北京师范大学
BEIJING NORMAL UNIVERSITY

2-loop anti-kT jet function

Hao-yu Liu(Beijing Normal University)
With Xiaohui Liu and Sven-Olaf Moch

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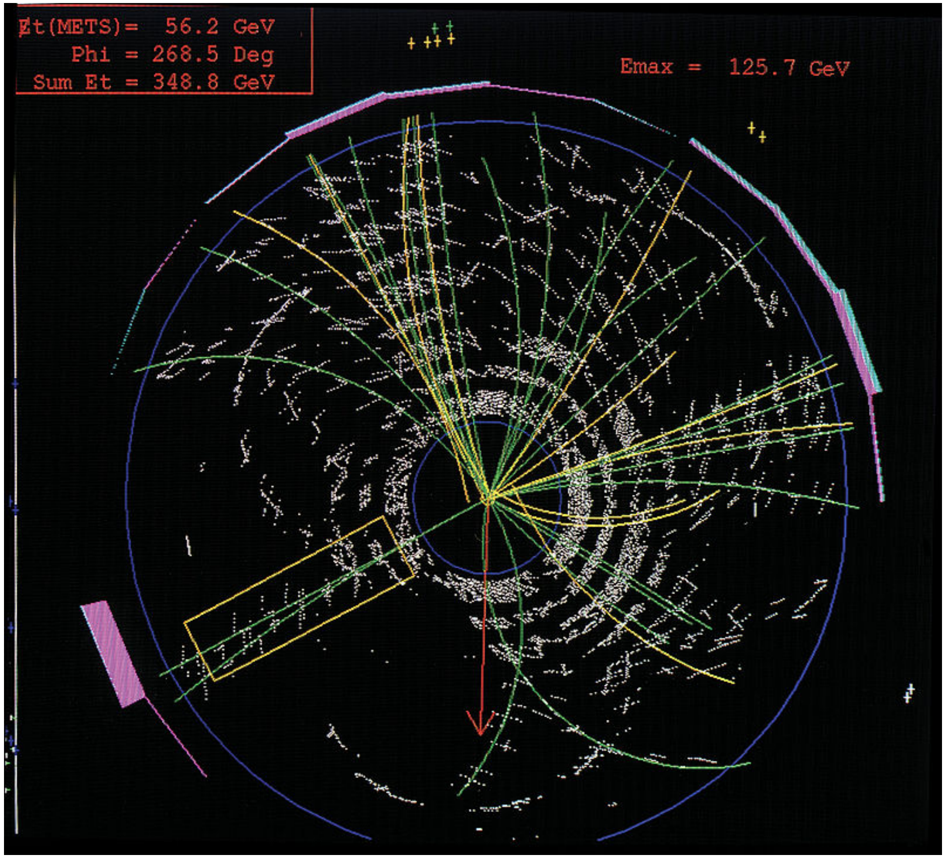
Email address: hyliu1@bnu.edu.cn

Outline

- Jet and factorization
- NLO jet function and method
- NNLO anti-kT jet function

What is a jet?

- More than half of the papers published by ATLAS and CMS make use of jets since 2010!
- Jet is a bunch of hadrons flying nearly in the same direction in high energy collider
- Jet algorithms are used to classify particles into jets



Why jets?

- Indispensable tools for precision test of TeV physics
  SM
  Beyond SM

- Unique probes of non-perturbative dynamics:
  PDFs
  TMDPDFs
  Intrinsic spin of the nucleon
  Hot medium effects of QGPs

- A further boost at the future Electron-Ion Collider (EIC)
  Including extracting TMD

High precision calculation is crucial for probes!

Factorization formula

$d\sigma$ with N exclusive jets for jet radius $R \ll 1$

$$d\sigma = \mathcal{F}_a \mathcal{F}_b \, \mathrm{Tr}[H S_G] \prod_c^N \sum_m \mathrm{Tr}[J_m^c \otimes_\Omega S_{cs,m}^c]$$

[Becher, Neubert, Rothen, and Shao, PRL, 2016] [Liu, Moch, and Ringer, PRL, 2017]

$\mathcal{F}$ The parton distributions, especially TMD

H The hard function

S_G Soft function

J_m Jet function

$S_{cs,m}$ Collinear-soft function

m Multiplicity

$\otimes_\Omega$ Angular convolution

Jet function includes recursive anti-kT algorithm

For Non-Global logs, threshold resummation and resummation when using TMD distribution.

