Two-loop helicity amplitudes for diphoton plus jet production in full colour

Federico Buccioni

Rudolf Peierls Centre for Theoretical Physics
University of Oxford

Radcor & LoopFest,
FSU, Tallahassee, FL, USA
19th May 2021

based on [2103.02671, 2105.04585]

in collaboration with: Bakul Agarwal, Andreas von Manteuffel and Lorenzo Tancredi

Outline

Main object of this talk:

First exact results for NNLO QCD corrections to a 2→3 massless scattering amplitude
(in all helicity configurations)

Scattering processes taken into account:

$q\bar{q} \to g\gamma\gamma$ $\quad$ $qg \to q\gamma\gamma$ $\quad$ $g\bar{q} \to \bar{q}\gamma\gamma$

How we got there:

- use latest advances in the calculation of multiloop multileg amplitudes
- reduce algebraic complexity throughout the calculation

Outline

Main object of this talk:

First exact results for NNLO QCD corrections to a 2→3 massless scattering amplitude (in all helicity configurations)

Scattering processes taken into account:

$q\bar{q} \to g\gamma\gamma$ $qg \to q\gamma\gamma$ $g\bar{q} \to \bar{q}\gamma\gamma$

How we got there:

- use latest advances in the calculation of multiloop multileg amplitudes
- reduce algebraic complexity throughout the calculation

Disclaimer:

I will not cover all the great results achieved so far in this field of research

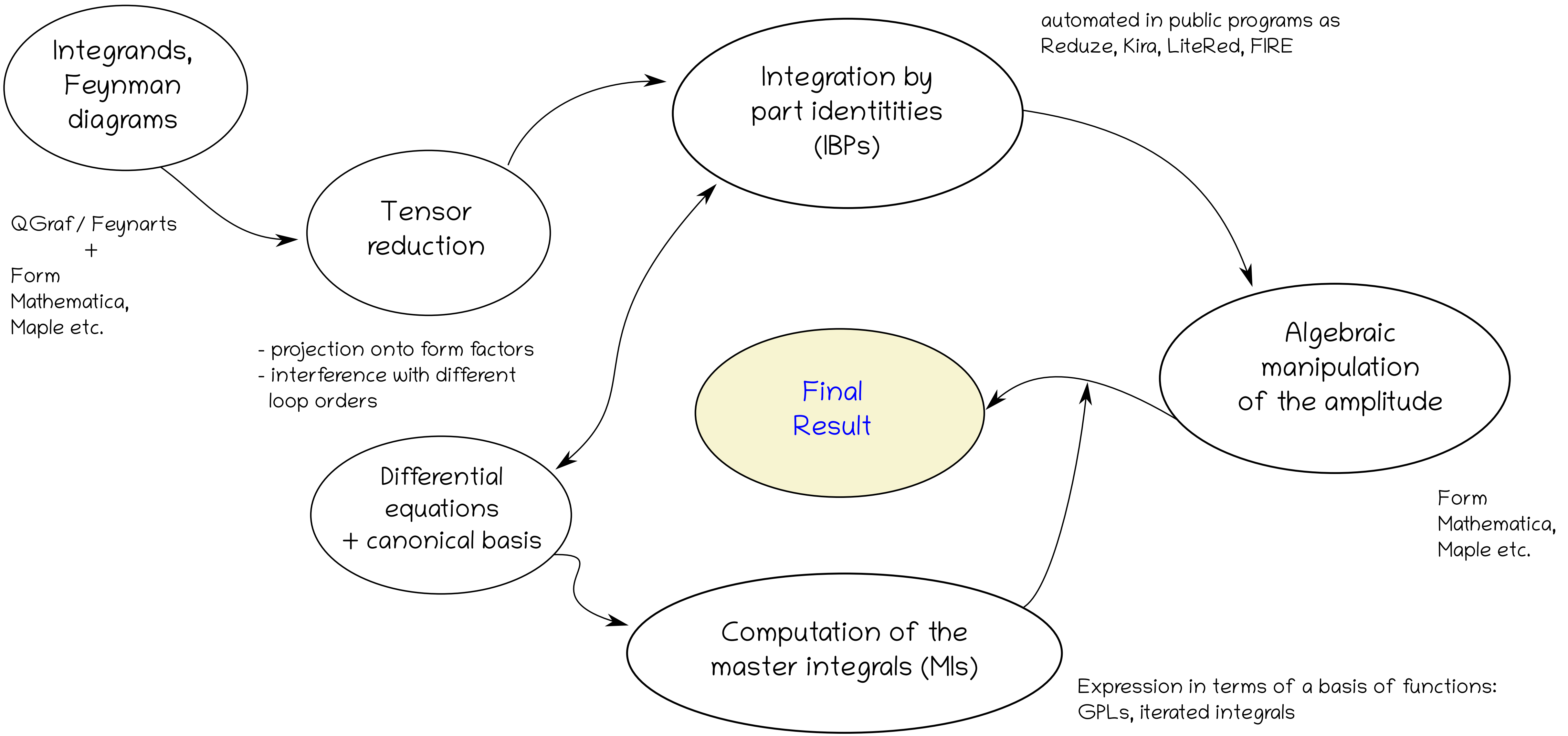
For more formal aspects and state-of-the-art: Vasily's talk on Monday

For pheno results and applications: Rene's talk on Friday

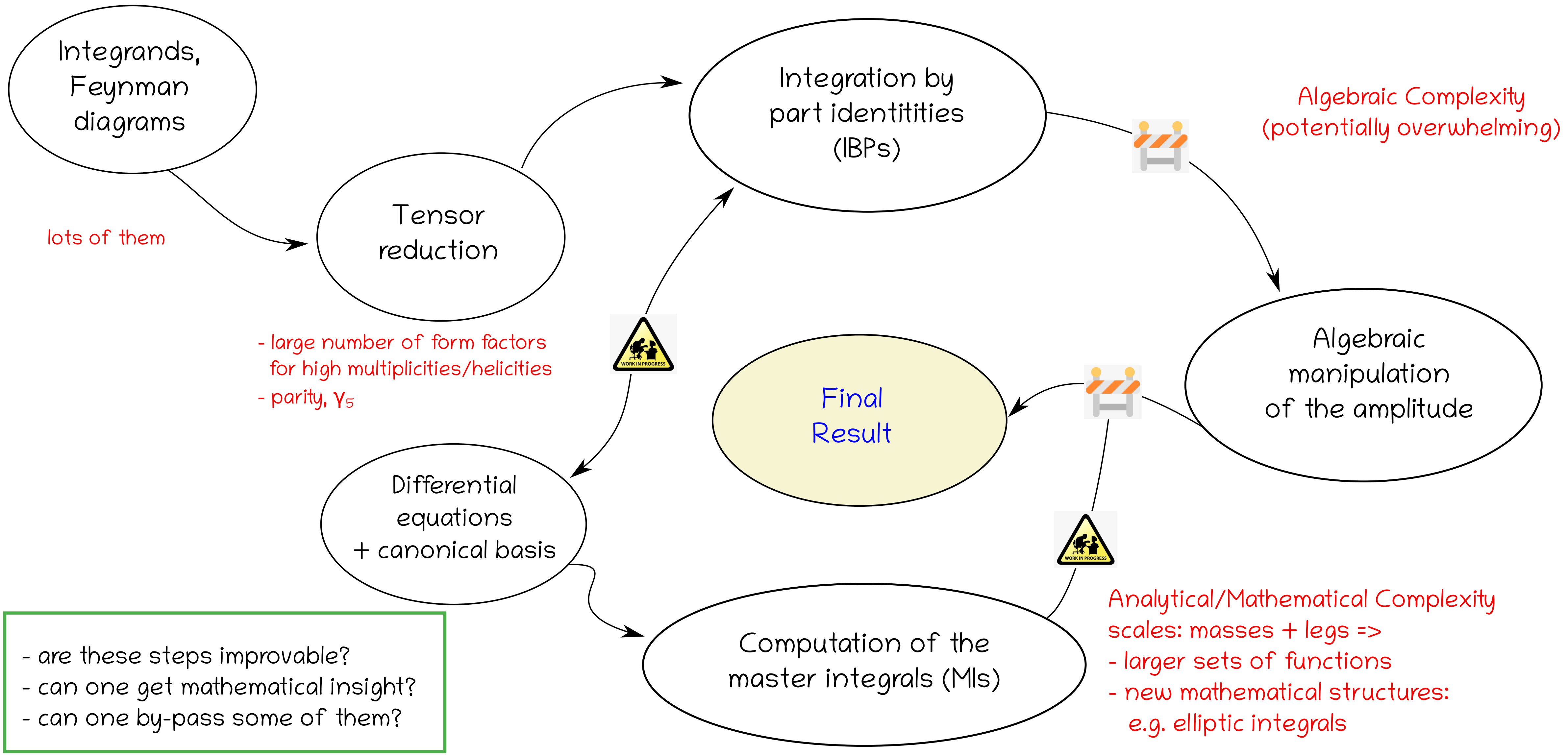
More on massless 5-pt amplitudes: Herschel's talk

