THREE-LOOP AMPLITUDES IN MASSLESS QCD

Radcor-Loopfest 2021
21/05/2021

based on work with *F. Caola, A. von Manteuffel, A. Chakraborty, G. Gambuti, P. Bargiela, T. Peraro*
[arXiv:2011.13946, and arXiv:1906.03298 + arXiv:2012.00820 *and more to come*]

Lorenzo Tancredi - University of Oxford

RADIATIVE CORRECTIONS

Developments in fixed-order calculations at the center of Radcor-Loopfest

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$

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In the last 10-20 years, much effort dedicated to understand **two-loop scattering amplitudes** in QCD with the goal of **breaking the NNLO frontier for 2 → 2 processes**

In parallel, first impressive results for N^3LO **for 2 → 1** (Higgs and Drell-Yan)

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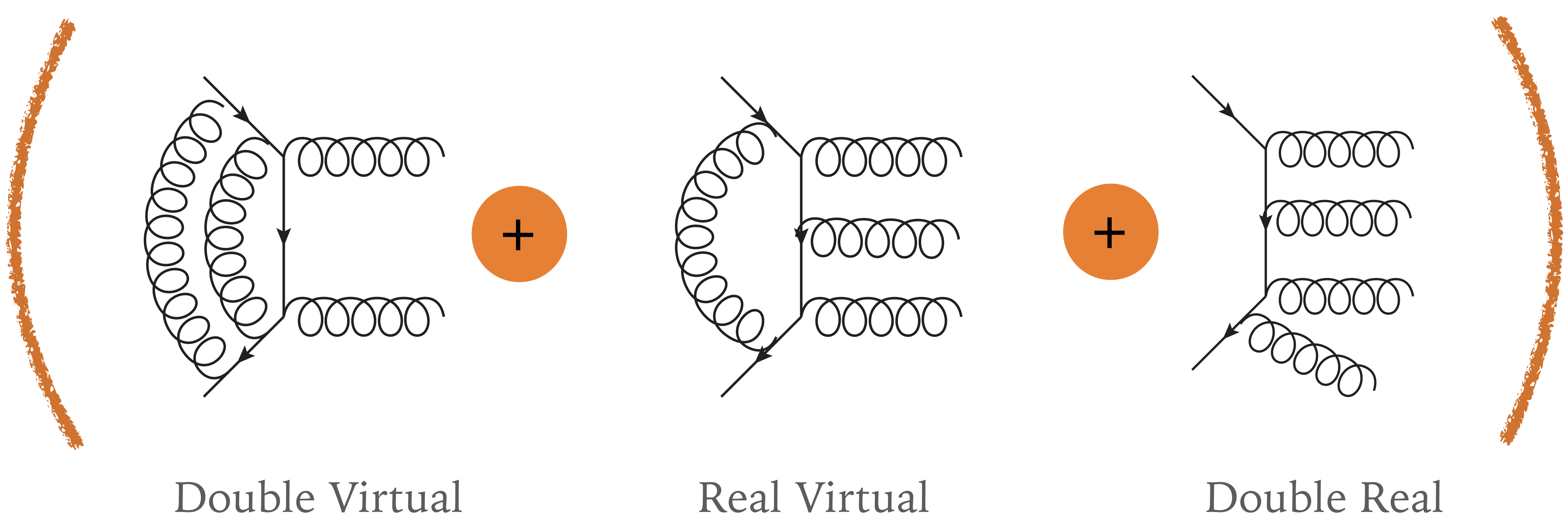
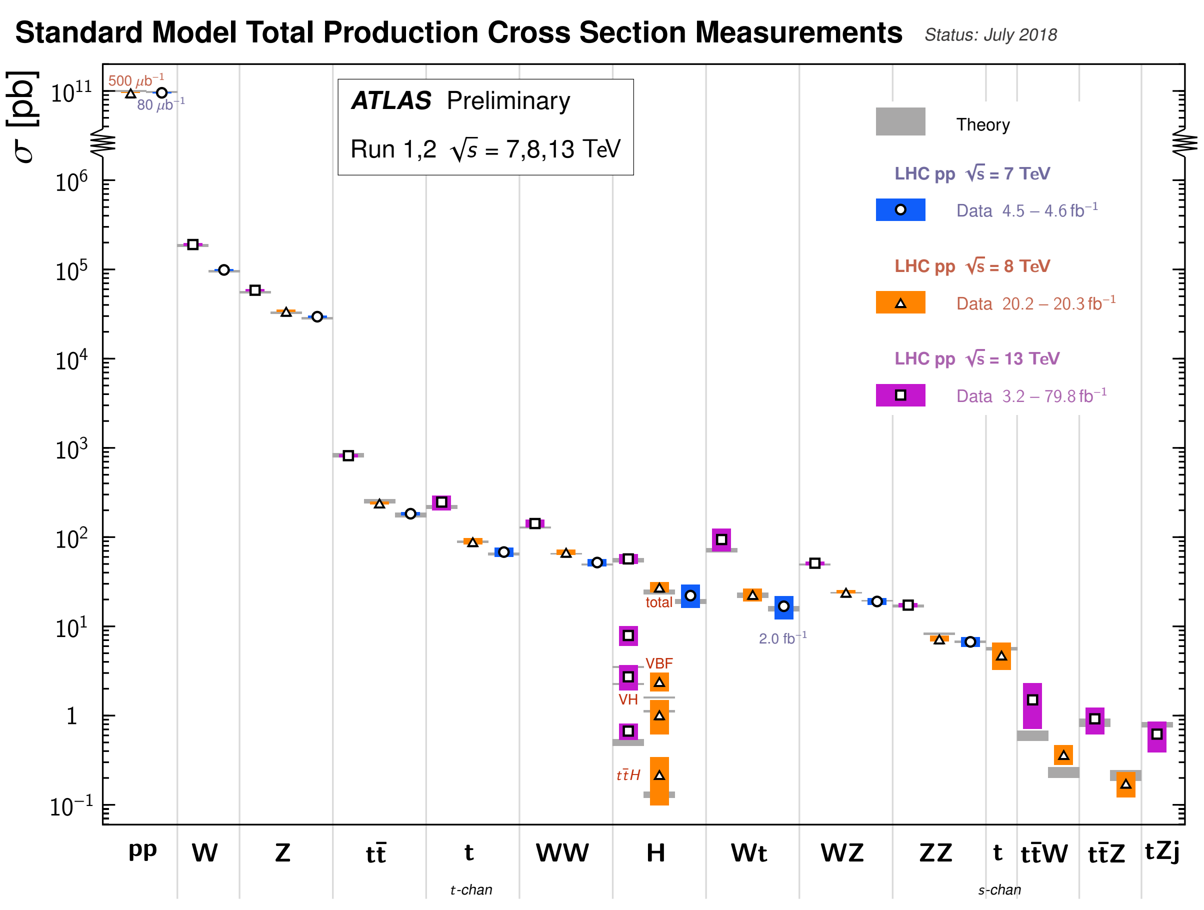
Complex **2 loop 4-point graphs** + **IR subtraction**



