Caltech

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From cross sections in colliders to gravitational wave observables

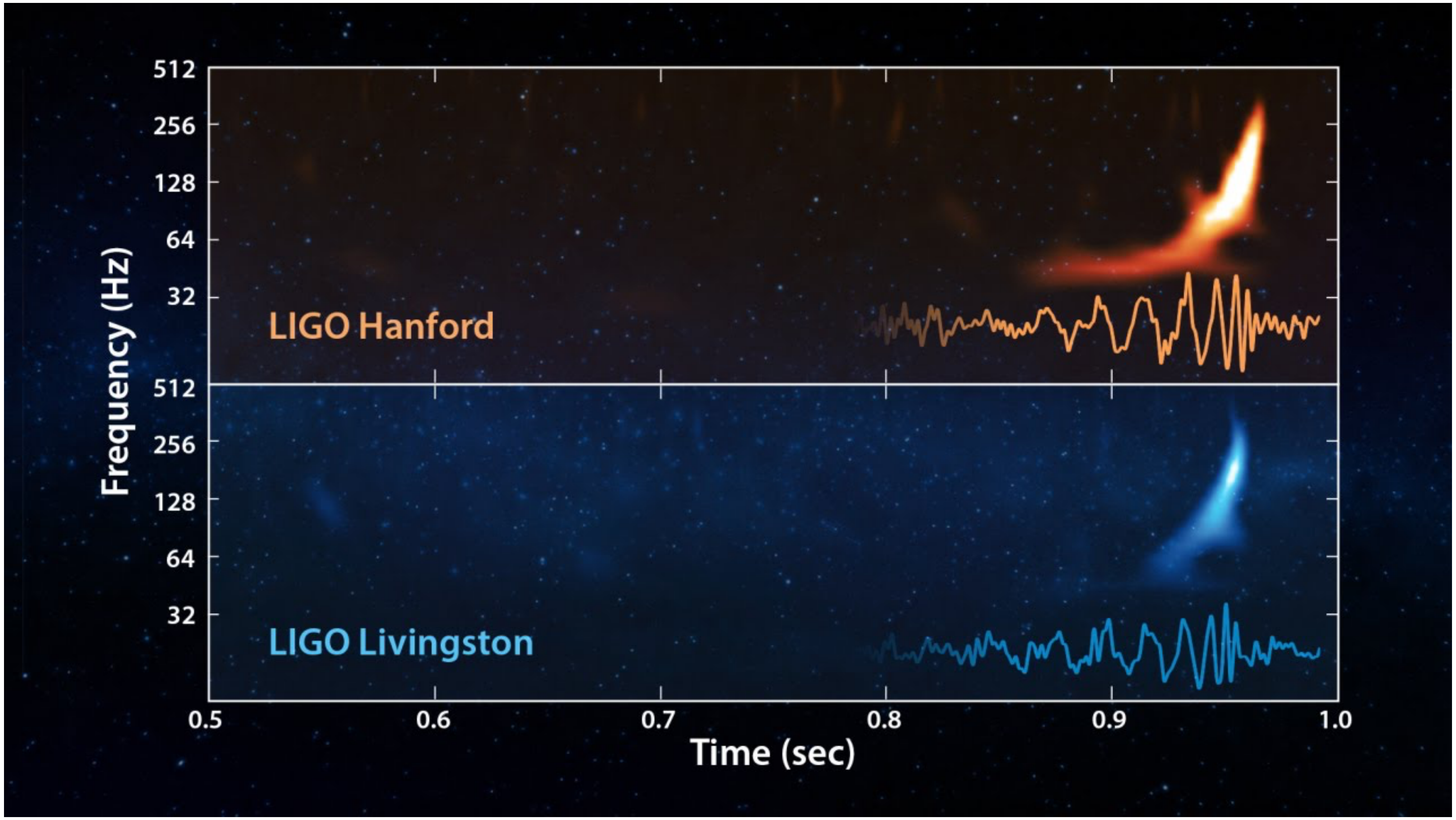
Julio Parra-Martinez

w/ Bern, Roiban, Ruf, Shen, Solon, Zeng [2101.07254, 210X.XXXXX]

w/ Ruf, Zeng [2005.04236] + Herrmann [2101.07255, 2104.03957]

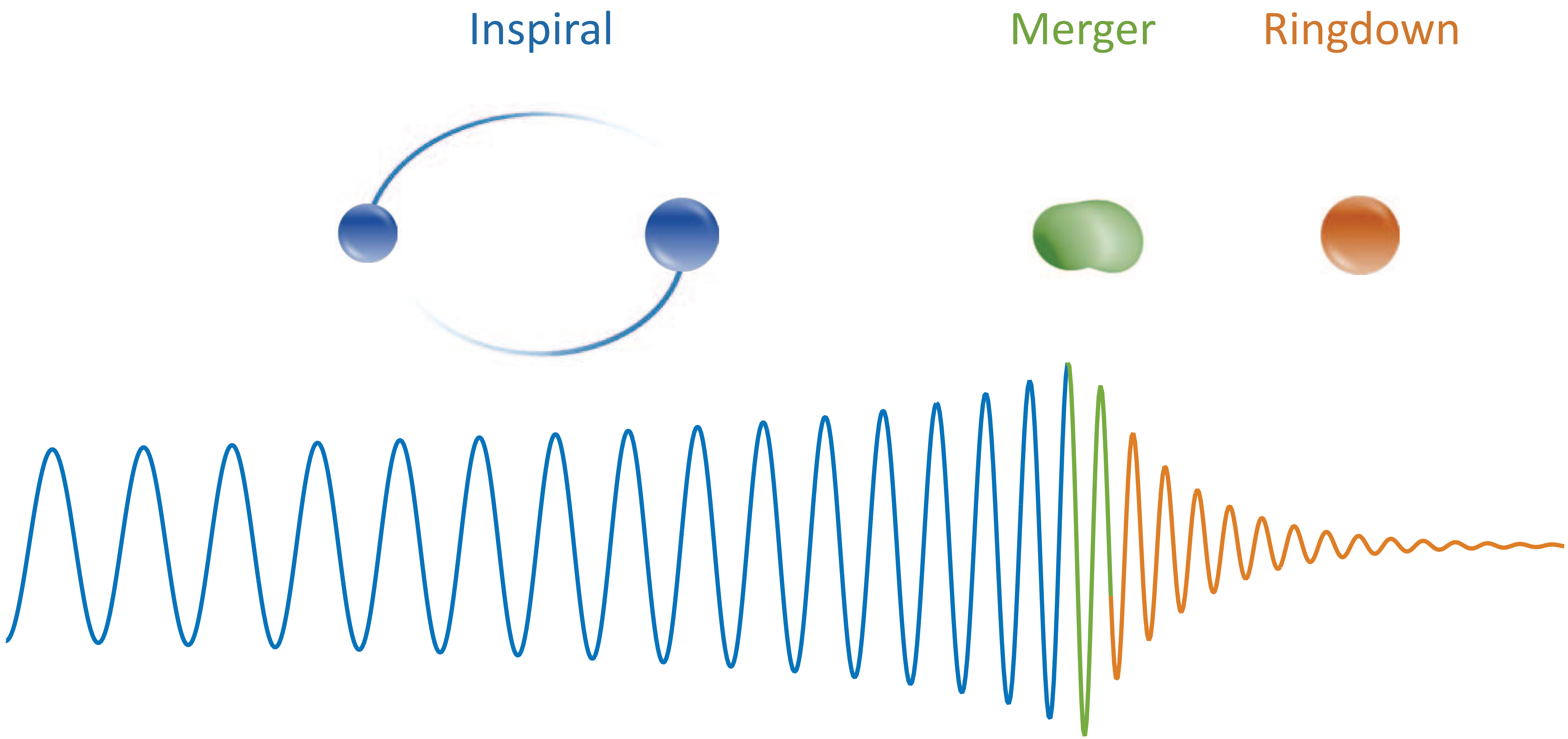
@ RadCor & Loopfest 2021, FSU

GW astronomy is here to stay



Can particle theorists help?

