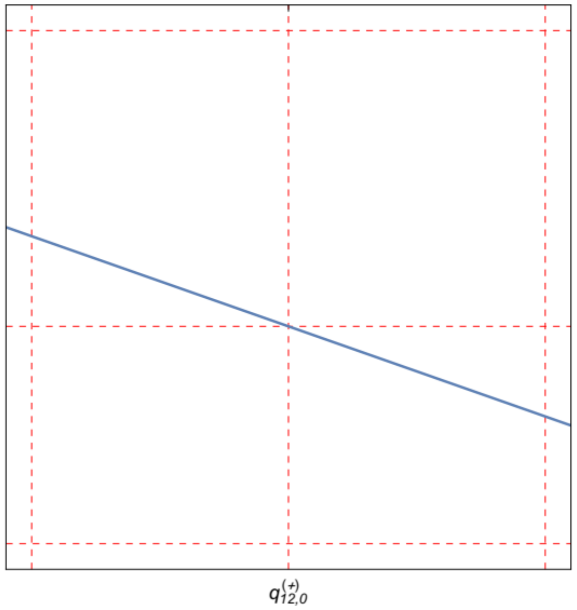
- **This is the Causal Representation and exists for any QFT amplitude!**
- **Advantages**

  1. Causal denominators have **same-sign combinations of on-shell energies** (positive numbers), thus are **more stable numerically!**
  2. **Only physical thresholds remain**; spurious un-physical instabilities are removed!

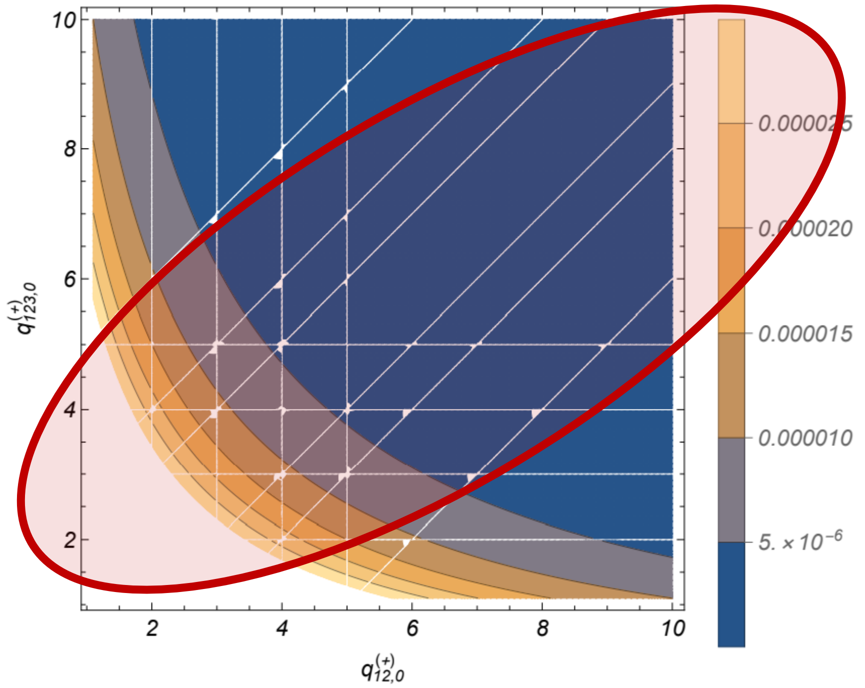
MORE DETAILED EXAMPLES IN WJTB's TALK!!



Without causal representation



With causal representation



White lines = Numerical instabilities

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]
Ramírez Uribe et al (2020) arXiv:2006.13818 [hep-ph]

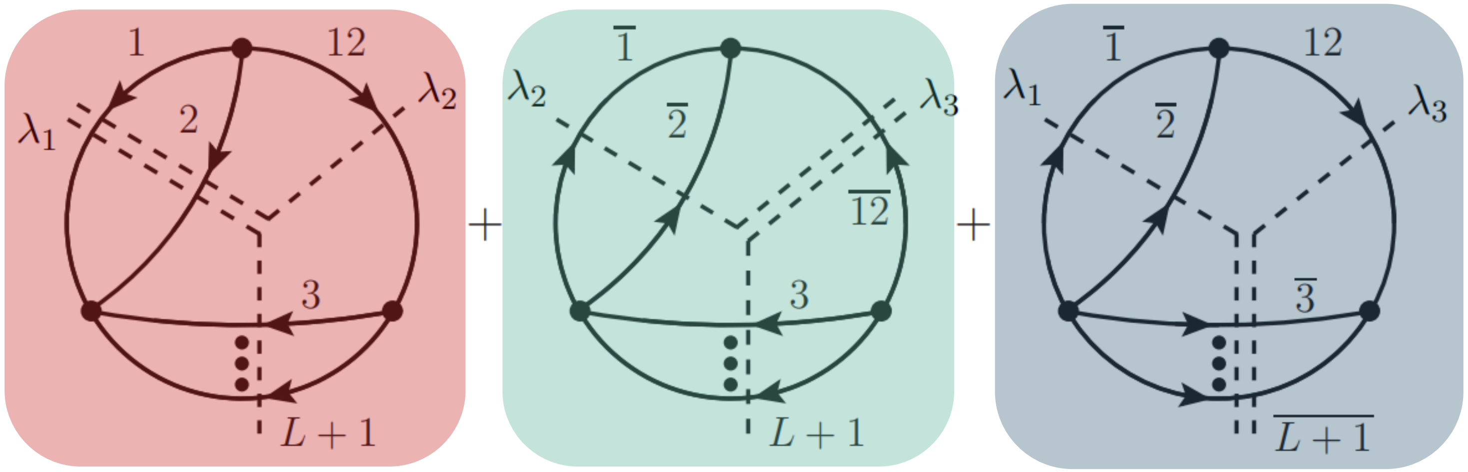
Manifestly Causal Representation

- Similarly compact representations were found for more complicated topologies!!

  JHEP 01 (2021) 069, JHEP 04 (2021) 129, JHEP 04 (2021) 183

- **Graphical interpretation in terms of entangled thresholds**
  1. **Each causal** propagator represents a **threshold** of the diagram
  2. **Each diagram** contains **several thresholds**
  3. **The causal representation involves products of (*compatible*) thresholds**

Causal denominators (λ) are associated to *cut lines* in the diagrams: momenta flow must be adjusted to be compatible



$$\mathcal{A}^{(L)}_{\rm NMLT}(1,2,\ldots,L+2) = \int_{\vec{\ell}_1,\ldots,\vec{\ell}_L} \frac{2}{x_{L+2}} \left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1} \right)$$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]
Ramírez Uribe et al (2020) arXiv:2006.13818 [hep-ph]

Geometric Algorithm for Causal Reconstruction

- **Causal representation** obtained directly after **summing over all the nested residues**

Master formula

$$\mathcal{A}_N^{(L)}(1,\ldots,L+k) = \sum_{\sigma\in\Sigma}\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{\mathcal{N}_\sigma(\{q_{r,0}^{(+)}\},\{p_{j,0}\})}{x_{L+k}} \times \prod_{i=1}^{k} \frac{1}{-\lambda_{\sigma(i)}} + (\sigma \leftrightarrow \bar{\sigma})$$

