# All-order renormalization of electric charge in the Standard Model and beyond

Stefan Dittmaier

Albert-Ludwigs-Universität Freiburg

mainly based on
S.D., Phys.Rev.D 103 (2021) 5, 053006 [arXiv:2101.05154];
A.Denner, S.D., Phys.Rept. 864 (2020) 1 [arXiv:1912.06823];
A.Denner, S.D., G.Weiglein, Nucl.Phys.B 440 (1995) 95-128 [hep-ph/9410338]

## Contents

# Introduction

# Introduction

## Electric unit charge $e$:

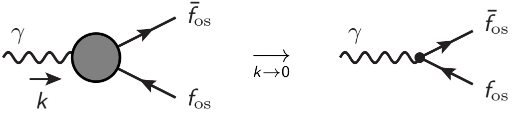
- crucial input parameters of QED and the SM(+beyond)
- defined in the *Thomson limit* (low-energy $\gamma$ interaction with $\mathrm{e}^-$)
- high-precision measurements of $\alpha(0) = \frac{e^2}{4\pi}$ at low energies
- links high-energy and low-energy physics
- enters analysis of running of couplings to high energies (unification?)

## Some features/theorems connected with electric charge $Qe$:

- charge quantization:
  all relative charges $Q$ are integers or integer multiple of $1/3$
  $\hookrightarrow$ only partial explanation in SM ($Q_\mathrm{e} = Q_\mathrm{W}$ ok, $Q_\mathrm{e} = 3Q_\mathrm{d}$ unexplained)
- charge universality:
  renormalization of $e$ does not depend on charged particle
  (=field-theoretical fact in QED and the SM)
  $\hookrightarrow$ any charged particle can be used for renormalization of $e$
- Thirring's low-energy theorem: Thirring '50 (SM: S.D. '97)
  $\sigma^{\mathrm{e}^+\gamma\to\mathrm{e}^-\gamma}_{\mathrm{Compton}}(E_\gamma) \underset{E_\gamma\to 0}{\longrightarrow} \sigma^{\mathrm{e}^+\gamma\to\mathrm{e}^-\gamma}_{\mathrm{Thomson}}$ to all orders!

## Renormalization of $e$:

- renormalization condition in Thomson limit
  $\hookrightarrow$ no corrections in the low-energy limit for on-shell (OS) electrons




$$\bar{u}(p')\,\Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(k,-p',p)\,u(p) \quad \underset{k\to 0}{\longrightarrow} \quad -\,Q_f\,e\,\bar{u}(p)\gamma_\mu u(p)$$

- renormalization transformation: $\underset{\text{bare}}{e_0} = Z_e \underset{\text{ren.}}{e} = (1+\delta Z_e)e$

  $\hookrightarrow$ charge ren. constant $Z_e$ fixed by vertex correction $\Gamma^{A\bar{f}f}_\mu(0,-p,p)$
- QED: $Z_e$ derived from $\gamma\gamma$ self-energy $\Sigma^{AA}$ via QED Ward identity
- SM(+beyond): $Z_e$ calculable from $\Sigma^{AA/AZ}$, but underlying gauge-invariance arguments much more complicated!
  $\hookrightarrow$ issue of this talk

Relevance of relating $Z_e$ to ($f$-independent) self-energies $\Sigma^{AA/AZ}$?
$\hookrightarrow$ General understanding, proving theorems, technical simplifications, ...

# On the history of charge renormalization in the SM

## Charge renormalization at NLO:

- pioneering works on electroweak renormalization:
  - $\delta Z_e$ from explicit 1-loop vertex correction
    Ross, Taylor '73; Sirlin '80; Bardin, Khristova, Fedorenko '80; Aoki et al. '82
  - $\delta Z_e$ via Ward identities from explicit calculations
    Fleischer, Jegerlehner '81; Böhm, Spiesberger, Hollik '86; Hollik '90; Denner '93
- more recently: derivation of $\delta Z_e$ via Lee identities Denner, S.D. '19

## Charge renormalization at NNLO:

- $\delta Z_e$ from explicit 2-loop vertex correction
  Degrassi, Vicini '03; Actis, Passarino et al. '06
- $\delta Z_e$ checked against BFM result (see below) Degrassi, Vicini '03

## Charge renormalization to all orders:

- Background-Field Method (BFM):
  generalization of QED-like result to SM Denner, S.D., Weiglein '94
- conventional $R_\xi$ gauge:
  - correct conjecture, but wrong proofs Bauberger '97; Freitas et al. '02; Awramik et al. '02
  - new approach via charge universality S.D. '21
    $\hookrightarrow$ confirmation of previous conjecture $\rightarrow$ this talk!

# Charge renormalization in QED

# Charge renormalization in QED

Ward identity for the unrenormalized $Af\bar{f}$ vertex function:

$$k^\mu \Gamma^{A\bar{f}f}_\mu(k,\bar{p},p) = -\,Q_f e_0 \left[\Gamma^{\bar{f}f}(\bar{p},-\bar{p}) - \Gamma^{\bar{f}f}(-p,p)\right]$$

Renormalization transformation: $\psi_{0,f}(x) = Z_f^{1/2}\,\psi_f(x),\; A_0^\mu(x) = Z_{AA}^{1/2}\,A^\mu(x)$

$$G^{Af\bar{f}}_\mu(k,\bar{p},p) = Z_{AA}^{1/2}\, Z_f\, G^{Af\bar{f}}_{\mathrm{R},\mu}(k,\bar{p},p) \qquad \text{(full, reducible Green function)}$$

$$\Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(k,\bar{p},p) = Z_{AA}^{1/2}\, Z_f\, \Gamma^{A\bar{f}f}_\mu(k,\bar{p},p) \qquad \text{(amputated, 1PI vertex function)}$$

