Two-loop helicity amplitudes for gg->ZZ with full top mass dependence

RADCOR-LoopFest 2021

Based on the work https://arxiv.org/abs/2011.15113 with S.P. Jones and A. von Manteuffel

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ZZ production at the LHC

- Significant contribution to off-shell Higgs production through interference [Kauer, Passarino (2012)]
- Constrain Higgs width [Caola, Melnikov (2013)]
- Measuring anomalous $t\bar{t}Z$ coupling; importance of longitudinal modes [Azatov, Grojean, Paul, Salvioni (2016)], [Cao, Yan, Yuan, Zhang (2020)]
- Important channel for BSM searches

- $gg \to ZZ$ formally NNLO at LHC
- High gluon luminosity => large contribution
- Provides $\sim 60\,\%$ of the total NNLO correction [Cascioli, German, Grazzini, Kallweit, Maierhöfer, von Manteuffel, Pozzorini, Rathlev, Tancredi, Weihs (2014)]
- Increase of $5\,\%$ to the full NNLO result from $gg \to ZZ$ at NLO [Grazzini, Kallweit, Wiesemann, Yook (2018)]

Status of the calculation

$gg \to ZZ$:

- Known exactly at 1-loop [Glover, van der Bij (1988)]
- Massless internal fermions at 2-loops [von Manteuffel, Tancredi (2015)], [Caola, Henn, Melnikov, Smirnov, Smirnov (2015)]
- Large top-mass approximation at 2-loops [Dowling, Melnikov (2015)], [Caola, Dowling, Melnikov, Röntsch, Tancredi (2016)] with Padé approximants [Campbell, Ellis, Czakon, Kirchner (2016)]
- Expansion around $t\bar{t}$ threshold with Padé approximants [Gröber, Maier, Raum (2019)]
- Small top-mass expansion with Padé approximants [Davies, Mishima, Steinhauser, Wellman (2020)] (See Go Mishima's talk)
- 2-loop amplitudes with full top-mass dependence [Agarwal, Jones, von Manteuffel (2020)], [Brønnum-Hansen, Wang (2021)]

Other similar gluon-induced calculations involving massive internal loops :

- HH production at 2-loops with full top-mass dependence [Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Zirke (2016)] and in small top-mass expansion [Davies, Mishima, Steinhauser, Wellmann (2018)] (See Joshua Davies' talk)
- ZH amplitudes at 2-loops with full top-mass dependence [Chen, Heinrich, Jones, Kerner, Klappert, Schlenk (2020)] and in small and large top-mass regions [Davies, Mishima, Steinhauser (2020)] (See Matthias Kerner's talk)
- WW amplitudes at 2-loop with 3rd generation quarks [Brønnum-Hansen, Wang (2020)] (See Chen-Yu Wang's talk, also for ZZ)

Multiloop calculations

Recipe for a multi-loop amplitude:

1. Generation of unreduced amplitude
2. IBP reduction
    - Major bottleneck for processes with many scales and/or legs
    - Significant progress with syzygy based approaches and finite-field methods
3. Insertion of IBP identities into the amplitude
    - Significant blow-up for intermediate results and final reduced amplitude
    - Numerical instabilities in final coefficients
    - Use of multivariate partial fractioning to tame the computational complexity and improve numerical performance
4. Evaluation of master integrals
    - Internal masses => Functions beyond multiple polylogarithms
    - Use of numerical methods instead, improved with the use of finite integrals

Syzygies

- Integration-By-Parts reduction to reduce all the integrals to a basis set
- Generate linear relations between integrals [Chetyrkin & Tkachov (1981)]
- Systematically construct and reduce a linear system to a basis set of master integrals -> Laporta's algorithm [Laporta (2000)] . Public codes available AIR, FIRE6, Kira, LiteRed, Reduze 2, etc.
- In Baikov representation [Baikov (1996)] :

$$0 = \int \left(\prod_i^L dz_i \right) \sum_i^N \frac{\partial}{\partial z_i} \left(f_i(z_1, \ldots, z_N) P^{(d-L-E-1)/2} \prod_i^N \frac{1}{z_i^{\nu_i}} \right)$$

$$0 = \int \left(\prod_i^L dz_i \right) \sum_i^N \left(\frac{\partial f}{\partial z_i} + \boxed{\frac{d-L-E-1}{2P} f_i \frac{\partial P}{\partial z_i}} - \boxed{\frac{\nu_i f_i}{z_i}} \right) P^{(d-L-E-1)/2}$$

Dimension shifting term   Doubled propagators

- Require:
    - No dimension-shifting terms
    - No integrals with doubled propagators

Syzygies

Disadvantages:

- Such integrals don't appear in amplitudes
- Significantly larger linear system to reduce for the appearance of auxiliary integrals

Avoiding doubled propagators:

- Generating vectors using Groebner basis [Gluza, Kajda, Kosower (2010)]
- Linear algebra based approach [Schabinger (2011)]
- Differential geometry [Zhang (2014)]

$$f_i \frac{\partial P}{\partial z_i} \sim P$$

Dimension shifting term

- Explicit solutions known [Boehm, Georgoudis, Larsen, Schulze, Zhang (2017)] [Abreu, Cordero, Ita, Page, Zeng (2017)]
- Polynomials of degree 1 in Baikov parameters
- Straightforward to write