Factorization for jet production

The simplified factorization formula

$$d\sigma = \mathcal{F}_a \mathcal{F}_b \, \mathrm{Tr}[H S_G] \prod_c^N J^c \, S_{cs}^c \, e^{L_{\rm ngl}}$$

Decouples the angular correlation between J_m and $S_{cs,m}$

$$J = \sum_m \langle J_m \rangle_\Omega$$

$$S_{cs} = \langle S_{cs,1} \rangle_\Omega$$

Leading NGLs are resumed into $e^{L_{\rm ngl}}$

Valid up to NLL

Motivation

$$d\sigma = \mathcal{F}_a\mathcal{F}_b\,\mathrm{Tr}[H S_G]\prod_c^N\sum_m \mathrm{Tr}[J_m^c \otimes_\Omega S_{cs,m}^c]$$

$$d\sigma = \mathcal{F}_a\mathcal{F}_b\,\mathrm{Tr}[H S_G]\prod_c J^c\, S_{cs}^c\, e^{L_{\mathrm{ngl}}}$$

In order to push the resumed accuracy of $d\sigma$ to beyond NLL

At least 2-loop level accuracy is required

Hard function	2-loop or beyond	[G. Heinrich,2009.00516]
Soft function	2-loop or beyond	[G. Heinrich,2009.00516]
Jet function	2-loop still missing	

Due to the complicated recursive clustering procedure

NLO quark-jet function

matrix element $q_a \to q_i g_j$ Dimensional regularization Collinear limit

$$J_{bare}^{(1)} = \frac{1}{4}\frac{1}{(2\pi)^{d-1}}\frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\int \mathrm{d}z\mathrm{d}s_{ij}s_{ij}^{-\epsilon}(z\bar{z})^{1-\epsilon}\frac{8\pi\alpha_s Z_\alpha\,\mu^{2\epsilon}e^{\gamma_E\epsilon}}{(4\pi)^\epsilon s_{ij}}C_F\left[\frac{1+\bar{z}^2}{z}-\epsilon z\right]\theta(R^2-\Delta R_{ij}^2)$$

phase space — LO splitting function — jet algorithm

Z_α Renormalization of α_s

$s_{ij} = 2p_i \cdot p_j$ invariant mass

z the momentum fraction of the gluon $\bar{z} = 1 - z$

Exclusive jet production

Smallest ←

$$\rho_{ij} = \min\left[p_{T,i}^{-2\alpha}, p_{T,j}^{-2\alpha}\right]\frac{\Delta R_{ij}^2}{R^2} \qquad \rho_i = p_{T,i}^{-2\alpha}$$

$$\Delta R_{ij}^2 = \Delta\eta_{ij}^2 + \Delta\phi_{ij}^2 \approx \frac{2p_i\cdot p_j}{p_{i,T}p_{j,T}} = \frac{s_{ij}}{z\bar{z}p_T^2} \qquad R \ll 1$$

$$x_1 \equiv \tilde{s}_{ij} = \frac{s_{ij}}{z\bar{z}(p_T R)^2} \leq 1 \qquad x_2 \equiv z \leq 1$$

NLO result of the quark-jet function

$$J_{bare}^{(1)} = e^{2\epsilon L}\frac{\alpha_s C_F}{2\pi}\frac{Z_\alpha\, e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)}\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2\, x_1^{-1-\epsilon} x_2^{-1-2\epsilon}(1-x_2)^{-2\epsilon}\left[1+(1-x_2)^2-\epsilon\, x_2^2\right]$$

All the singularities — Finite when $x_i \to 0$

$$L = \log\frac{\mu}{p_T R} \qquad x_1 \equiv \tilde{s}_{ij} = \frac{s_{ij}}{z\bar{z}(p_T R)^2} \leq 1 \qquad x_2 \equiv z \leq 1$$

$$x_i^{-1-a_i\epsilon} = -\frac{1}{a_i\epsilon}\delta(x_i) + \sum_{n=0}\frac{(-a_i\epsilon)^n}{n!}\left[\frac{\log^n x_i}{x_i}\right]_+$$

Laurent expansion

Matrix element also expanded by ϵ

The coefficients of ϵ series are finite and numerical calculable

$$J_{bare}^{(1)} = e^{2\epsilon L} Z_\alpha \frac{\alpha_s}{2\pi} C_F\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+\frac{13}{2}-\frac{3\pi^2}{4}+\left[26-\frac{9\pi^2}{8}-\frac{49}{3}\zeta_3\right]\epsilon+\left[104-\frac{39}{8}\pi^2-\frac{49}{2}\zeta_3-\frac{11}{32}\pi^4\right]\epsilon^2\right)$$