WHAT HAVE WE LEARNED?

phenomenology and SM physics



Rediscovered the SM, **discovered the Higgs**, and started doing **precision physics**

Vector bosons, top quarks, Higgs couplings, jets, heavy flavours…

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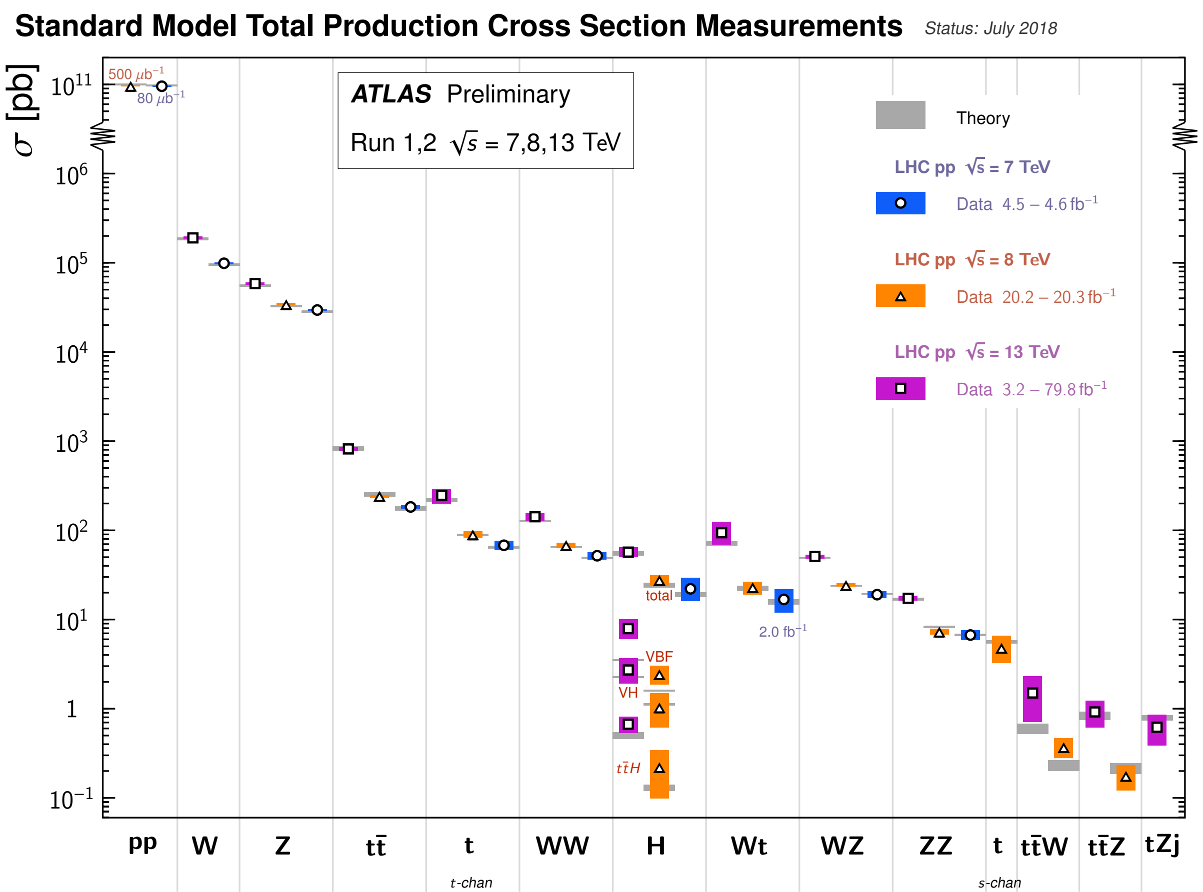

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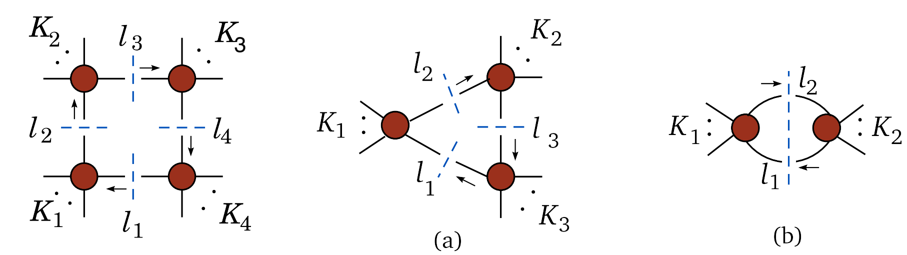
formal developments

structure of scattering amplitudes:

- unitarity, recursion relations, spinor helicity, color ordering, IBPs, DEs,…



Special functions in pQFT:

- connections with algebraic geometry and number theory, polylogs, elliptic stuff, CYs, iterated integrals…

$$G(c_1, c_2, \dots, c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, \dots, c_n, t_1)$$

$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \dots \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

IR divergences, factorisation in QCD, resummation, effective field theory…

BEYOND NNLO FOR 2->2 THERE IS STILL A LOT TO LEARN

Just scratching the surface…!