First 5-pt one-mass results: Ben's, Konstantinos' and Bayu's talks

Traditional approach to multiloop amplitudes

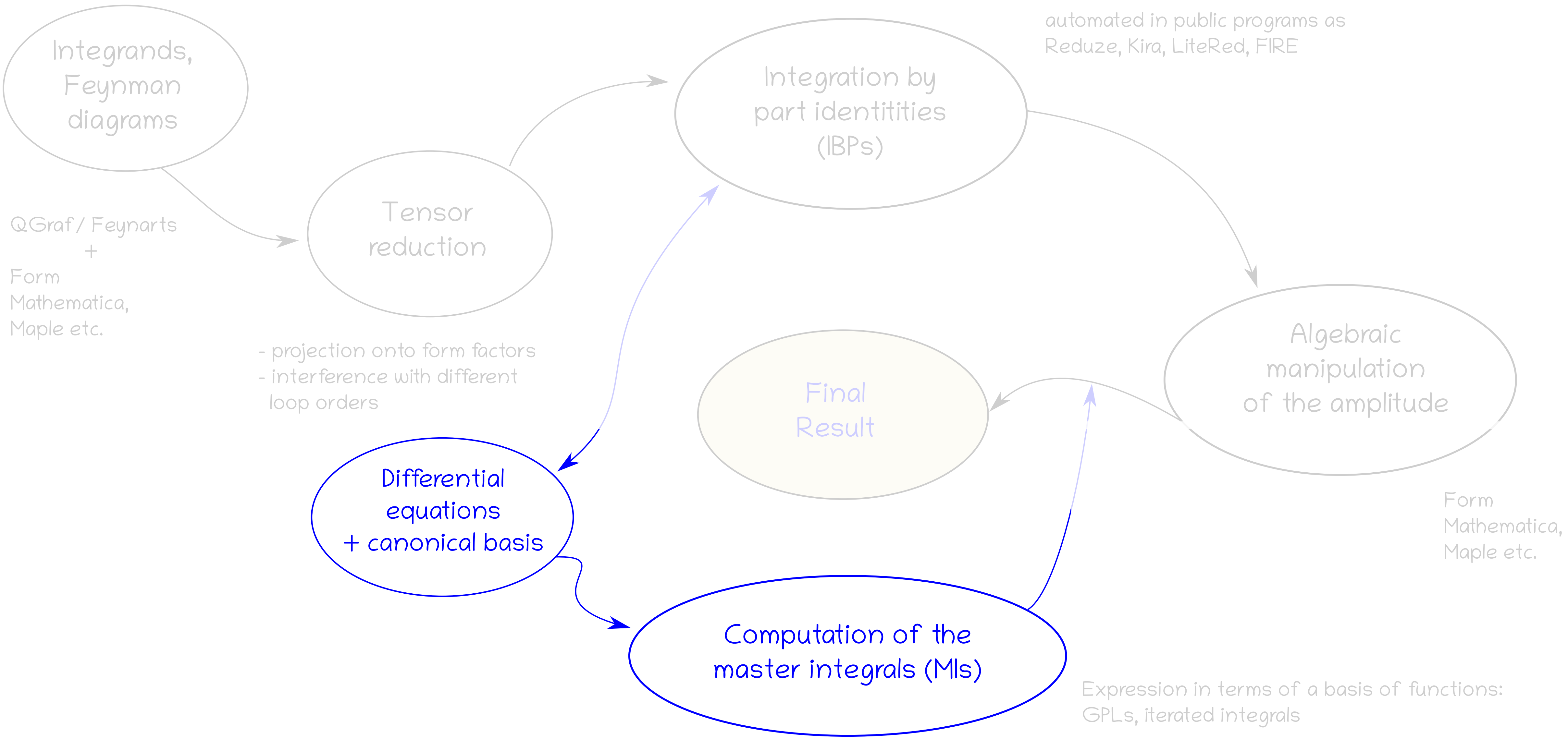


Challenges and complexity



- are these steps improvable?
- can one get mathematical insight?
- can one by-pass some of them?

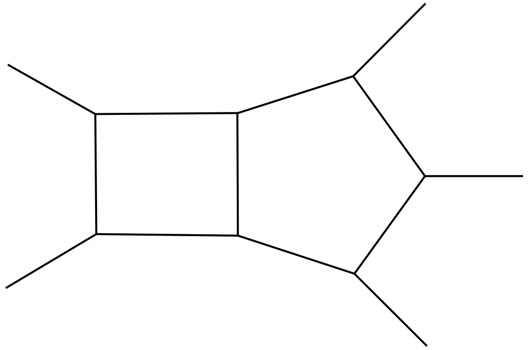
Traditional approach to multiloop amplitudes



Pentagon functions for 2→3 masless scattering amplitudes

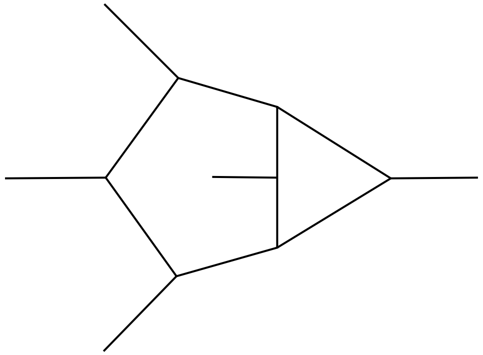
All master integrals known

Pentagon-Box



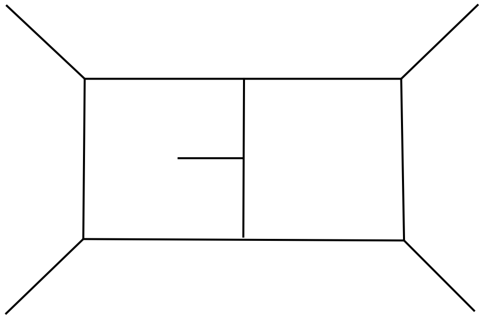
[Gehrmann, Henn, Lo Presti 1511.05409, 1807.09812],
[Papadopoulos, Tommasini, Wever 1511.09404]

Hexagon-Box



[Boehm, Georgoudis, Larsen, Schoenemann, Zhang],
[Abreu, Page, Zeng, 1807.11522]
[Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 1809.06240]

Double-Pentagon



[Abreu, Dixon, Herrmann, Page, Zeng 1901.08563],
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia 1812.11160]

MIs through Pentagon Functions
Expressed (and evaluated) as iterated Chen integrals along a path γ

$$f^{(\omega)}(\vec{x}) = \int_\gamma \mathrm{d}\log W_{i_1} \ldots \mathrm{d}\log W_{i_n}$$

ω integrations

Full set made available recently
[Chicherin, Sotnikov 2009.07803]

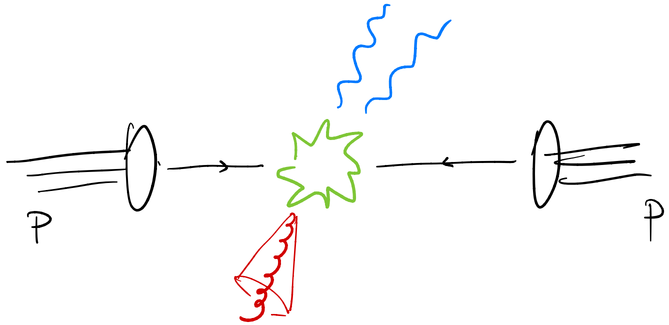
Results in the whole physical region

They can be used for *all* massless 5-pt amplitudes

For a comprehensive description, see Vasily's talk from Monday

Diphoton+jet production in pp collisions

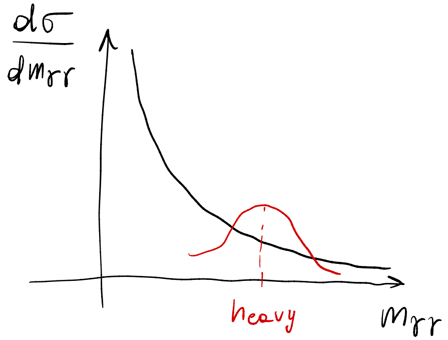
Diphoton production: important class of processes at the LHC