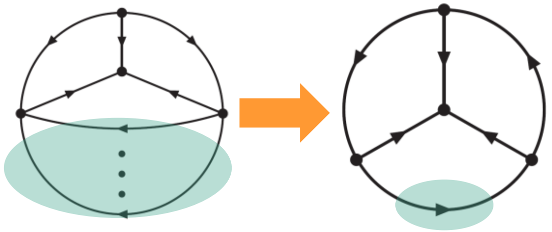
Set of entangled thresholds

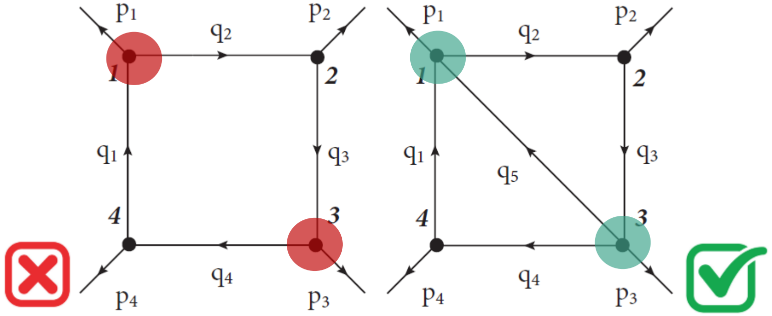
Products of k causal propagators

- *Is it possible to do it in other way?* → YES! Geometrical reconstruction & Algebraic reconstruction

WJTB's TALK!!

- Previous concepts
  1. **Diagrams** are made of **vertices** and **edges** *(bunches of propagators, connecting two given vertices)*
  2. **Edges** define a **basis of momenta**, that lead to the **"vertex matrix"** → **Defines the casual structure!**
  3. **Binary partitions** are given by **subsets of vertices** that **splits in two** the original diagram → **Connected partitions!**

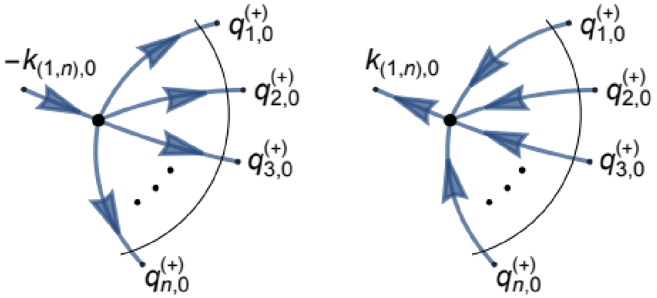


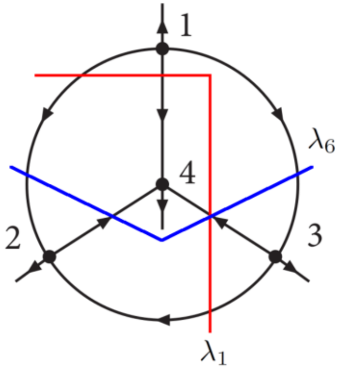


Sborlini (2021) arXiv:2102.05062 [hep-ph]

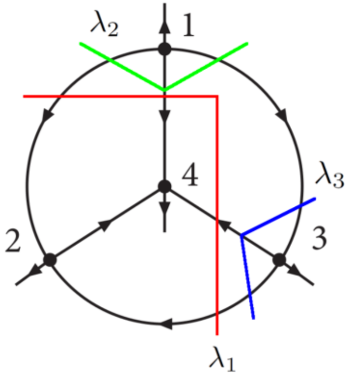
More detailed explanation arXiv:2102.05062 [hep-ph]

1. <u>Generate causal propagators</u>
   - Causal propagators are associated to **binary connected partitions** of the diagram, namely *"connected sub-blocks of the diagram"*
   - They encode the possible **physical thresholds**
   - Involve a **consistent (aligned) energy flow** through the cut lines
2. <u>Order of a diagram:</u> it quantifies the complexity of a given topology
   - *k=1 for MLT, k=2 for NMLT and so on* → ***k = vertices - 1***
   - A diagram of **order k** involves **products of k causal propagators**
3. <u>Geometric compatibility rules:</u> determine the entangled thresholds
   a) **All the edges are cut at least once**
   b) **Causal propagators do no intersect;** i.e. they are associated to disjoint or extended partitions of the diagram
   c) **All the edges** involved in a causal threshold must carry **momenta flowing in the same direction** → Distinction λ^+ / λ^-





No intersections



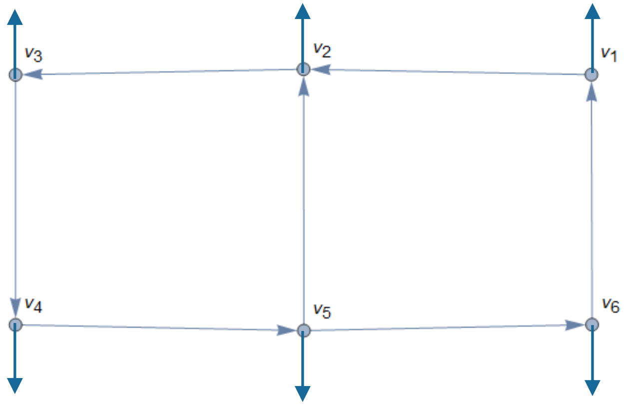
Compatible causal flux

Sborlini (2021) arXiv:2102.05062 [hep-ph]

Geometric Algorithm for Causal Reconstruction

Implemented in a Mathematica package
https://github.com/gfsborlini/tLTD
(ongoing development, not public -yet-)

- **Example:** 2-loop hexagon (7 edges, 6 vertices, 1 external leg per vertex)




```
Orden = 5; NumeroVertices = 6;
Eq[1] = {-q[1] + q[2] + p[1]};
Eq[2] = {-q[2] + q[3] - q[7] + p[4]};
Eq[3] = {-q[3] + q[4] + p[2]};
Eq[4] = {-q[4] + q[5] + p[3]};
Eq[5] = {-q[5] + q[6] + q[7] + p[5]};
Eq[6] = {-q[6] + q[1] - (p[1] + p[2] + p[3] + p[4] + p[5])};
```

Input: vertex definition, i.e. labelling & momentum conservation

```
SetDirectory[NotebookDirectory[]];
<< tLTDtoolsv4.m;

++++++++++++++++++++++++++++++++++++++++++++++
++++++      tLTD tools - version 4.0     ++++++
++++++    last update: 09-May-2021       ++++++
++++++ based on arXiv:2102.05062[hep-ph] ++++++
++++++ improved geometric reconstruction ++++++
++++++++++++++++++++++++++++++++++++++++++++++
```

$$\left(\begin{array}{ccccccc|ccccc}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1
\end{array}\right)$$