$\Rightarrow$ Ward identity for the renormalized $Af\bar{f}$ vertex function:

$$k^\mu \Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(k,\bar{p},p) = -\,Q_f e\, Z_e\, Z_{AA}^{1/2} \left[\Gamma^{\bar{f}f}_{\mathrm{R}}(\bar{p},-\bar{p}) - \Gamma^{\bar{f}f}_{\mathrm{R}}(-p,p)\right]$$

$\hookrightarrow$ Expansion for $k \to 0$ yields

$$\underbrace{\bar{u}(p)\,\Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(0,-p,p)\,u(p)}_{\stackrel{!}{=}\,-Q_f e\,\bar{u}(p)\gamma_\mu u(p)} = -\,Q_f e\, Z_e\, Z_{AA}^{1/2}\, \underbrace{\bar{u}(p)\,\frac{\partial \Gamma^{\bar{f}f}_{\mathrm{R}}(-p,p)}{\partial p^\mu}\,u(p)}_{=\,\bar{u}(p)\,\gamma_\mu\,u(p)}$$

(wave-fct. ren. of $\psi_f$)

$\Rightarrow\; Z_e = Z_{AA}^{-1/2}$ = independent of $f$ $\Rightarrow$ Charge universality of QED!

# Charge renormalization in the SM @ NLO

# From QED to the SM

| QED | SM |
|---|---|
| **Gauge symmetry:** | |
| $\mathrm{U(1)_{em}}$ exact | $\mathrm{U(1)}_Y$ spontaneously broken,<br>$\mathrm{U(1)_{em}}$ mixes $\mathrm{SU(2)}_I$ and $\mathrm{U(1)}_Y$ trafos |
| **Neutral gauge bosons:** | |
| photon field $A_\mu$ | 2 neutral gauge fields $W^3_\mu, B_\mu$,<br>mass basis via Weinberg rotation:<br>$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} c_\mathrm{W} & -s_\mathrm{W} \\ s_\mathrm{W} & c_\mathrm{W} \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}$<br>$\hookrightarrow$ mixing of $A_\mu, Z_\mu$ in higher orders |
| **BRS symmetry:** | |
| fully decoupling FP ghost fields<br>$\hookrightarrow$ ghost-free Ward identities | FP ghosts non-decoupling<br>$\hookrightarrow$ ghost contributions in Slavnov–Taylor (ST) ids. for Green fcts. $G^{\Psi_1\Psi_2\dots}$ and Lee ids. for 1PI vertex fcts. $\Gamma^{\Psi_1\Psi_2\dots}$ |

# Charge renormalization in the SM @ NLO in $R_\xi$ gauge

1-loop Ward identity: derived via Lee identities (Denner, S.D., 1912.06823) or ST identities (see also backup slides)

$$\bar{u}(p)\,\Gamma_\mu^{A\bar{f}f}(0,-p,p)\,u(p) = -\,Q_f e_0\,\bar{u}(p)\,\frac{\partial\Gamma^{\bar{f}f}(-p,p)}{\partial p^\mu}\,u(p) - \frac{I^3_{\mathrm{W},f} e_0}{s_{\mathrm{W},0} c_{\mathrm{W},0}}\,\frac{\Sigma_{\mathrm{T}}^{AZ}(0)}{M_Z^2}\,\bar{u}(p)\,\gamma_\mu\omega_-\,u(p), \qquad (*) \qquad \omega_\pm = \tfrac{1}{2}(1\pm\gamma_5)$$

Generalization beyond 1-loop level not available!

Renormalization:

$$\begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} Z_{ZZ}^{1/2} & Z_{ZA}^{1/2} \\ Z_{AZ}^{1/2} & Z_{AA}^{1/2} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}, \qquad \omega_\sigma f_0 = f_0^\sigma = (Z^{f,\sigma})^{1/2} f^\sigma$$

$$\underbrace{\bar{u}(p)\,\Gamma_{\mathrm{R},\mu}^{A\bar{f}f}(0,-p,p)\,u(p)}_{\stackrel{!}{=} -Q_f e\,\bar{u}(p)\,\gamma_\mu\,u(p)} = \bar{u}(p)\,\Gamma_\mu^{A\bar{f}f}(0,-p,p)\,u(p) + \tfrac{1}{2}\delta Z_{ZA}\,\bar{u}(p)\,\Gamma_{\mu,0}^{Z\bar{f}f}\,u(p) + \tfrac{1}{2}\left(\delta Z_{AA} + \delta Z^{f,+} + \delta Z^{f,-}\right)\bar{u}(p)\,\Gamma_{\mu,0}^{A\bar{f}f}\,u(p)$$

$$\underset{(*)}{=} -\,Q_f e\left(1 + \delta Z_e + \frac{1}{2}\delta Z_{AA} + \frac{s_{\mathrm{W}}}{2c_{\mathrm{W}}}\delta Z_{ZA}\right)\bar{u}(p)\,\gamma_\mu\,u(p) + \frac{I^3_{\mathrm{W},f} e}{s_{\mathrm{W}} c_{\mathrm{W}}}\underbrace{\left(\frac{1}{2}\delta Z_{ZA} - \frac{\Sigma_{\mathrm{T}}^{AZ}(0)}{M_Z^2}\right)}_{=0\ \text{(OS renormalization)}}\bar{u}(p)\,\gamma_\mu\omega_-\,u(p)$$

$$\Rightarrow\ \delta Z_e = -\frac{1}{2}\delta Z_{AA} - \frac{s_{\mathrm{W}}}{2c_{\mathrm{W}}}\delta Z_{ZA}$$

# Renormalization in the background-field method

# Basics of the Background-Field Method (BFM)

DeWitt '67,'80; 't Hooft '75; Boulware '81; Abbott '81   SM: Denner, S.D., Weiglein '94