Irreducible background to SM and BSM processes. Most notably H -> γγ

First pheno studies in LC [Chawdhry et al. 2105.06940]. See Rene's talk for further details

Invariant mass:
relevant for direct searches of resonances



N^{3}LO QCD corrections to the cross section for pp -> γγ
di-photon + jet amplitudes necessary ingredient

pT distribution:
unique probe to investigate the properties of the decaying particles

Recoil against hard QCD radiation

Relevance of higher-order QCD corrrections for target accuracy

Partonic channels:

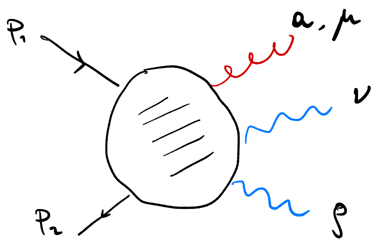
qq(g) -> γγg(q) ← this talk

gg -> γγg ← loop induced

Structure of the scattering amplitude

Let us consider the scattering process:

$$q(p_1) + \bar{q}(p_2) \to g(p_3) + \gamma(p_4) + \gamma(p_5)$$



The helicity amplitudes are given by:

$$A^a_{ij}(\boldsymbol{\lambda}) = i\,(4\pi\alpha)Q_q^2\sqrt{4\pi\alpha_s}\mathbf{T}^a_{ij}\mathcal{A}(\boldsymbol{\lambda})$$

Three independent helicity configurations

$$\{L,+,+,+\}\,, \qquad \{L,-,+,+\}\,, \qquad \{L,-,-,+\}$$

Factor out spinor phase as

$$\mathcal{A}(\boldsymbol{\lambda}) = \Phi(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda})$$

Separate then into an even and an odd component and expand perturbatively

$$\mathcal{B}(\boldsymbol{\lambda}) = \mathcal{B}^E(\boldsymbol{\lambda}) + \epsilon_5\mathcal{B}^O(\boldsymbol{\lambda})$$

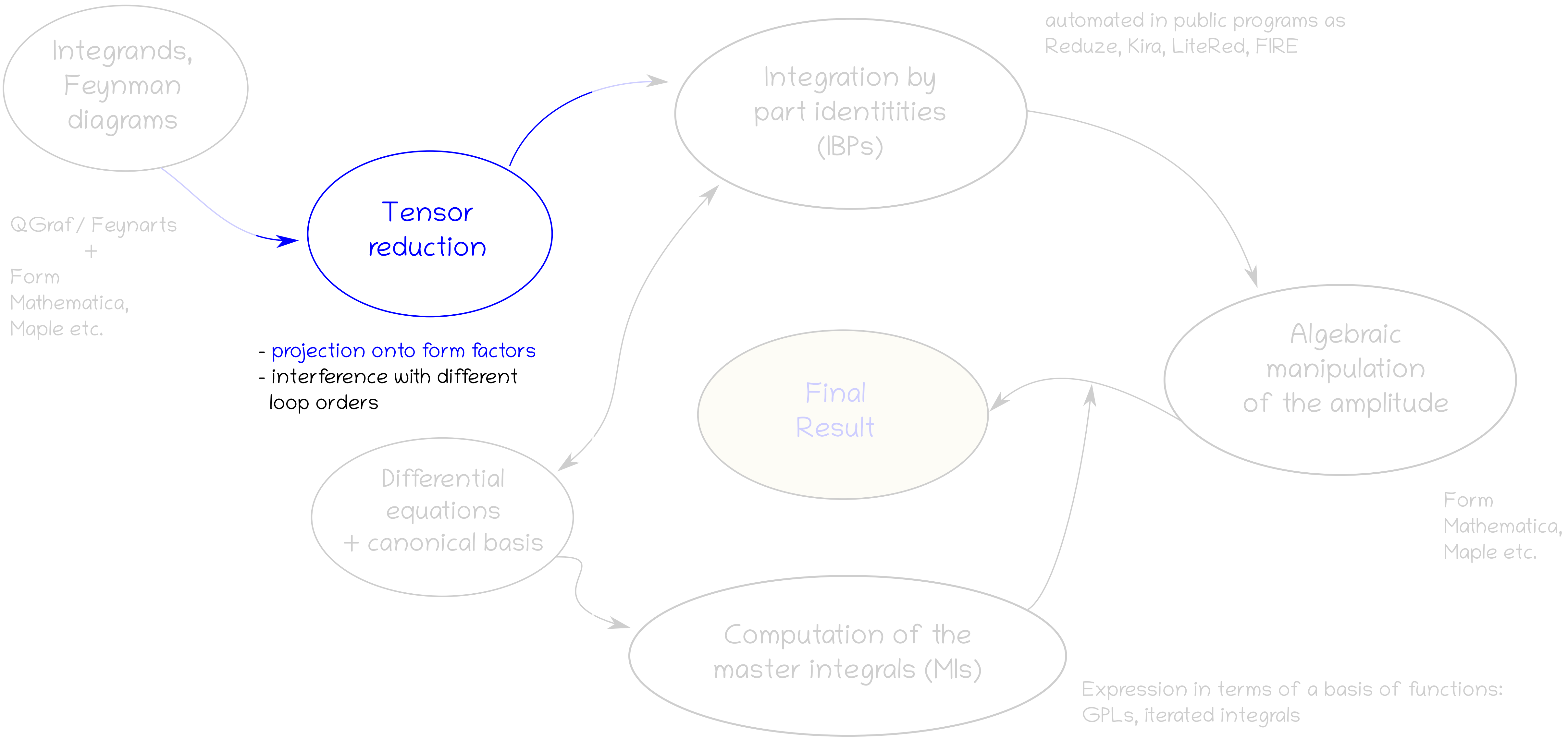
$$\mathcal{B}^P(\boldsymbol{\lambda}) = \sum_{k=0}^{2}\left(\frac{\alpha_s^b}{2\pi}\right)^k \mathcal{B}^{P,(k)}(\boldsymbol{\lambda}) + \mathcal{O}((\alpha_s^b)^3)$$

After UV renormalisation and IR factorisation (a 'la Catani)

Extract 2-loop finite remainders

$$\mathcal{R}^{P,(k)}(\boldsymbol{\lambda})$$

Traditional approach to multiloop amplitudes



Physical projectors

Project out amplitude onto form factors as suggested in [Peraro, Tancredi 1906.03298,2012.00820]

avoid evanescent form factors throughout

Main idea: decompose the amplitude into Lorentz structure which are independent in d=4

In practice: calculate only the physical helicity amplitudes in the t'Hooft-Veltman scheme

Generic tensor structure for our amplitudes:

$$\mathcal{T}_j \sim \bar{u}(p_2)\not{p}_{3,4}u(p_1)p^{\mu}_{i_3}p^{\nu}_{i_4}p^{\rho}_{i_5}\epsilon^*_{\mu}(p_3)\epsilon^*_{\nu}(p_4)\epsilon^*_{\rho}(p_5)$$

Transversality of on-shell bosons + gauge fixing: # of independent tensors in (d=4) = # of helicity configurations

$$\mathcal{A}(\boldsymbol{\lambda}) = \sum_{j=1}^{16} \mathcal{F}_j \mathcal{T}_j(\boldsymbol{\lambda})$$

Each form factor F_j can then be extracted via projectors:

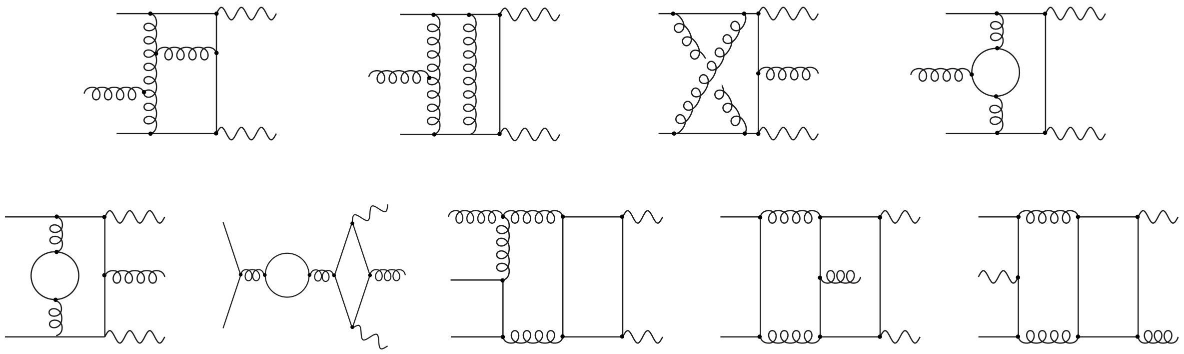
$$\mathcal{P}_j = \sum_{j=1}^{16} c^j_k \mathcal{T}^{\dagger}_k \qquad \mathcal{F}_j = \sum_{\text{pol}} \mathcal{P}_j \mathcal{A}$$

no d-dependence in c^j_k for $n > 4$

Colour structure

Colour structure of the (UV and IR renormalised) 2-loop finite remainder

$$\mathcal{R}^{P,(2)}(\boldsymbol{\lambda}) = \sum_{i=1}^{10} \tilde{c}_i \, \mathcal{R}_i^{P,(2)}(\boldsymbol{\lambda})$$



$$n_f^{\gamma\gamma} = \frac{1}{Q_q^2} \sum_i^{n_f} Q_i^2 \qquad n_f^{\gamma} = \frac{1}{Q_q} \sum_i^{n_f} Q_i,$$

Just for reference (no speculation about actual magnitude of contributions)

$d\bar{d} \to g\gamma\gamma$: $N = 3$ $n_f = 5$ $n_f^{\gamma\gamma} = 11$ $n_f^{\gamma} = -1$

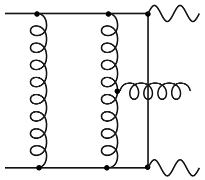
Complexity

$\tilde{c}_3 = N^{-2}$

$\tilde{c}_2 = 1$

$\tilde{c}_7 = N\, n_f^{\gamma\gamma}$

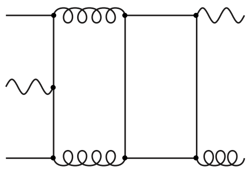
DP



$\tilde{c}_9 = d_{abc} d_{abc} n_f^{\gamma}$

$\tilde{c}_8 = N^{-1}\, n_f^{\gamma\gamma}$

HB

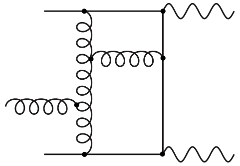


$\tilde{c}_1 = N^2$

$\tilde{c}_4 = N n_f$

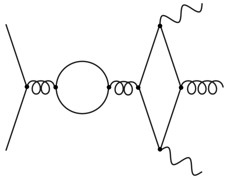
$\tilde{c}_5 = N^{-1} n_f$

PB

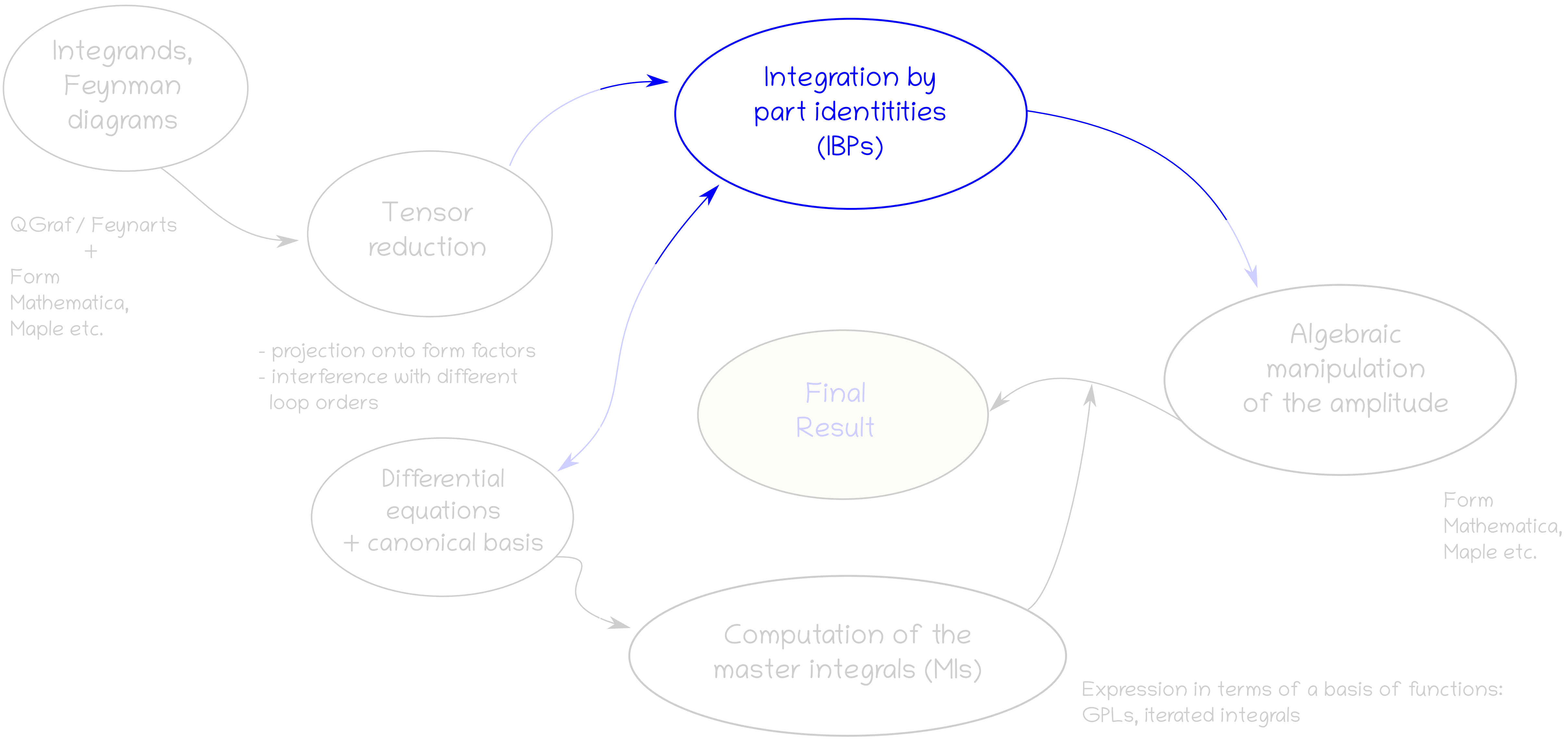


$\tilde{c}_6 = n_f^{\gamma\gamma} n_f$

1-loop



Traditional approach to multiloop amplitudes



Reduction to master integrals

IBP identities obtained using FinRed: capable of reducing the non-planar integral families completely

Made it possible thanks to Finred!