Progress towards NNLO QCD corrections to $2 \to 3$ processes dominated this conference

See talks by: *B. Page, H. Chawdhry, F. Buccioni, V Sotnikov, N. Syrrakos, C. Papadopoulos, R. Poncelet, H. Hartanto,…*

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IR singularities and new sources for possible **factorisation breaking** (di-jet / $t\bar{t}$ @ N^3LO…)

New challenges from pushing methods to compute **scattering amplitudes** from two to **three loops**:

Higher combinatorial complexity, *new special functions and new geometries*, discontinuities (bootstrap?)…

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SCATTERING AMPLITUDES AT 3 LOOPS

Some results for 3 loop amplitudes in **SUSY** known (N=4, N=8 SUGRA, etc..)

[Henn, Mistlberger '19,'20]

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Simplest, **non-trivial** place to start investigations of three loop amplitudes in QCD

$q\bar{q} \to \gamma\gamma$ non trivial for various reasons:

- Relatively large number of Feynman diagrams ($\sim$3000)
- Very non trivial IBP reduction needed *(rank-6 10 propagator NPL integrals)*

But still relatively simple

- *Simple functions:* 4 point massless @ 3 loops can be expressed in terms of **HPLs**
  [Henn, Mistlberger, Smirnov, Wasser '19]
- simpler color correlations and simpler IR structure than, say, $gg \to gg$

DI-PHOTON AS OF TODAY

The **production of two photons** has received a lot of attention [Anastasiou et al ’00; Bern et al ’00,’01,’03; Glover et al. ’00,’01,’03, …]

- One- and Two-loop scattering amplitudes known for 20 years

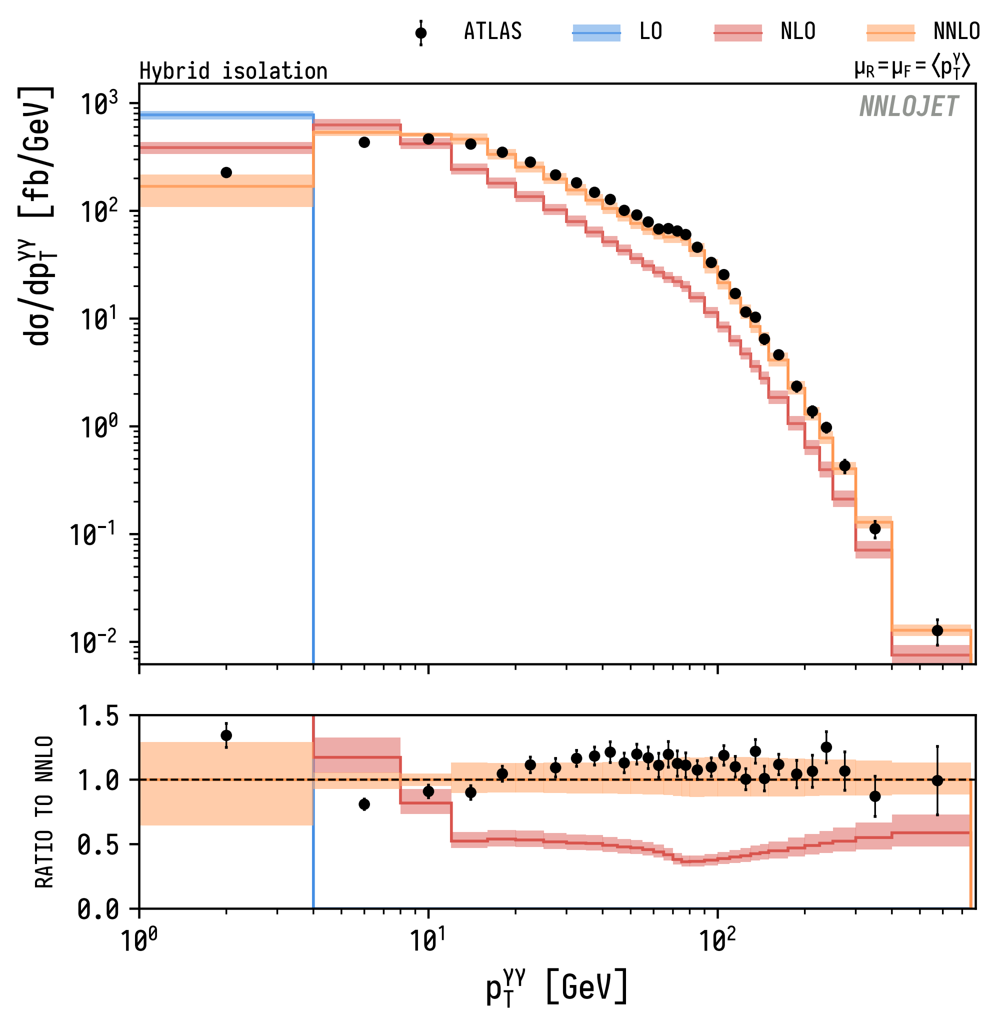
- NNLO *inclusive* and *recently exclusive* over final state radiation [Catani, et al ’11, ’13, Campbel et al ’16] [Chawdhry et al ’21]

- Various degrees of sophistication (resummation, PS, etc) [Alioli, et al ’10 …] [Gehrmann et al ’20]

Important background for **Higgs** + **New Physics**

<u>Clean final state</u>, <u>high production rate</u>, etc

Interesting theory/exp questions: (IR sensitivity cone isolation…) [Gehrmann et al ’20]



TOWARDS DIPHOTON AT 3 LOOPS (AND N3LO)

Consider the production of 2 photons in quark-antiquark annihilation

$$q(p_1) + \bar{q}(p_2) \to \gamma(p_3) + \gamma(p_4)\,, \quad \text{with} \quad p_i^2 = 0$$

$$s = (p_1 + p_2)^2\,, \quad t = (p_1 - p_3)^2\,, \quad \text{and} \quad x = -\,t/s \quad \longrightarrow \quad s > 0\,, \quad t < 0 \quad 0 < x < 1$$