Not only the integrated jet function,
distributions differential in x_i are also able to be generated

NNLO real-virtual contribution

The phase space is identical to the NLO

Matrix element is the one-loop splitting function

[Kosower and Uwer, NPB, 1999]

The result is

$$J_{rv}^{(2)} = \frac{\alpha_s^2 e^{4\epsilon L}}{(2\pi)^2} C_F \left(C_F \mathcal{K}_{C_F}^{rv} + C_A \mathcal{K}_{C_A}^{rv} \right)$$

where

$$\mathcal{K}_{C_A}^{rv} = -\frac{1}{4\epsilon^4} - \frac{3}{4\epsilon^3} + \left(-5 + \frac{11\pi^2}{24}\right)\frac{1}{\epsilon^2} + \left(-\frac{63}{2} + \frac{13\pi^2}{8} + \frac{26}{3}\zeta_3\right)\frac{1}{\epsilon} - \frac{781}{4} + 11\pi^2 + \frac{85}{2}\zeta_3 - \frac{67}{1440}\pi^4$$

$$\mathcal{K}_{C_F}^{rv} = \left(-\frac{5}{4} + \frac{\pi^2}{3}\right)\frac{1}{\epsilon^2} + \left(-\frac{31}{2} + \frac{\pi^2}{2} + 22\zeta_3\right)\frac{1}{\epsilon} - \frac{575}{4} + \frac{137}{24}\pi^2 + 33\zeta_3 + \frac{10}{9}\pi^4$$

The double-real correction

The double-real correction contains three parts

$$\int d\Phi_3(\{x_i\})|M(\{x_i\})|^2\theta_{\text{jet}}(\{x_i\})$$

Encodes the anti-kT jet clustering algorithm

Kinematics in terms of $\{x_i\}$

We hope:

Laurent expansion

$$\int d\Phi_3|\mathcal{M}|^2 = \int_0^1 \prod_i \mathrm{d}x_i\, x_i^{-1-a_i\epsilon}\, F(\{x_i\},\epsilon)$$

Problem: Fractional power

The three-body phase space

The three-body phase space in the collinear limit $\int d\Phi_3 |M|^2 \theta_{\text{jet}}$

$$d\Phi_3 = 4 \frac{ds_{ij} ds_{ik} ds_{jk} dz_i dz_j}{(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)} \Delta^{-\frac{1}{2}-\epsilon}$$

Gram determinant $\Delta = 4 z_i z_j s_{ik} s_{jk} - (z_k s_{ij} - z_i s_{jk} - z_j s_{ik})^2 > 0$

$$p_{T,i} = z_i p_T \qquad z_k = 1 - z_i - z_j$$

We introduce the angular variables

$$\tilde{s}_{ij} \equiv \frac{s_{ij}}{z_i z_j} \frac{1}{p_T^2 R^2} \qquad \tilde{s}_{ik} = (\sqrt{\tilde{s}_{jk}} - \sqrt{\tilde{s}_{ij}})^2 + 4\sqrt{\tilde{s}_{ij}\tilde{s}_{jk}}\, t \qquad t \in [0,1]$$

The phase space becomes

$$d\Phi_3 = (2p_T R)^{4-4\epsilon} \frac{d\tilde{s}_{ij} d\tilde{s}_{jk} dt dz_i dz_j}{2(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)} (z_i z_j z_k)^{1-2\epsilon} (\tilde{s}_{ij}\tilde{s}_{jk})^{-\epsilon} t^{-\frac{1}{2}-\epsilon} (1-t)^{-\frac{1}{2}-\epsilon}$$

The three-body phase space

$\tilde{s}_{ik}$ in the denominator of the matrix element

$$\int d\Phi_3 |M|^2 \theta_{\text{jet}}$$

Linear divergence

We used this non-linear transformation $$x_5' = \frac{(\sqrt{\tilde{s}_{ij}} - \sqrt{\tilde{s}_{jk}})^2 (1-t)}{\tilde{s}_{ik}}$$