Opportunities for theory



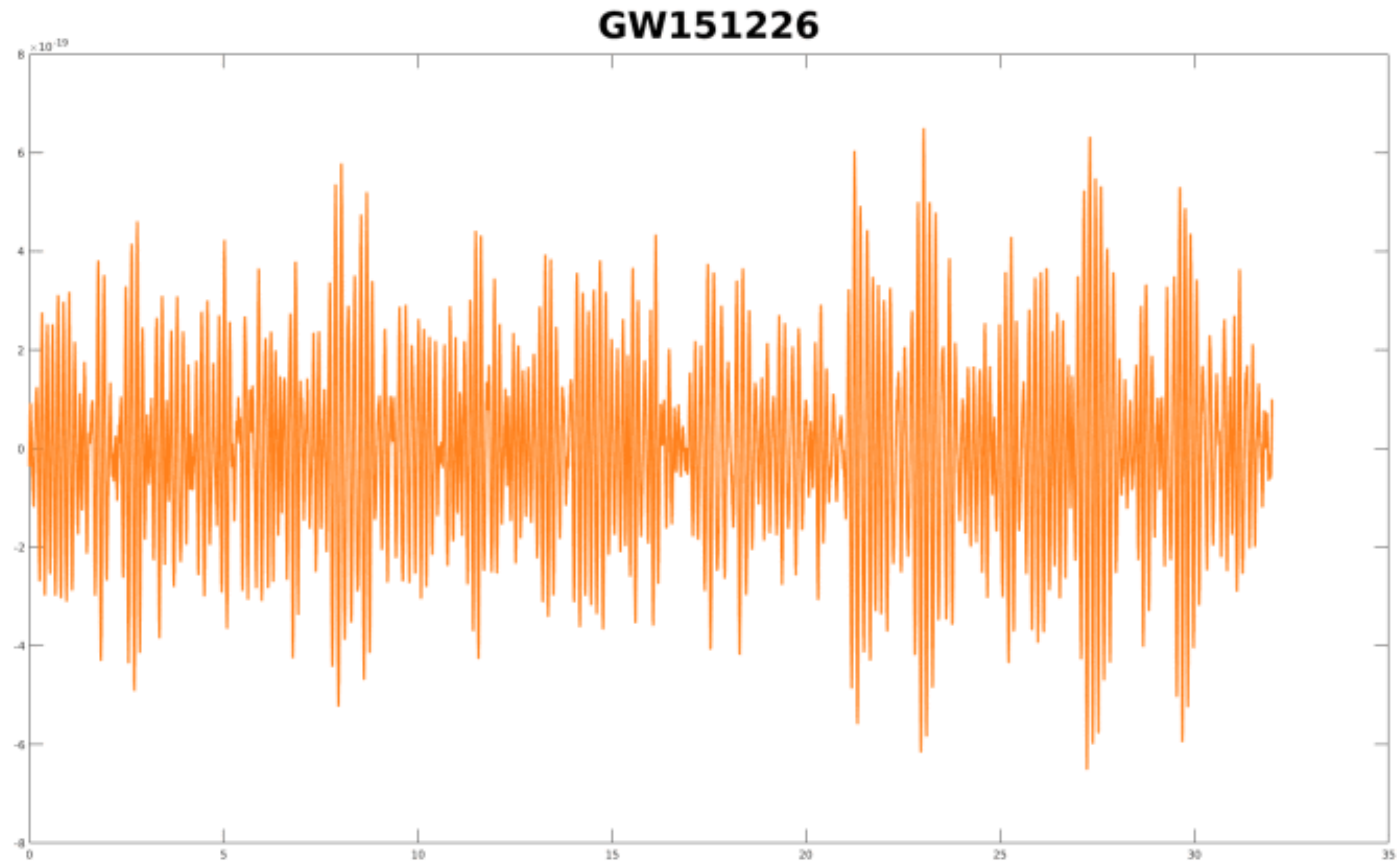
Two-body perturbation theory

Numerical GR

Black Hole perturbation theory

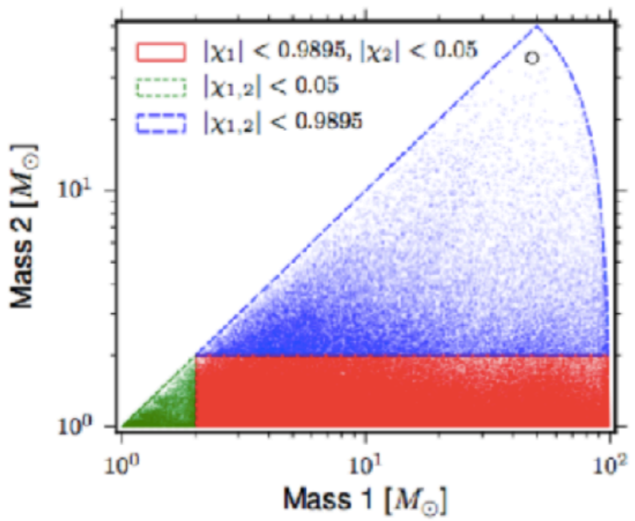
Figure from 1610.03567

Theoretical input essential

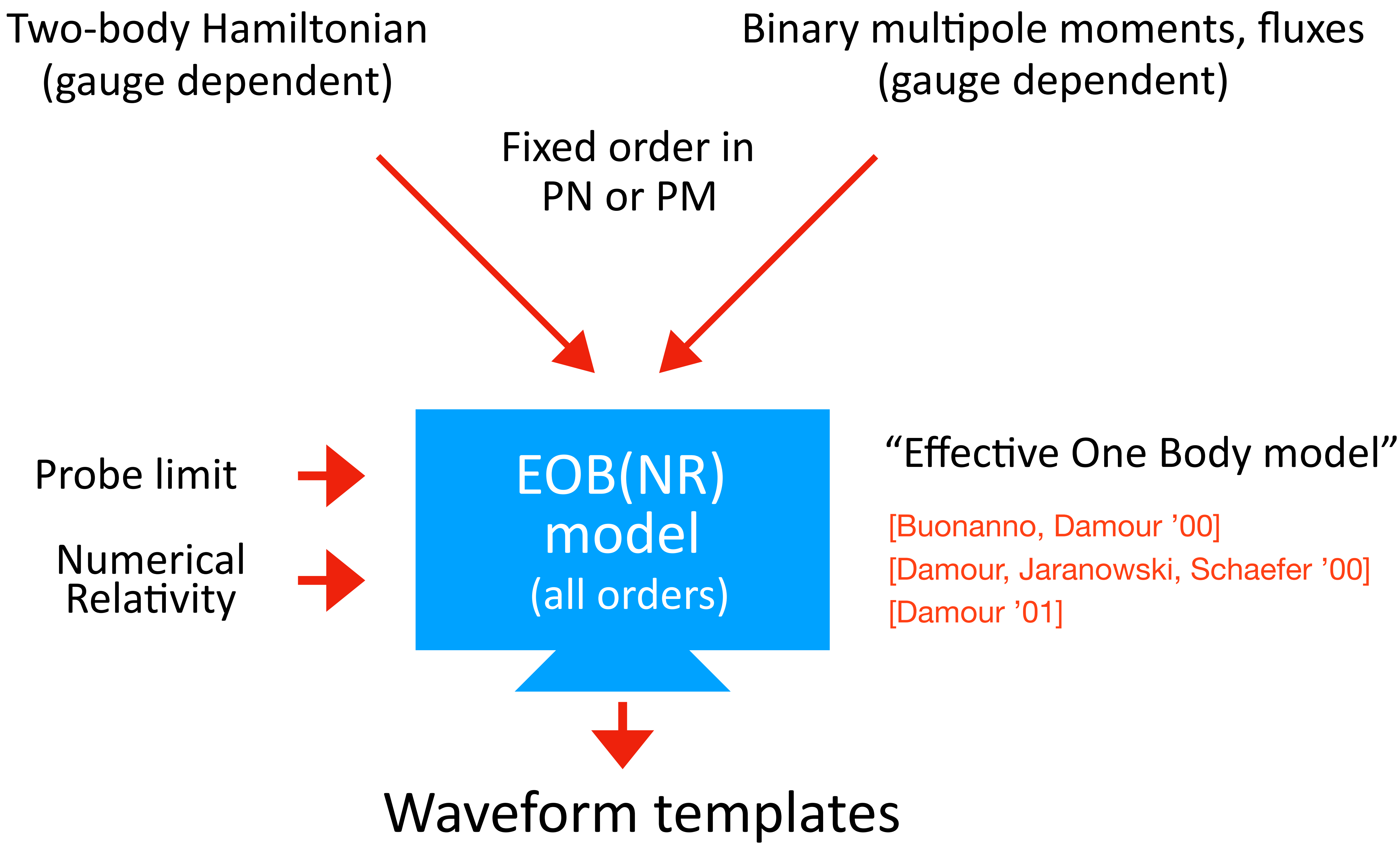


Detection via matched filtering against template bank

$$\int \frac{h_{\mu\nu}^{\text{Data}} h_{\mu\nu}^{\text{Template}}}{S}$$



Current pipeline to LIGO



PN vs. PM

Post-Newtonian expansion - bound system (virial)

$$\frac{v^2}{c^2} \sim \frac{GM}{r} << 1$$

0PN (1687) Newton

1PN (1938) Einstein, Infeld, Hoffman

2PN (1973) Ohta, Okamura, Kimura, Hiida

3PN (2000) Damour, Jaranowski, Schafer; Blanchet, Faye

4PN (2014) Damour, Jaranowski, Schafer

5PN (2020) Blumlein, Marquard, Maier, Schafer

...

$$
\begin{aligned}
G(1 &+ v^2 + v^4 + v^6 + v^8 + \cdots) \\
G^2(1 &+ v^2 + v^4 + v^6 + \cdots) \\
G^3(1 &+ v^2 + v^4 + \cdots) \\
G^4(1 &+ v^2 + \cdots) \\
&\vdots
\end{aligned}
$$

State of the art
partial 6PN (2020)

See Maier's talk!