$$f_i \sim z_i$$

Doubled propagator term

- Trivial to write explicit solutions

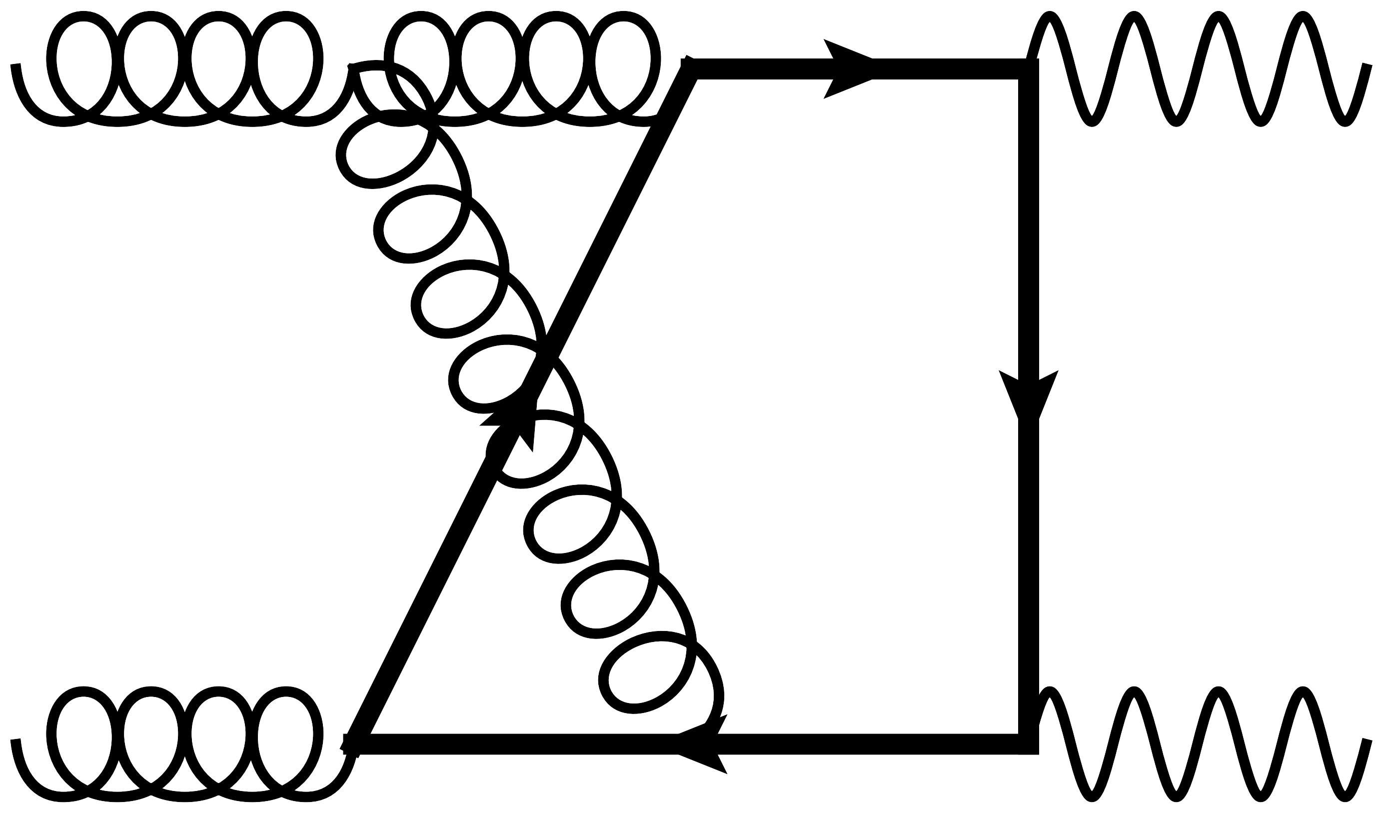
Syzygies

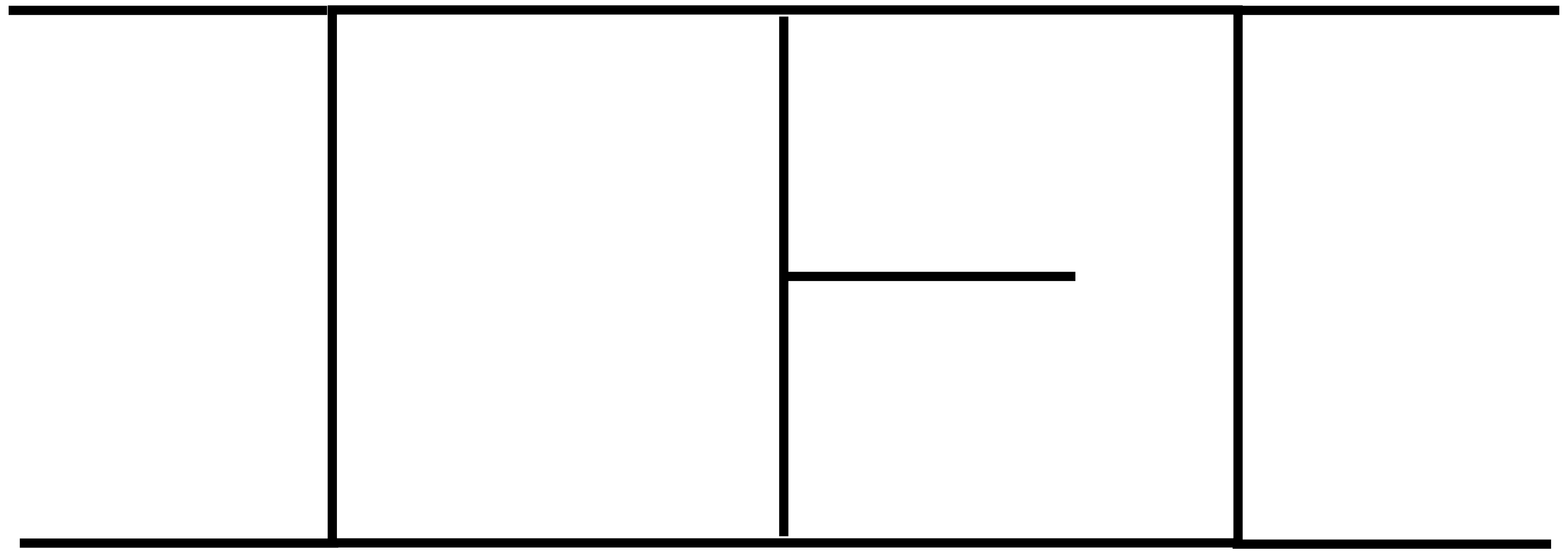
- Simultaneous solution for the two constraints highly non-trivial
- Compute module intersection of the two syzygy modules
- Conventional approaches insufficient [Larsen, Zhang (2015)] [Boehm, Georgoudis, Larsen, Schoenemann, Zhang (2018)]
  - Syzygies for top-level topologies inaccessible

- Developed a new **linear algebra approach based on finite fields** [Agarwal, Jones, von Manteuffel (2020)]
  - Map the problem of module intersection to row reduction of a matrix; Finred - finite field based solver for the linear algebra
  - Solutions produced up to a requested degree in z_i
  - **Much faster** for our purpose than the Groebner basis approach; can run in a highly distributed manner
  - Able to generate the required syzygies for this calculation

- Use Finred - finite field based solver, to compute the required IBP reductions
- Also use this approach for the 2-loop amplitudes for diphoton+jet production [Agarwal, Buccioni, von Manteuffel, Tancredi (2021)], [Agarwal, Buccioni, von Manteuffel, Tancredi (2021)] (see Federico Buccioni's talk)

Syzygies





- Total size of syzygies $\sim 2GB$
- Largest syzygy $\sim 230MB$
- Up to $s = 4$ integrals
- 2 scales s, t $(m_t, m_Z$ set to numbers)
- Extremely complicated due to internal masses

- Total size of syzygies $\sim 1GB$
- Largest syzygy $\sim 40MB$
- Up to $s = 5$ integrals
- 4 scales $s_{23}, s_{34}, s_{45}, s_{51}$ $(s_{12} = 1)$

Finite integrals

- Feynman integrals often have UV and IR divergences
- Sector decomposition standard method to resolve IR poles [Binoth, Heinrich (2000)] [Bogner, Weinzierl (2007)]

Public codes: Fiesta4, pySecDec, etc.

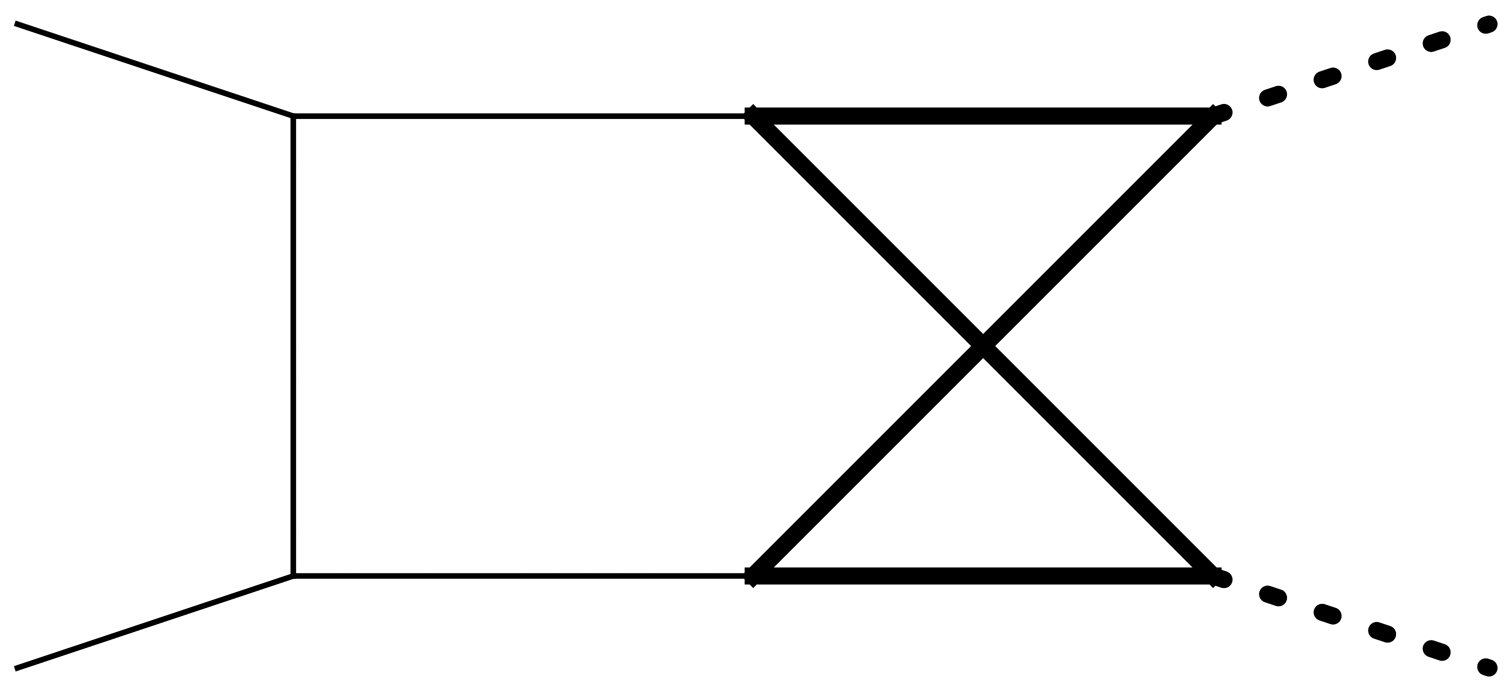
Why use finite integrals instead?