Vertex matrix: Basic object to generate the causal representation

Generate causal propagators

```
tmpSALIDA = AbsoluteTiming[SALIDA = GeneraLambdas[MatrizVertices]];
Print["Time: ", tmpSALIDA[[1]]]

Numero de lambdas: 21
Time: 0.0184136
```

```
λm[1] → ·
λm[2] → -p[·
λm[3] → ·
λm[4] → ·
λm[5] → -p[
λm[6] → -p[1] - p[2] -
λm[7] → -p[1] -
λm[8] → -p[2] - p[
λm[9] → -p[2] -
λm[10] → -p[4] - p[
λm[11] → -p[
λm[12] → -p[1] - p[3] - p[
λm[13] → -p[3]
λm[14] → -p[1] - p[2] - p[
λm[15] → -p[1] - p[2]
λm[16] → -p[1] - p[
λm[17] → -p[1] - p[4]
λm[18] → -p[2] - p[
λm[19] → -p[3] - p[4]
λm[20] → -p[2] - p[4] - p[5] + q[2] + q[4] + q[5] + q[6]
λm[21] → -p[2] - p[3] - p[4] + q[2] + q[5] + q[7]
```

```
λp[1] → p[1] + q[1] + q[2]
λp[2] → p[4] + q[2] + q[3] + q[7]
λp[3] → p[2] + q[3] + q[4]
λp[4] → p[3] + q[4] + q[5]
λp[5] → p[5] + q[5] + q[6] + q[7]
λp[6] → p[1] + p[2] + p[3] + p[4] + p[5] + q[1] + q[6]
λp[7] → p[1] + p[4] + q[1] + q[3] + q[7]
λp[8] → p[2] + p[3] + p[4] + p[5] + q[2] + q[6]
λp[9] → p[2] + p[4] + q[2] + q[4] + q[7]
λp[10] → p[4] + p[5] + q[2] + q[3] + q[5] + q[6]
λp[11] → p[2] + p[3] + q[3] + q[5]
λp[12] → p[1] + p[3] + p[4] + p[5] + q[1] + q[3] + q[4] + q[6]
λp[13] → p[3] + p[5] + q[4] + q[6] + q[7]
λp[14] → p[1] + p[2] + p[4] + p[5] + q[1] + q[4] + q[5] + q[6]
λp[15] → p[1] + p[2] + p[3] + p[4] + q[1] + q[5] + q[7]
λp[16] → p[1] + p[2] + p[4] + q[1] + q[4] + q[7]
λp[17] → p[1] + p[4] + p[5] + q[1] + q[3] + q[5] + q[6]
λp[18] → p[2] + p[3] + p[5] + q[3] + q[6] + q[7]
λp[19] → p[3] + p[4] + p[5] + q[2] + q[3] + q[4] + q[6]
λp[20] → p[2] + p[4] + p[5] + q[2] + q[4] + q[5] + q[6]
λp[21] → p[2] + p[3] + p[4] + q[2] + q[5] + q[7]
```

Generate entangled thresholds (using selection rules)

```
tmpSALIDA3 = AbsoluteTiming[
    SALIDA3 = GeneraCausal[SALIDA, MomentosBASICOS,
      Orden];];
Print["Time: ", tmpSALIDA3[[1]]]

++++ Armado de lista de combinaciones ++++
  Construccion combinaciones - paso 1: 21
  Construccion combinaciones - paso 2: 148
  Construccion combinaciones - paso 3: 535
  Construccion combinaciones - paso 4: 1152
  Construccion combinaciones - paso 5: 1571
++++ Aplicacion de criterios de seleccion ++++
  *Despues de Criterio 1: 797
  *Despues de Criterio 2: 194
Numero total de lambdas signados: 42
Representacion causal obtenida: 388 terminos
Time: 1.46145
```

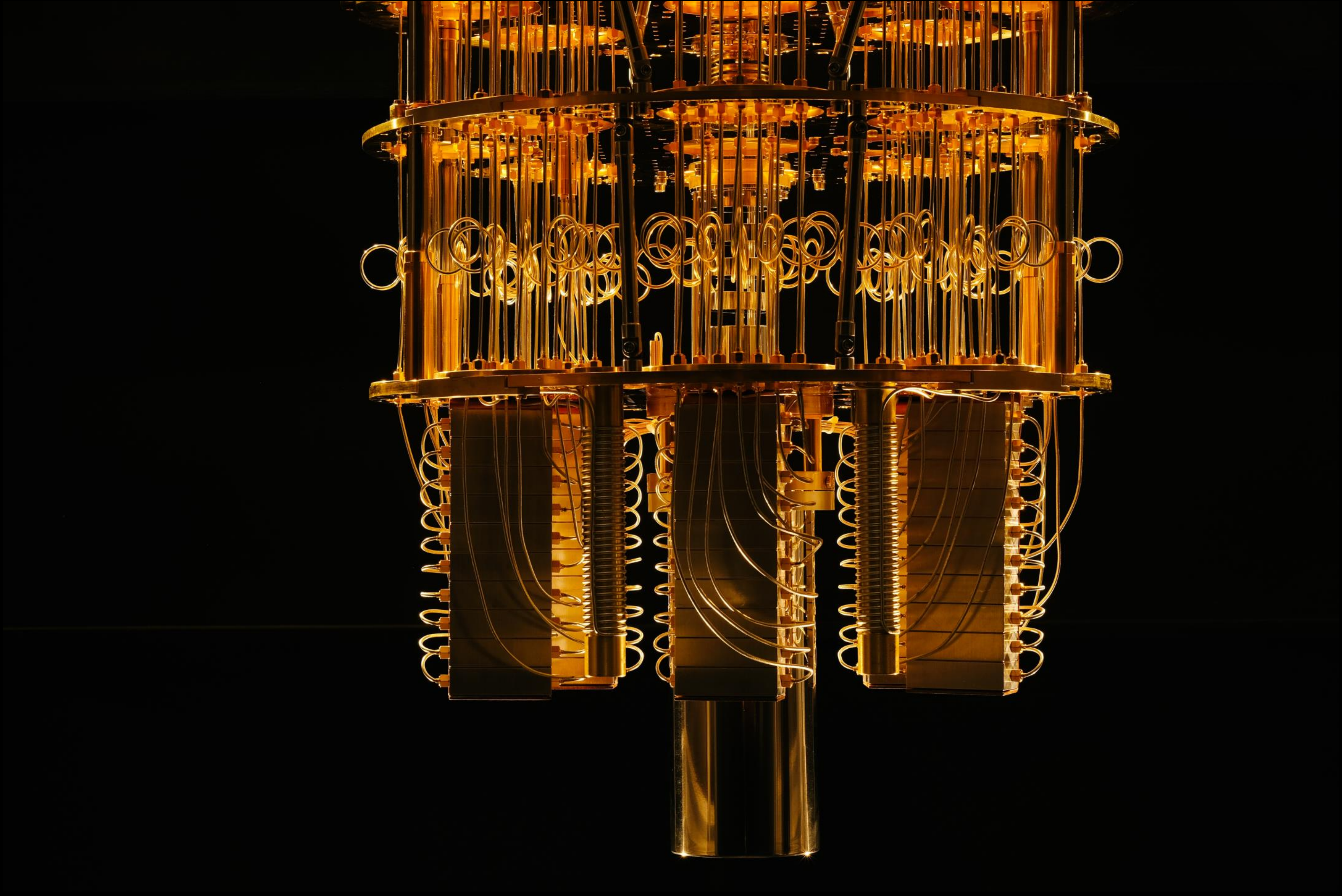
Causal representation

SALIDA3[[5]]

$$\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[12]}+\frac{1}{\lambda m[2]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[12]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[12]\times\lambda m[13]}+\frac{1}{\lambda m[1]\times\lambda m[5]\times\lambda m[6]\times\lambda m[12]\times\lambda m[13]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[5]\times\lambda m[6]\times\lambda m[14]}+\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[15]}+$$
$$\frac{1}{\lambda m[2]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[15]}+\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[6]\times\lambda m[11]\times\lambda m[15]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[11]\times\lambda m[15]}+\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[6]\times\lambda m[14]\times\lambda m[16]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[6]\times\lambda m[7]\times\lambda m[14]\times\lambda m[16]}+\frac{1}{\lambda m[2]\times\lambda m[6]\times\lambda m[7]\times\lambda m[14]\times\lambda m[16]}+$$
$$\frac{1}{\lambda m[2]\times\lambda m[6]\times\lambda m[9]\times\lambda m[14]\times\lambda m[16]}+\frac{1}{\lambda m[3]\times\lambda m[6]\times\lambda m[9]\times\lambda m[14]\times\lambda m[16]}+$$