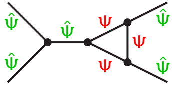
Fields $\Psi$ split into background and quantum parts: $\Psi \to \hat{\Psi} + \Psi$

- Background fields $\hat{\Psi}$:
  - sources of the BFM effective action $\hat{\Gamma}[\hat{\Psi}]$
  - external and tree lines in Feynman diagrams
  - gauge of background gauge fields $\hat{A}^a_\mu$ not fixed in $\hat{\Gamma}[\hat{\Psi}]$
- Quantum fields $\Psi$:
  - integration variables of the functional integral $\int \mathcal{D}\Psi \exp\left\{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}\right\}$
  - loop lines of Feynman diagrams
  - $R_\xi$-type gauge-fixing to support gauge invariance of $\hat{\Gamma}[\hat{\Psi}]$

BFM gauge invariance and gauge fixing of $\hat{\Psi}$:

- $\hat{\Gamma}[\hat{\Psi}]$ fully invariant under "ordinary" gauge transformations of $\hat{\Psi}$

  $\hookrightarrow$ "ghost-free" QED-like Ward identities for BFM vertex fcts. $\hat{\Gamma}^{\hat{\Psi}\cdots}$
- Reducible Green fcts. $\hat{G}^{\hat{\Psi}\cdots}$ and S-matrix elements:
  - gauge-fixing of $\hat{\Psi}$ required for bkg. propagators $\hat{G}^{\hat{\Psi}\hat{\Psi}}$
  - formed from trees with vertex fcts. $\hat{\Gamma}^{\hat{\Psi}\cdots}$



# BFM version of charge renormalization in the SM

Denner, S.D., Weiglein '94

BFM Ward identity for the unrenormalized $\hat{A}\bar{f}f$ vertex function:

$$k^\mu \hat{\Gamma}^{\hat{A}\hat{V}}_{\mu\nu}(k,-k) = 0, \qquad \hat{V} = \hat{A}, \hat{Z},$$
$$k^\mu \hat{\Gamma}^{\hat{A}\bar{f}_i f_j}_{\mu}(k,\bar{p},p) = -\,Q_f e_0 \left[\hat{\Gamma}^{\bar{f}_i f_j}(\bar{p},-\bar{p}) - \hat{\Gamma}^{\bar{f}_i f_j}(-p,p)\right]$$

Renormalization transformation: $\begin{pmatrix}\hat{Z}_0\\ \hat{A}_0\end{pmatrix} = \begin{pmatrix} Z^{1/2}_{\hat{Z}\hat{Z}} & Z^{1/2}_{\hat{Z}\hat{A}} \\ Z^{1/2}_{\hat{A}\hat{Z}} & Z^{1/2}_{\hat{A}\hat{A}}\end{pmatrix}\begin{pmatrix}\hat{Z}\\ \hat{A}\end{pmatrix}$, $\quad f^\sigma_{0,n} = \sum_j \left(Z^{f,\sigma}_{nj}\right)^{1/2} f^\sigma_j$

$$\hat{\Gamma}^{\bar{f}_i f_j}_{\mathrm{R}}(-p,p) = \sum_{l,n} \left(Z^{f,\sigma *}_{li}\right)^{1/2} \left(Z^{f,\sigma}_{nj}\right)^{1/2} \hat{\Gamma}^{\bar{f}_l f_n}(-p,p),$$
$$\hat{\Gamma}^{\hat{A}\bar{f}_i f_j}_{\mathrm{R},\mu}(k,\bar{p},p) = \sum_{\hat{V}=\hat{A},\hat{Z}} \sum_{l,n} \underbrace{Z^{1/2}_{\hat{V}\hat{A}}}_{Z_{\hat{Z}\hat{A}}=0 \text{ due to BFM gauge invariance}} \left(Z^{f,\sigma *}_{li}\right)^{1/2} \left(Z^{f,\sigma}_{nj}\right)^{1/2} \hat{\Gamma}^{\hat{V}\bar{f}_l f_n}_{\mu}(k,\bar{p},p)$$

BFM Ward identity for the renormalized $\hat{A}\bar{f}f$ vertex function:

$$k^\mu \hat{\Gamma}^{\hat{A}\bar{f}_i f_j}_{\mathrm{R},\mu}(k,\bar{p},p) = -\,Q_f\, e\, Z_e\, Z^{1/2}_{\hat{A}\hat{A}} \left[\hat{\Gamma}^{\bar{f}_i f_j}_{\mathrm{R}}(\bar{p},-\bar{p}) - \hat{\Gamma}^{\bar{f}_i f_j}_{\mathrm{R}}(-p,p)\right]$$

$\Rightarrow\;$ $Z_e = Z^{-1/2}_{\hat{A}\hat{A}}$ analogously to QED $\;\Rightarrow\;$ Charge universality of the SM!

# Charge renormalization in the SM to all orders in $R_\xi$ gauge

# Charge renormalization with a fake fermion

Idea: exploit charge universality and
introduce fake fermion $\eta$ with infinitesimal weak hypercharge:

$$\tfrac{1}{2} Y_{\mathrm{W},\eta} = Q_\eta \to 0, \qquad I_{\mathrm{W},\eta} = 0, \qquad m_\eta = \text{arbitrary}$$

Lagrangian:

$$\mathcal{L}_\eta = \bar\eta \Big( \mathrm{i}\partial\!\!\!/ - \tfrac{1}{2} g_1 Y_{\mathrm{W},\eta} B\!\!\!/ - m_\eta \Big) \eta = \bar\eta \left[ \mathrm{i}\partial\!\!\!/ - Q_\eta e \left( A\!\!\!/ + \frac{s_\mathrm{W}}{c_\mathrm{W}} Z\!\!\!/ \right) - m_\eta \right] \eta$$