[Anastasiou,Melnikov and Petriello, PRD,2004]

then

$$\tilde{s}_{ik} = (\tilde{s}_{ij} - \tilde{s}_{jk})^2 \left((\sqrt{\tilde{s}_{ij}} - \sqrt{\tilde{s}_{jk}})^2 + 4\sqrt{\tilde{s}_{ij}\tilde{s}_{jk}}\, x_5' \right)^{-1}$$

and

$$d\Phi_3 = (2p_T R)^{4-4\epsilon} \frac{\pi\, d\tilde{s}_{ij} d\tilde{s}_{jk} dz_i dz_j dx_5}{2(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)} (z_i z_j z_k)^{1-2\epsilon}$$
$$\times (\tilde{s}_{ij}\tilde{s}_{jk})^{-\epsilon}\, x_5'^{\,-\epsilon} (1-x_5')^{-\epsilon} |\tilde{s}_{ij} - \tilde{s}_{jk}|^{1-2\epsilon}$$
$$\times \left((\sqrt{\tilde{s}_{ij}} - \sqrt{\tilde{s}_{jk}})^2 + 4\sqrt{\tilde{s}_{ij}\tilde{s}_{jk}}\, x_5' \right)^{-1+2\epsilon}$$

The three-body phase space

Without loss of generality, we assume

$$\int d\Phi_3 |M|^2 \theta_{\text{jet}}$$

$$z_i \leq z_j \qquad \tilde{s}_{ij} \leq \tilde{s}_{jk}$$

then

$$\tilde{s}_{jk} = x_1 \,, \quad \tilde{s}_{ij} = x_1 x_2 \,, \quad z_j = x_3 \,, \quad z_i = x_3 x_4 \quad x_5' = \sin^2\left(\frac{\pi}{2} x_5\right)$$

The parameterization of the three-body phase space

$$\begin{aligned} d\Phi_3 = & (2p_T R)^{-4\epsilon} \frac{\pi \, dx_1 dx_2 dx_3 dx_4 dx_5}{2(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)} \times z_k \\ & \times \left[z_k^2 \, x_5'(1-x_5') \left((\sqrt{x_2}-1)^2 + 4\sqrt{x_2}\, x_5' \right)^{-2} \right]^{-\epsilon} \\ & \times x_1^{-1-2\epsilon} \, x_2^{-1-\epsilon} \, (1-x_2)^{-1-2\epsilon} \, x_3^{-1-4\epsilon} \, x_4^{-1-2\epsilon} \\ & \times \left[x_1^2 x_2 (1-x_2)^2 x_3^4 x_4^2 ((1-\sqrt{x_2})^2 + 4\sqrt{x_2} x_5')^{-1} \right] \end{aligned}$$

→ Laurent expansion

Matrix element

$$\int d\Phi_3 |M|^2 \theta_{\text{jet}}$$

The tree level $a \to ijk$ splitting kernel

$$|\mathcal{M}|^2 = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon} \frac{64\pi^2\alpha_s^2}{s_{ijk}^2} P_{a\to ijk}(z_i, z_j, z_k)$$

[Catani and Grazzini,NPB,2000]

Contains several parts of contributions $P_{\bar{q}_1' q_2' q_3}\,,\quad P^{(\text{id})}_{\bar{q}_1 q_2 q_3}\,,\quad P^{(\text{ab})}_{g_1 g_2 q_3}\,,\quad \text{and} \quad P^{(\text{nab})}_{g_1 g_2 q_3}$

For example $P_{\bar{q}'q'q} = C_F T_F \frac{s_{123}}{2s_{12}}\left[-\frac{[z_1(s_{12}+2s_{23})-z_2(s_{12}+2s_{13})]^2}{(z_1+z_2)^2 s_{12}s_{123}} + \frac{4z_3+(z_1-z_2)^2}{z_1+z_2} + (1-2\epsilon)\left(z_1+z_2-\frac{s_{12}}{s_{123}}\right)\right]$

$$|M(1,2,3)|^2$$

$$1 \to i, 2 \to j, 3 \to k \text{ and permutations}$$

$$|M(i,j,k)|^2$$

$$\tilde{s}_{jk} = x_1\,, \quad \tilde{s}_{ij} = x_1 x_2\,, \quad z_j = x_3\,, \quad z_i = x_3 x_4$$

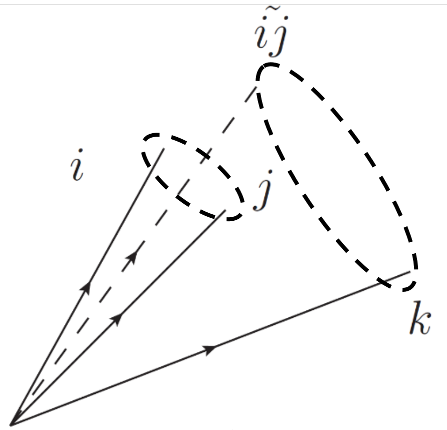
$$|M(\{x_i\})|^2$$

Clustering condition and subtraction

$$\int d\Phi_3 |M|^2 \theta_{\text{jet}}$$

Anti-kT jet algorithm

$$\rho_{ij} = \min\left[p_{T,i}^{-2\alpha}, p_{T,j}^{-2\alpha}\right] \frac{\Delta R_{ij}^2}{R^2} \qquad \rho_i = p_{T,i}^{-2\alpha}$$