PN vs. PM

Post-Minkowskian (PM) expansion - (non virial) $\frac{GM}{r} << 1$

Arbitrary velocities, natural fit with relativistic amplitudes

Modern approach, fast progress

$$G(1 + v^2 + v^4 + v^6 + v^8 + \cdots)$$ 1PM

$$G^2(1 + v^2 + v^4 + v^6 + \cdots)$$ 2PM (1985) Westphal

$$G^3(1 + v^2 + v^4 + \cdots)$$ 3PM (2019) Bern, Cheung, Roiban, Shen, Solon, Zeng

$$G^4(1 + v^2 + \cdots)$$ 4PM (2021) Bern, **JPM**, Roiban, Ruf, Shen, Solon, Zeng

$$\vdots$$

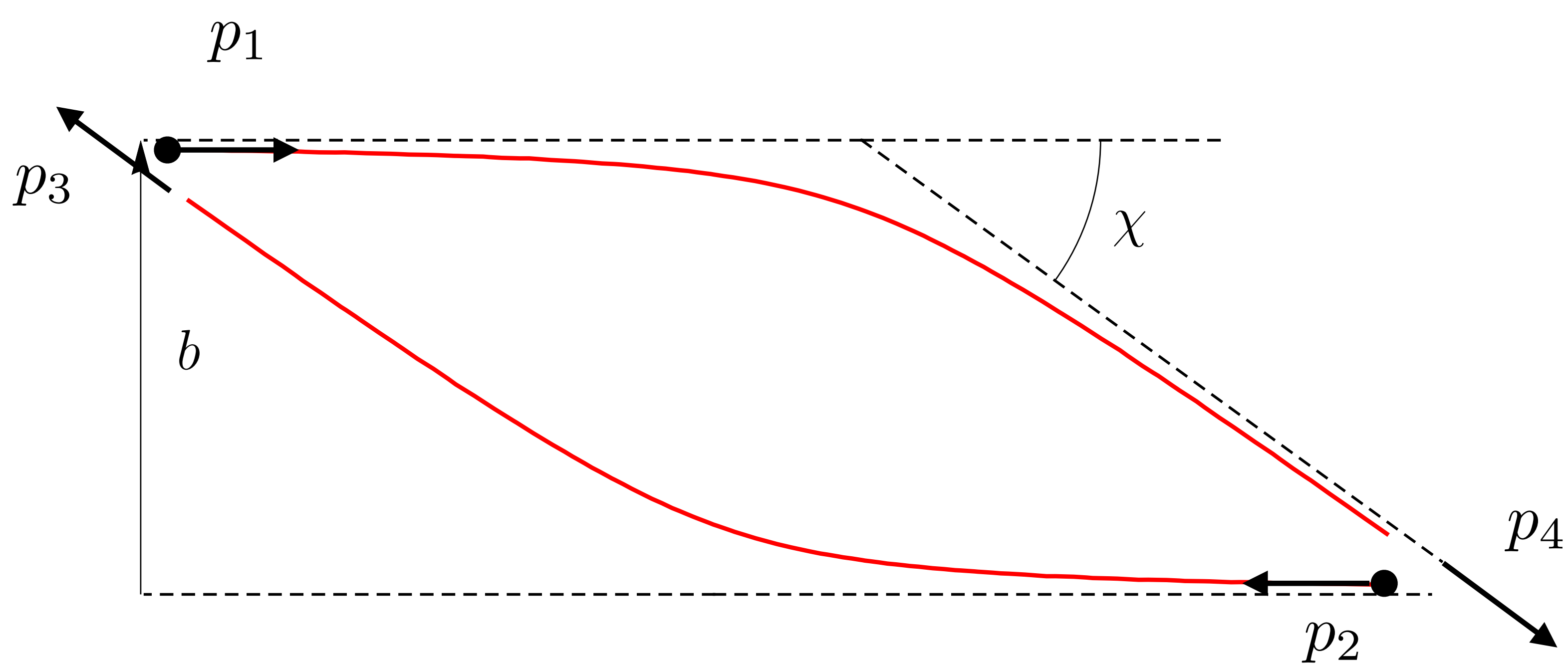
PM potential
(Isotropic gauge)

$$V(\boldsymbol{p}, r) = \sum_n c_n(\boldsymbol{p}^2) \left(\frac{GM}{r}\right)^n$$

See Mao Zeng's talk!

A theoretical experiment

Start from scattering dynamics….



…and determine the basic ingredients, $H(p, r), \cdots$

We will do this by calculating scattering amplitudes in QFT, leveraging modern perturbative methods, and then take $\hbar \to 0$.

Point particle EFT

- Black holes, Neutron stars, … ⟶ minimally coupled scalar

$$\mathcal{L} = \frac{1}{16\pi G} R + \sum_i \nabla_\mu \phi_i \nabla^\mu \phi_i^\dagger + m_i |\phi_i|^2$$

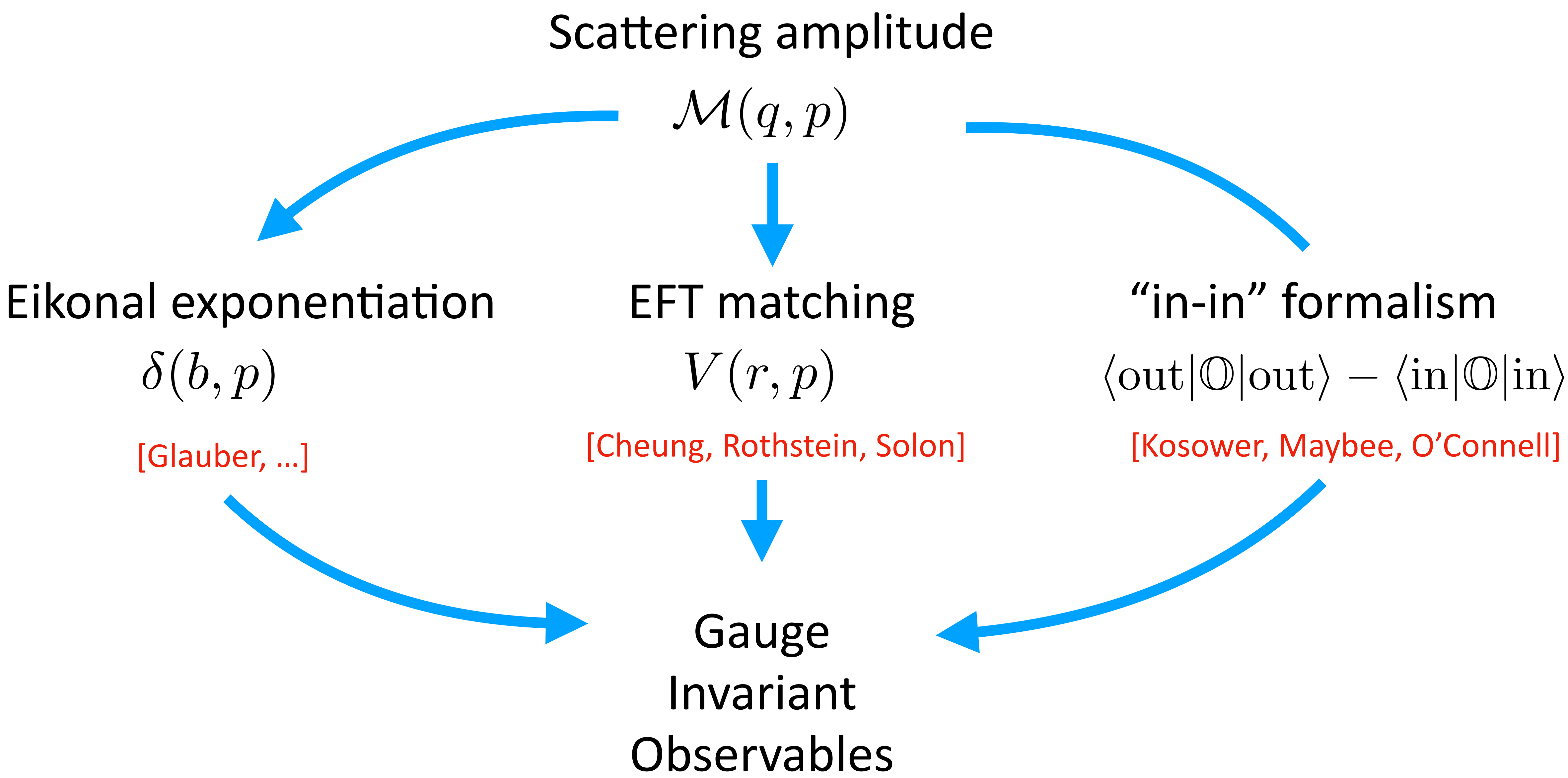
- Finite size effects via tidal operators [Cheung, Solon; Haddad, Helset; Bern, **JPM**, Roiban, Sawyer, Shen;…]

$$\Delta\mathcal{L} = \sum c_{a,b,c} \phi_i \nabla^a \phi_i \nabla^b R^c$$

- Spin d.o.f by making field massive higher spin [Cachazo, Guevara; + Ochirov, Vines, Bern, Luna, Roiban, Shen, Zeng]