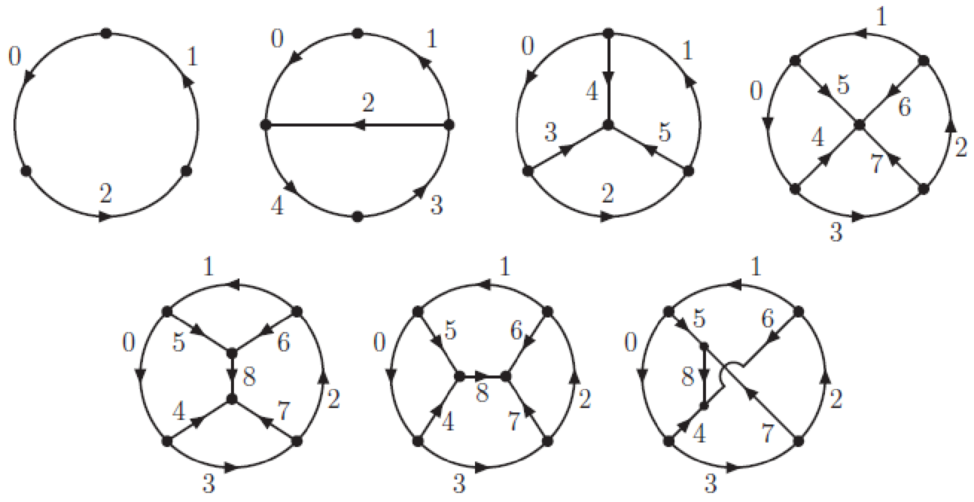
(+ similar terms …)

Sborlini (2021) arXiv:2102.05062 [hep-ph]



NEW PAPER!!
arXiv:2105.08703 [hep-ph]

- New technology based on **Grover's algorithm** to identify causal flux!
- We assign **1 qubit to each edge**, and impose logical conditions to select configurations without closed cycles → **Non-cyclical configurations = Causal flux**
- **Important: "loop"** refers to **integration variables; "eloop"** to loops in the **graph**



Total number of orderings (n = nº of edges)

$$N = 2^n \rightarrow |q\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

Quantum superposition of N flux configurations

$$|q\rangle = \cos\theta\,|q_\perp\rangle + \sin\theta\,|w\rangle$$

$$|w\rangle = \frac{1}{\sqrt{r}} \sum_{x \in w} |x\rangle$$

"Winning state" (causal flow)

$$|q_\perp\rangle = \frac{1}{\sqrt{N-r}} \sum_{x \notin w} |x\rangle$$

States with non-causal flow

- Grover's algorithm **enhances** the probability of the **winning state** by using two operators:

$$U_w = \boldsymbol{I} - 2|w\rangle\langle w|$$

Oracle operator (changes sign of winning states)

$$U_q = 2|q\rangle\langle q| - \boldsymbol{I}$$

Diffusion operator (reflects with respect to initial state)

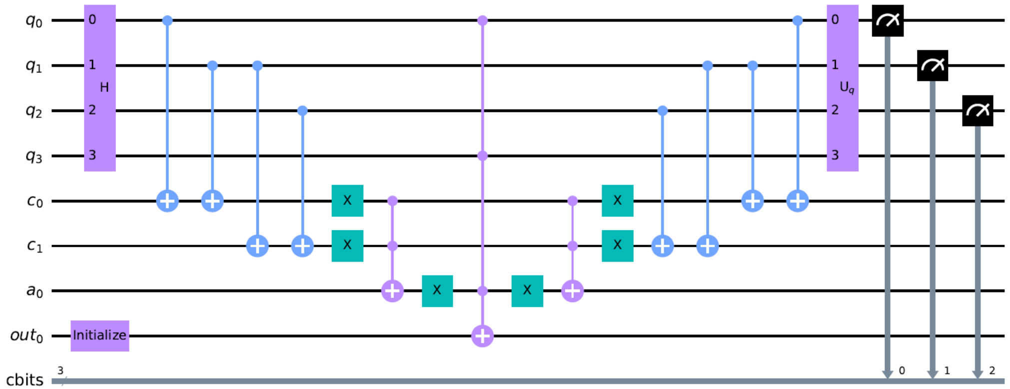
$$(U_q U_w)^t |q\rangle = \cos\theta_t\,|q_\perp\rangle + \sin\theta_t\,|w\rangle$$

with $\sin^2\theta_t \sim 1$

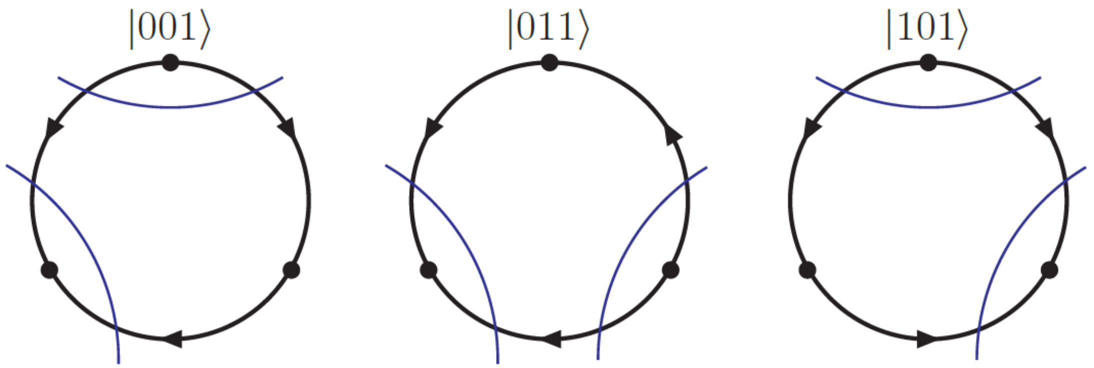
Ramírez-Uribe et al (2021) arXiv:2105.08703 [hep-ph]

NEW PAPER!!
arXiv:2105.08703 [hep-ph]