Exclusive jet production

i and j clustered first, then $\tilde{ij}$ clustered with k

Small R limit $\Delta R_{\tilde{ij},k}^2 = \Delta\eta_{\tilde{ij},k}^2 + \Delta\phi_{\tilde{ij},k}^2 \approx 2(\cosh\Delta\eta_{\tilde{ij},k} - \cos\Delta\phi_{\tilde{ij},k}) \leq R^2$

Collinear limit $p_{T,i} = z_i\, p_T \qquad p_{\tilde{ij}}^\mu = p_i^\mu + p_j^\mu$

$$2p_{\tilde{ij}} \cdot p_k = 2p_{k,T}(m_{\tilde{ij},T}\cosh\Delta\eta_{\tilde{ij},k} - p_{\tilde{ij},T}\cos\Delta\phi_{\tilde{ij},k}) = 2p_{k,T}\left(\sqrt{p_{\tilde{ij},T}^2 + s_{ij}}\cosh\Delta\eta_{\tilde{ij},k} - p_{\tilde{ij},T}\cos\Delta\phi_{\tilde{ij},k}\right)$$
$$\approx 2p_T^2(z_i + z_j)z_k(\cosh\Delta\eta_{\tilde{ij},k} - \cos\Delta\phi_{\tilde{ij},k})$$

then $\dfrac{z_i\tilde{s}_{ik} + z_j\tilde{s}_{jk}}{z_i + z_j} \leq 1 \qquad \tilde{s}_{ij} \equiv \dfrac{s_{ij}}{z_i z_j}\dfrac{1}{p_T^2 R^2}$

Clustering condition and subtraction

i and j clustered first requires $\int d\Phi_3 |M|^2 \theta_{\rm jet}$

$$\rho_{ij} < \min\left[\rho_{ik}\,, \rho_{jk}\,, \rho_i\,, \rho_j\,, \rho_k\right]$$

$$\rho_i = p_{T,i}^{-2\alpha} \quad \Rightarrow \quad z_i^{-2\alpha}$$

$$\rho_{ij} = \min\left[p_{T,i}^{-2\alpha}, p_{T,j}^{-2\alpha}\right] \frac{\Delta\eta_{ij}^2 + \Delta\phi_{ij}^2}{R^2} \quad \Rightarrow \quad \min[z_i^{-2\alpha}, z_j^{-2\alpha}]\tilde{s}_{ij}$$

$$\min[z_i^{-2\alpha}, z_j^{-2\alpha}]\tilde{s}_{ij} < \min[\min[z_i^{-2\alpha}, z_k^{-2\alpha}]\tilde{s}_{ik}, \min[z_j^{-2\alpha}, z_k^{-2\alpha}]\tilde{s}_{jk}, z_i^{-2\alpha}, z_j^{-2\alpha}, z_k^{-2\alpha}]$$

Case I: $\tilde{s}_{ij} < z_j^{2\alpha} z_k^{-2\alpha} \min[\tilde{s}_{ik}, \tilde{s}_{jk}, 1]$, for $z_i \le z_j \le z_k$

Case II: $\tilde{s}_{ij} < \min[z_j^{2\alpha} z_i^{-2\alpha} \tilde{s}_{ik}, \tilde{s}_{jk}, 1]$, for $z_k \le z_i \le z_j$

Case III: $\tilde{s}_{ij} < \min[z_k^{-2\alpha} z_j^{2\alpha} \tilde{s}_{ik}, \tilde{s}_{jk}, 1]$, for $z_i \le z_k \le z_j$

Clustering condition and subtraction

For case I:

$$\int d\Phi_3 |M|^2 \theta_{\text{jet}}$$

$$d\Phi_3^{\text{I}} = d\Phi_3 \theta\left(\tilde{s}_{ij} < \left(\frac{z_j}{z_k}\right)^{2\alpha} \tilde{s}_{jk}\right) \theta\left(\tilde{s}_{ij} < \left(\frac{z_j}{z_k}\right)^{2\alpha} \tilde{s}_{ik}\right) \theta\left(\frac{z_i}{z_j}\tilde{s}_{ik} + \tilde{s}_{jk} \leq 1 + \frac{z_i}{z_j}\right) \theta(z_i \leq z_j \leq z_k)$$