Charge renormalization condition for $\eta$:

$$\bar u(p)\, \Gamma^{A\bar\eta\eta}_{\mathrm{R},\mu}(0,-p,p)\, u(p) \Big|_{p^2=m_\eta^2} = -Q_\eta e\, \bar u(p) \gamma_\mu u(p)$$

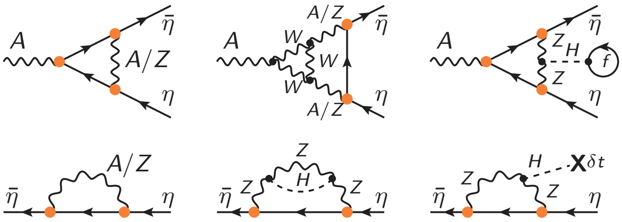
Renormalization transformation: $\eta_0 = Z_\eta^{1/2}\, \eta$

$$\Gamma^{A\bar\eta\eta}_{\mathrm{R},\mu}(k,\bar p,p) = Z_\eta Z_{AA}^{1/2} \underbrace{\Gamma^{A\bar\eta\eta}_{\mu}(k,\bar p,p)}_{= -Q_\eta e_0 \gamma_\mu + \text{h.o.}} + Z_\eta Z_{ZA}^{1/2} \underbrace{\Gamma^{Z\bar\eta\eta}_{\mu}(k,\bar p,p)}_{= -Q_\eta e_0 \frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}} \gamma_\mu + \text{h.o.}}$$

But: higher-order contributions to $Z_\eta$ and $\Gamma^{V\bar\eta\eta}_\mu$ are of $\mathcal{O}(Q_\eta^2)$!

## Charge renormalization with a fake fermion (continued)

Some sample diagrams for corrections to $Z_\eta$ and $\Gamma_\mu^{V\bar\eta\eta}$:




$\mathcal{O}(Q_\eta^2)$

$\Rightarrow$ Direct calculation of $\Gamma_{\mathrm{R},\mu}^{A\bar\eta\eta}(0,-p,p)$ (without Ward identities, etc.):

$$\Gamma_{\mathrm{R},\mu}^{A\bar\eta\eta}(0,-p,p) = -Q_\eta e\gamma_\mu\, Z_e \left[Z_{AA}^{1/2} + Z_{ZA}^{1/2}\,\frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}}\right] + \mathcal{O}(Q_\eta^2)$$
$$\stackrel{!}{=} -Q_\eta e\gamma_\mu$$

$$\Rightarrow\ Z_e = \left[Z_{AA}^{1/2} + Z_{ZA}^{1/2}\,\frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}}\right]^{-1} = \left[Z_{AA}^{1/2} + Z_{ZA}^{1/2}\sqrt{\frac{s_\mathrm{W}^2-\delta c_\mathrm{W}^2}{c_\mathrm{W}^2+\delta c_\mathrm{W}^2}}\right]^{-1}$$

$=$ function of gauge-boson self-energies only

and in agreement with previously "conjectured" results of Bauberger '97; Freitas et al. '02; Awramik et al. '02

# Generalization to gauge groups $G \times U(1)$

## The class of G×U(1)$_Y$ models

- Gauge group G×U(1)$_Y$: G = any Lie group of rank $r$,
  U(1)$_Y$ = "weak hypercharge" subgroup as in the SM,
  so that U(1)$_{\rm em}$ = mixture of G and U(1)$_Y$
- Neutral gauge fields: $B^\mu$ and $C_k^\mu$ $(k=1,\dots,r)$,
  ($C_k$ correspond to diagonal generators of G)

  Mass basis of neutral gauge fields: photon $A^\mu$, $r$ bosons $Z_k^\mu$

$$\begin{pmatrix} B^\mu \\ C_1^\mu \\ \vdots \\ C_r^\mu \end{pmatrix} = R \begin{pmatrix} A^\mu \\ Z_1^\mu \\ \vdots \\ Z_r^\mu \end{pmatrix}, \qquad R = \begin{pmatrix} R_{BA} & R_{BZ_1} & \cdots & R_{BZ_r} \\ R_{C_1A} & R_{C_1Z_1} & \cdots & R_{C_1Z_r} \\ \vdots & \vdots & \ddots & \vdots \\ R_{C_rA} & R_{C_rZ_1} & \cdots & R_{C_rZ_r} \end{pmatrix},$$

generalization of Weinberg rotation

- Examples:
  - SU(2)$_I$×U(1)$_Y$ → SM
  - SU(3)×U(1)$_Y$ Pisano, Pleitez '91; ...
  - SU(2)×U(1)×U(1)$_Y$ Babu, Kolda, March-Russell '97; ...

    Special feature: kinetic mixing

    $\mathcal{L}_{\rm YM} \supset \kappa C_1^{\mu\nu} B_{\mu\nu}$ for U(1) gauge fields $C_1^\mu$, $B_1^\mu$

# Renormalization of the G$\times$U(1)$_Y$ model

## OS renormalization in the photon–Z-boson sector

$$\begin{pmatrix} A_0^\mu \\ Z_{0,1}^\mu \\ \vdots \\ Z_{0,r}^\mu \end{pmatrix} = \begin{pmatrix} Z_{AA}^{1/2} & Z_{AZ_1}^{1/2} & \cdots & Z_{AZ_r}^{1/2} \\ Z_{Z_1A}^{1/2} & Z_{Z_1Z_1}^{1/2} & \cdots & Z_{Z_1Z_r}^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{Z_rA}^{1/2} & Z_{Z_rZ_1}^{1/2} & \cdots & Z_{Z_rZ_r}^{1/2} \end{pmatrix} \begin{pmatrix} A^\mu \\ Z_1^\mu \\ \vdots \\ Z_r^\mu \end{pmatrix}$$