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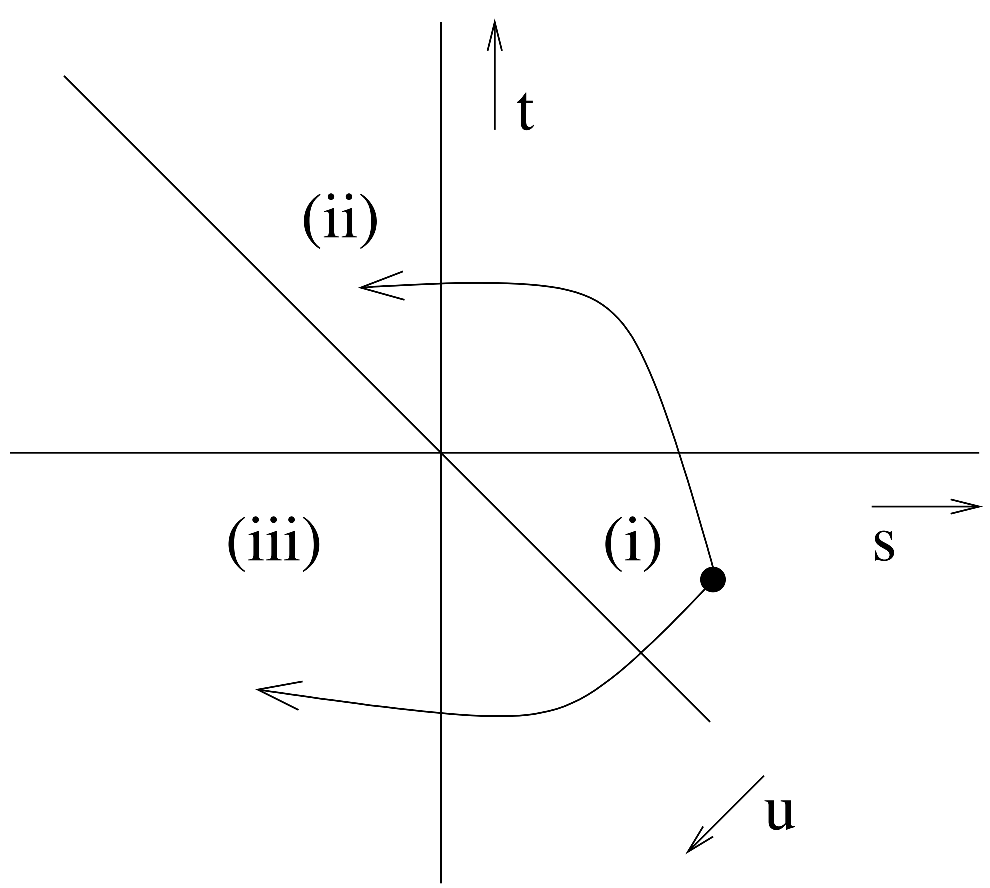

Interesting analytic structure,
no Euclidean region

[Smirnov ’99; Smirnov, Veretin ’00; Tausk ’00]

[Anastasiou, Gehrmann, Oleari, Remiddi, Tausk ’00]

THE HELICITY AMPLITUDES IN 'THV

To compute helicity amplitudes, start from generic tensor decomposition in d-dim

$$\mathscr{A}(s,t) = \sum_{i=1}^{5} \mathscr{F}_i(s,t) T_i$$

$$T_i = \bar{u}(p_2)\, \Gamma_i^{\mu\nu}\, u(p_1) \epsilon_{3,\mu} \epsilon_{4,\nu}$$

$$\Gamma_1^{\mu\nu} = \gamma^\mu p_2^\nu\,, \quad \Gamma_2^{\mu\nu} = \gamma^\nu p_1^\mu\,,$$
$$\Gamma_3^{\mu\nu} = \not{p}_3\, p_1^\mu p_2^\nu\,, \quad \Gamma_4^{\mu\nu} = \not{p}_3\, g^{\mu\nu}$$
$$\Gamma_5^{\mu\nu} = \gamma^\mu \not{p}_3\, \gamma^\nu\,.$$

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Helicity amplitudes in tHV can be computed by fixing helicities on the tensors in d=4

$$\mathscr{A}_{\lambda_q \lambda_3 \lambda_4}(s,t) = \sum_{i=1}^{5} \mathscr{F}_i(s,t) \left[T_i\right]_{\lambda_q \lambda_3 \lambda_4, d=4}$$

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One of five tensors is not independent in d=4

$$\lim_{d\to 4}\left(T_5 - \frac{u}{s}T_1 + \frac{u}{s}T_2 - \frac{2}{s}T_3 + T_4\right) = 0$$

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We can identify one **evanescent tensor structure**

$$\overline{T}_i = T_i \,, \quad i = 1, \ldots, 4 \,,$$

$$\overline{T}_5 = T_5 - \frac{u}{s} T_1 + \frac{u}{s} T_2 - \frac{2}{s} T_3 + T_4$$

Various approaches to exploit this, see also [Chen ’19] [Davies et al ’20]

useful also in other context, see for example [Abreu et al ’18]

PROJECTORS IN 'T HOOFT-VELTMAN

In new basis of tensors, **by definition** only first four contribute to hel amplitudes

$$\mathscr{A}_{\lambda_q\lambda_3\lambda_4}(s,t) = \sum_{i=1}^{5} \overline{\mathscr{F}}_i(s,t)\left[\overline{T}_i\right]_{\lambda_q\lambda_3\lambda_4,d=4} = \sum_{i=1}^{4} \overline{\mathscr{F}}_i(s,t)\left[\overline{T}_i\right]_{\lambda_q\lambda_3\lambda_4,d=4} + \mathscr{O}(\epsilon)$$

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Derive **projector operators** for these tensors

$$\mathcal{P}_i = \sum_{k=1}^{5} c_{ik}\overline{T}_k^\dagger \qquad \sum_{pol} \mathcal{P}_i \mathcal{A}(s,t) = \overline{\mathcal{F}}_i(s,t) \qquad M_{ij} = \sum_{pol} \overline{T}_i^\dagger \overline{T}_j \qquad c_{ik} = \left(M^{-1}\right)_{ik}$$

$$M^{-1} = \frac{1}{(d-3)(s+u)}\begin{pmatrix} X & 0 \\ 0 & -\frac{1}{2u(d-4)}\end{pmatrix} \qquad X = \begin{pmatrix} -\frac{u}{2s^2} & 0 & -\frac{u}{2s^2(s+u)} & 0 \\ 0 & -\frac{u}{2s^2} & \frac{u}{2s^2(s+u)} & 0 \\ -\frac{u}{2s^2(s+u)} & \frac{u}{2s^2(s+u)} & -\frac{du^2+4s^2+4su}{2s^2u(s+u)^2} & \frac{2s+u}{2su(s+u)} \\ 0 & 0 & \frac{2s+u}{2su(s+u)} & -\frac{1}{2u} \end{pmatrix}$$