- Finite-fields arithmetics [von Manteuffel, Schabinger 1406.4513; Peraro 1905.08019]
- Syzygy techniques [Gluza, Kadja, Kosower 1009.0472, Ita 1510.05626; Larsen, Zhang, 1511.01071, Agarwal, Jones, von Manteuffel 2011.15113] ← See Bakul's talk
- Denominators guessing [Abreu, Dormans, Febres Cordero, Ita, Page 1812.04586; Heller, von Manteuffel 2101.08283]

Most complicated integrals: 8-line denominators + 5 scalar products, (t=8, s=5) for DP topology

A good choice of MIs basis is crucial: Canonical basis/UT weight integrals

we use the canonical basis provided in [Chicherin, Sotnikov 2009.07803].

$$I(s_{ij};d) = \sum_{k=1}^{M} a_k(s_{ij},d)\mathcal{J}(s_{ij};d)$$

← rational function, over a common denominator

$$a_k(s_{ij};d) = \frac{\mathcal{N}(s_{ij};d)}{\mathcal{Q}(d)\mathcal{D}(s_{ij})} \qquad \mathcal{D}(s_{ij}) = \prod_{n=1}^{N_d} \mathcal{D}_n^{p_n}(s_{ij})$$

Pros:

Exposes physical cuts of the integrals

simpler rational coefficients

extra bonus: d-dependence factorised

Natural to make the association:

Rational function → partial-fraction decomposition

1) univariate partial-fraction decomposition wrt d (trivial)

2) multivariate partial-fraction decomposition wrt s_{ij} (hard)

Multivariate partial fraction decomposition (MVPFD)

It has been long known that a MVPFD simplifies significantly the IBP reductions

$$a_k(s_{ij};d) = \frac{\mathcal{N}(s_{ij};d)}{\mathcal{Q}(d)\mathcal{D}(s_{ij})} \xrightarrow[\text{pf in d}]{\text{after}} a_k(s_{ij};d) = \sum_l g_l(d)\mathcal{R}_l(s_{ij})$$

How should we go about this?

Proposals/approaches for MVPFD:
[Pak 1111.0868], [Abreu et al, 1904.00945],
[Boehm, Wittmann, Wu, Xu, Zhang, 2008.13194]

Systematic study of reduction of IBPs: [2008.13194; Bendle et al 2104.06866]

We employ the algorithm implemented in the recently published package MultivariateApart [Heller, von Manteuffel, 2101.08283]

Big advantages of MultivariateApart:

1) systematically avoids spurious denominator factors
2) produces unique results also when applied to terms of a sum separately

← we exploit both

Drastic reduction of algebraic complexity. IBPs tractable in a fully symbolic fashion

Examples:

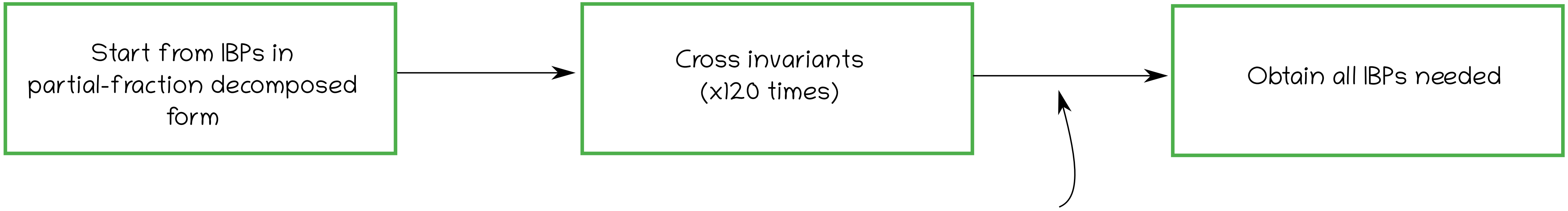
		on GCD		MVPFD
PB:	INT[TA,8,255,8,5,{1,1,1,1,1,1,1,1,-5,0,0}]	162 mb	→	3.9 mb
HB:	INT[TB,8,255,8,5,{1,1,1,1,1,1,1,1,-4,0,-1}]	513 mb	→	9.9 mb
DP:	INT[TB,8,510,8,5,{0,1,1,1,1,1,1,1,1,0,-5}]	1.2 gb	→	12 mb

The largest simplifications occur for the most complicated integrals: up to a factor ~ 100 in reduction size!

Crossing of IBP identities

For the complete reduction we need (potentially) all permutations of the external momenta

Being able to treat the IBPs in a fully symbolic fashion, this becomes extremely cheap (wrt other steps)



You don't want to spoil what done so far

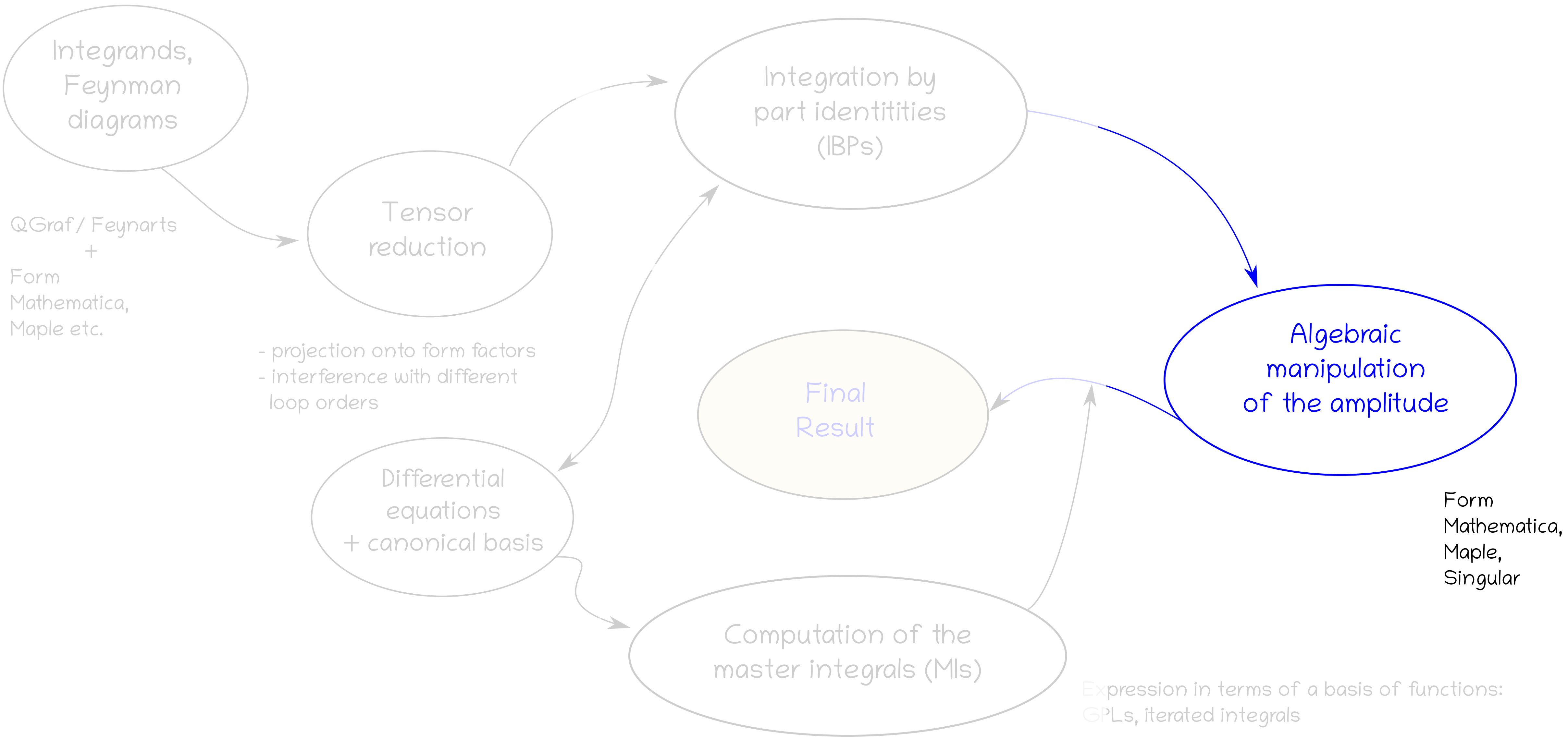
After crossing the invariants: second partial fraction decomposition according to a prefixed global Groebner basis

In practice: all terms in the sum decomposed locally but a unique representation of the rational functions across all IBP identities guaranteed

Crucial for the many (very many indeed) cancellations in the final result.