$\hookrightarrow$ order-by-order calculation of OS field ren. constants:

$$Z_{Z_kA}^{1/2} = \frac{\Sigma_{\mathrm{SR,T}}^{Z_kA}(0) - \sum_{l\,(l\neq k)} Z_{Z_lZ_k}^{1/2} Z_{Z_lA}^{1/2}(M_{Z_l}^2 + \delta M_{Z_l}^2)}{Z_{Z_kZ_k}^{1/2}\,(M_{Z_k}^2 + \delta M_{Z_k}^2)} = \mathcal{O}(\alpha)$$

$$Z_{AA} = 1 - \sum_k Z_{Z_kA} - \Sigma_{\mathrm{SR,T}}^{AA\,\prime}(0) = 1 + \mathcal{O}(\alpha), \quad \ldots$$

## BFM renormalization $\rightarrow$ in full analogy to the SM!

- gauge-invariant BFM invariant action $\hat{\Gamma}$
- QED-like Ward identities, $Z_{\hat{Z}_k\hat{A}} = 0$
- charge renormalization constant: $Z_e = Z_{\hat{A}\hat{A}}^{-1/2}$
- charge universality $\Rightarrow$ method of fake fermion applicable!

# Charge renormalization of the $G\times U(1)_Y$ model with a fake fermion $\eta$

## Properties of $\eta$:

- gauge charges: $\frac{1}{2} Y_{\mathrm{W},\eta} = Q_\eta \to 0, \quad e = g_1 R_{BA}$
- Yukawa couplings to Higgs fields $S_i$: $\quad y_i \to 0$

## Lagrangian:

$$\mathcal{L}_\eta = \overline{\eta}\left[\mathrm{i}\partial\!\!\!/ - Q_\eta e\left(A\!\!\!/ + \sum_k \frac{R_{BZ_k}}{R_{BA}} Z\!\!\!/_k\right) - m_\eta - \sum_i y_i S_i\right]\eta$$

## Renormalized $A\bar{\eta}\eta$ vertex function:

$$\begin{aligned}
\Gamma^{A\bar{\eta}\eta}_{\mathrm{R},\mu}(k,\bar{p},p) &= Z_\eta Z_{AA}^{1/2}\,\Gamma^{A\bar{\eta}\eta}_{\mu}(k,\bar{p},p) + \sum_k Z_\eta Z_{Z_kA}^{1/2}\,\Gamma^{Z_k\bar{\eta}\eta}_{\mu}(k,\bar{p},p) \\
&= -\mathrm{i}Q_\eta e\gamma_\mu\, Z_e\left[Z_{AA}^{1/2} + \sum_k Z_{Z_kA}^{1/2}\frac{R_{BZ_k,0}}{R_{BA,0}}\right] + \mathcal{O}(Q_\eta^2) \\
&\stackrel{!}{=} -\mathrm{i}Q_\eta e\gamma_\mu \quad \text{for } k\to 0 \text{ and } p^2 = m_\eta^2
\end{aligned}$$

## Charge renormalization:

$$Z_e = \left[Z_{AA}^{1/2} + \sum_k Z_{Z_kA}^{1/2}\,\frac{R_{BZ_k} + \delta R_{BZ_k}}{R_{BA} + \delta R_{BA}}\right]^{-1}$$

# Conclusions

# Conclusions

## Electric charge in QED, the SM, and beyond:

- Unit charge $e$ is a crucial model parameter
  $\hookrightarrow$ defined (renormalized) in Thomson's low-energy limit
- Charge quantization $\to$ grand unification required for full explanation
- Charge universality $\to$ charge ren. does not distinguish specific particle

## Renormalization of electric charge (ren. constant $Z_e$):

$\hookrightarrow$ desirable: express $Z_e$ in terms of (gauge-boson) self-energies $\Sigma^{VV'}$

- QED: $Z_e$ fixed by $\Sigma^{AA}$ to all orders (Ward identity)
- SM: situation more complicated (AZ-mixing, BRS invariance)
  - NLO: $Z_e$ fixed by $\Sigma^{AA/AZ}$ (laborious proof via Slavnov–Taylor/Lee ids.)
  - all orders: situation "QED-like" in BFM renormalization,
    new: exploit charge universality in conventional $R_\xi$ gauge and
    employ "fake fermion" $\eta$ with $Q_\eta \to 0$ for renormalization
    (concept adapted from ren. of mixing angles) Denner, S.D., Lang '18
- $G\times U(1)_Y$ gauge theories: all-order renormalization as in SM
- Outlook: general gauge theories?
  BFM renormalization similar to SM/QCD; $R_\xi$ gauge laborious?!

# Backup slides

# Derivation of the QED Ward identity for the $Af\bar{f}$ vertex

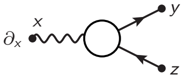
Becchi–Rouet–Stora (BRS) invariance of Green fcts.

+ decoupling of U(1) Faddeev–Popov fields $u(x)$, $\bar{u}(x)$:

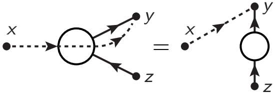
$$\delta_{\mathrm{BRS}}\psi_f(y) = -\mathrm{i}Q_f e_0 u(y)\psi_f(y), \quad \delta_{\mathrm{BRS}}\bar{u}(x) = -\tfrac{1}{\xi}\partial A(x) \quad (R_\xi \text{ gauge})$$

$$\Rightarrow \; 0 = \delta_{\mathrm{BRS}}\langle 0|\, T\, \bar{u}(x)\,\psi_f(y)\,\bar{\psi}_f(z)\,|0\rangle$$