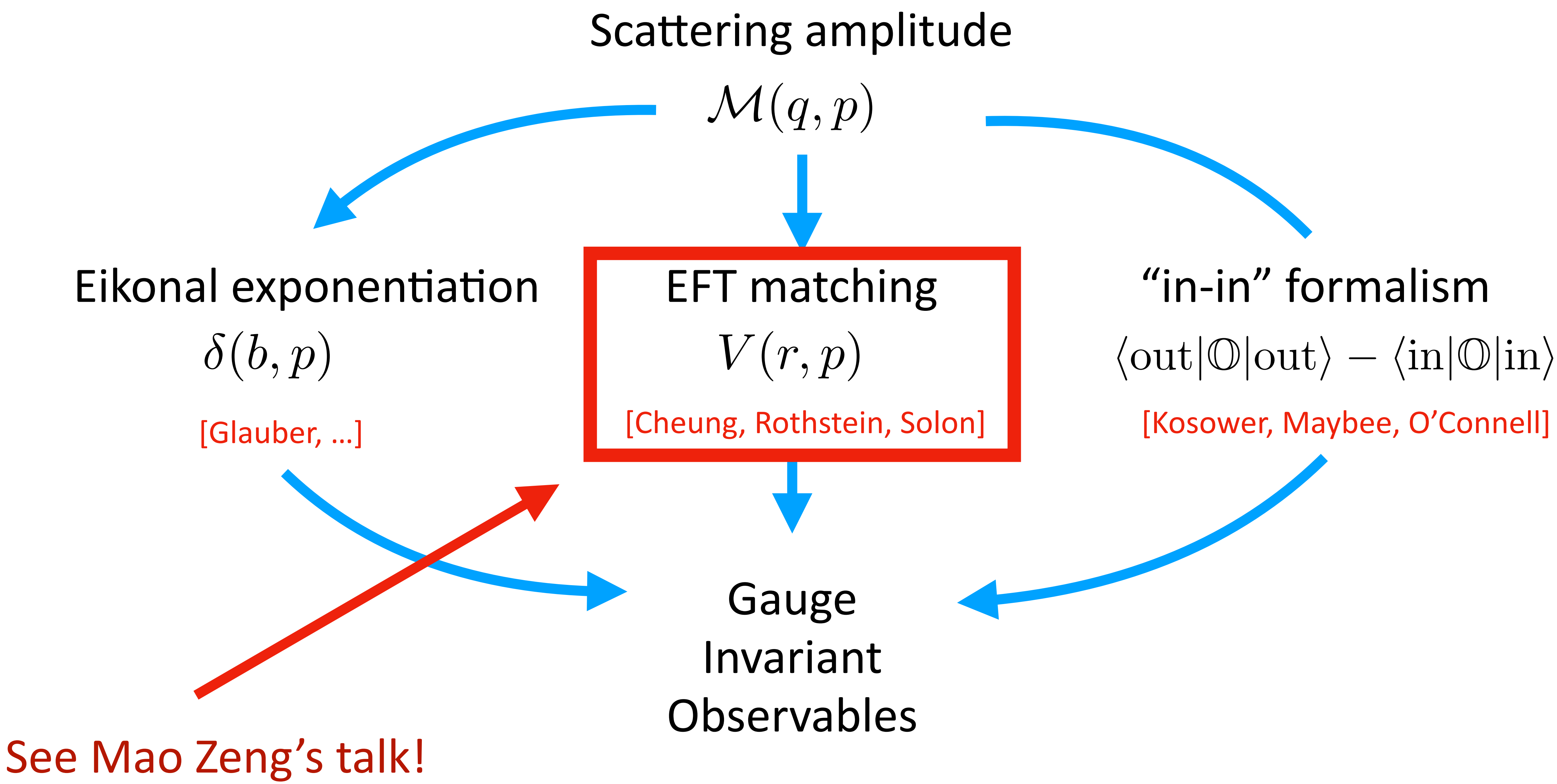
$$\phi_i \to \phi_i^{\mu(s)}$$

Multiple approaches



χ $\quad \Delta p^{\mu}$ $\quad E^{\rm rad}$ $\quad J^{\rm rad}$ and ultimately $h_{\mu\nu}$

Multiple approaches



χ Δp^{μ} E^{rad} J^{rad} and ultimately $h_{\mu\nu}$

Radiative observables

[Kosower, Maybee, O'Connell]

- Real time dynamics "in-in"

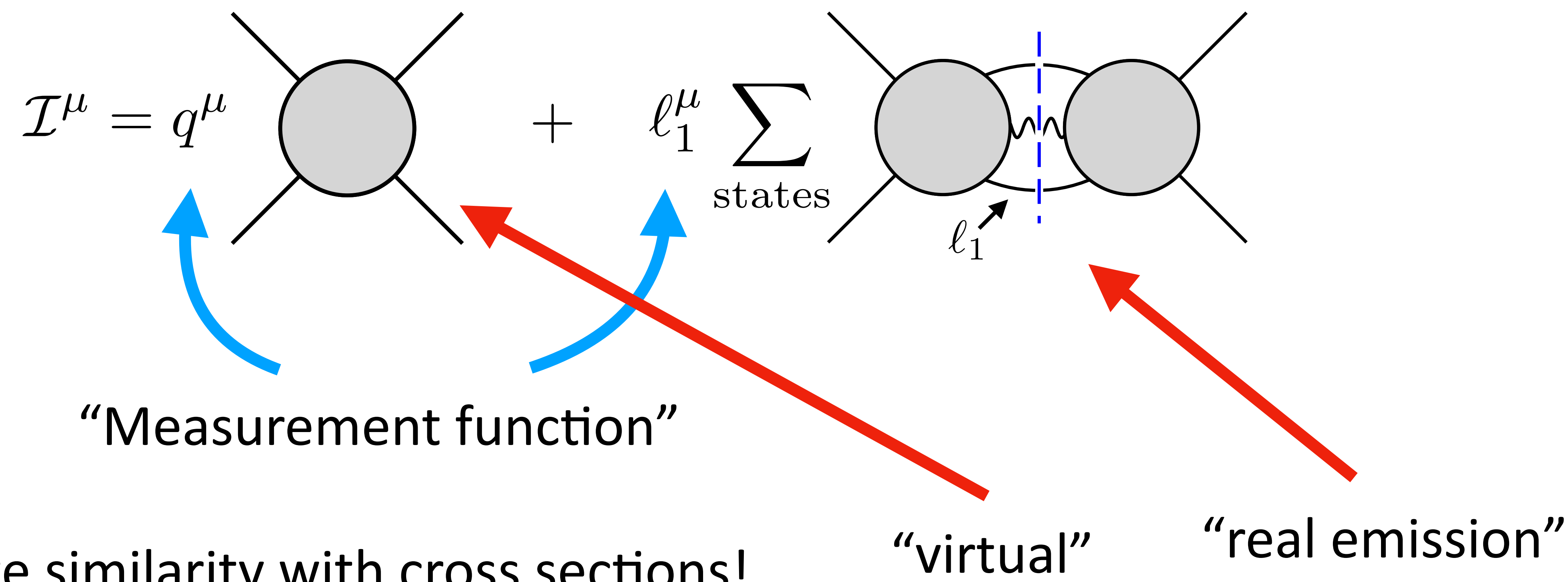
  $$\Delta\mathcal{O} = \langle \text{out}|\mathbb{O}|\text{out}\rangle - \langle \text{in}|\mathbb{O}|\text{in}\rangle$$

  states related by S-matrix

  $$|\text{out}\rangle = \mathbb{S}|\text{in}\rangle = (1 + i\mathcal{M})|\text{in}\rangle$$

- Example 1: Impulse

  $$\Delta p_1^\mu = \int d^D q\, \delta(2u_1\cdot q)\delta(2u_2\cdot q)\, e^{ib\cdot q}\, \mathcal{I}^\mu$$



- Note similarity with cross sections!