No need for expensive GCD operations

Traditional approach to multiloop amplitudes



Finite remainder of the amplitude

Insert IBPs into the amplitude, then further partial fraction decomposition: Multivariate Apart + Singular [Decker, Greuel, Pfister, Schoenemann] as backend

No GCD needed to see cancellations!

think of a linear combination with f_k elements of the basis

$$\mathcal{R}(\lambda) = \sum_k r_k(\{s_{ij}, \epsilon_5\}) f_k(\{s_{ij}, \epsilon_5\})$$

in partial fraction decomposed form: i.e. sum of a large number of monomials

Already observed in:
[Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 1904.00945]
[De Laurentis, Maitre, arXiv:2010.14525]
[Chawdhry, Czakon, Mitov, Poncelet, 2012.13553, 2103.04319]
[Abreu, Cordero, Ita, Page,Sotnikov, 2102.13609]

rational functions are not independent

$$r_k = \sum_{m_1+\ldots+m_n \leq p} a_{k,m_1\ldots m_{30}} M_{m_1\ldots m_{30}}, \qquad M_{m_1\ldots m_{30}} \equiv q_1^{m_1}\cdots q_{25}^{m_{25}} s_{12}^{m_{26}} \cdots s_{51}^{m_{30}}$$

monomials are independent objects

We look for linear relations among the various rational functions:

$$0 = \sum_k r_k b_k$$

as many equations as independent monomials

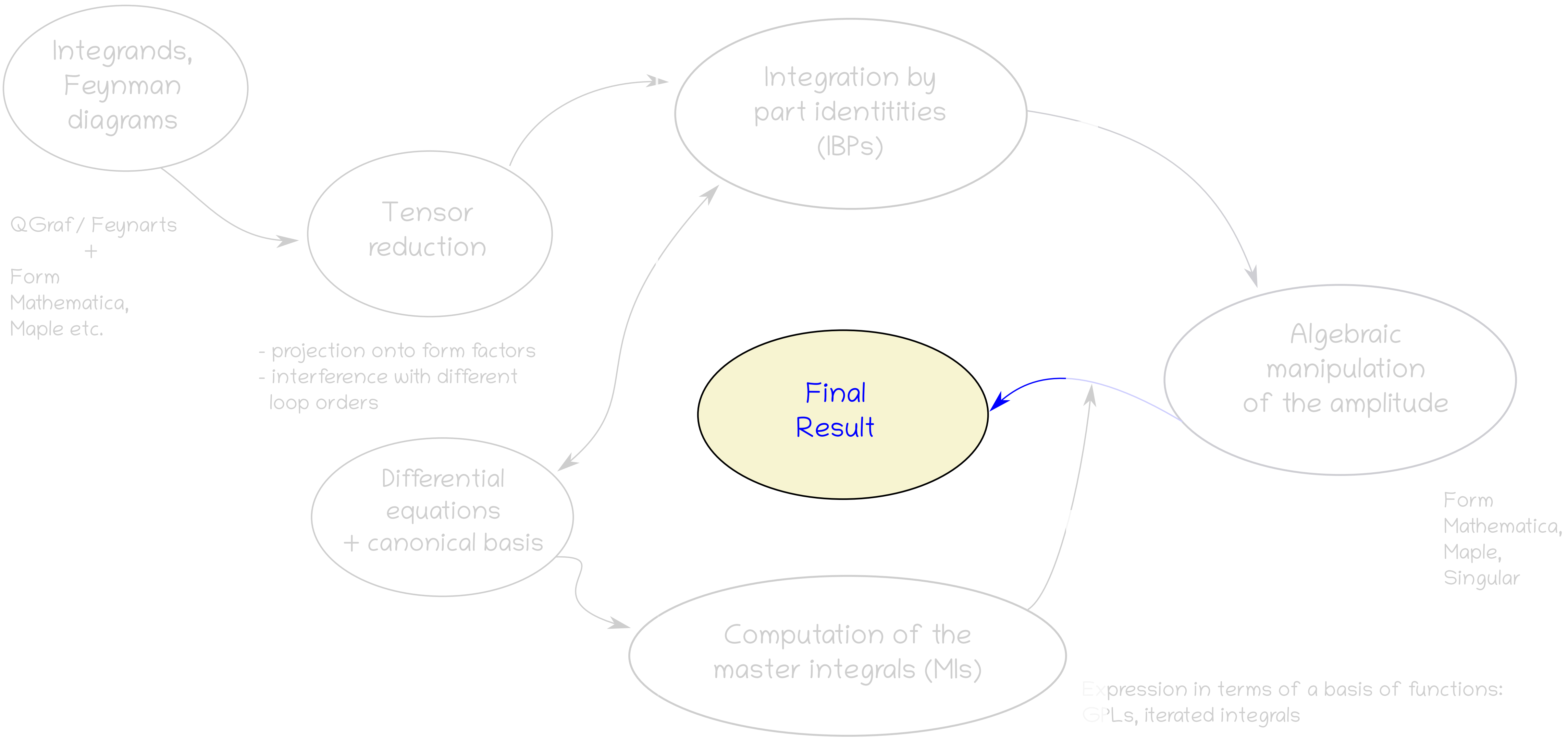
$$0 = \sum_k a_{k,m_1\ldots m_{30}} b_k$$

\# monomials >> # rational functions
but this linear system is over constrained thus admits a solution

(similar in spirit to IBP reduction)

drastic reduction of final expressions

Traditional approach to multiloop amplitudes



Final form of the results

think of a linear combination
with r_k elements of the basis

$$\mathcal{R}(\lambda) = \sum_k r_k(\{s_{ij}, \epsilon_5\}) f_k(\{s_{ij}, \epsilon_5\})$$

A global Groebner basis exposes all cancellations, but might introduce spurious denominators. Easy (but important) fix:

1) Look for all and only the "physical" denominators and the exponents thereof

2) move to a representation which avoids spurious denominators/exponents

→

- More compact results
- Improved numerical stability of rational functions

In the position to derive results for crossed partonic channels: qg and $g\bar{q}$ channels

No need to perform any heavy step again: only need 1↔3 and 2↔3 permutations

Crossing r_k is trivial

Crossing f_k more involved →

1) Express 2-loop MIs and crossings thereof in terms of pentagon functions

2) Exploit the fact that the full set of MIs is mapped onto itself under permutations

3) Obtain a formal system of linear equations for crossed pentagon functions

4) Solve the system (using FinRed). Solutions are enough to cross the whole amplitude

Checks on the finite remainders of the helicity amplitudes

- We checked that the IR poles of the UV-renormalised helicity amplitudes reproduce those predicted by Catani's factorisation formula

- Check against the LC part of the amplitude published in [Chawhdhry et al. 2103.04319] finding complete agreement.

- Strongest of all checks: we performed an independent calculation of the tree-two-loop interference

  Independent as in:

  - No projectors are used: direct interference of the 2-loop amplitude with the tree-level one summed over polarisations
  - Calculation of the interference fully in CDR
  - The qg channel is derived by crossing the $q\bar{q}$ interference prior to IBP reduction (not at the final level of pentagon functions)

  After UV renormalisation and IR factorisation: finite remainder in CDR and t'Hooft-Veltman are equivalent

Direct interference vs interference from amplitudes: complete agreement for all colour factors

Full color results for the helicity amplitudes

Benchmark results for the complete helicity amplitudes

	$u\bar{u} \to g\gamma\gamma$	$ug \to u\gamma\gamma$
$\mathcal{R}^{(1)}(\boldsymbol{\lambda}_A)$	$0.08637873 + 0.6505825\,i$	$-0.05575262 + 1.282163\,i$
$\mathcal{R}^{(1)}(\boldsymbol{\lambda}_B)$	$4.812087 + 0.8811173\,i$	$-5.332701 - 6.518506\,i$
$\mathcal{R}^{(1)}(\boldsymbol{\lambda}_C)$	$0.05297897 - 4.432186\,i$	$-2.497722 - 22.42864\,i$
$\mathcal{R}^{(2)}(\boldsymbol{\lambda}_A)$	$-2.385158 + 18.22971\,i$	$-28.12588 + 26.67761\,i$
$\mathcal{R}^{(2)}_{\mathrm{LC}}(\boldsymbol{\lambda}_A)$	$0.4123777 + 22.64313\,i$	$-1.450073 + 7.396238\,i$
$\mathcal{R}^{(2)}(\boldsymbol{\lambda}_B)$	$115.9528 + 18.71704\,i$	$17.16557 - 102.3377\,i$
$\mathcal{R}^{(2)}_{\mathrm{LC}}(\boldsymbol{\lambda}_B)$	$144.2892 - 3.600533\,i$	$33.14649 - 134.9655\,i$
$\mathcal{R}^{(2)}(\boldsymbol{\lambda}_C)$	$-36.87656 - 153.3540\,i$	$-26.92189 - 508.2138\,i$
$\mathcal{R}^{(2)}_{\mathrm{LC}}(\boldsymbol{\lambda}_C)$	$-55.57522 - 190.2039\,i$	$76.13565 - 214.1456\,i$