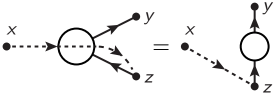
$$= -\frac{1}{\xi}\,\langle 0|\, T\, \partial_x^\mu A_\mu(x)\,\psi_f(y)\,\bar{\psi}_f(z)\,|0\rangle$$




$$-\,\mathrm{i}Q_f e_0\,\langle 0|\, T\, u(y)\,\bar{u}(x)\,\psi_f(y)\,\bar{\psi}_f(z)\,|0\rangle$$




$$+\,\mathrm{i}Q_f e_0\,\langle 0|\, T\, u(z)\,\bar{u}(x)\,\psi_f(y)\,\bar{\psi}_f(z)\,|0\rangle$$




$$= -\frac{1}{\xi}\,\partial_x^\mu\, G_\mu^{Af\bar{f}}(x,y,z) - \mathrm{i}Q_f e_0\, G_0^{u\bar{u}}(y,x)\, G^{f\bar{f}}(y,z) + \mathrm{i}Q_f e_0\, G_0^{u\bar{u}}(z,x)\, G^{f\bar{f}}(y,z)$$

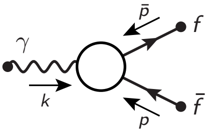
$\hookrightarrow$ Ward in. for $G_\mu^{Af\bar{f}}$ in momentum space:

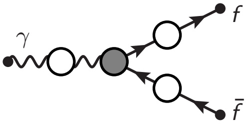
$$0 = -\frac{1}{\xi}\, k^\mu\, G_\mu^{Af\bar{f}}(k,\bar{p},p) - \frac{\mathrm{i}Q_f e_0}{k^2}\left[G^{f\bar{f}}(-p,p) - G^{f\bar{f}}(\bar{p},-\bar{p})\right]$$

## Derivation of the QED Ward identity for the $Af\bar{f}$ vertex (continued)

Amputation of external propagators: Green fcts. $G \;\to\;$ 1PI vertex fcts. $\Gamma$

$$G^{Af\bar{f}}_{\mu}(k,\bar{p},p) \quad = \quad G^{AA}_{\mu\nu}(k,-k)\, G^{f\bar{f}}(\bar{p},-\bar{p})\, \mathrm{i}\Gamma^{A\bar{f}f,\nu}(k,\bar{p},p)\, G^{f\bar{f}}(-p,p)$$




$$\mathrm{i}\Gamma^{AA}_{\mu\nu}(k,-k) = -\left[G^{AA}_{\mu\nu}(k,-k)\right]^{-1}, \qquad \mathrm{i}\Gamma^{\bar{f}f}(-p,p) = -\left[G^{f\bar{f}}(-p,p)\right]^{-1}$$

+ use Ward identity for $\gamma$ propagator: $\quad k^{\mu}\, G^{AA}_{\mu\nu}(k,-k) = -\dfrac{\mathrm{i}\xi k_{\nu}}{k^2}$

$\Rightarrow$ Ward id. for $\Gamma^{A\bar{f}f,\nu}(k,\bar{p},p)$:

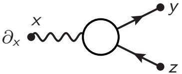
$$k^{\mu}\Gamma^{A\bar{f}f}_{\mu}(k,\bar{p},p) = -\,Q_f e_0 \left[\Gamma^{\bar{f}f}(\bar{p},-\bar{p}) - \Gamma^{\bar{f}f}(-p,p)\right]$$

## Deriving Ward identity $(*)$ for the $Af\bar{f}$ vertex in the SM from BRS invariance

$$\delta_{\mathrm{BRS}}\psi_f(y) = -\mathrm{i}Q_f e_0 u^A(y)\psi_f(y) + \dots, \quad \delta_{\mathrm{BRS}}\bar{u}^A(x) = -\frac{1}{\xi_A}\partial A(x) \quad (R_\xi \text{ gauge})$$

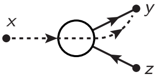
$$\Rightarrow \; 0 = \delta_{\mathrm{BRS}}\langle 0|\, T\, \bar{u}^A(x)\, \psi_f(y)\, \bar{\psi}_f(z)\, |0\rangle$$

$$= -\frac{1}{\xi_A}\,\partial_x^\mu G_\mu^{Af\bar{f}}(x,y,z)$$



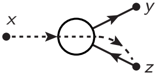

$$+\,\mathrm{i}e_0\left[-\frac{Q_f}{c_{\mathrm{W},0}}\,G^{u^B\bar{u}^A f\bar{f}}(y,x,y,z) + \frac{I^3_{\mathrm{W},f}}{s_{\mathrm{W},0}c_{\mathrm{W},0}}\,\omega_-\,G^{u^Z\bar{u}^A f\bar{f}}(y,x,y,z)\right.$$

$$\left. + \frac{1}{\sqrt{2}s_{\mathrm{W},0}}\,\omega_-\,G^{u^\rho\bar{u}^A f'\bar{f}}(y,x,y,z)\right]$$




$$+\,\mathrm{i}e_0\left[\frac{Q_f}{c_{\mathrm{W},0}}\,G^{u^B\bar{u}^A f\bar{f}}(z,x,y,z) - \frac{I^3_{\mathrm{W},f}}{s_{\mathrm{W},0}c_{\mathrm{W},0}}\,G^{u^Z\bar{u}^A f\bar{f}}(z,x,y,z)\omega_+\right.$$

$$\left. - \frac{1}{\sqrt{2}s_{\mathrm{W},0}}\,G^{u^{-\rho}\bar{u}^A f\bar{f}'}(z,x,y,z)\omega_+\right]$$




$I_{\mathrm{W},f}$ = weak isospin, $\quad \rho = \mathrm{sgn}(I^3_{\mathrm{W},f})$, $\quad u^B = c_{\mathrm{W}}u^A + s_{\mathrm{W}}u^Z$