- Much better behaved numerically
- Require fewer orders in epsilon expansion in general
- Poles drop out into the coefficients => Easier to take $d \to 4$ limit

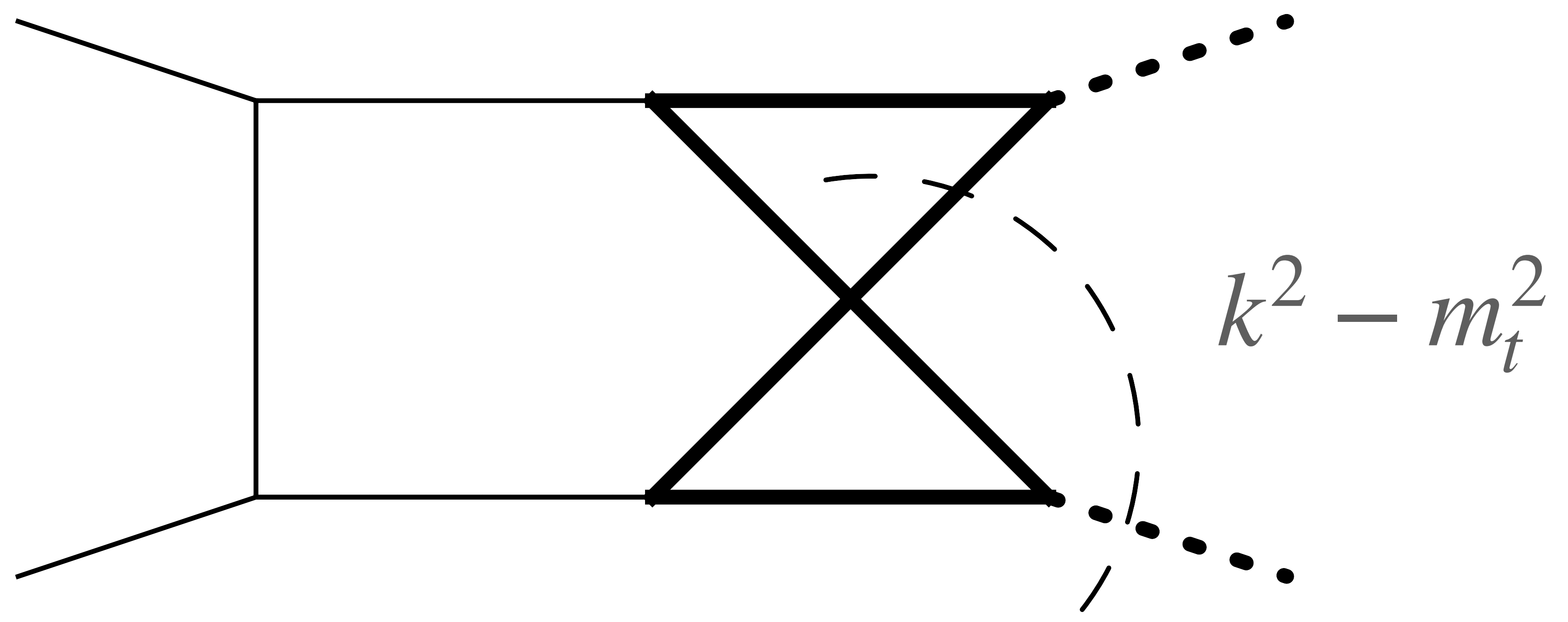
Constructing finite integrals:

- Dimension shifted integrals [Bern, Dixon, Kosower (1992)]
- Existence of a finite basis [Panzer (2014)] [von Manteuffel, Panzer, Schabinger (2014)]
- Reduze 2 to find such integrals, usually involving doubled propagators (dots) and dimension shifts

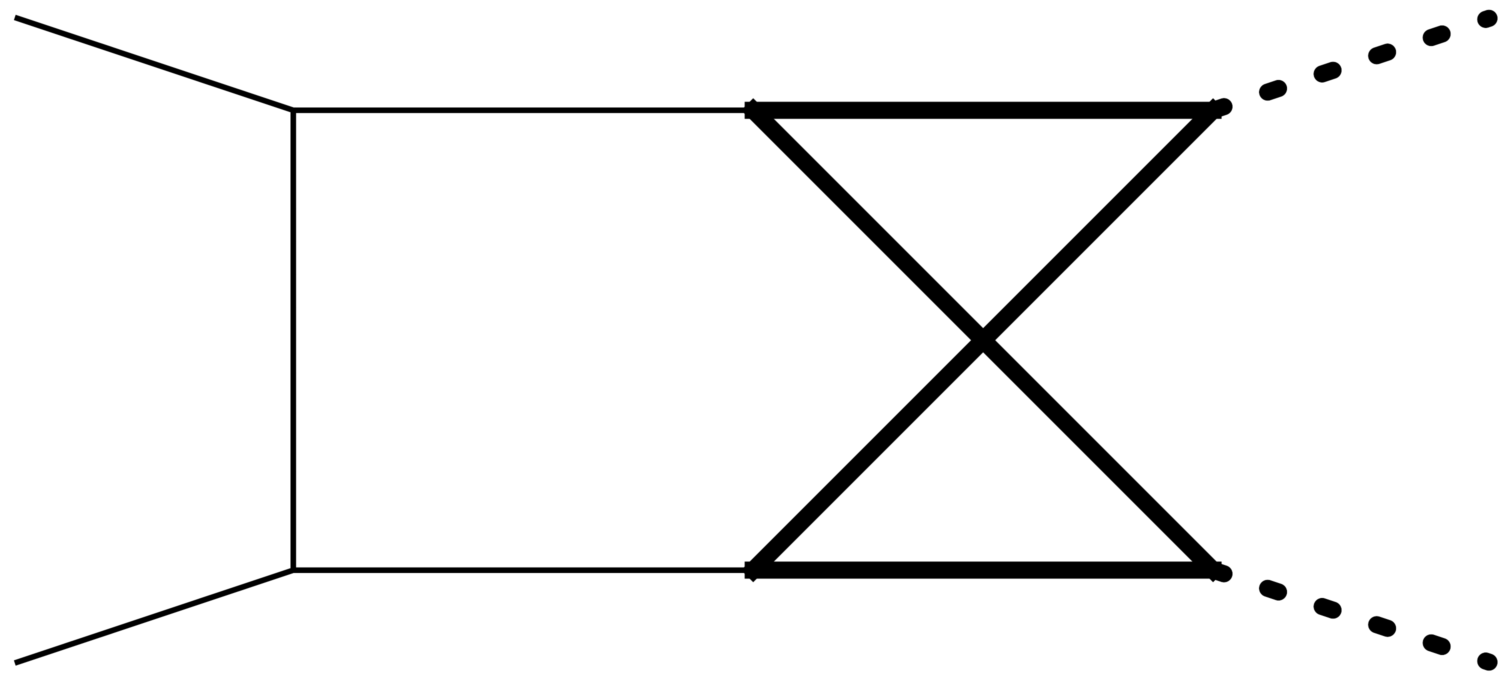
Finite integrals



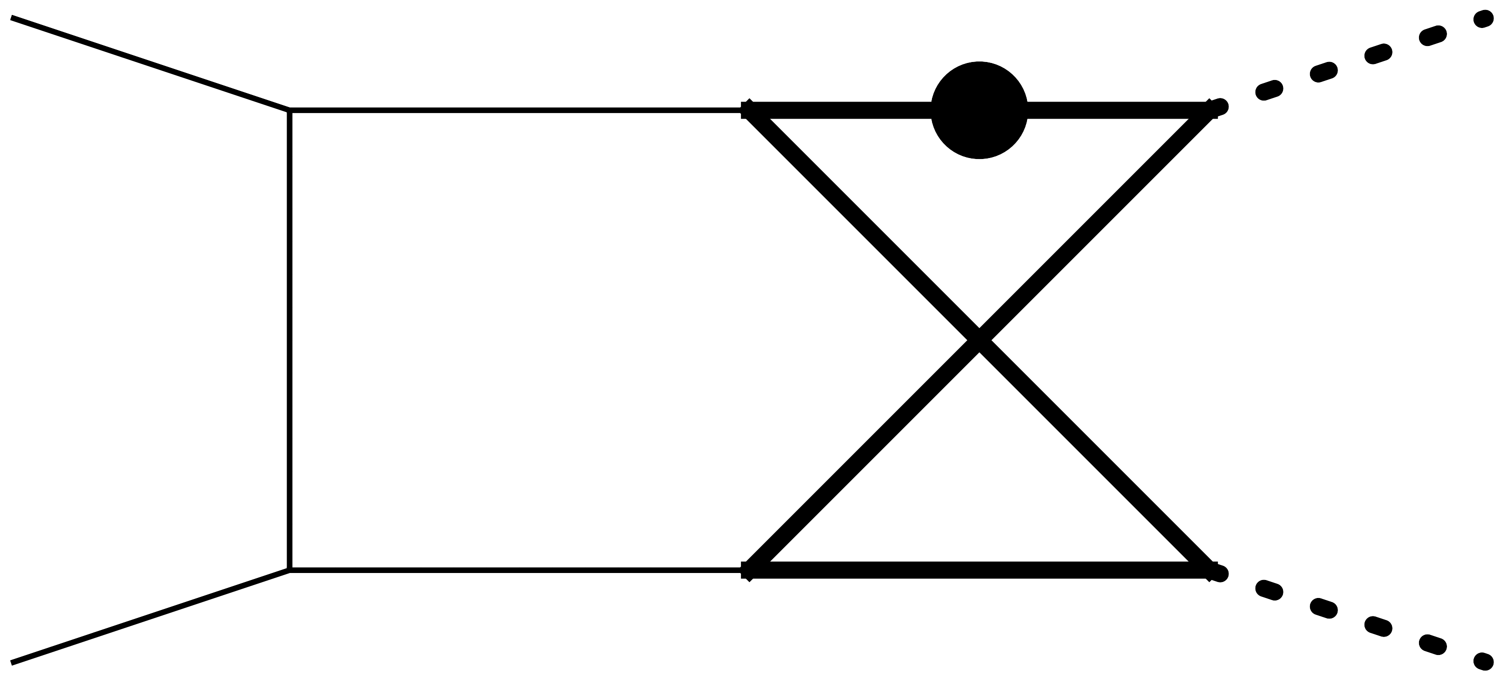
Divergent integral in $d = 4 - 2\epsilon$