Radiative observables

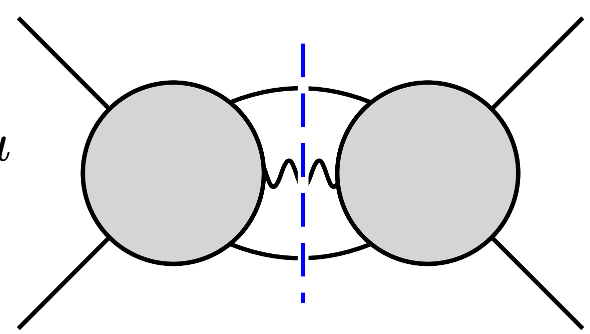
[Kosower, Maybee, O'Connell]

- Example 2: Radiated momentum

$$R^\mu = \int d^D q \, \delta(2u_1 \cdot q)\delta(2u_2 \cdot q) \, e^{ib\cdot q} \, \mathcal{R}^\mu$$

"Radiation kernel"

$$\mathcal{R}^\mu = \sum_{\text{states}} k^\mu$$



$$\sim \int k^\mu |h_{\alpha\beta}|^2$$

Waveform

- Simply a measurement of the momentum of the final state radiation

Classical vs. quantum & PM

- Scales and the classical limit

$$\ell_c \quad \ll \quad R_s \quad \ll r$$

"Compton" "Schwarzschild" "Separation"

$$\frac{\hbar}{m} \quad \ll \quad Gm \quad \ll \quad \frac{\hbar}{q}$$

- Two small parameters $\frac{q}{m} \ll Gmq << 1 \qquad (\hbar = 1)$

Quantum vs. classical PM

- Not ordinary perturbation theory! $GE^2 \sim Gm^2 \ll 1$

Classical vs. quantum (regions)

- Classical ≠ tree-level (e.g. due to masses)

- Classical = long distance = large angular momentum

$$\frac{GM}{r} \sim \frac{R_s}{r} << 1 \qquad \frac{m_i^2}{-q^2} \sim \frac{s}{-q^2} \sim J^2 \gg 1$$

- Method of regions

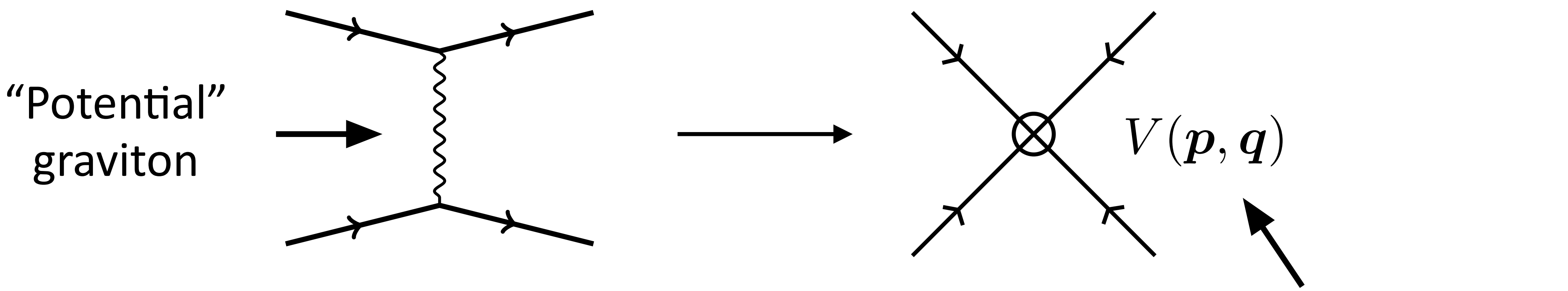
[Beneke, Smirnov]

"Matter" (hard):	$p \sim m$	(analytic)
gravitons (soft):	$\ell \sim q \sim \frac{m}{J}$	(non-analytic)

Nonlinear in G corrections encoded in loops with soft gravitons.

Conservative vs. dissipative

- Conservative interactions mediated by off-shell gravitons



- Velocity regions (c.f. NRQCD)

Potential: $(\omega, \boldsymbol{\ell}) \sim (v, 1)$

Radiation: $(\omega, \boldsymbol{\ell}) \sim (v, v)$

Conservative dynamics encoded in loops with potential gravitons.

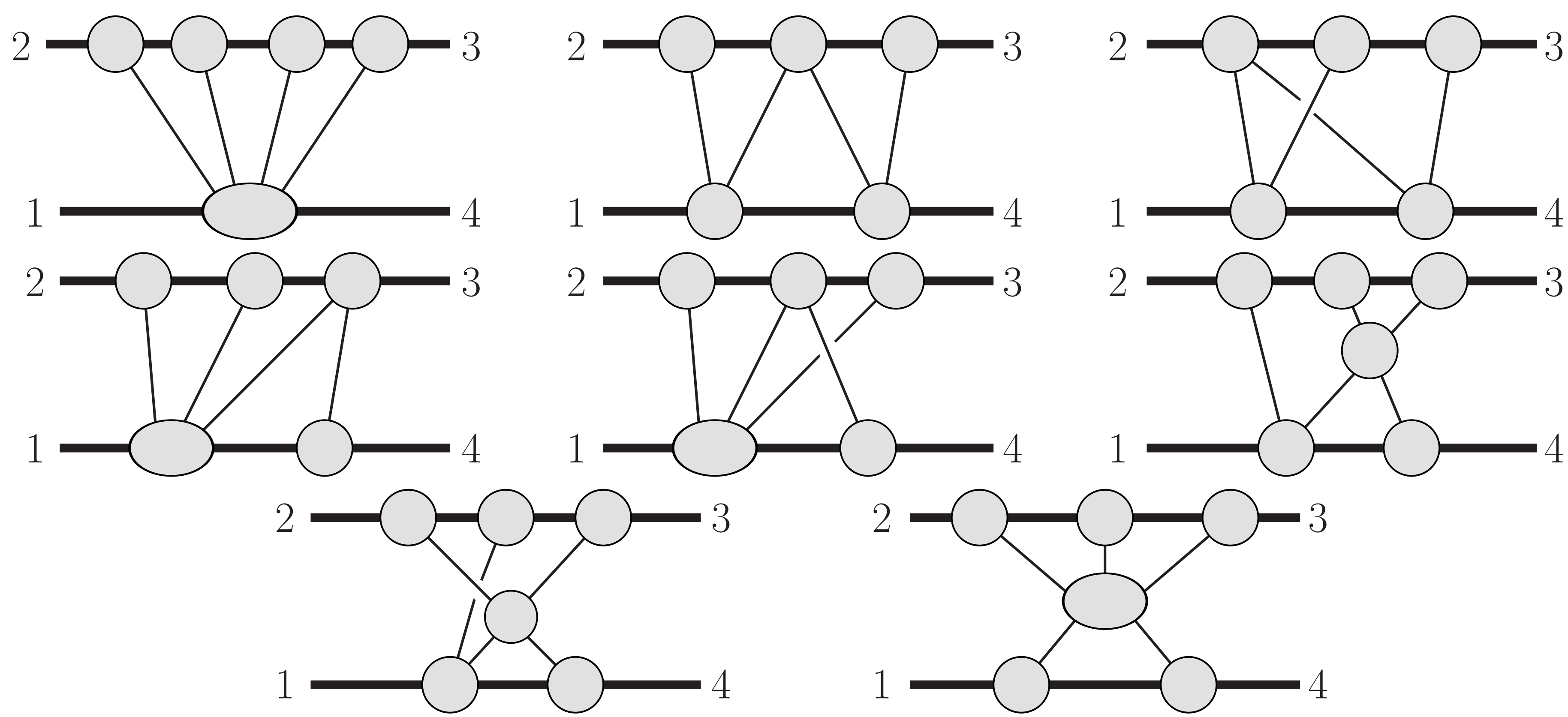
Goal: calculate amplitude & cuts in relevant regions, rest can be extracted from it!