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evanescent tensor structure

$$X = \begin{pmatrix} -\frac{u}{2s^2} & 0 & -\frac{u}{2s^2(s+u)} & 0 \\ 0 & -\frac{u}{2s^2} & \frac{u}{2s^2(s+u)} & 0 \\ -\frac{u}{2s^2(s+u)} & \frac{u}{2s^2(s+u)} & -\frac{du^2+4s^2+4su}{2s^2u(s+u)^2} & \frac{2s+u}{2su(s+u)} \\ 0 & 0 & \frac{2s+u}{2su(s+u)} & -\frac{1}{2u} \end{pmatrix}$$

'THV VS CDR

With this choice, fifth tensor required to recover **CDR result** starting at $\mathcal{O}(\epsilon)$

$$\frac{1}{N_c}\sum_{pol,col}\mathcal{A}^{(n)}\mathcal{A}^{(m)*} = \frac{2(s-t)t}{u}\overline{\mathcal{F}}_4^{(n)}\left[-s\overline{\mathcal{F}}_1^{(m)*} + s\overline{\mathcal{F}}_2^{(m)*} - st\overline{\mathcal{F}}_3^{(m)*} - \frac{2\overline{\mathcal{F}}_4^{(m)*}\left(-s^2-t^2+u^2\epsilon\right)}{(s-t)}\right]$$
$$+\frac{2st}{u}\overline{\mathcal{F}}_1^{(n)}\left[-2s(\epsilon-1)\overline{\mathcal{F}}_1^{(m)*} - s\overline{\mathcal{F}}_2^{(m)*} + st\overline{\mathcal{F}}_3^{(m)*} - \overline{\mathcal{F}}_4^{(m)*}(s-t)\right]$$
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$$+\,4tu\,\epsilon\,(2\epsilon-1)\overline{\mathcal{F}}_5^{(m)*}\overline{\mathcal{F}}_5^{(n)}\,,$$

↓

$\mathscr{F}_5$ only contributes starting at $\mathcal{O}(\epsilon)$

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with this approach we are guaranteed to compute only **tensor structures** that are **relevant** for helicity amplitudes in tHV

becomes **crucial for** $2 \to 3$ **processes**
see F. Buccioni's talk

[Peraro, Tancredi ’19,’20]

THE HELICITY AMPLITUDES

Fixing the helicities on the remaining tensors, we find in spinor-helicity

$$\mathcal{A}_{L--} = \frac{2[34]^2}{\langle 13\rangle[23]}\alpha(x)\,, \quad \mathcal{A}_{L-+} = \frac{2\langle 24\rangle[13]}{\langle 23\rangle[24]}\beta(x)\,,$$

$$\mathcal{A}_{L+-} = \frac{2\langle 23\rangle[41]}{\langle 24\rangle[32]}\gamma(x)\,, \quad \mathcal{A}_{L++} = \frac{2\langle 34\rangle^2}{\langle 31\rangle[23]}\delta(x)\,.$$

$$\alpha(x) = \frac{t}{2}\left(\overline{\mathcal{F}}_2 - \frac{t}{2}\overline{\mathcal{F}}_3 + \overline{\mathcal{F}}_4\right),$$

$$\beta(x) = \frac{t}{2}\left(\frac{s}{2}\overline{\mathcal{F}}_3 + \overline{\mathcal{F}}_4\right),$$

$$\gamma(x) = \frac{s\,t}{2u}\left(\overline{\mathcal{F}}_2 - \overline{\mathcal{F}}_1 - \frac{t}{2}\overline{\mathcal{F}}_3 - \frac{t}{s}\overline{\mathcal{F}}_4\right)$$

$$\delta(x) = \frac{t}{2}\left(\overline{\mathcal{F}}_1 + \frac{t}{2}\overline{\mathcal{F}}_3 - \overline{\mathcal{F}}_4\right).$$

$$\alpha^{(0)}(x) = \delta^{(0)}(x) = 0$$

$$\beta^{(0)}(x) = \gamma^{(0)}(x) = 1$$

$$\gamma(x) = \beta(1-x),\ \delta(x) = -\alpha(x),\ \alpha(1-x) = -\alpha(x)$$

3LOOP INFRARED POLES AFTER UV RENORMALISATION

UV-ren **helicity amplitudes** and **form factors** can be computed as series in α_s

$$\overline{\mathcal{F}}_i = \delta_{kl}(4\pi\alpha)\, e_q^2 \sum_{k=0}^{3} \left(\frac{\alpha_s(\mu)}{2\pi}\right)^k \overline{\mathcal{F}}_i^{(k)}$$

IR poles follow general **factorisation formula** [Catani '99; Becher, Neubert '13,...]

$$\begin{aligned}
\overline{\mathcal{F}}_i^{(1)} &= \mathcal{I}_1 \overline{\mathcal{F}}_i^{(0)} + \overline{\mathcal{F}}_i^{(1,\text{fin})}, \\
\overline{\mathcal{F}}_i^{(2)} &= \mathcal{I}_2 \overline{\mathcal{F}}_i^{(0)} + \mathcal{I}_1 \overline{\mathcal{F}}_i^{(1)} + \overline{\mathcal{F}}_i^{(2,\text{fin})}, \\
\overline{\mathcal{F}}_i^{(3)} &= \mathcal{I}_3 \overline{\mathcal{F}}_i^{(0)} + \mathcal{I}_2 \overline{\mathcal{F}}_i^{(1)} + \mathcal{I}_1 \overline{\mathcal{F}}_i^{(2)} + \overline{\mathcal{F}}_i^{(3,\text{fin})}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_1 &= \frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}, \\
\mathcal{I}_2 &= -\frac{\mathcal{I}_1^2}{2} - \frac{\beta_0}{2\epsilon}\left(\mathcal{I}_1 + \frac{\Gamma_0'}{8\epsilon^2}\right) + \frac{\Gamma_1'}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}, \\
\mathcal{I}_3 &= -\frac{\mathcal{I}_1^3}{3} - \mathcal{I}_1\mathcal{I}_2 + \frac{\beta_0^2\Gamma_0'}{36\epsilon^4} - \frac{\beta_0}{3\epsilon}\left(\mathcal{I}_1^2 + 2\mathcal{I}_2 + \frac{\Gamma_1'}{12\epsilon^2}\right) - \frac{\beta_1}{3\epsilon}\left(\mathcal{I}_1 + \frac{\Gamma_0'}{12\epsilon^2}\right) + \frac{\Gamma_2'}{36\epsilon^2} + \frac{\Gamma_2}{6\epsilon},
\end{aligned}$$