$s_{12} = 157$, $s_{23} = -43$, $s_{34} = 83$, $s_{45} = 61$, $s_{15} = -37$, $\mu^2 = 100$

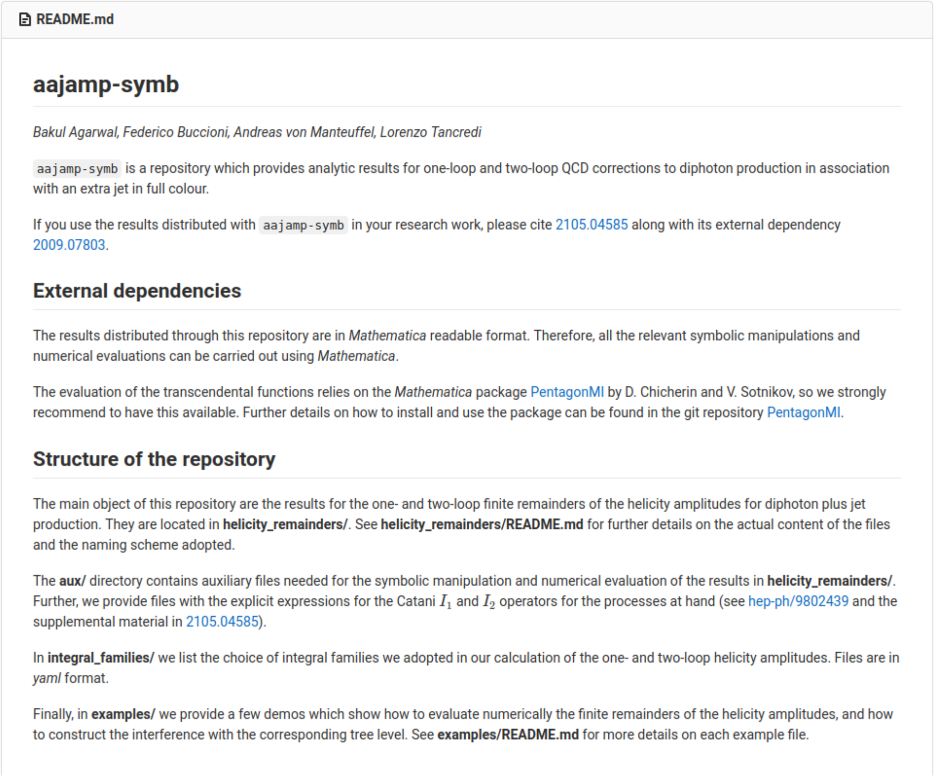
Numerical evaluation performed using PentagonMI [Chicherin, Sotnikov 2009.07803]

Here, just for comparison/reference: full vs LC

My point of view:

for a reliable assessment of impact of sub-LC,
need to look into a (statistically) large set of MC events

Our analytic results are publicly available at
https://gitlab.msu.edu/vmante/aajamp-symb


README.md

aajamp-symb

Bakul Agarwal, Federico Buccioni, Andreas von Manteuffel, Lorenzo Tancredi

`aajamp-symb` is a repository which provides analytic results for one-loop and two-loop QCD corrections to diphoton production in association with an extra jet in full colour.

If you use the results distributed with `aajamp-symb` in your research work, please cite 2105.04585 along with its external dependency 2009.07803.

External dependencies

The results distributed through this repository are in *Mathematica* readable format. Therefore, all the relevant symbolic manipulations and numerical evaluations can be carried out using *Mathematica*.

The evaluation of the transcendental functions relies on the *Mathematica* package PentagonMI by D. Chicherin and V. Sotnikov, so we strongly recommend to have this available. Further details on how to install and use the package can be found in the git repository PentagonMI.

Structure of the repository

The main object of this repository are the results for the one- and two-loop finite remainders of the helicity amplitudes for diphoton plus jet production. They are located in **helicity_remainders/**. See **helicity_remainders/README.md** for further details on the actual content of the files and the naming scheme adopted.

The **aux/** directory contains auxiliary files needed for the symbolic manipulation and numerical evaluation of the results in **helicity_remainders/**. Further, we provide files with the explicit expressions for the Catani I_1 and I_2 operators for the processes at hand (see hep-ph/9802439 and the supplemental material in 2105.04585).

In **integral_families/** we list the choice of integral families we adopted in our calculation of the one- and two-loop helicity amplitudes. Files are in *yaml* format.

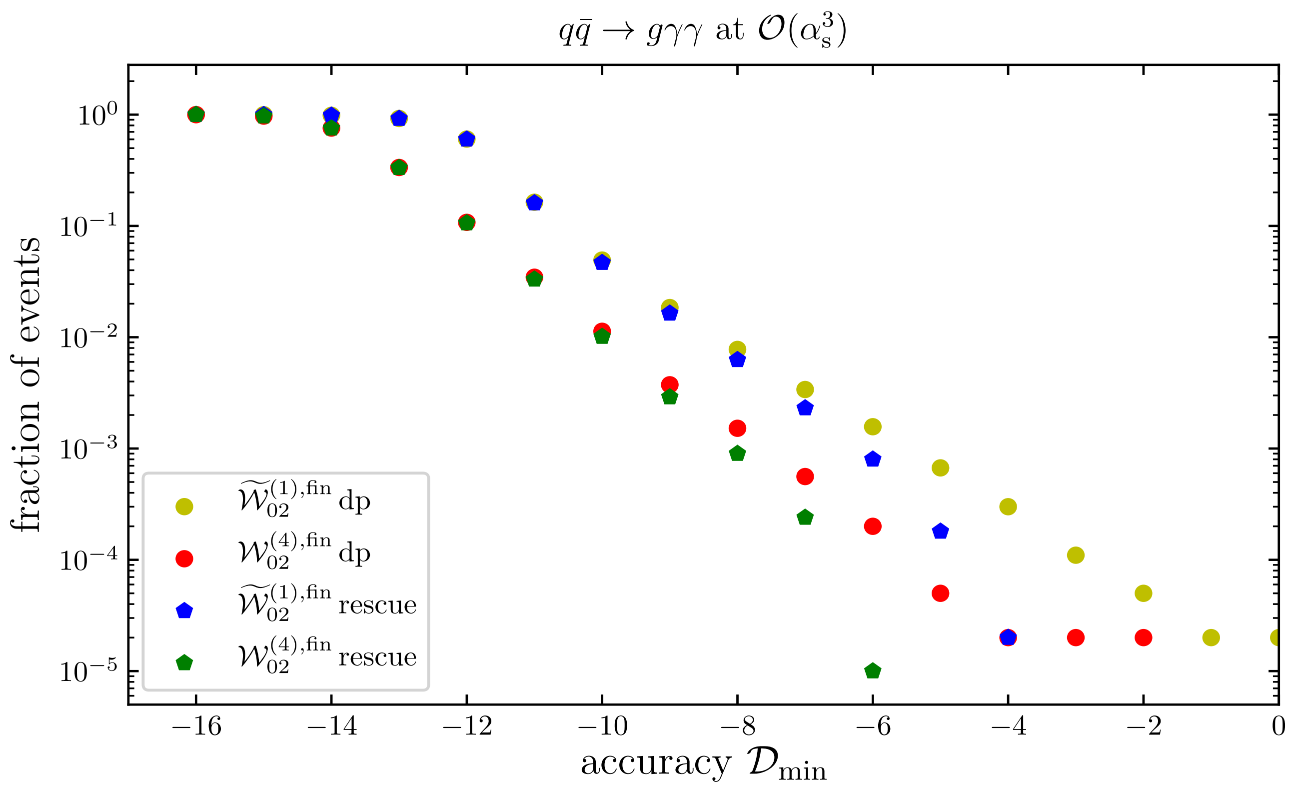
Finally, in **examples/** we provide a few demos which show how to evaluate numerically the finite remainders of the helicity amplitudes, and how to construct the interference with the corresponding tree level. See **examples/README.md** for more details on each example file.