Change the fractional power of ϵ when $z_j \to 0$, invalidate the Laurent expansion

$$I_1 = \int_0^1 \mathrm{d}x_3\, \mathrm{d}x_2\, x_2^{-1-a_2\epsilon} x_3^{-1-a_3\epsilon} \Rightarrow (-\frac{1}{a_3\epsilon})(-\frac{1}{a_2\epsilon})$$

$$I_2 = \int_0^1 \mathrm{d}x_3 \mathrm{d}x_2\, x_2^{-1-a_2\epsilon} x_3^{-1-a_3\epsilon} \theta(x_3^\alpha - x_2) \Rightarrow [-\frac{1}{(a_3 + a_2\alpha)\epsilon}](-\frac{1}{a_2\epsilon})$$

Direct Laurent expansion of $x_2^{-1-a_2\epsilon} x_3^{-1-a_3\epsilon}$ is not allowed

We introduce $d\Phi^{\text{I}}_{3,sub.} = d\Phi_3\, \theta\left(\tilde{s}_{ij} < z_j^{2\alpha} \tilde{s}_{jk}\right) \theta\left(\tilde{s}_{jk} \leq 1\right) \theta\left(\tilde{s}_{ik} \leq 1\right) \theta(z_i \leq z_j \leq z_k)$

$(d\Phi^I_3 - d\Phi^I_{3,sub.})|\mathcal{M}|^2$ No $z_j \to 0$ poles   No fractional power problem

$d\Phi^I_{3,sub.}\ |\mathcal{M}|^2$ Still have $z_j \to 0$ poles   Cancel when sum 3 cases

Clustering condition and subtraction

For case II and III, relabeled to $z_i \leq z_j \leq z_k$

$$\int d\Phi_3 |M|^2 \theta_{\text{jet}}$$

i and k clustered first

$$d\Phi_3^{\text{II}} = d\Phi_3 \left[1 - \theta\left(\tilde{s}_{ij} < \left(\frac{z_j}{z_k}\right)^{2\alpha} \tilde{s}_{ik}\right)\right] \theta(\tilde{s}_{ik} < 1)\,\theta(\tilde{s}_{ik} \leq \tilde{s}_{jk})\theta\left(\frac{z_i}{z_k}\tilde{s}_{ij} + \tilde{s}_{jk} \leq 1 + \frac{z_i}{z_k}\right)\theta(z_i \leq z_j \leq z_k)$$

j and k clustered first

$$d\Phi_3^{\text{III}} = d\Phi_3 \left[1 - \theta\left(\tilde{s}_{ij} < \left(\frac{z_j}{z_k}\right)^{2\alpha} \tilde{s}_{jk}\right)\right] \theta(\tilde{s}_{jk} < 1)\ \theta(\tilde{s}_{jk} < \tilde{s}_{ik})\theta\left(\frac{z_j}{z_k}\tilde{s}_{ij} + \tilde{s}_{ik} \leq 1 + \frac{z_j}{z_k}\right)\theta(z_i \leq z_j \leq z_k)$$

Subtraction terms for removing $z_j \to 0$ poles

$$d\Phi^{\text{II}}_{3,sub.} = d\Phi_3 \left[1 - \theta\left(\tilde{s}_{ij} < z_j^{2\alpha}\tilde{s}_{jk}\right)\right] \theta(\tilde{s}_{ik} \leq \tilde{s}_{jk})\theta(\tilde{s}_{ik} < 1)\theta\left(\tilde{s}_{jk} < 1\right)\theta(z_i < z_j < z_k)$$

$$d\Phi^{\text{III}}_{3,sub.} = d\Phi_3 \left[1 - \theta\left(\tilde{s}_{ij} < z_j^{2\alpha}\tilde{s}_{jk}\right)\right] \theta(\tilde{s}_{jk} < \tilde{s}_{ik})\ \theta(\tilde{s}_{jk} < 1)\theta\left(\tilde{s}_{ik} < 1\right) \times \theta(z_i < z_j < z_k)$$

$$\sum_{i=\text{I}}^{\text{III}} d\Phi^i_{3,sub.} = d\Phi_3\theta(\tilde{s}_{jk} < 1)\theta\left(\tilde{s}_{ik} < 1\right)\ \theta(z_i < z_j < z_k)$$

No $z_j \to 0$ poles

$$d\Phi_{\text{alg.}}|\mathcal{M}|^2 = \sum_{i=\text{I}}^{\text{III}} \left(d\Phi_3^i - d\Phi^i_{3,sub.}\right)|\mathcal{M}|^2 + d\Phi^i_{3,sub.}|\mathcal{M}|^2 + (\text{all } z_i,\ z_j,\ z_k \text{ orders})$$