Divergent integral in $d = 4 - 2\epsilon$ with a numerator



Finite integral in $d = 6 - 2\epsilon$



Finite integral in $d = 6 - 2\epsilon$ with a dot

Finite integrals

However:

- Integrals with dots and dimension-shifts often hard to reduce e.g. need reductions for integrals with 4 dots for the required finite integrals
- Higher dots implies higher powers of $\mathscr{F}$ polynomial in the denominator => worse contour deformation which leads to numerical instabilities

Alternate approach - combining divergent integrals into **finite linear combinations**. Advantages:

- Integrals often already appearing in the amplitude => avoid computing extra reductions
- More "natural" $d = 4$ representation
- Finite at the integrand level i.e. integrand free of non-integrable divergences

- In general a highly non-trivial task to find these numerators
- Algorithmically construct finite linear combinations in $d = 4$ from a list of seed integrals [Agarwal, Jones, von Manteuffel (2020)]
- Arbitrary integrals with numerators, dots, dimension shifts, subsector integrals etc allowed as seed integrals

Finite integrals

$$\text{Integrand} = a_1 \frac{1}{D_1 \ldots D_N} + a_2 \frac{D_{N+1}}{D_1 \ldots D_N} + a_3 \frac{D_j}{D_1 \ldots D_j \ldots D_N} + \ldots$$

Corner integral — Numerator integral — Subsector integral

- Combine over a common denominator using the general formula for Feynman parametric representation [Agarwal, Jones, von Manteuffel (2020)]

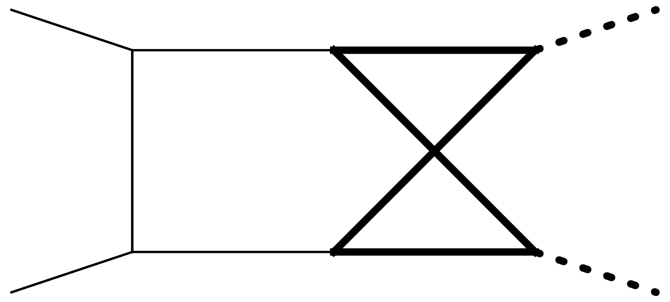
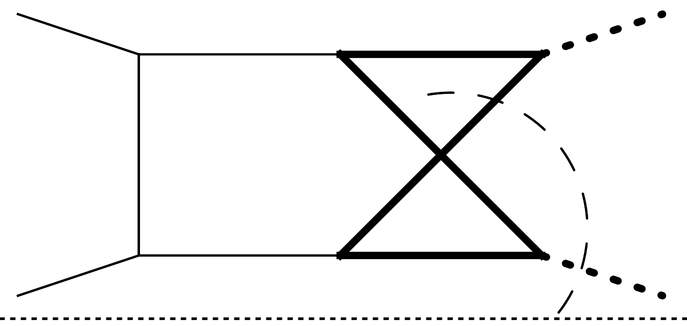
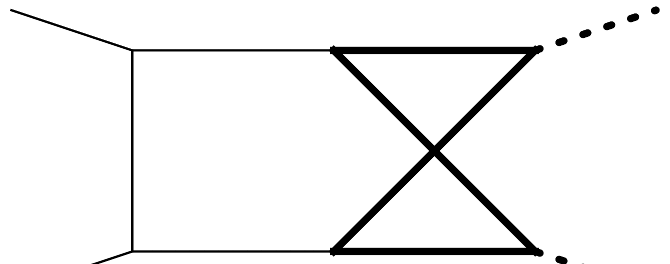
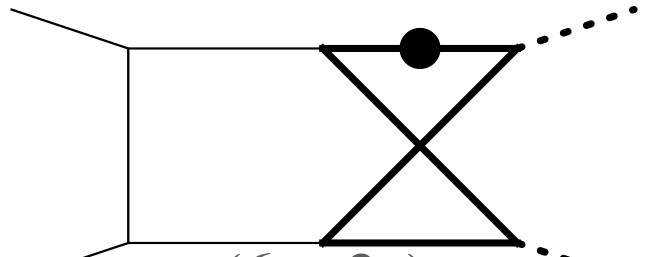
$$I(\nu_1, \ldots, \nu_N) = (-1)^{r+\Delta t}\,\Gamma(\nu - L\,d/2) \int \left(\prod_{j \in \mathcal{N}_T} dx_j \right) \left(\prod_{j \in \mathcal{N}_t} \frac{x^{\nu_j - 1}}{\Gamma(\nu_j)} \right) \delta\left(1 - \sum_{j \in \mathcal{N}_T} x_j \right)$$

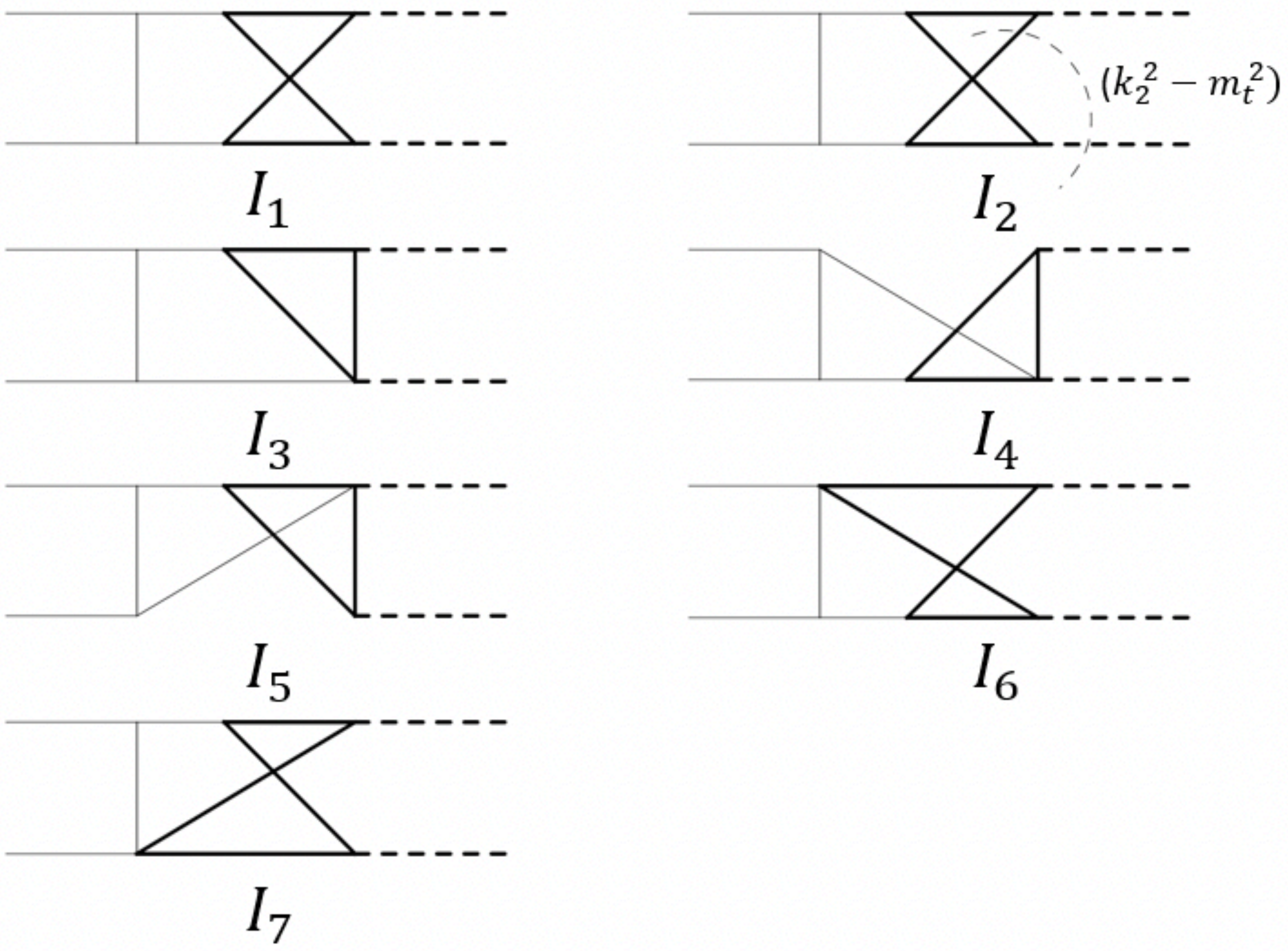
$$\left[\left(\prod_{j \in \mathcal{N}_{\setminus T}} \frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}} \right) \left(\prod_{j \in \mathcal{N}_{\Delta t}} \frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}} \right) \frac{\mathcal{U}^{\,\nu-(L+1)d/2}}{\mathcal{F}^{\,\nu - L\,d/2}} \right]_{x_j = 0\ \forall\ j \in \mathcal{N}_{\setminus T}} \qquad (\nu_j \in \mathbb{Z})$$