A LOOK AT THE RESULTS

Reduction to Master Integrals very non-trivial ***(10 denominators, rank 6)***

- performed with Finred, private implementation by A. von Manteuffel

Helicity amplitudes in "d dimensions" (tHV) expressed in terms of **486 masters integrals**

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MIs are "simple", can be computed in terms *HPLs with indices {0,1}*

- *BUT* large number, *non trivial boundary conditions and canonical basis*

[Henn, Mistlberger, Smirnov, Wasser '20]

Interesting observation: when expanding in d~4, finite remainder expressed in terms of weight 6 HPLs —> there are at most **146 such functions**

Impressive generalisation of **n point -> boxes,triangles,bubbles and tadpoles** @ 1 loop!

A LOOK AT THE RESULTS

Three-loop corrections helicity amplitudes can be written, schematically, as

$$\mathscr{A}_{\lambda_q\lambda_3\lambda_4} = N_f^2 C_F\, A_1(x) + N_f\left[C_F C_A A_2(x) + C_F^2 A_3(x)\right] + N_{\gamma\gamma}\left[C_A C_F A_4(x) + C_F^2 A_5(x) + C_F N_f A_6(x) + \frac{d_{abc}d_{abc}}{N_c} A_7(x)\right]$$
$$+ C_A^2 C_F A_8(x) + C_A C_F^2 A_9(x) + C_F^3 A_{10}(x)$$

Where each of the $A_i(x)$ is has the expansion

$$A_i(x) = A_i^{[0]}(x) + \frac{1}{x} B_i^{[-1]}(x) + x\, B_i^{[1]}(x) + x^2\, B_i^{[2]}(x) + \frac{1}{1-x} C_i^{[-1]}(x) + \frac{1}{(1-x)^2} C_i^{[-2]}(x)$$

A LOOK AT THE RESULTS

Three-loop corrections helicity amplitudes can be written, schematically, as

$$\mathscr{A}_{\lambda_q\lambda_3\lambda_4} = N_f^2 C_F\, A_1(x) + N_f\left[C_F C_A A_2(x) + C_F^2 A_3(x)\right] + N_{\gamma\gamma}\left[C_A C_F A_4(x) + C_F^2 A_5(x) + C_F N_f A_6(x) + \frac{d_{abc}d_{abc}}{N_c} A_7(x)\right]$$
$$+ C_A^2 C_F A_8(x) + C_A C_F^2 A_9(x) + C_F^3 A_{10}(x)$$