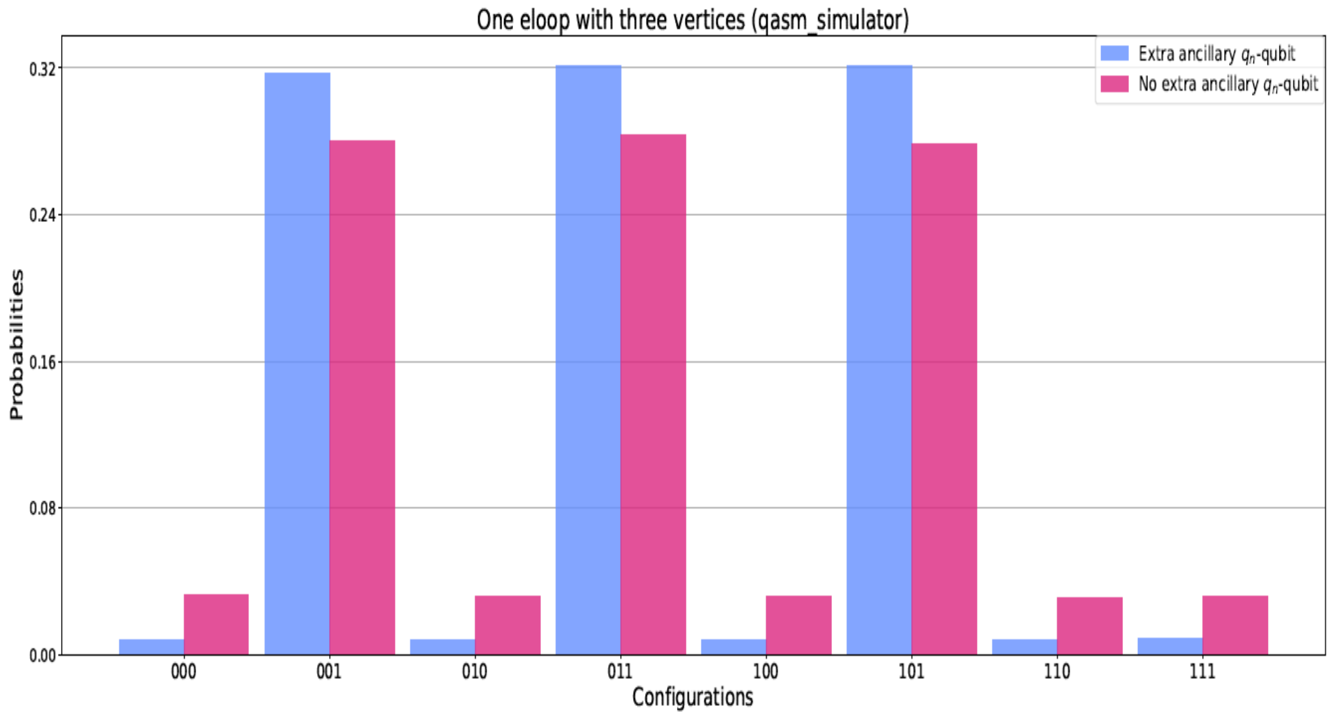
- Implemented with Qiskit and run in **IBM Q** (simulator & real QC)
- Several topologies studied!! **Enhanced performance** with extra-qubits



Quantum circuit



Causal configurations



The selected configurations are exactly |001>, |011>, |101>

The algorithm identifies the causal fluxes, relying on geometrical concepts!

Ramírez-Uribe et al (2021) arXiv:2105.08703 [hep-ph]

Conclusions and outlook

- Use LTD to cleverly rewrite Feynman integrals: **Minkowski to Euclidean**
- **Novel LTD approach** based on **nested residues leads to manifestly causal representations** of multiloop scattering amplitudes!
- Very compact formulae **with strong physical/conceptual** motivation

- **Geometrical rules** select **entangled thresholds. Complete reconstruction** of multiloop amplitudes!
- **Quantum algorithms** to speed-up **causal flux selection**. *Exploring new disruptive tools for breaking the precision frontier!!*

- **Outlook:**

1. Deepen into the **interpretation** of entangled causal propagators
2. Find the connection between **residues and graph theory**
3. Generalize the use of **Quantum Algorithms** to speed-up calculations in HEP
4. Tackle the **calculation of physical observables** with this new representation
5. Test the **efficiency for cross-section calculations**

An index of submitted letters can be viewed using this direct URL link. The letters will be stored permanently in the Fermilab archive Doc.db shortly after August 31, 2020. The current LOIs files organized in the directories corresponding to the primary frontiers used during submissions are shown here.

Manifestly Causal Scattering Amplitudes

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August 30, 2020

LoI for Snowmass 2021 (sent on 30.08.2020)

Great progress made since then!!!


THANKS!

BACKUP SLIDES.

Nested residues: Displaced poles

- Practical (mathematical) example:

$$f(\vec{x}) = \frac{1}{(x_1^2 - y_1^2)\ldots(x_L^2 - y_L^2)\,(z_{L+1}^2 - y_{L+1}^2)}$$

to calculate

$$I = \left(\prod_{i=1}^{L}\int\frac{dx_i}{2\pi\imath}\right) f(\vec{x})$$

Complex coefficients $y_i \to \tilde{y}_i = \sqrt{y_i^2 - \imath 0}$

$$z_{L+1} = -\sum_{j=1}^{L} x_j + k_{L+1}$$

Sum of integration variables (real)

- **1st step:** Apply C.R.T. in x_1, by promoting $x_1 \in \mathbb{R} \to \mathbb{C}$ (the other *x*'s remain real)

$$I = -\left(\prod_{i=2}^{L}\int\frac{dx_i}{2\pi\imath}\right)\sum_{x_{1,j}\in \mathrm{Poles}[f,x_1]} \mathrm{Res}\left(f(\vec{x}),\{x_1,x_{1,j}\}\right)\theta(-\mathrm{Im}(x_{1,j})) \;\Rightarrow\; I = -\left(\prod_{i=2}^{L}\int\frac{dx_i}{2\pi\imath}\right)\sum_{x_{1,j}\in \mathrm{Poles}^{(+)}[f,x_1]} \mathrm{Res}\left(f(\vec{x}),\{x_1,x_{1,j}\}\right)$$

Theta functions removed

$$\mathrm{Poles}^{(+)}[f,x_1] = \{y_1, y_{L+1} - k_{L+1} - x_2 - \ldots - x_L\}$$

Subset of poles with negative imaginary part
IMPORTANT! x's are real, y's are complex

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

Nested residues: Displaced poles

- Practical (mathematical) example:

$$I = -\left(\prod_{i=2}^{L}\int \frac{dx_i}{2\pi\imath}\right) \sum_{x_{1,j}\in \mathrm{Poles}^{(+)}[f,x_1]} \mathrm{Res}\left(f(\vec{x}), \{x_1, x_{1,j}\}\right)$$

$$\mathrm{Poles}^{(+)}[f,x_1] = \{y_1, y_{L+1} - k_{L+1} - x_2 - \ldots - x_L\}$$

$$\mathrm{Res}(f,\{x_1,\mathrm{Im}(x_1)<0\}) = \frac{1}{2y_1\left(x_2^2-y_2^2\right)\ldots\left(x_L^2-y_L^2\right)\left((y_1+x_2+\ldots+x_L-k_{L+1})^2-y_{L+1}^2\right)} + \frac{1}{2y_{L+1}\left((y_{L+1}+k_{L+1}-x_2-\ldots-x_L)^2-y_1^2\right)(x_2^2-y_2^2)\ldots(x_L^2-y_L^2)}$$

Sum of the residues in x_1 (negative imaginary part)

- **2nd step:** Apply C.R.T. in x_2, by promoting $x_2 \in \mathbb{R} \to \mathbb{C}$ (the other *x*'s remain real)

$$\mathrm{Res}(\,\mathrm{Res}(f,\{x_1,\mathrm{Im}(x_1)<0\})\,,\{x_2,\mathrm{Im}(x_2)<0\})$$

$$= \sum_{x_{2,l}\in \mathrm{Poles}[f,x_1,x_2]} \mathrm{Res}(\,(\mathrm{Res}(f,\{x_1,\mathrm{Im}(x_1)<0\}),\{x_2,x_{2,l}\}\,)\,\theta(-\mathrm{Im}(x_{2,l}))$$

Theta functions remain!

$$\mathrm{Poles}[f,x_1;x_2] = \{\pm y_2, \pm y_1 + y_{L+1} - x_3 - \ldots - x_L + k_{L+1}, \pm y_{L+1} - y_1 - x_3 - \ldots - x_L + k_{L+1}\}$$

All the possible poles:
SIGN OF IMAGINARY PART + or - !!!