Jet algorithm and clustering condition

$x_1^{-1-2\epsilon}\, x_2^{-1-\epsilon}\,(1-x_2)^{-1-2\epsilon}\, x_3^{-1-4\epsilon}\; x_4^{-1-2\epsilon}$ may be still not sufficient to isolate all singularities, especially for $\sum_{i=\mathrm{I}}^{\mathrm{III}} d\Phi^i_{3,sub}$

So we introduce

$$|\mathcal{M}|^2_{sub.} = \lim_{x_4\to 0}|\mathcal{M}|^2 = T_j\cdot T_k\,\frac{s_{jk}}{s_{ij}s_{ik}}P^{(0)}_{a\to jk}$$

which is just the product of the LO eikonal factor and the LO splitting kernel

The final result is given by [Catani and Grazzini,NPB,2000]

$$d\Phi_{\mathrm{alg.}}|\mathcal{M}|^2 = \sum_{i=\mathrm{I}}^{\mathrm{III}} d\Phi^i_{3,sub.}|\mathcal{M}|^2_{sub.} + \left(d\Phi^i_3 - d\Phi^i_{3,sub.}\right)|\mathcal{M}|^2$$
$$+\sum_{i=\mathrm{I}}^{\mathrm{III}} d\Phi^i_{3,sub.}(|\mathcal{M}|^2 - |\mathcal{M}|^2_{sub.}) + (\mathrm{all}\ z_i,\ z_j,\ z_k\ \mathrm{orders})$$

$$\int d\Phi_3|\mathcal{M}|^2 = \int_0^1 \prod_i \mathrm{d}x_i\, x_i^{-1-a_i\epsilon}\, F(\{x_i\},\epsilon)$$

The double-real result

$$J_{rr}^{(2)} = \frac{\alpha_s^2 e^{4\epsilon L}}{(2\pi)^2} C_F \left(C_F \mathcal{K}_{C_F}^{rr} + C_A \mathcal{K}_{C_A}^{rr} + N_F T_F \mathcal{K}_{N_F T_F}^{rr} \right)$$

where

$$\mathcal{K}_{C_F}^{rr} = \frac{1}{2\epsilon^4} + \frac{3}{2\epsilon^2} - \frac{1.8171(3)}{\epsilon^2} - \frac{20.899(2)}{\epsilon} - 73.09(1)$$

$$\mathcal{K}_{C_A}^{rr} = \frac{1}{4\epsilon^4} + \frac{1.20833}{\epsilon^3} + \frac{1.5484(2)}{\epsilon^2} - \frac{7.304(2)}{\epsilon} - 63.64(1)$$

$$\mathcal{K}_{N_F T_F}^{rr} = -\frac{1}{6\epsilon^3} - \frac{7}{9\epsilon^2} + \frac{0.1067(3)}{\epsilon} + 16.688(5)$$

Check by RG Equation

The leading poles up to ϵ^{-2} of the NNLO RR+RV result can be predicted by solving the RGE up to $\alpha_s^2 L^2$

For C_F^2 terms, RGE gives $\mathcal{K}^{rr}_{C_F}\Big|_{\epsilon^{-4}} = \frac{1}{2\epsilon^4}$ $\mathcal{K}^{rr}_{C_F}\Big|_{\epsilon^{-3}} = \frac{3}{2\epsilon^3}$

$$\left(\mathcal{K}^{rv}_{C_F} + \mathcal{K}^{rr}_{C_F}\right)\Big|_{\epsilon^{-2}} = \left(\frac{61}{8} - \frac{3\pi^2}{4}\right)\frac{1}{\epsilon^2} \approx \frac{0.222797}{\epsilon^2} = \frac{0.2228(3)}{\epsilon^2}$$

Our calculation gives $\mathcal{K}^{rr}_{C_F} = \frac{1}{2\epsilon^4} + \frac{3}{2\epsilon^2} - \frac{1.8171(3)}{\epsilon^2}$ $\mathcal{K}^{rv}_{C_F} = \left(-\frac{5}{4} + \frac{\pi^2}{3}\right)\frac{1}{\epsilon^2}$

For $C_F N_F T_F$ terms, RGE gives $\mathcal{K}^{rr}_{N_F T_F}\Big|_{\epsilon^{-2}} = -\frac{7}{9\epsilon^2}$ $\mathcal{K}^{rr}_{N_F T_F}\Big|_{\epsilon^{-3}} = -\frac{1}{6\epsilon^3}$