Collider methods for classical amplitudes

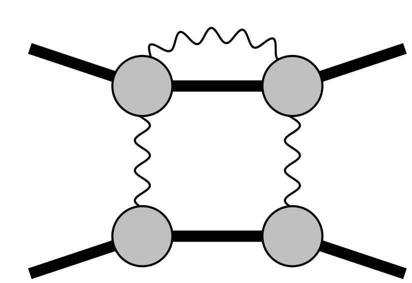
Integrands from generalized unitarity

- Classical restrictions = “no matter contacts”
  potential restriction = “one matter propagator per loop”

  e.g. 4PM [Bern, **JPM**, Roiban, Ruf, Shen, Solon, Zeng]

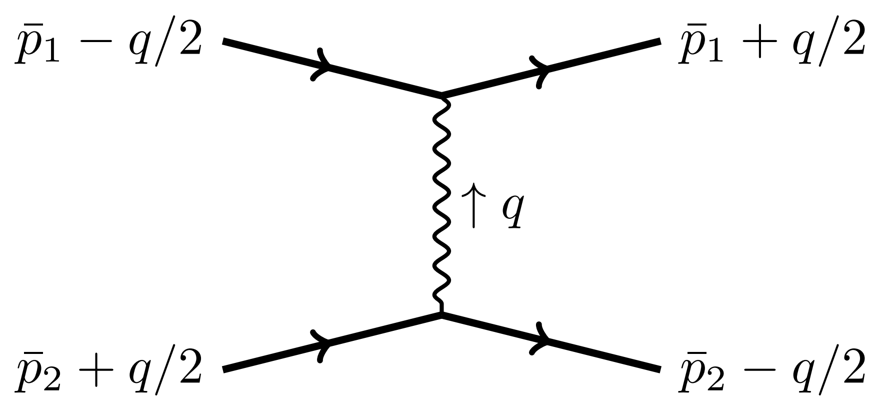


- Additional contributions needed for dissipative sector



Classical loop integrals

[**JPM**, Ruf, Zeng]



- Special variables [Sudakov] $\bar{p}_i \cdot q = 0$

- Soft integrals (method of regions, c.f. HQET)

$$\ell^2 \to \ell^2$$

"Eikonalized" $$(\ell + p_i)^2 - m_i^2 \to 2\ell \cdot u_i\,, \qquad u_i = \bar{p}_i / \bar{m}_i$$

$$I(q, \bar{p}_i, \bar{m}_i) = (-q^2)^a\, I(y) \qquad\qquad y = \tfrac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2} = \sigma + \mathcal{O}(q^2)$$

Single scale problem to all PM (or loop) orders! [**JPM**, Ruf, Zeng]

- IBP reduction & velocity differential equations! canonical form [Henn]

$$\mathrm{d}\vec{I}(y) = \epsilon \sum_i A_i \mathrm{dlog}\, \alpha_i(y)\, \vec{I}(y)$$

Disclaimer: Elliptic integrals enter at three loops. DE still useful!

Potential boundary conditions

[**JPM**, Ruf, Zeng]

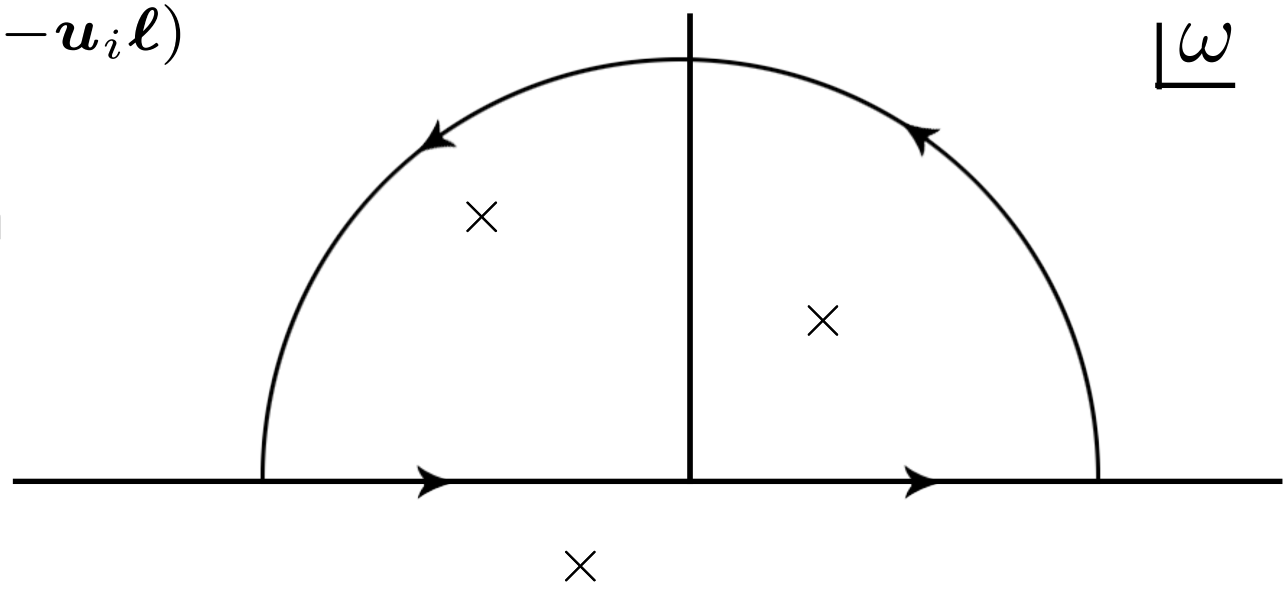
- Radiation and potential regions split in near-static limit $v \ll 1$
- Integrals in potential region satisfy same differential equations as soft integrals! Only need to calculate appropriate boundary conditions.
- Potential region:

Graviton: $\frac{1}{\ell^2} = \frac{1}{\omega^2 - \boldsymbol{\ell}^2} = -\frac{1}{\boldsymbol{\ell}^2} - \frac{\omega^2}{(\boldsymbol{\ell}^2)^2} - \frac{\omega^4}{(\boldsymbol{\ell}^2)^3} + \cdots$

Matter: $\frac{1}{2u_i \cdot \ell} = \frac{1}{2(u_i^0\, \omega - \boldsymbol{u}_i \boldsymbol{\ell})}$

- Evaluated by residue prescription

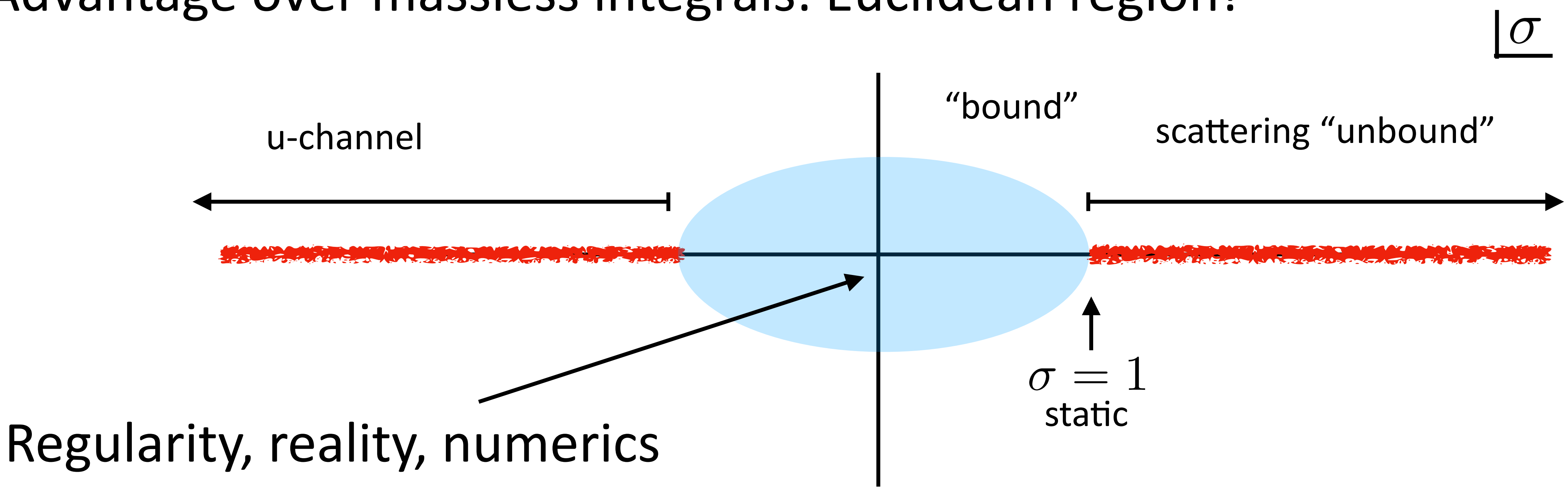
(Similar to NRQCD/NRGR)



Soft boundary conditions

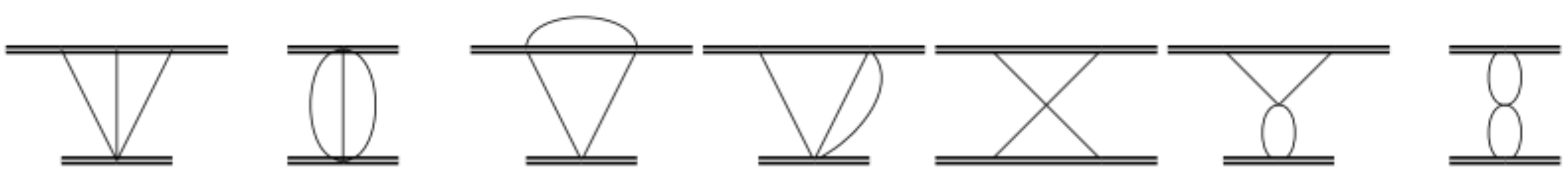
[Herrmann, **JPM**, Ruf, Zeng]

- Advantage over potential region: analyticity in velocity
- Advantage over massless integrals: Euclidean region!