$\mathcal{N}_T$: Parent sector
$\mathcal{N}_t$: Current integral propagators
$\mathcal{N}_{\setminus T}$: Numerators
$\mathcal{N}_{\Delta t}$: Pinched propagators

- Constrain a_i requiring absence of non-integrable divergences in the integrand

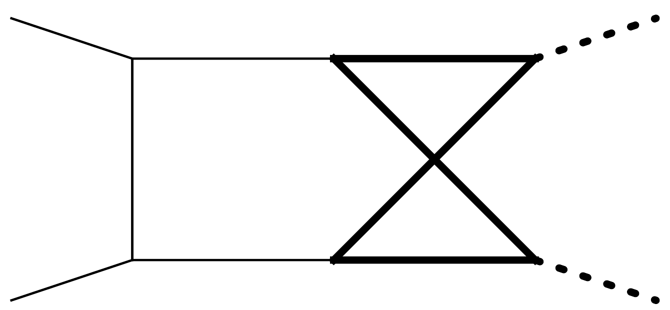
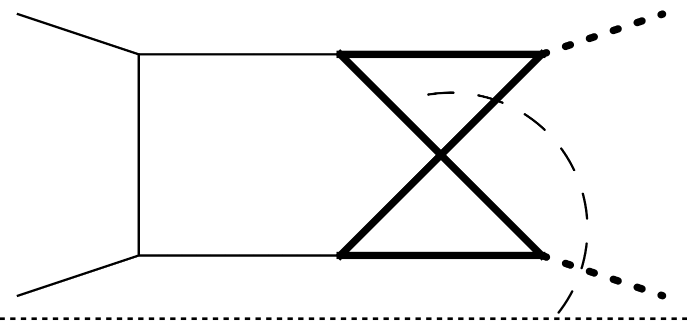
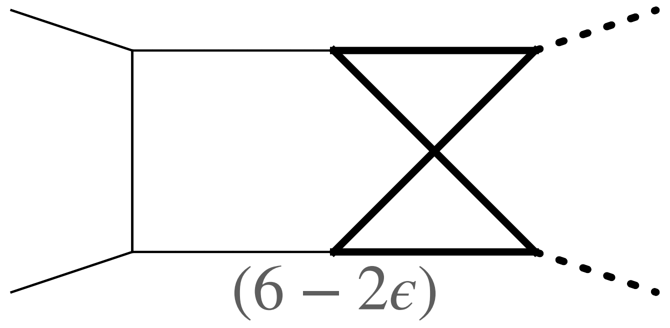
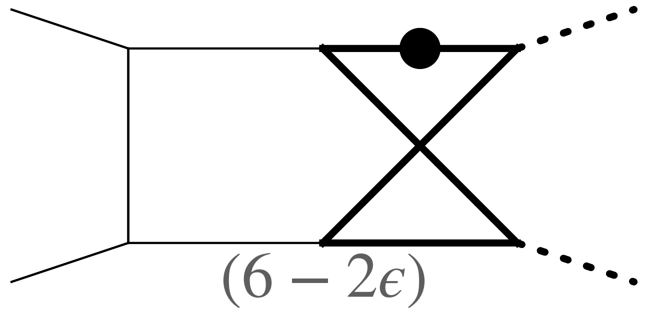
Finite integrals

Integral	Rel. err.	Timing (s)	
	~2*10^-3	45	
	~4*10^-2	63	
	~8*10^-6	55	$\sim \frac{1}{\mathscr{F}}$
	~8*10^-4	60	$\sim \frac{1}{\mathscr{F}^2}$
Linear Combination	~1*10^-4	18	$\sim \frac{1}{\mathscr{F}^3}$

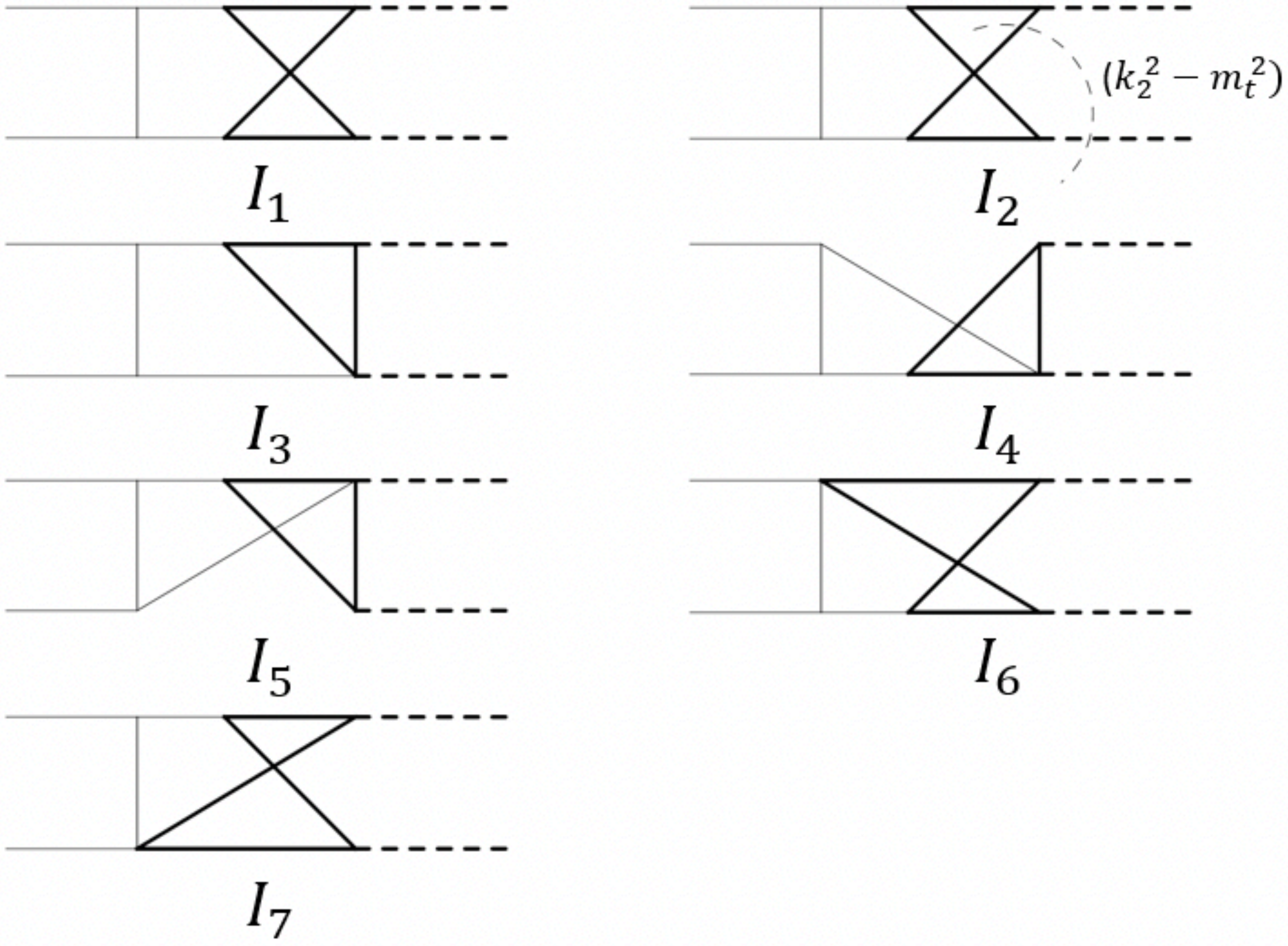


$$I = (m_Z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_Z^2 - t)I_7$$

Finite integrals

Integral	Rel. err.	Timing (s)	
	~2*10^-3	45	
	~4*10^-2	63	
	~8*10^-6	55	$\sim \frac{1}{\mathcal{F}}$
	~8*10^-4	60	$\sim \frac{1}{\mathcal{F}^2}$
Linear Combination	~1*10^-4	18	$\sim \frac{1}{\mathcal{F}^3}$