Where each of the $A_i(x)$ is has the expansion

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14 classical polylogs

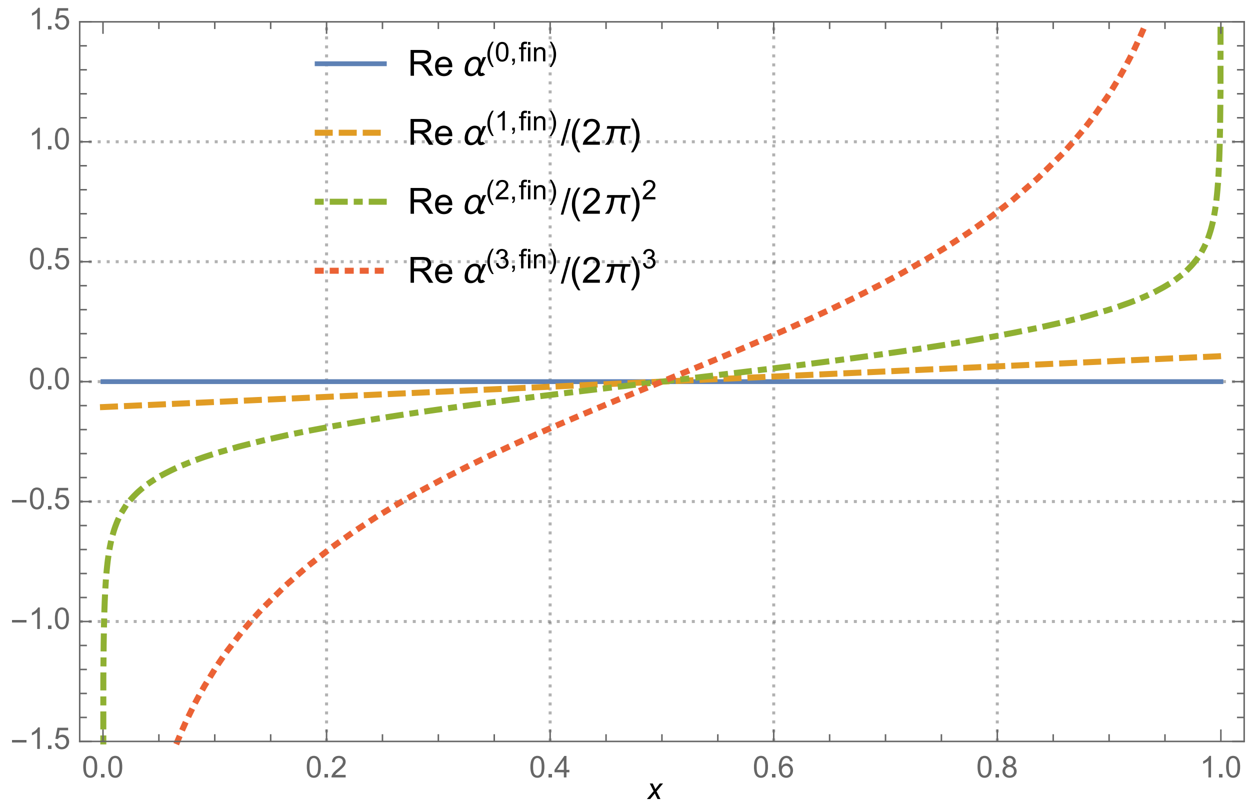
9 extra MPLs

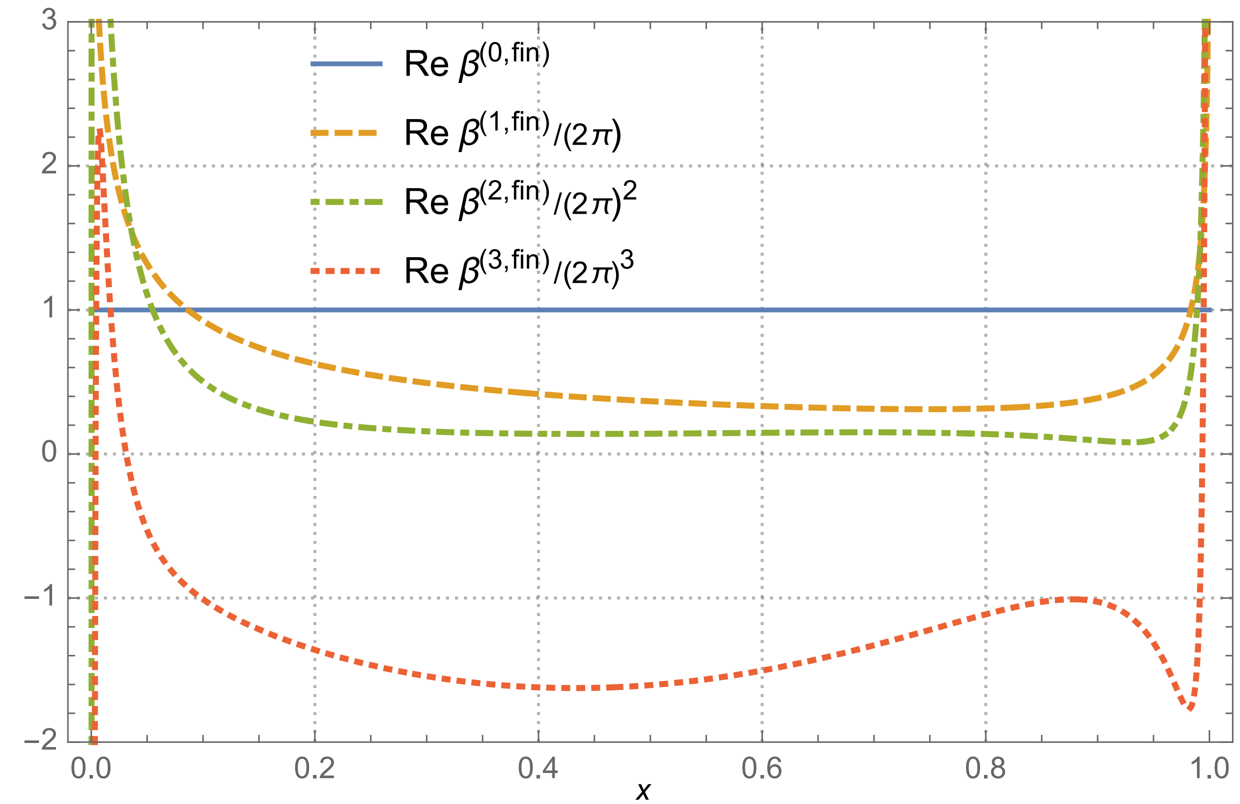
$A_i^{[0]}(x)\,,\ B_i^{[n]}(x)\,,\ C_i^{[n]}(x)$ → $\mathrm{Li_n(r(x))}$

$$\mathrm{Li}_{3,2}(x,1),\ \mathrm{Li}_{3,2}(1-x,1),\ \mathrm{Li}_{3,2}(1,x),$$
$$\mathrm{Li}_{3,3}(x,1),\ \mathrm{Li}_{3,3}(1-x,1),\ \mathrm{Li}_{3,3}(x/(x-1),1),$$
$$\mathrm{Li}_{4,2}(x,1),\ \mathrm{Li}_{4,2}(1-x,1),\ \mathrm{Li}_{2,2,2}(x,1,1),$$

NUMERICAL RESULTS

In particular written in this second form, numerical evaluation is **instantaneous**





CONCLUSIONS

- In the last 20 years, impressive results for 2 and 3 loop amplitudes in QCD
- In particular, understanding $2 \to 2$ **@ 2 loops crucial for LHC**
- With control of **IR divergences** $\to$ **NNLO in QCD possible**
- Interest has moved towards $2 \to 3$ @ 2 loops, impressive results both for **amplitudes and pheno**

- $2 \to 2$ @ **3 loops equally interesting**, could help elucidate new properties of scattering amplitudes
- Calls for more for more profound understanding of pQFT to control **intermediate expressions** and avoid **extra complexity** <u>purely d dimensional</u>

THANK YOU FOR YOUR ATTENTION,

AND THANKS TO THE ORGANISERS FOR THE GREAT CONFERENCE!

BACK UP

3LOOP INFRARED POLES AFTER UV RENORMALISATION

All **anomalous dimensions** at the relevant order are known

$$\gamma_{c,0} = 2\,,$$
$$\gamma_{c,1} = \left(\frac{67}{9} - \frac{\pi^2}{3}\right) C_A - \frac{20 n_f T_R}{9}\,,$$
$$\gamma_{c,2} = C_A^2 \left(\frac{11\zeta_3}{3} + \frac{245}{12} - \frac{67\pi^2}{27} + \frac{11\pi^4}{90}\right) + C_A n_f T_R \left(-\frac{28\zeta_3}{3} - \frac{209}{27} + \frac{20\pi^2}{27}\right)$$
$$+ C_F n_f T_R \left(8\zeta_3 - \frac{55}{6}\right) - \frac{8 n_f^2 T_R^2}{27}\,;$$