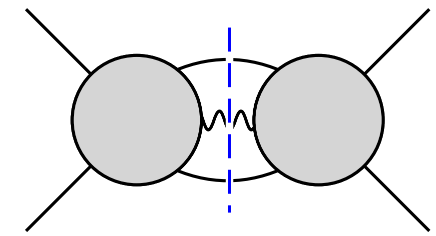


- Only a few boundary conditions are independent.

e.g. 3PM



Tools for cut integrals

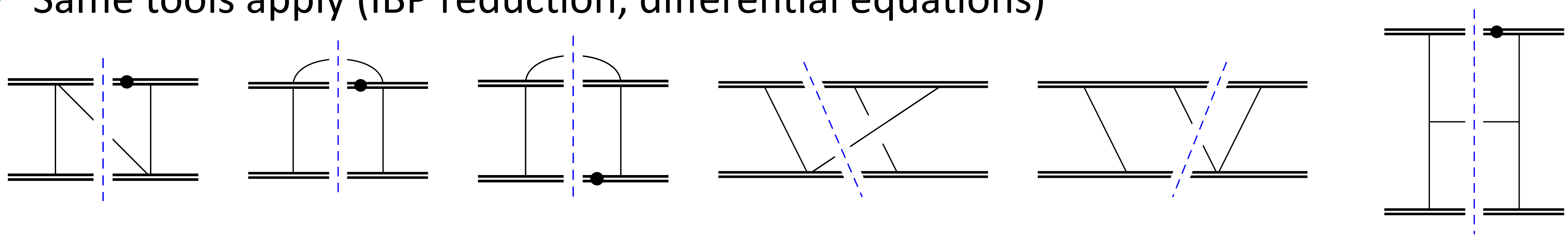


[Herrmann, **JPM**, Ruf, Zeng]

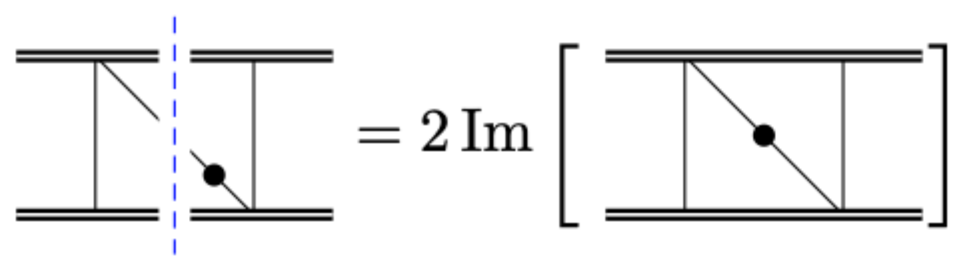
- Reverse unitarity for cut integrals
  [Anastasiou, Melnikov]

$$2\pi i\,\delta(2u_1\cdot\ell_1) = \frac{1}{2u_1\cdot\ell_1 - i\epsilon} - \frac{1}{2u_1\cdot\ell_1 + i\epsilon}$$

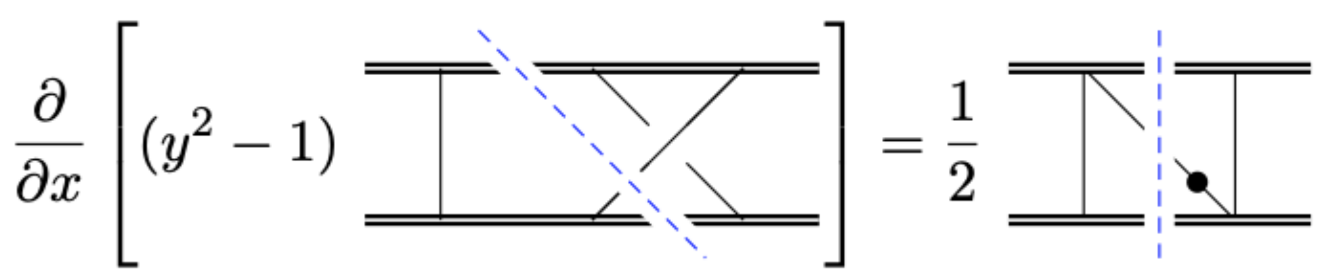
- Same tools apply (IBP reduction, differential equations)



- Many given via optical theorem/ Cutkosky cutting rules

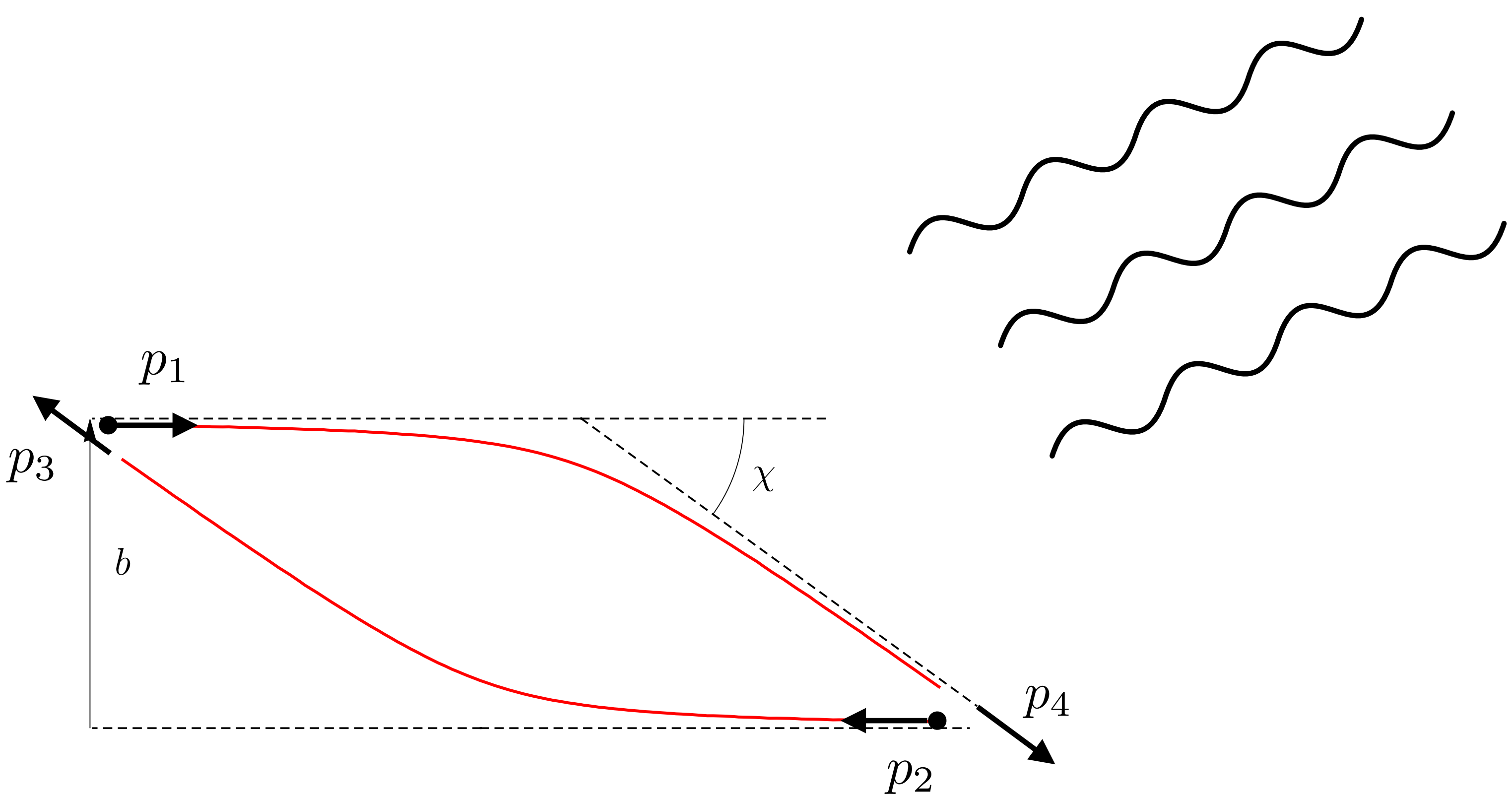


- Others related by same differential equations

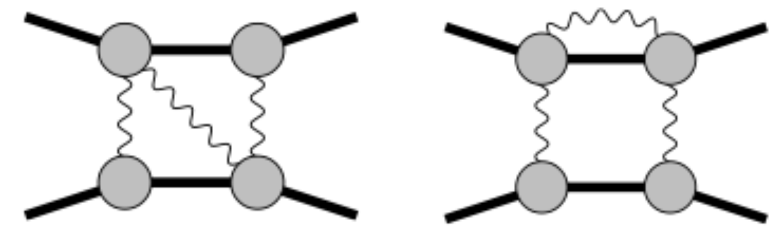


b.c. from velocity power counting!