Naively expected to be much worse



$$I = (m_Z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_Z^2 - t)I_7$$

Multivariate partial fractioning

- All unreduced integrals expressed in terms of the optimised finite basis
- Need to insert these identities into the amplitude to obtain the “reduced” amplitude
- Resulting coefficients are coefficients in kinematics and d

Challenges:

- This is computationally very difficult; IBPs size of over 200 GB
- Intermediate steps require TB of disk space and computationally very expensive
- Numerical performance issues due to presence of spurious poles

Multivariate partial fractioning

- Certain choices lead to spurious poles with denominators depending on both kinematics and d; want to avoid such poles, e.g.

$$\frac{1}{1250 - 500\,d - 9000\,t + 3600\,d\,t + 16200\,t^2 - 6480\,d\,t^2 - 4050\,s + 1575\,d\,s + 19440\,s\,t - 8100\,d\,s\,t - 52488\,s\,t^2 + 20412\,d\,s\,t^2 - 29160\,s^2\,t + 11664\,d\,s^2\,t}$$

In $d \to 4$ this becomes : $\dfrac{1}{-125 + 375\,s + 900\,t - 2160\,s\,t + 2916\,s^2\,t - 1620\,t^2 + 4860\,s\,t^2}$

- Spurious poles lead to numerical instabilities
- Choose d-factoring basis to avoid such denominators [Smirnov, Smirnov (2020)], [Usovitsch(2020)]
- Employ multivariate partial fractioning

Multivariate partial fractioning

- Multivariateapart [Heller, von Manteuffel (2021)]. Also see [Pak (2011)], [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov (2019)], [Boehm, Wittman, Wu, Xu, Zhang (2020)], [Bendle, Böhm, Heymann, Ma, Rahn, Ristau, Wittmann, Wu, Zhang (2021)].
- Use Singular to perform partial fractioning using a Groebner basis to prevent new denominators from appearing. E.g. naive partial fractioning in Mathematica:

$$\frac{1}{25-270\,t+324\,s\,t}\,\frac{1}{-5+18\,t+9\,s} = \frac{-1}{\boxed{(5+18\,t)(-5+36\,t)}(-5+18\,t+9\,s)} + \frac{36\,t}{\boxed{(5+18\,t)(-5+36\,t)}(25-270\,t+324\,s\,t)}$$

New denominators

- Instead use a Groebner basis approach; Find relations between all appearing denominators to reduce them to simpler ones
- Unique decomposition for a chosen ordering of denominator polynomials
- Handle nasty degree 6 denominators:

$$\begin{aligned}&105625-468000\,t-797850\,t^2+3863700\,t^3+2001105\,t^4-5904900\,t^5+2125764\,t^6-3676500\,s+17309700\,s\,t\\&-19260180\,s\,t^2+25850340\,s\,t^3-35901792\,s\,t^4+8503056\,s\,t^5+25891650\,s^2-73614420\,s^2\,t^2-75149694\,s^2\,t^3\\&+12754584\,s^2\,t^4-50490540\,s^3+80752788\,s^3\,t-60466176\,s^3\,t^2+8503056\,s^3\,t^3+29452329\,s^4-18187092\,s^4\,t\\&+2125764\,s^4\,t^2\end{aligned}$$

Multivariate partial fractioning

- Drastic simplification of coefficients after partial fractioning

Intermediate size: O(TB)

Size after partial fractioning : < 1 MB per coefficient

E.g. Complexity reduces significantly for one of the hardest coefficients in the amplitude

$$\text{coefficient} = \frac{num(s,t,d)}{den(s,t,d)}$$

$$\{deg(num,s) + deg(den,s), deg(num,t) + deg(den,t), deg(num,d) + deg(den,d)\} = \{107{,}117{,}38\}$$

After partial fractioning, worst term = {20, 15, 9}

Total number of terms after partial fractioning = 10842

Multivariate partial fractioning

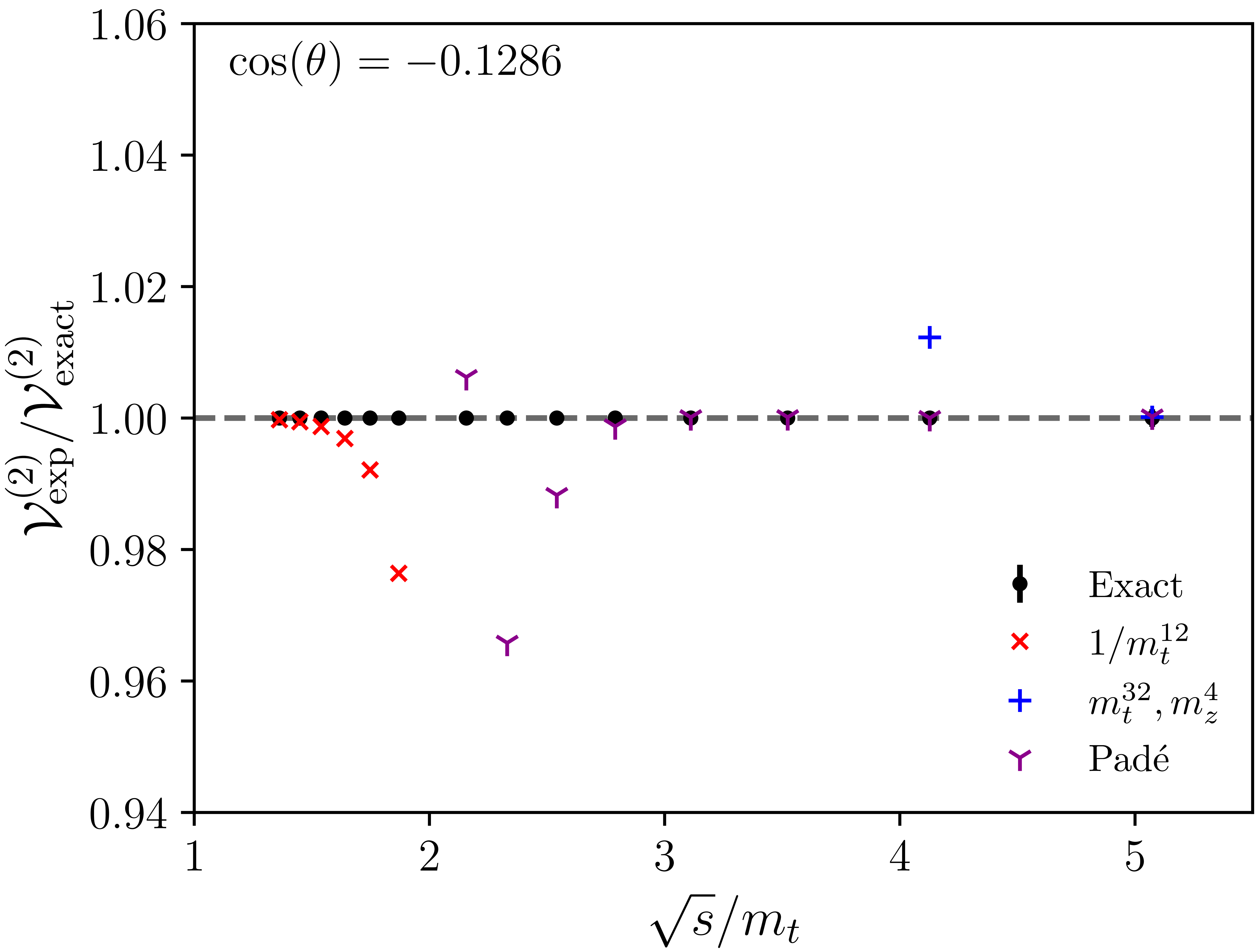
- Partial fraction in d to separate the poles
- Set $d = 4$. Allowed since the basis is finite

Factorised form: $$\frac{1}{(-1+d)(-3+d)^2(-4+d)(-7+2d)} = (\frac{1}{3}+\frac{2\epsilon}{9})(1+2\epsilon)^2(\frac{-1}{2\epsilon})(1+4\epsilon)$$ ~16 terms

Partial fractioned: $$\frac{1}{3(-4+d)} + \frac{5}{4(-3+d)} + \frac{1}{2(-3+d)^2} + \frac{1}{60(-1+d)} + \frac{-16}{5(-7+2d)} = \frac{-1}{6\epsilon} + \frac{-13}{9}$$ 2 terms