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

- Practical (mathematical) example:

$$\text{Res}(\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,\text{Im}(x_2)<0\}) = \sum_{x_{2,l}\in\text{Poles}[f,x_1,x_2]} \text{Res}((\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,x_{2,l}\})\,\theta(-\text{Im}(x_{2,l}))$$

- **3rd step:** Collect the different contributions according to $\theta(-\text{Im}(x_{2,l}))$:

$$\text{Res}((\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,y_2\})$$
$$= \frac{1}{4y_1y_2\,(x_3^2-y_3^2)\ldots(x_L^2-y_L^2)((y_1+y_2+x_3+\ldots+x_L-k_{L+1})^2-y_{L+1}^2)}$$
$$+\frac{1}{4y_{L+1}y_2\,((y_{L+1}-y_2-x_3-\ldots-x_L+k_{L+1})^2-y_1^2)\ldots(x_L^2-y_L^2)}$$

$$\text{Res}(\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,y_1+y_{L+1}-x_3-\ldots-x_L+k_{L+1}\})$$
$$=\frac{1}{4y_1y_3\,((y_1+y_{L+1}-x_3-\ldots-x_L+k_{L+1})^2-y_2^2)(x_3^2-y_3^2)\ldots(x_L^2-y_L^2)}$$

Theta functions are trivially 1: y's have negative imaginary part, x's are real → **Only sums of y's!!! ALIGNED CONTRIBUTIONS**

$$[\text{Res}(\text{Res}(f,\{x_1,y_1\}),\{x_2,y_{L+1}-y_1-x_3-\ldots-x_L+k_{L+1}\})$$
$$+\text{Res}(\text{Res}(f,\{x_1,y_{L+1}-x_2-\ldots-x_L+k_{L+1}\}),$$
$$\{x_2,y_{L+1}-y_1-x_3-\ldots-x_L+k_{L+1}\})]\;\theta(\text{Im}(y_1-y_{L+1}))$$

Different-sign combinations of y's: **NON-TRIVIAL THETA!** → **DISPLACED POLES: VANISH!!**

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

- *Theorem*: Given a generic* rational function $F(x_i, x_j) = \frac{P(x_i, x_j)}{((x_i - a_i)^2 - y_i^2)^{\gamma_i}((x_i + x_j - a_{ij})^2 - y_k^2)^{\gamma_k}}$

then: $$\text{Res}(\text{Res}(F(x_i,x_j), \{x_i, y_i + a_i\}), \{x_j, y_k - y_i + a_{ij} - a_i\})$$
$$= -\text{Res}\left(\text{Res}\left(F(x_i, x_j), \{x_i, y_k - x_j + a_{ij}\}\right), \{x_j, y_k - y_i + a_{ij} - a_i\}\right)$$

- **Mathematical consequences:**
  1. In each iteration of C.R.T., contributions with **different sign combinations of *y*'s vanish**
  2. Thus, after iterating over all integration variables, **only same-sign combinations of *y*'s remain**

Example:
$L = 2$

$$\text{Res}(\text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, \text{Im}(x_2) < 0\})$$
$$= \frac{1}{4y_1y_2((y_1 + y_2 - k_3)^2 - y_3^2)} + \frac{1}{4y_2y_3((y_3 + y_1 + k_3)^2 - y_2^2)}$$
$$+ \frac{1}{4y_1y_3((y_3 - y_2 + k_3)^2 - y_1^2)}$$
$$= -\frac{1}{8y_1y_2y_3}\left(\frac{1}{y_1 + y_2 + y_3 - k_3} + \frac{1}{y_1 + y_2 + y_3 + k_3}\right)$$

Connection to QFT

$y_i \leftrightarrow q_{i,0}^{(+)} = \sqrt{\boldsymbol{q}_i^2 + m_i^2 - \imath 0}$

$x_i \leftrightarrow q_{i,0}$

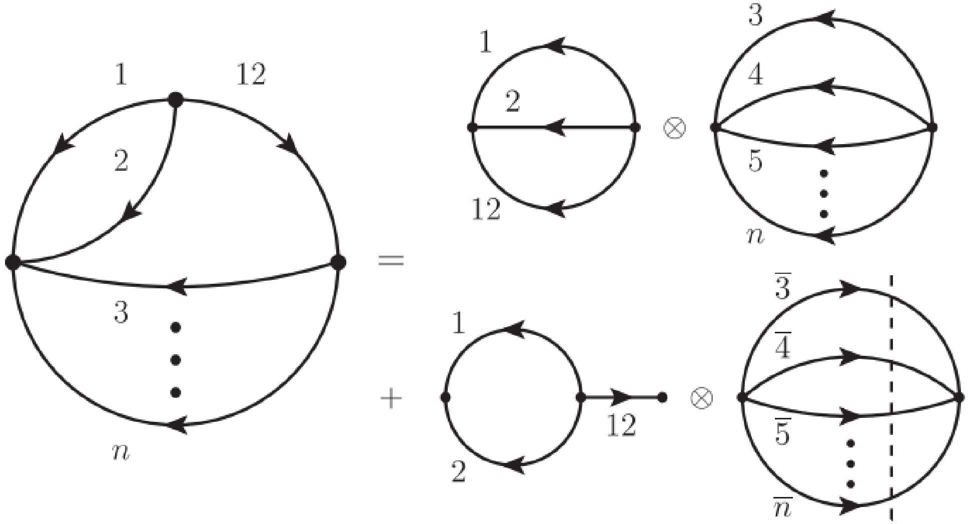
$a_i \leftrightarrow \{k_{m,0}\}$

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

Nested residues: Compact representations

- More complicated topologies can be described by convolutions with MLT-like diagrams

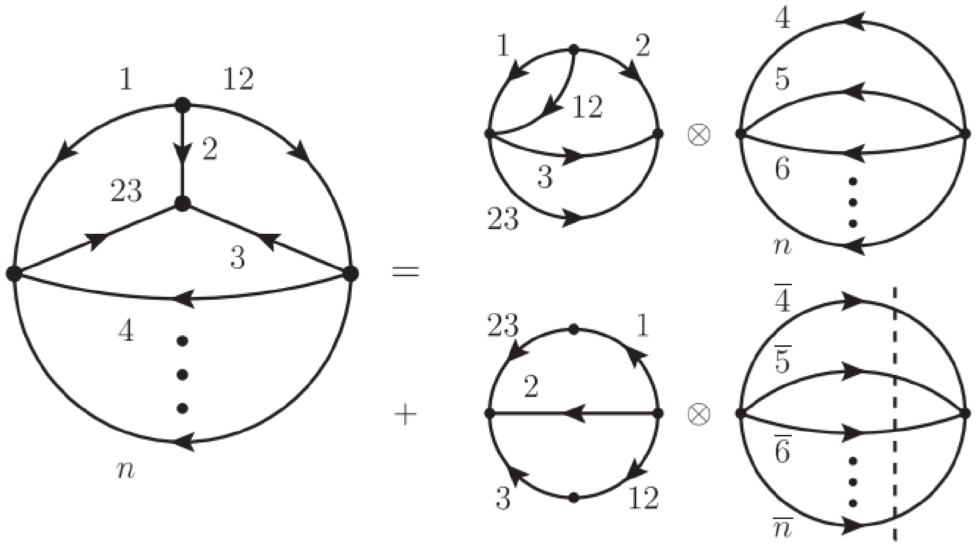
Next-to Maximal Loop Topology (3 vertices, L+2 lines)