Results 1: Gravitational Bremsstrahlung

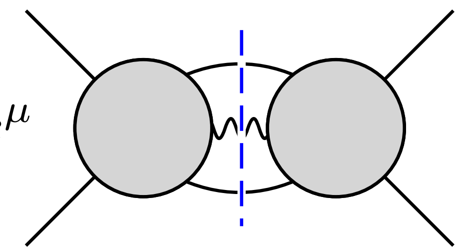


Radiative observables

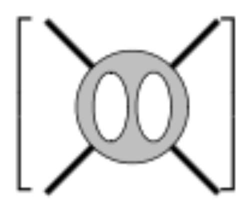
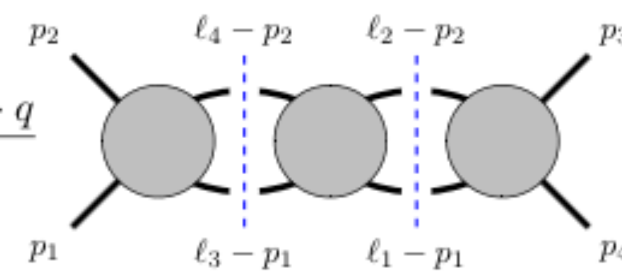


[Herrmann, **JPM**, Ruf, Zeng]

- Integrand via generalized unitarity, integrals via reverse unitarity/cutting rules

- Radiated momentum $\mathcal{R}^\mu = \sum_{\text{states}} k^\mu$ 

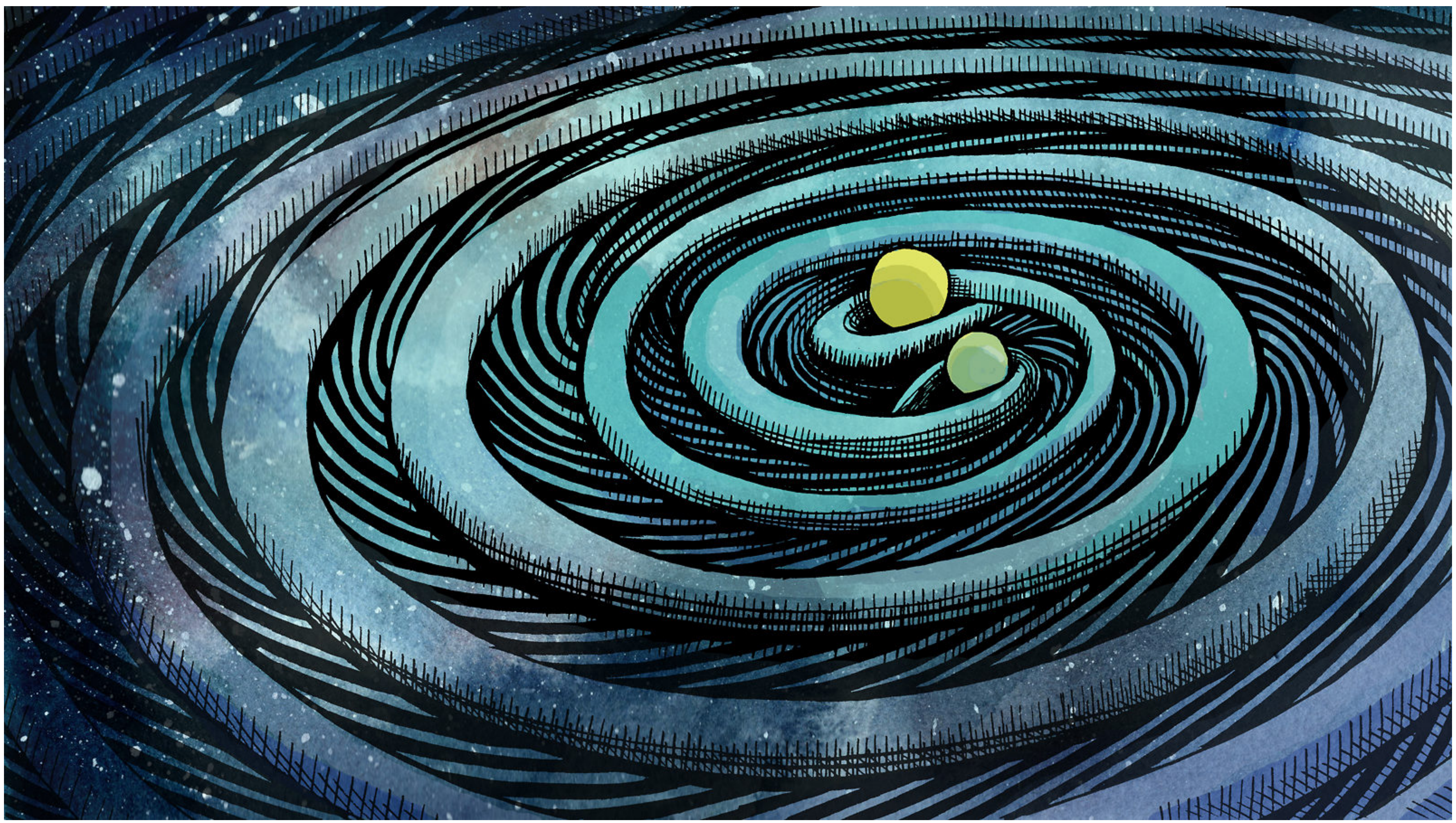
$$R^\mu = \frac{G^3 m_1^2 m_2^2}{4|b|^3}\frac{u_1^\mu + u_2^\mu}{\sigma+1}\left(h_1 + h_2 \log\left(\frac{\sigma+1}{2}\right) + h_3 \frac{\text{arccosh}\sigma}{\sqrt{\sigma^2-1}}\right) + \mathcal{O}(G^4)$$

- Impulse on a particle $\mathcal{I}_\perp^{(2)} = \text{Re}$  $- i\int d\tilde{\Phi}_2^2 \frac{(\ell_1-\ell_3)\cdot q}{2q^2}$ 

$$\Delta p_1^\mu = -\frac{16G^3 m_1^2 m_2^2 \sigma^4}{\sigma^2-1}\frac{b^\mu}{b^4}\left(-\frac{2\sigma^2}{\sigma^2-1} + \left(4 - \frac{4\sigma\left(\sigma^2-2\right)}{\left(\sigma^2-1\right)^{3/2}}\right)\text{arcsinh}\sqrt{\frac{\sigma-1}{2}} + \frac{\sigma^2 s}{m_1 m_2\left(\sigma^2-1\right)^{3/2}}\right)$$
$$-\frac{4\pi G^3 m_1^2 m_2^2 \sigma^4}{(\sigma^2-1)^{3/2}}\frac{\sigma u_2^\mu - u_1^\mu}{|b|^3}\left(-\frac{2\sigma^2}{\sigma^2-1} + \left(4 - \frac{4\sigma\left(\sigma^2-2\right)}{\left(\sigma^2-1\right)^{3/2}}\right)\text{arcsinh}\sqrt{\frac{\sigma-1}{2}} + 4\log\left(\frac{1}{2}(1+\sigma-\sqrt{\sigma^2-1})\right)\right)$$

(N=8 result, GR lengthier, but same story)

Results 2: Fourth post-Minkowskian Potential



4PM "potential" amplitude

[Bern, **JPM**, Roiban, Ruf, Shen, Solon, Zeng]

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}}\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1},$$

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \qquad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}},$$

$$\begin{aligned}
\mathcal{M}_4^{\mathrm{f}} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \mathrm{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \mathrm{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right),
\end{aligned}$$

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \qquad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

- Three loops. Appearance of classical elliptic integrals.
- Infrared divergence/tail effect - (splitting of potential/radiation) coefficient matches radiated energy!

Conclusions/future

- Modern tools offer an efficient scalable way to compute classical potentials and radiation observables in general relativity

integration by parts

method of regions

differential equations

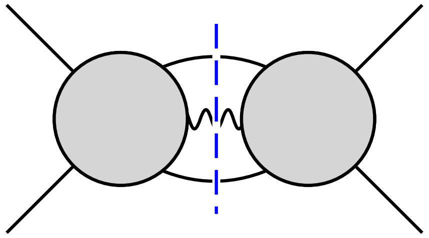
generalized unitarity

reverse unitarity

- We have reached the state of the art in the computation of these quantities. The methods are not close to being exhausted

- Other observables in the works e.g: Power-spectrum ~ p_T , rapidity distributions

$$\frac{dR^\mu}{d\omega} = \sum_{\text{states}} k^\mu \quad \delta(k^0 - \omega)$$



Also angular distribution ~ event shapes, radiated angular momentum, …

- Future challenges (multiloop, multiscale, elliptic…) fit for RADCOR/LOOPFEST community, **come join the effort!**

Thank you!