Numerical implementation (LC)

Our LC results implemented in the public code aajamp, available @ https://gitlab.msu.edu/vmante/aajamp

It relies on the external library Pentagon Functions++ https://gitlab.com/pentagon-functions/PentagonFunctions-cpp [Chicherin, Sotnikov]

Performances: ~ 1.2 s/point.



Allows for evaluations in both double and quadruple precision arithmetic

We have implemented a (simple) rescue system: automatic QP activation if

$$\mathcal{D}_i < \chi\, s_{12}$$

$$E_{\mathrm{com}} = 1\,\mathrm{TeV}, \quad p_{\mathrm{T},g} > 30\,\mathrm{GeV},$$
$$p_{\mathrm{T},\gamma_1} > 30\,\mathrm{GeV}, \quad p_{\mathrm{T},\gamma_2} > 30\,\mathrm{GeV}$$

Summary and final remarks

- We presented the NNLO QCD corrections to diphoton + jet amplitudes in full colour. Focus on amplitudes with a fermionic pair

  Analytic results are publicly available.

First time a massless 5-pt 2-loop amplitude computed exactly for all helicity configurations

- Made it possible thanks to very recent advances:

  physical projectors, pentagon functions, IBP reduction, multivariate partial fraction decomposition

- Main focus of this talk: how to go about algebraic complexity and how to reduce and tame it.

Some more technical remarks/ideas:

except for IBP reduction, the whole calculation has been carried out symbolically

This method can be applied to any massless 5-point 2-loop amplitude

great advantages from MVPFD at basically any step

- drastic reduction of complexity of IBP identities
- choice of a unique Groebner basis: cancellations immediate, never need to do expensive GCD operations
- natural way to look for independent rational functions and physical set of denominators (again, no GCD needed!)

Backup

Integral families

Prop. den.	Family A	Family B
D_1	k_1^2	k_1^2
D_2	$(k_1+p_1)^2$	$(k_1-p_1)^2$
D_3	$(k_1+p_1+p_2)^2$	$(k_1-p_1-p_2)^2$
D_4	$(k_1+p_1+p_2+p_3)^2$	$(k_1-p_1-p_2-p_3)^2$
D_5	k_2^2	k_2^2
D_6	$(k_2+p_1+p_2+p_3)^2$	$(k_2-p_1-p_2-p_3-p_4)^2$
D_7	$(k_2+p_1+p_2+p_3+p_4)^2$	$(k_1-k_2)^2$
D_8	$(k_1-k_2)^2$	$(k_1-k_2+p_4)^2$
D_9	$(k_1+p_1+p_2+p_3+p_4)^2$	$(k_2-p_1)^2$
D_{10}	$(k_2+p_1)^2$	$(k_2-p_1-p_2)^2$
D_{11}	$(k_2+p_1+p_2)^2$	$(k_2-p_1-p_2-p_3)^2$

TC/DP can be obtained from TB as:

TC = TBx12435 + $\{k_2 \to k_1+p_1+p_3,\ k_1 \to p_1+k_2\}$

UV renormalisation and IR factorisation

$$\mathcal{A}(\boldsymbol{\lambda}) = \Phi(\boldsymbol{\lambda})\left(\mathcal{B}^{(0)}(\boldsymbol{\lambda}) + \left(\frac{\alpha_s}{2\pi}\right)\mathcal{B}^{(1)}(\boldsymbol{\lambda}) + \left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{B}^{(2)}(\boldsymbol{\lambda})\right) + \mathcal{O}(\alpha_s^3)$$

We renormalise our results in $\overline{\text{MS}}$

$$\alpha_s^b \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu^{2\epsilon}\left[1 - \frac{\beta_0}{\epsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right]$$

The IR structure is completely determined [Catani 9802439; Garland, Gehrmann, Glover, Koukoutsakis, Remiddi 0206067]

We subtract the IR poles according to Catani's scheme [Catani 9802439]

Barred objects are UV renormalised

$$\bar{\mathcal{B}}^{(1)}(\boldsymbol{\lambda}) = I_1(\epsilon,\mu^2)\mathcal{B}^{(0)}(\boldsymbol{\lambda}) + \mathcal{R}^{(1)}(\boldsymbol{\lambda})$$

$$\bar{\mathcal{B}}^{(2)}(\boldsymbol{\lambda}) = I_2(\epsilon,\mu^2)\mathcal{B}^{(0)}(\boldsymbol{\lambda}) + I_1(\epsilon,\mu^2)\bar{\mathcal{B}}^{(1)}(\boldsymbol{\lambda}) + \mathcal{R}^{(2)}(\boldsymbol{\lambda})$$

What we are interested in

MultivariateApart, example

Demo on PF decomposition with Multivariate Apart

```
In[•]:= Get[HomeDirectory[] <> "/hep_tools/MultivariateApart.wl"]
MultivariateApart -- Multivariate partial fractions. By Matthias Heller (maheller@students.uni-mainz.de) and Andreas von Manteuffel (vmante@
```

Univariate PF

In[•]:= fx = $\frac{x^2+3x-2}{x^2 (x-1)(x+1)^2}$;
Apart[fx]

Out[•]= $\frac{1}{2(-1+x)} + \frac{2}{x^2} - \frac{5}{x} + \frac{2}{(1+x)^2} + \frac{9}{2(1+x)}$

Multivariate PF

Spurious poles

In[•]:= fxy = $\frac{2x+y}{y(x-y)(x+y)}$;
Apart[fxy]

Out[•]= $\frac{2}{x y} - \frac{3}{2x(-x+y)} - \frac{1}{2x(x+y)}$

In[•]:= Apart[fxy, x] (* Treat y as constant: no spurious poles *)
Apart[fxy, y] (* Treat x as constant: introduce spurious poles *)

Out[•]= $\frac{3}{2(x-y)y} + \frac{1}{2y(x+y)}$

Out[•]= $\frac{2}{x y} - \frac{3}{2x(-x+y)} - \frac{1}{2x(x+y)}$

Employ a multivariate partial fraction decomposition

In[•]:= MultivariateApart[fxy]

Out[•]= $\frac{3}{2(x-y)y} + \frac{1}{2y(x+y)}$

Unique representation when applied to terms in a sum (commutes with summation)

In[•]:= gxy = $\frac{1}{x y} + \frac{1}{2x(x-y)} - \frac{1}{2x(x+y)}$;

(* The following GCD operation can be extremely expensive for large and complicated rational functions *)
gxy // Together

Out[•]= $\frac{x}{(x-y)y(x+y)}$

In[•]:= (* We could try to apply the PF to each term individually and expand the sum, but with univariate PF:
- different answers,
- spurious poles,
- cancellation not complete *)
Map[Apart[#, x] &, gxy] // Expand
Map[Apart[#, y] &, gxy] // Expand

Out[•]= $\frac{1}{2(x-y)y} + \frac{1}{2y(x+y)}$

Out[•]= $\frac{1}{x y} - \frac{1}{2x(-x+y)} - \frac{1}{2x(x+y)}$

Multivariate Apart

In[•]:= DenominatorFactors = {x, y, x - y, x + y};
qis = {q1, q2, q3, q4};
DenominatorsToQs = $\{\frac{1}{x} \to q1, \frac{1}{y} \to q2, \frac{1}{x-y} \to q3, \frac{1}{x+y} \to q4\}$;
QsToDenominators = Map[Reverse, DenominatorsToQs];

Gxy = gxy /. DenominatorsToQs

Out[•]= $q1\,q2 + \frac{q1\,q3}{2} - \frac{q1\,q4}{2}$

Ordering choice 1

```
In[•]:= ord = {{q4}, {q3}, {q2}, {q1}, {x, y}};
GB = ApartBasis[DenominatorFactors, qis, ord];
Map[ApartReduce[#, GB, ord] &, Gxy] /. QsToDenominators // Expand
```

Out[•]= $\frac{1}{2x(x-y)} + \frac{1}{x y} - \frac{1}{2x(x+y)}$

Ordering. Choice 2

```
In[•]:= ord = {{q1}, {q2}, {q3}, {q4}, {x, y}};
GB = ApartBasis[DenominatorFactors, qis, ord];
Map[ApartReduce[#, GB, ord] &, Gxy] /. QsToDenominators // Expand
```

Out[•]= $\frac{1}{(x-y)(x+y)} + \frac{1}{y(x+y)}$