$$\mathcal{A}^{(L)}_{\mathrm{NMLT}}(1,\dots,n,12) = \mathcal{A}^{(2)}_{\mathrm{MLT}}(1,2,12) \otimes \mathcal{A}^{(L-2)}_{\mathrm{MLT}}(3,\dots,n) + \mathcal{A}^{(1)}_{\mathrm{MLT}}(1,2) \otimes \mathcal{A}^{(0)}(12) \otimes \mathcal{A}^{(L-1)}_{\mathrm{MLT}}(\bar{3},\dots,\bar{n})$$

IMPORTANT FACTORIZATION FORMULAE
Singular and causal structure is determined by the corresponding sub-topologies

Next-to-Next-to Maximal Loop Topology (4 vertices, L+3 lines)

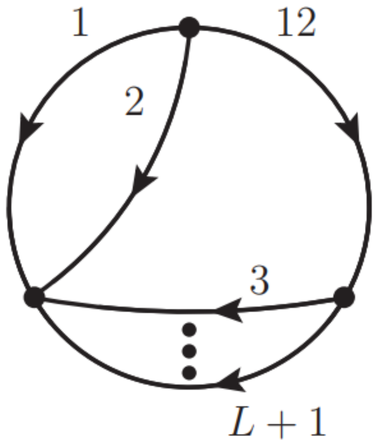


$$\mathcal{A}^{(L)}_{\mathrm{NNMLT}}(1,\dots,n,12,23) = \mathcal{A}^{(3)}_{\mathrm{NMLT}}(1,2,3,12,23) \otimes \mathcal{A}^{(L-3)}_{\mathrm{MLT}}(4,\dots,n) + \mathcal{A}^{(2)}_{\mathrm{MLT}}(1\cup 23, 2, 3\cup 12) \otimes \mathcal{A}^{(L-2)}_{\mathrm{MLT}}(\bar{4},\dots,\bar{n})$$

Inductive proofs of these formulae to all-loop orders available in JHEP 02 (2021) 112

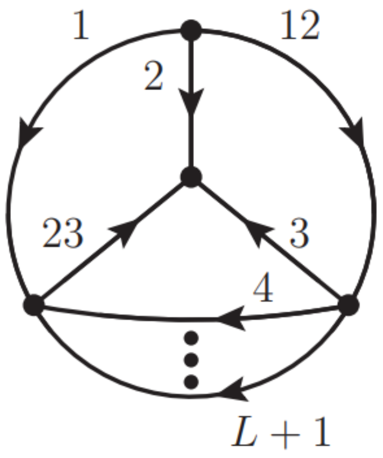
Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

- Similar formulae can be found for NMLT and NNMLT to all loop orders!



$$\mathcal{A}^{(L)}_{\mathrm{NMLT}}(1,2,\ldots,L+2) = \int_{\vec{\ell}_1,\ldots,\vec{\ell}_L} \frac{2}{x_{L+2}} \left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1} \right)$$

with $\lambda_1 = \sum_{i=1}^{L+1} q_{i,0}^{(+)}$ $\quad \lambda_2 = q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{L+2,0}^{(+)}$ $\quad \lambda_3 = \sum_{i=3}^{L+2} q_{i,0}^{(+)}$



$$\mathcal{A}^{(L)}_{\mathrm{N^2MLT}}(1,2,\ldots,L+3) = -\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{2}{x_{L+3}} \left[\frac{1}{\lambda_1} \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \left(\frac{1}{\lambda_4} + \frac{1}{\lambda_5} \right) + \frac{1}{\lambda_6} \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_4} \right) \left(\frac{1}{\lambda_3} + \frac{1}{\lambda_5} \right) + \frac{1}{\lambda_7} \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_5} \right) \left(\frac{1}{\lambda_3} + \frac{1}{\lambda_4} \right) \right]$$

with $\lambda_4 = q_{2,0}^{(+)} + q_{3,0}^{(+)} + q_{L+3,0}^{(+)}$ $\quad \lambda_6 = q_{1,0}^{(+)} + q_{3,0}^{(+)} + q_{L+2,0}^{(+)} + q_{L+3,0}^{(+)}$

$\lambda_5 = q_{1,0}^{(+)} + q_{L+3,0}^{(+)} + \sum_{i=4}^{L+1} q_{i,0}^{(+)}$ $\quad \lambda_7 = q_{2,0}^{(+)} + \sum_{i=4}^{L+3} q_{i,0}^{(+)}$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]

Manifestly Causal representation: Implementation

- We profit from compact causal formulae for integrals with higher-powers:

$$\mathcal{A}^{(L)}_{\mathrm{N}^{k-1}\mathrm{MLT}}(1^2, 2^2, \ldots, L^2, L+1, \ldots, L+k) = \prod_{i=1}^{L} \frac{\partial}{\partial (q_{i,0}^{(+)})^2} \mathcal{A}^{(L)}_{\mathrm{N}^{k-1}\mathrm{MLT}}(1, 2, \ldots, L+1, \ldots, L+k)$$

Is also causal by construction! (*derivatives preserve denominators*)