$$\gamma_{q,0} = -\frac{3C_F}{2}\,,$$
$$\gamma_{q,1} = C_A C_F \left(\frac{13\zeta_3}{2} - \frac{961}{216} - \frac{11\pi^2}{24}\right) + C_F^2 \left(-6\zeta_3 - \frac{3}{8} + \frac{\pi^2}{2}\right) + \left(\frac{65}{54} + \frac{\pi^2}{6}\right) C_F n_f T_R\,,$$
$$\gamma_{q,2} = C_F^3 \left(-\frac{17\zeta_3}{2} + \frac{2\pi^2\zeta_3}{3} + 30\zeta_5 - \frac{29}{16} - \frac{3\pi^2}{8} - \frac{\pi^4}{5}\right)$$
$$+ C_A C_F^2 \left(-\frac{211\zeta_3}{6} - \frac{\pi^2\zeta_3}{3} - 15\zeta_5 - \frac{151}{32} + \frac{205\pi^2}{72} + \frac{247\pi^4}{1080}\right)$$
$$+ C_A^2 C_F \left(\frac{1763\zeta_3}{36} - \frac{11\pi^2\zeta_3}{18} - 17\zeta_5 - \frac{139345}{23328} - \frac{7163\pi^2}{3888} - \frac{83\pi^4}{720}\right)$$
$$+ C_F^2 n_f T_R \left(\frac{64\zeta_3}{9} + \frac{2953}{216} - \frac{13\pi^2}{36} - \frac{7\pi^4}{54}\right) + C_F n_f^2 T_R^2 \left(-\frac{4\zeta_3}{27} + \frac{2417}{1458} - \frac{5\pi^2}{27}\right)$$
$$+ C_A C_F n_f T_R \left(-\frac{241\zeta_3}{27} - \frac{8659}{2916} + \frac{1297\pi^2}{972} + \frac{11\pi^4}{180}\right)\,.$$

MORE 2->2 @ 3 LOOPS

Similar approach works for all $2 \to 2$ massless scattering amplitudes in 3 loop QCD

Particularly interesting $q\bar{q} \to Q\bar{Q}$, where standard *projector/form factor approach* becomes very cumbersome since **d-dimensional γ-algebra does not close**

$$
\begin{aligned}
\mathcal{P}(A_2) = {} & \frac{1}{32 s_{13}^2 s_{23}^2 s_{12}^2 (d-5)(d-7)(d-3)(d-4)} \times \Big(\\
& - s_{13}(35 s_{23}^2 d^3 - 55 s_{13} s_{23} d^3 + 1046 s_{13} s_{23} d^2 - 1872 s_{13}^2 d + 2432 s_{13}^2 - 454 s_{23}^2 d^2 \\
& \quad -6040 s_{13} s_{23} d - 2688 s_{23}^2 + 368 s_{13}^2 d^2 + 1928 s_{23}^2 d - 20 s_{13}^2 d^3 + 11136 s_{13} s_{23}) \mathcal{D}_1^\dagger \\
& + 2 s_{13}(-2 s_{13}^2 d^2 - 9 s_{13} s_{23} d^2 + 142 s_{13} s_{23} d - 448 s_{13} s_{23} + 7 s_{23}^2 d^2 + 136 s_{23}^2 - 48 s_{13}^2 \\
& \quad +28 s_{13}^2 d - 62 s_{23}^2 d) \mathcal{D}_3^\dagger \\
& + (-340 s_{13}^2 d^3 + 11008 s_{13}^2 - 740 s_{13} s_{23} d^3 + 44032 s_{13} s_{23} - 260 s_{23}^2 d^3 - 4144 s_{23}^2 d + 3712 s_{23}^2 \\
& \quad +15 s_{13}^2 d^4 + 2852 s_{13}^2 d^2 - 28864 s_{13} s_{23} d + 1604 s_{23}^2 d^2 + 6944 s_{13} s_{23} d^2 - 9968 s_{13}^2 d \\
& \quad +30 s_{13} s_{23} d^4 + 15 s_{23}^2 d^4) \mathcal{D}_2^\dagger \\
& - s_{13} s_{23}(12 s_{13} + s_{23} d - 4 s_{23} - s_{13} d) \mathcal{D}_5^\dagger \\
& + (-6 s_{23}^2 d + 24 s_{13}^2 + 2 s_{13} s_{23} d^2 - 40 s_{13} s_{23} d - 14 s_{13}^2 d + s_{13}^2 d^2 + 8 s_{23}^2 + s_{23}^2 d^2 + 192 s_{13} s_{23}) \mathcal{D}_6^\dagger \\
& - 2(5 s_{13}^2 d^3 + 5 s_{23}^2 d^3 + 10 s_{13} s_{23} d^3 - 240 s_{13} s_{23} d^2 - 100 s_{13}^2 d^2 - 56 s_{23}^2 d^2 + 580 s_{13}^2 d \\
& \quad +1832 s_{13} s_{23} d + 196 s_{23}^2 d - 208 s_{23}^2 - 800 s_{13}^2 - 4224 s_{13} s_{23}) \mathcal{D}_4^\dagger \Big),
\end{aligned}
$$

[Glover '00]

One of 6 **projectors** in d dimensions, valid up to **2 loops**

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&\quad +1832 s_{13} s_{23} d + 196 s_{23}^2 d - 208 s_{23}^2 - 800 s_{13}^2 - 4224 s_{13} s_{23}) \mathcal{D}_4^\dagger \Big),
\end{aligned}
$$

[Glover '00]

One of 6 **projectors** in d dimensions, valid up to **2 loops**

$$
\overline{P}_i = \sum_{j=1}^{2} \left(M_{ij}^{(2\times 2)} \right)^{-1} \overline{T}_j^\dagger
$$

$$
X_{ij} = \frac{1}{4\, s_{12}^2} \begin{pmatrix} 1 & \dfrac{s_{12} + 2 s_{23}}{s_{23}(s_{12} + s_{23})} \\ \dfrac{s_{12} + 2 s_{23}}{s_{23}(s_{12} + s_{23})} & \dfrac{(d-2) s_{12}^2 + 4 s_{23}(s_{12} + s_{23})}{s_{23}^2 (s_{12} + s_{23})^2} \end{pmatrix}
$$

$$
\left(M^{2\times 2} \right)_{ij}^{-1} = \frac{1}{d-3} X_{ij}
$$

Only 2 projectors at any order in d=4