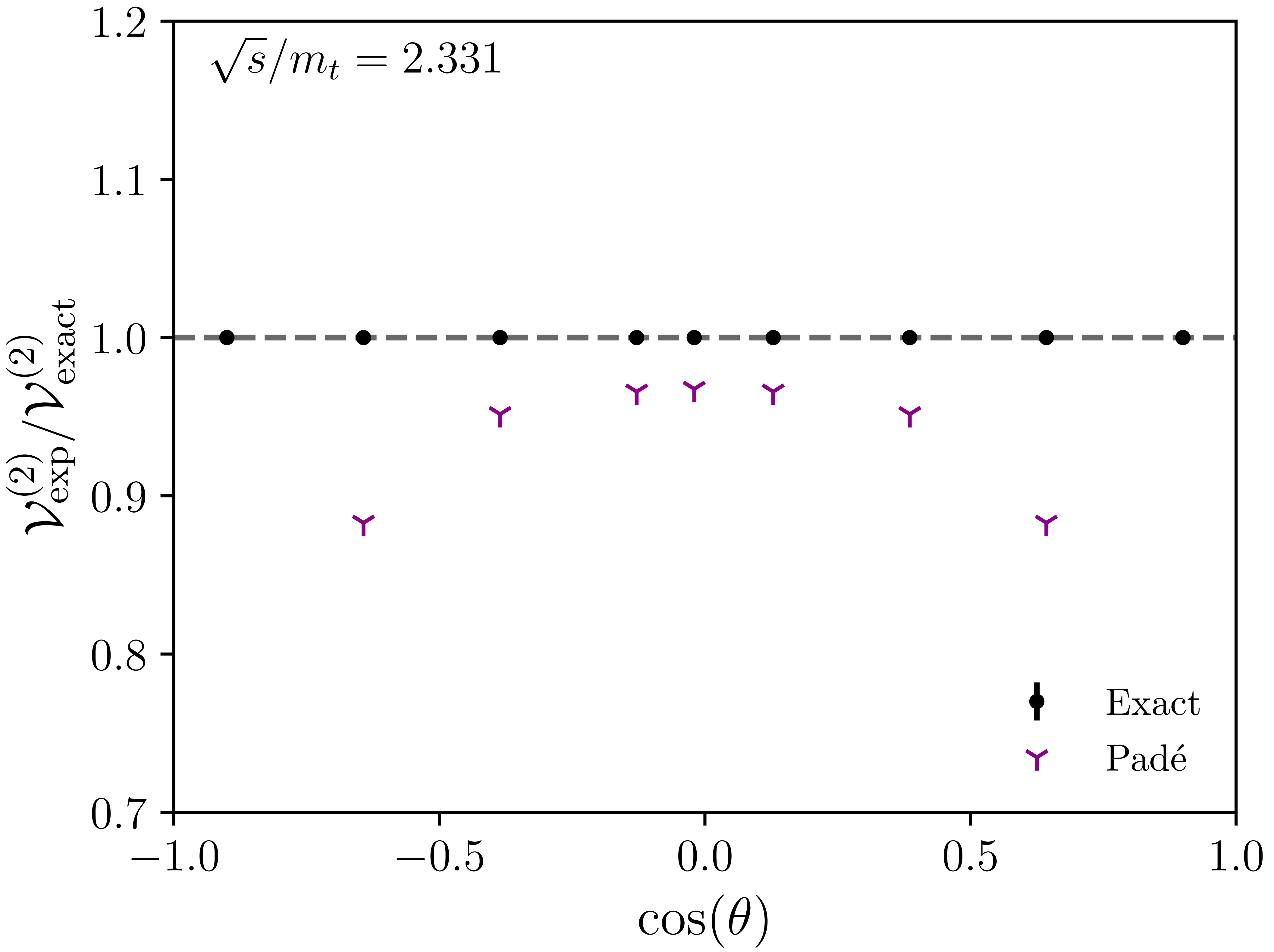
- Prevents proliferation of terms
- Partial fraction in kinematics to arrive at final form
- Resulting coefficients smaller than 1MB in size. Total size of all coefficients $O(100)$ MB
- Very fast numerical evaluation

Results

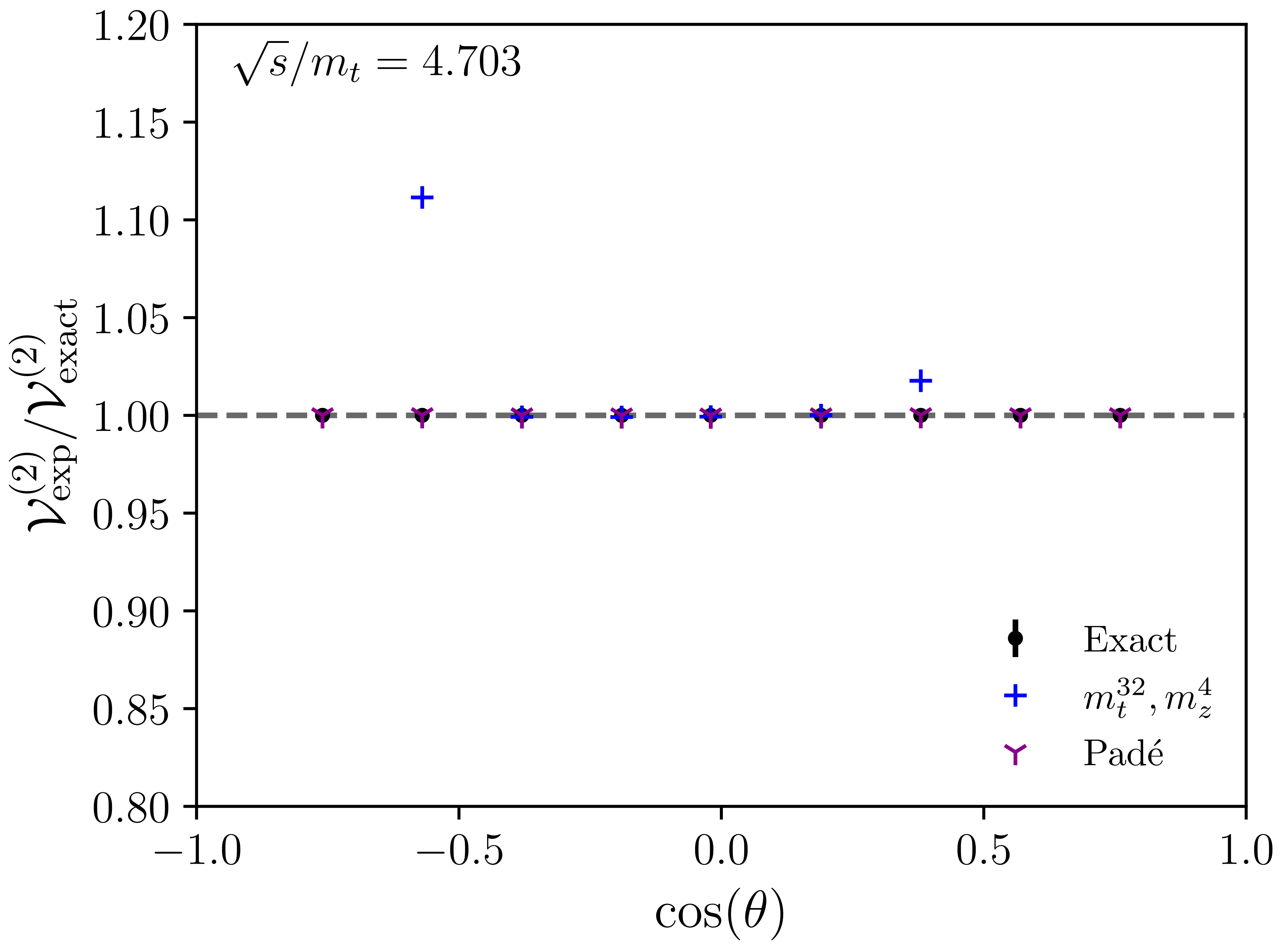


Comparison of $\sqrt{s}$ dependence of the unpolarised interference with expansion results at fixed $\cos\theta = -0.1286$. Exact results from [Agarwal, Jones, von Manteuffel (2020)] . Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)]. Error bars for the exact result are plotted but they are too small to be visible.

Results



Comparison of $\cos\theta$ dependence of the unpolarised interference with expansion results at fixed energy $\sqrt{s} = 403$ GeV. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)].



Comparison of $\cos\theta$ dependence of the unpolarised interference with expansion results at fixed energy $\sqrt{s} = 814$ GeV. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)].