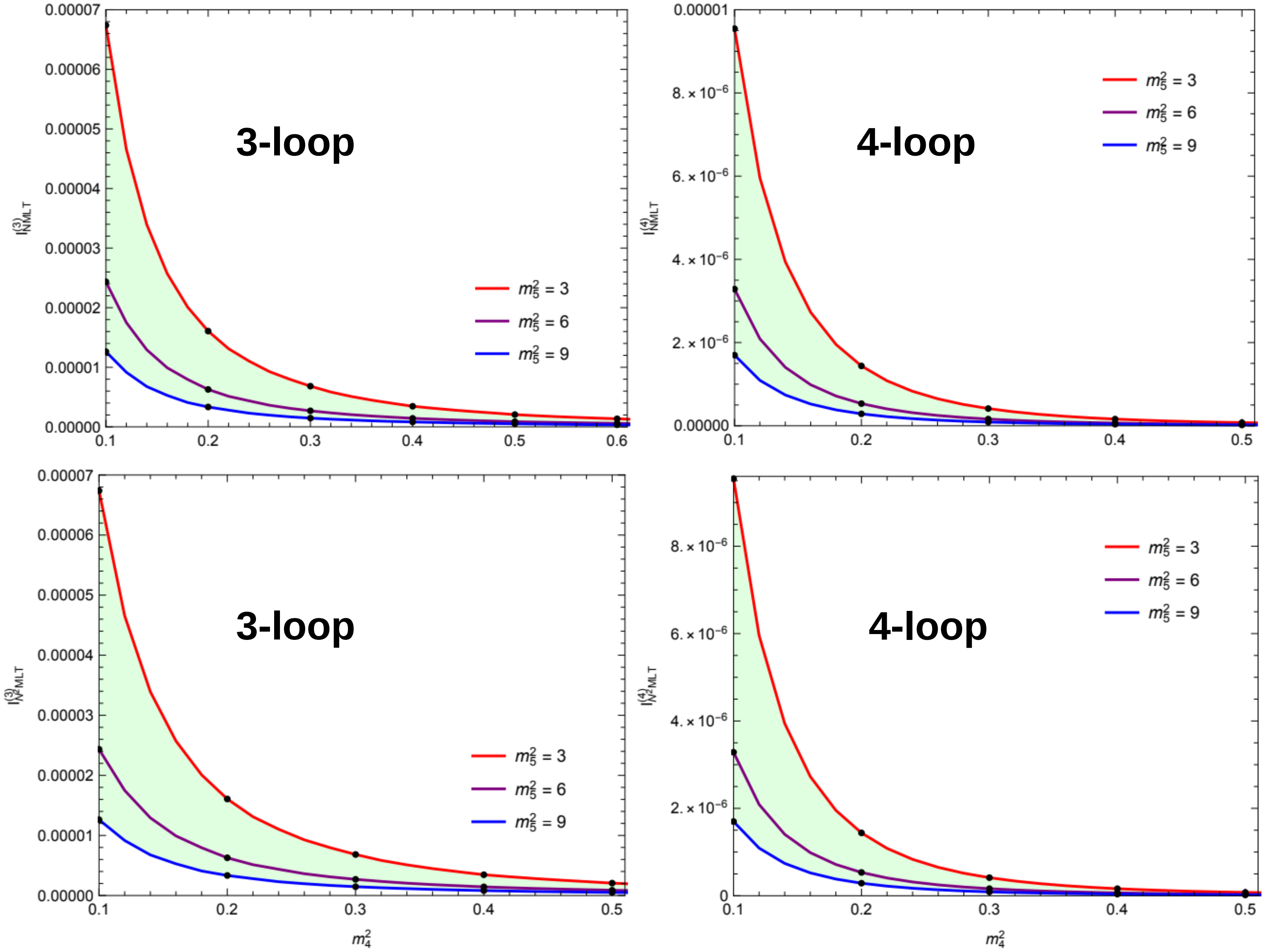
Causal representation available!

- *Setup of the numerical implementation:*

1. Tested for MLT, NMLT and NNMLT integrals, at 3 and 4 loops
2. Arbitrary masses, and with different numbers of space-time dimensions (D=2,3,4)
3. Compared with numerical results from FIESTA 4.2 and SecDec 3.0

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]

- **Numerical results in D=3:**



NMLT

Solid lines: LTD
Dots: FIESTA

NNMLT

Setup:

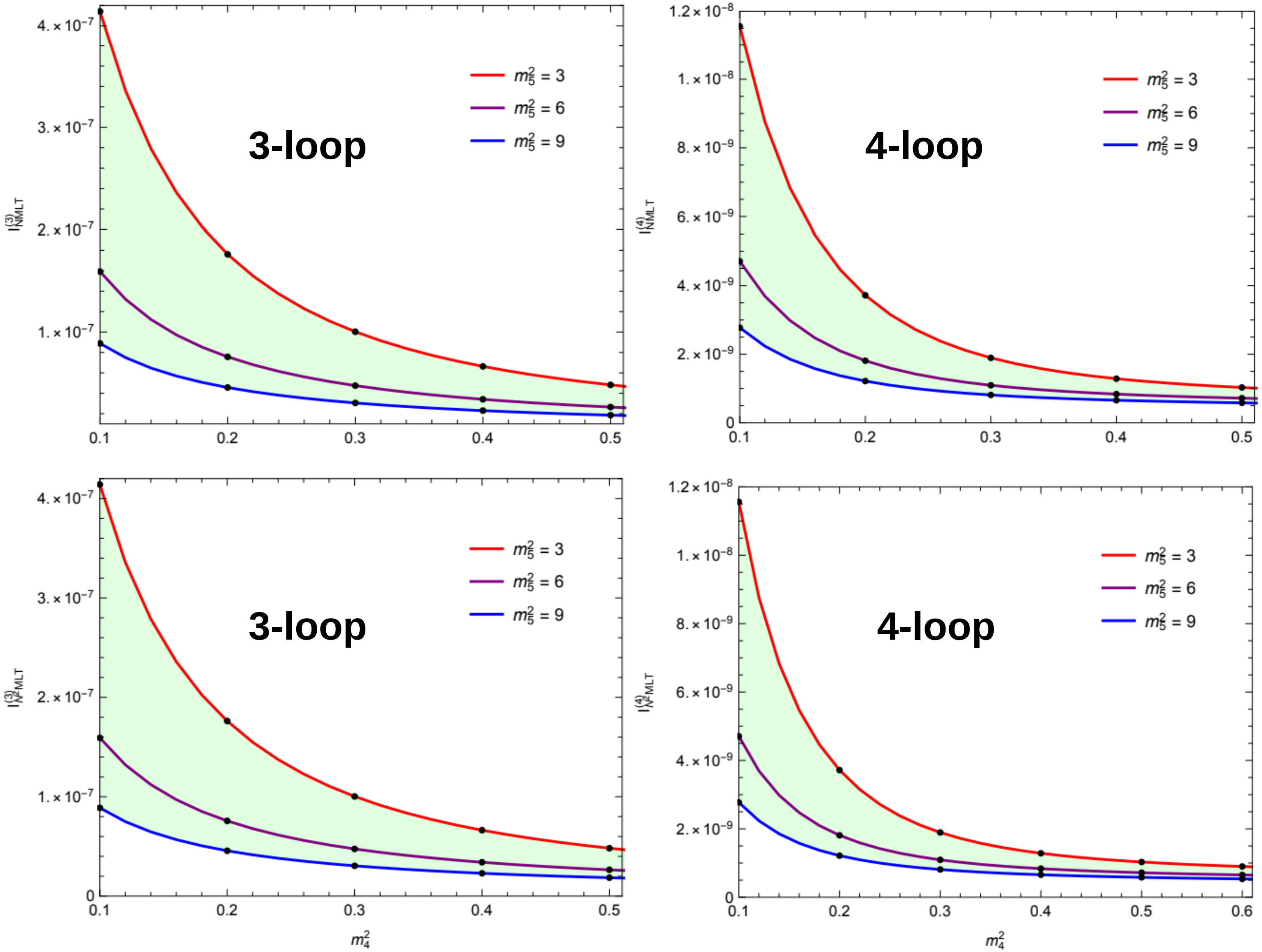
$$\mathcal{A}^{(L)}_{\mathrm{N}^{k-1}\mathrm{MLT}}(1^2, 2^2, \ldots, L^2, L+1, \ldots, L+k)$$

Mases: $\{1, 2, \ldots, L\} \Leftrightarrow m_4^2$

$\{L+1, \ldots, L+k\} \Leftrightarrow m_5^2$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]

- **Numerical results in D=4:**



NMLT

NNMLT

Solid lines: LTD
Dots: FIESTA

Setup:

$$\mathcal{A}^{(L)}_{\text{N}^{k-1}\text{MLT}}(1^2, 2^2, \ldots, L^2, L+1, \ldots, L+k)$$

Mases:

$$\{1, 2, \ldots, L\} \leftrightarrow m_4^2$$

$$\{L+1, \ldots, L+k\} \leftrightarrow m_5^2$$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]