$(f, f')_{\mathrm{L}}$ = weak isospin doublet, $\quad \omega_\pm = \frac{1}{2}(1 \pm \gamma_5)$

# Sketch of the evaluation of the ST identity

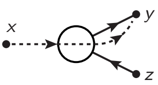
## Further manipulations:

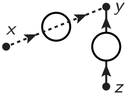
- "Project away" weak isospin part: $G_\mu^{Af\bar{f}}(x,y,z) \to \omega_+ G_\mu^{Af\bar{f}}(x,y,z)\omega_-$

$\hookrightarrow$ $G^{u^B\bar{u}^A f\bar{f}}$ terms are the only surviving ghost contributions:

  - no 1-loop graphs with links between $\langle u^B \bar{u}^A \rangle$ and $\langle f\bar{f} \rangle$ lines

    $\hookrightarrow$ $G^{u^B\bar{u}^A f\bar{f}}(y,x,y,z) \;=\; G^{u^B\bar{u}^A}(y,x)\, G^{f\bar{f}}(y,z)$ at one loop




  - non-vanishing contributions are major obtacle beyond 1-loop level!

- Transition to momentum space $\to$ projected 1-loop identity

$$
\begin{aligned}
0 &= \omega_+ k^\mu G_\mu^{Af\bar{f}}(k,\bar{p},p)\omega_- \\
&\quad + \xi_A \frac{Q_f e_0}{c_{W,0}}\, G^{u^B\bar{u}^A}(k^2)\,\omega_+ \left[G^{f\bar{f}}(-p,p) - G^{f\bar{f}}(\bar{p},-\bar{p})\right]\omega_- \qquad (**)
\end{aligned}
$$

## Sketch of the evaluation of the ST identity (continued)

### Next steps:

- Determination of $G^{u^B\bar{u}^A}(k^2)$ via ST identity: $(\delta_{\mathrm{BRS}} B^\nu = \partial^\nu u^B)$

$$0 = \delta_{\mathrm{BRS}} \langle 0|\, T\, \bar{u}^A(x)\, B_\nu(y)\, |0\rangle = -\frac{1}{\xi_A}\, \partial_x^\mu\, G^{AB}_{\mu\nu}(x,y) + \partial_y^\nu\, G^{u^B\bar{u}^A}(x,y)$$

$$\Rightarrow\ G^{u^B\bar{u}^A}(k^2) = -\frac{k^\mu k^\nu}{\xi_A k^2}\, G^{AB}_{\mu\nu}(k,-k) = -\frac{1}{\xi_A}\Big[c_{\mathrm{W},0}\, \underbrace{G^{AA}_{\mathrm{L}}(k^2)}_{=-\mathrm{i}\xi_A/k^2 \text{ (Slavnov identity)}} + s_{\mathrm{W},0}\, G^{AZ}_{\mathrm{L}}(k^2)\Big]$$

$$= \frac{\mathrm{i} c_{\mathrm{W},0}}{k^2}\left[1 - \frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}}\, \frac{\xi_Z \Sigma^{AZ}_{\mathrm{L}}(k^2)}{k^2 - \xi_Z M_Z^2}\right] \quad \text{at one loop}$$

where $$G^{VV'}_{\mu\nu}(k,-k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) G^{VV'}_{\mathrm{T}}(k^2) + \frac{k_\mu k_\nu}{k^2}\, G^{VV'}_{\mathrm{L}}(k^2)$$

at one loop: $$G^{AZ}_{\mathrm{T}}(k^2) = \frac{\mathrm{i}\Sigma^{AZ}_{\mathrm{T}}(k^2)}{k^2(k^2 - M_Z^2)}, \qquad G^{AZ}_{\mathrm{L}}(k^2) = \frac{\mathrm{i}\xi_A \xi_Z \Sigma^{AZ}_{\mathrm{L}}(k^2)}{k^2(k^2 - \xi_Z M_Z^2)}$$

$$\Sigma^{AZ}_{\mathrm{T}}(0) = \Sigma^{AZ}_{\mathrm{L}}(0) \quad \text{(analyticity!)}$$

## Sketch of the evaluation of the ST identity (continued)

- Amputation of the photon leg: (underlining = amputation; $\chi$ = Goldstone boson)

$$k^\mu G_\mu^{Af\bar f}(k,\bar p,p) = \sum_{V=A,Z} \underbrace{k^\mu G_{\mu\nu}^{AV}}_{=\,k_\nu G_{\rm L}^{AV}} G^{\underline{V}f\bar f,\nu} + \underbrace{k^\mu G_\mu^{A\chi}}_{=\,k^2 G_{\rm L}^{AZ}/(\mathrm{i}\xi_Z M_Z)} G^{\underline{\chi}f\bar f} \quad \text{(ST identity)}$$

$$= -\frac{\mathrm{i}\xi_A}{k^2} k^\nu G_{\underline{\nu}}^{\underline{A}f\bar f} + G_{\rm L}^{AZ}\left[k^\nu G_{\underline{\nu}}^{\underline{Z}f\bar f} - \frac{\mathrm{i}k^2}{\xi_Z M_Z} G^{\underline{\chi}f\bar f}\right]$$

$\Rightarrow$ Intermediate 1-loop result from $(**)$:

$$\omega_+ k^\mu G_{\underline{\mu}}^{\underline{A}f\bar f}\omega_- = Q_f e_0 \omega_+\left[G^{f\bar f}(-p,p) - G^{f\bar f}(\bar p,-\bar p)\right]\omega_-$$
$$- \frac{\mathrm{i}k^2}{\xi_A} G_{\rm L}^{AZ}\,\omega_+\left\{k^\mu G_{0,\mu}^{\underline{Z}f\bar f} - \frac{\mathrm{i}k^2}{\xi_Z M_Z} G_0^{\underline{\chi}f\bar f}\right.$$
$$\left. - Q_f e_0 \frac{s_{\rm W,0}}{c_{\rm W,0}}\left[G_0^{f\bar f}(-p,p) - G_0^{f\bar f}(\bar p,-\bar p)\right]\right\}\omega_-$$