Our calculation gives $\mathcal{K}^{rr}_{N_F T_F} = -\frac{1}{6\epsilon^3} - \frac{7}{9\epsilon^2}$

Check by RG Equation

For $C_F C_A$ terms, RGE gives

an additional $-\frac{\pi^2}{12\epsilon^2}$ by the non-global contribution is needed for the ϵ^{-2} term

[Becher et al ,Phys.Rev. Lett. ,2016]

[Schwartz and Zhu, Phys. Rev. D,2014.]

$$\left(\mathcal{K}^{rv}_{C_A} + \mathcal{K}^{rr}_{C_A}\right)\Big|_{\epsilon^{-4}} = 0$$

$$\left(\mathcal{K}^{rv}_{C_A} + \mathcal{K}^{rr}_{C_A}\right)\Big|_{\epsilon^{-3}} = \frac{11}{24\epsilon^3} \approx \frac{0.45}{\epsilon} = \frac{0.45833(5)}{\epsilon^3}$$

$$\left(\mathcal{K}^{rv}_{C_A} + \mathcal{K}^{rr}_{C_A}\right)\Big|_{\epsilon^{-2}} = \left(\frac{83}{36} - \frac{\pi^2}{8}\right)\frac{1}{\epsilon^2} \approx \frac{1.07186}{\epsilon^2} = \frac{1.0720(2)}{\epsilon^2}$$

Our calculation gives

$$\mathcal{K}^{rv}_{C_A} = -\frac{1}{4\epsilon^4} - \frac{3}{4\epsilon^3} + \left(-5 + \frac{11\pi^2}{24}\right)\frac{1}{\epsilon^2} \qquad \mathcal{K}^{rr}_{C_A} = \frac{1}{4\epsilon^4} + \frac{1.20833}{\epsilon^3} + \frac{1.5484(2)}{\epsilon^2}$$

The 2-loop anti-kT quark-jet function

$$J_{bare} = 1 + J^{(1)}_{bare} + \frac{\alpha_s^2 e^{4\epsilon L}}{4\pi^2} C_F \Big(C_F \mathcal{J}_{C_F} + C_A \mathcal{J}_{C_A} + N_F T_F \mathcal{J}_{N_F T_F} \Big)$$

The new two-loop result reads

$$\mathcal{J}_{C_F} = \frac{1}{2\epsilon^4} + \frac{3}{2\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{61}{8} - \frac{3\pi^2}{4}\right) - \frac{5.019(2)}{\epsilon} - 12.60(1)$$

$$\mathcal{J}_{C_A} = \frac{11}{24\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{83}{36} - \frac{\pi^2}{8}\right) - \frac{12.348(2)}{\epsilon} - 103.77(1)$$

$$\mathcal{J}_{N_F T_F} = -\frac{1}{6\epsilon^3} - \frac{7}{9\epsilon^2} + \frac{0.1067(3)}{\epsilon} + 16.688(5)$$

The renormalized jet function

$$J = 1 + \frac{\alpha_s}{2\pi} C_F \left(\frac{13}{2} - \frac{3\pi^2}{4}\right) + \frac{\alpha_s^2}{4\pi^2} \left(1.55(1) C_F^2 - 95.08(1) C_A C_F + 13.530(5) C_F N_F T_F\right)$$

Comments before summary

- There is no difficulty to provide also the angular dependent jet function J_m up to $m=3$

- $J^{(0)}_{m=2}$, $J^{(1)}_{m=2}$ and $J^{(0)}_{m=3}$ are already encoded in our calculation, which can be easily seen from the fully differential nature of the phase space sector decomposition

- The simplified factorization theorem only valid up to the single logarithmic level $\left(\mathcal{O}\left(e^{\alpha_s^n L^n}\right)\right)$

- Our method is also available for

$$d\sigma = \mathcal{F}_a \mathcal{F}_b \, \mathrm{Tr}[H S_G] \prod_c^N \sum_m \mathrm{Tr}[J_m^c \otimes_\Omega S^c_{cs,m}]$$

Summary

- We have developed a method to calculate the jet functions for exclusive jets with small R suitable for 2-loop level
- The first explicit results of quark-jet function with the anti-kT jet algorithm at NNLO
- Provides the missing input for the cross section by factorization at beyond NLL
- The computational framework is not limited to anti-kT E-scheme jets, and applicable to semi-inclusive jet function, the WTA jet, the soft-drop groomed jet,etc.

Thank You!