IR scheme dependence

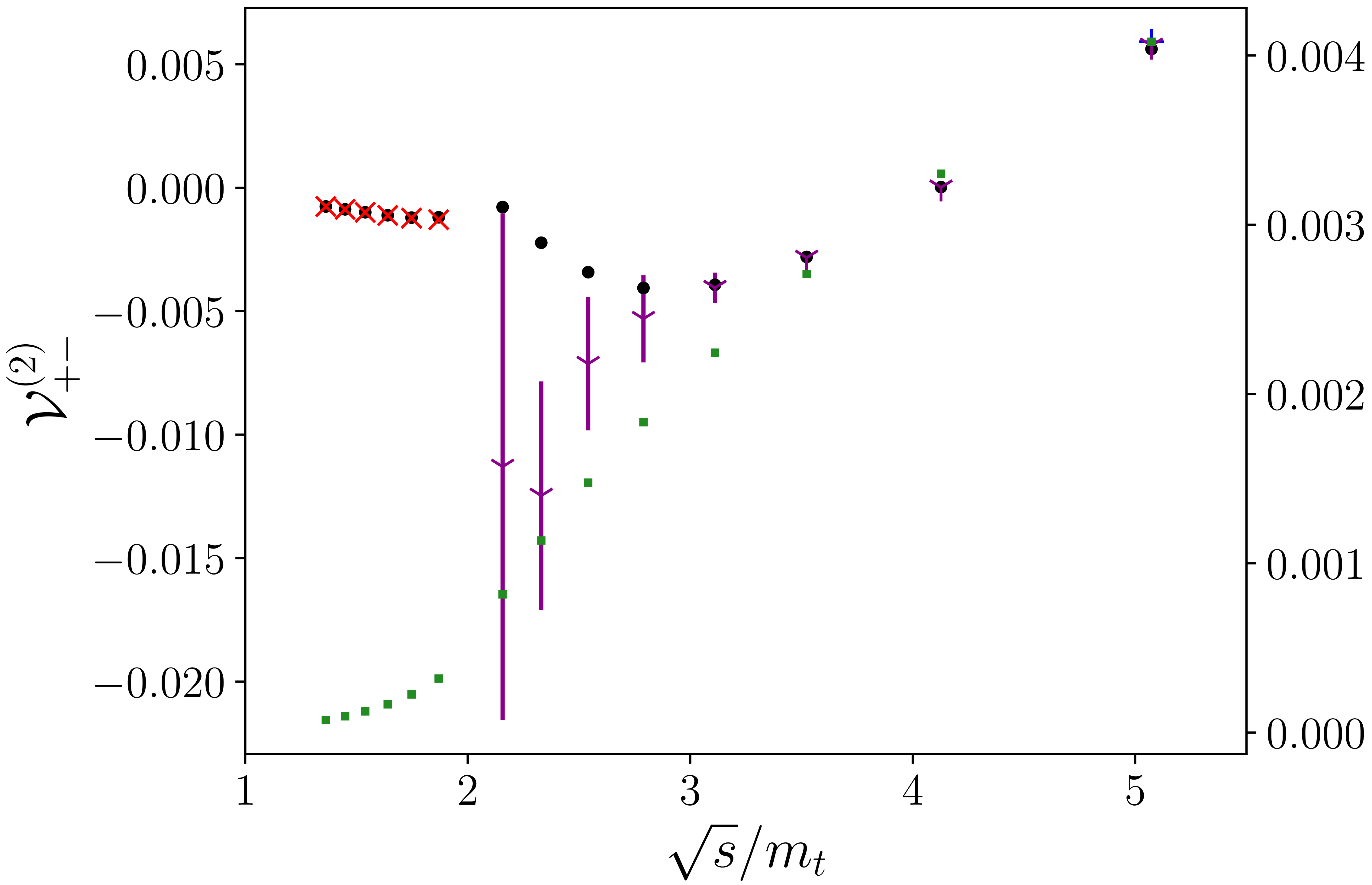
- For previous results, "q_T" subtraction scheme
- Transformation between Catani's original scheme and q_T scheme

$$A_i^{(2),fin,Catani} = A_i^{(2),fin,q_T} + \Delta I_1 A_i^{(1),fin}$$

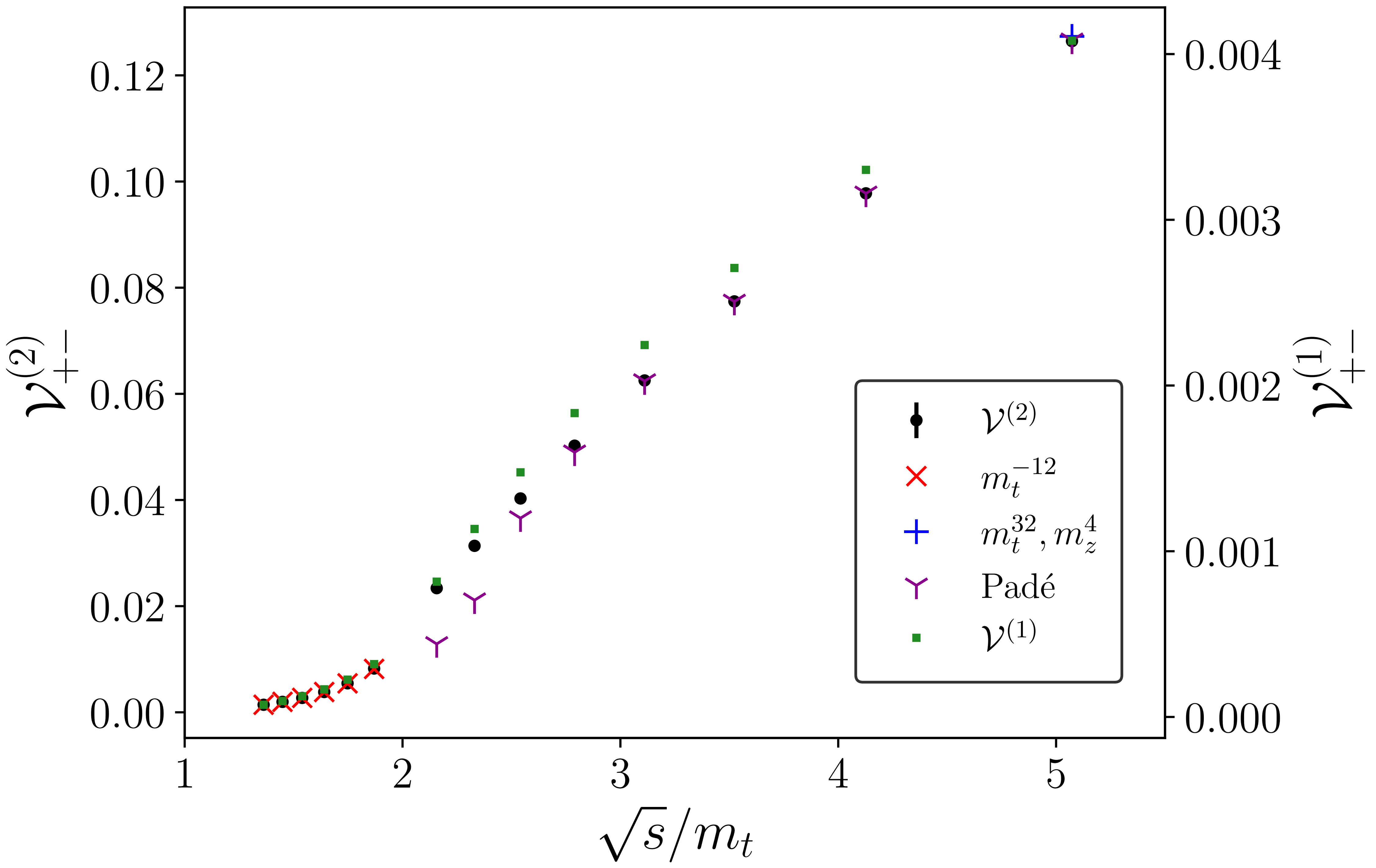
$$\Delta I_1 = -\frac{1}{2}\pi^2 C_A + i\pi\beta_0 \quad \sim 15$$

- For interference terms, 1-loop result multiplied by ~ 30 => Leads to a very different qualitative behaviour
- Relative comparisons highly dependent on IR scheme

IR scheme dependence



Catani's Scheme



q_T scheme

Comparison of $\sqrt{s}$ dependence of the polarised interference with expansion results at fixed $\cos\theta = -0.1286$. Exact results from [Agarwal, Jones, von Manteuffel (2020)]. Expansion and Padé results from [Davies, Mishima, Steinhauser, Wellmann (2020)].

Conclusions

- Results for two-loop corrections for $gg \to ZZ$ with full top mass dependence
- Use of syzygies and finite field methods for IBP reduction including presenting our **new algorithm for constructing syzygies**
- Method of finite integrals with **new general approach to construct finite integrals**
- Multivariate partial fractioning to drastically simplify amplitude coefficients
- IR scheme dependence of qualitative comparisons between the exact calculation and expansion results