## Sketch of the evaluation of the ST identity (continued)

### Final cumbersome steps:

- Low-energy limit:

calculate $\frac{\partial}{\partial k^\mu}\left[\omega_+ k^\nu G_\nu^{Af\bar{f}}(k,-p-k,p)\omega_-\right]_{k=0}$ using $G_{\mathrm{L}}^{AZ}(k^2) = \frac{\mathrm{i}\xi_A\xi_Z\Sigma_{\mathrm{L}}^{AZ}(k^2)}{k^2(k^2-\xi_Z M_Z^2)}$

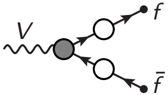
$$\Rightarrow \omega_+ G_\mu^{Af\bar{f}}(0,-p,p)\omega_- = -Q_f e_0\,\omega_+ \frac{\partial G^{f\bar{f}}}{\partial p^\mu}(-p,p)\omega_- - \frac{\Sigma_{\mathrm{L}}^{AZ}(0)}{M_Z^2}\,\omega_+\left[G_{0,\mu}^{Zf\bar{f}}(0,-p,p) + Q_f e_0\,\frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}}\,\frac{\partial G_0^{f\bar{f}}}{\partial p^\mu}(-p,p)\right]\omega_-$$

- Amputation of fermion lines:

$$\Gamma^{\bar{f}f}(-p,p) = \mathrm{i}\left[G^{f\bar{f}}(-p,p)\right]^{-1} = \not{p} - m_f + \text{1-loop corrections}$$

$$\frac{\partial G^{f\bar{f}}}{\partial p^\mu}(-p,p) = G^{f\bar{f}}(-p,p)\,\mathrm{i}\frac{\partial \Gamma^{\bar{f}f}}{\partial p^\mu}(-p,p)\,G^{f\bar{f}}(-p,p)$$

$$G_\mu^{Vf\bar{f}}(0,-p,p) = G^{f\bar{f}}(-p,p)\,\mathrm{i}\Gamma_\mu^{V\bar{f}f}(0,-p,p)\,G^{f\bar{f}}(-p,p)$$




- On-shell limit: isolate terms $\propto (p^2-m_f^2)^{-2}$ for $p^2 \to m_f^2$

($m_f$ = renormalized OS mass!)

# BFM renormalization of the SM – $A/Z$ sector

Denner, S.D., Weiglein '94

2-point vertex functions:

$$\hat\Gamma^{\hat V'\hat V}_{\mu\nu}(-k,k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\hat\Gamma^{\hat V'\hat V}_{\mathrm{T}}(k^2) + \frac{k_\mu k_\nu}{k^2}\hat\Gamma^{\hat V'\hat V}_{\mathrm{L}}(k^2), \qquad \hat V, \hat V' = \hat A, \hat Z$$

BFM Ward identity for the unrenormalized $\hat A \hat V$ vertex functions:

$$k^\mu \hat\Gamma^{\hat A\hat V}_{\mu\nu}(k,-k) = 0 \quad \Rightarrow \quad \hat\Gamma^{\hat A\hat V}_{\mathrm{L}}(k^2) = 0$$

$$\hat\Gamma^{\hat A\hat V}_{\mu\nu} = \text{regular at } k^2{=}0 \quad \Rightarrow \quad \hat\Gamma^{\hat A\hat V}_{\mathrm{T}}(0) = \hat\Gamma^{\hat A\hat V}_{\mathrm{L}}(0) = 0$$

OS renormalization transformation:

$$\begin{pmatrix}\hat Z_0\\ \hat A_0\end{pmatrix} = \begin{pmatrix} Z^{1/2}_{\hat Z\hat Z} & Z^{1/2}_{\hat Z\hat A}\\ Z^{1/2}_{\hat A\hat Z} & Z^{1/2}_{\hat A\hat A}\end{pmatrix}\begin{pmatrix}\hat Z\\ \hat A\end{pmatrix}, \qquad M^2_{\mathrm{Z},0} = M^2_{\mathrm{Z}} + \delta M^2_{\mathrm{Z}}$$

$$\Gamma^{\hat V'\hat V}_{\mathrm{R,T}}(k^2) = -Z^{1/2}_{\hat A\hat V'}Z^{1/2}_{\hat A\hat V}k^2 - Z^{1/2}_{\hat Z\hat V'}Z^{1/2}_{\hat Z\hat V}(k^2 - M^2_{\mathrm{Z}} - \delta M^2_{\mathrm{Z}}) - \underbrace{\Sigma^{\hat V'\hat V}_{\mathrm{SR,T}}(k^2)}_{\text{“subgraph-ren.” self-energy}}$$

OS renormalization conditions:

- $AA$ wave-fct. ren.: $\quad 1 + \Gamma^{\hat A\hat A\,\prime}_{\mathrm{R,T}}(0) = 0 \;\Rightarrow\; Z_{\hat A\hat A} = 1 - Z_{\hat Z\hat A} - \Sigma^{\hat A\hat A\,\prime}_{\mathrm{SR,T}}(0)$
- no $AZ$ mixing for $k^2{=}0$: $\quad \Gamma^{\hat Z\hat A}_{\mathrm{R,T}}(0) = 0 \;\Rightarrow\; Z^{1/2}_{\hat Z\hat A} = \dfrac{\Sigma^{\hat Z\hat A}_{\mathrm{SR,T}}(0)}{Z^{1/2}_{\hat Z\hat Z}(M^2_{\mathrm{Z}}+\delta M^2_{\mathrm{Z}})} = 0$