Two-loop leading colour QCD helicity amplitudes for $t\bar{t}gg$ in the gluon fusion channel

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Physics motivation

- Precise understanding of top quark.
- Analytic form of 2-loop $t\bar{t}gg$ amplitudes is important. What are the mathematical properties?
- How complicated is the use of these amplitudes, i.e. numerical evaluation, stability etc?

State of the art

- Fully differential NNLO predictions for $t\bar{t}$ production available for comparison with experimental data. [Bärnreuther, Czakon, Mitov, '12] [Czakon, Mitov, '12][Czakon, Fiedler, Mitov, '13] [Czakon, Heymes, Mitov, '15].

- Complete 2-loop amplitudes have been computed only numerically [Czakon, '08] [Bärnreuther, Czakon, Fiedler, '13] [Chen, Czakon, Poncelet, '17] .

- More complicated class of special functions begin to appear with amplitudes containing internal masses.

- These functions have been identified to involve integrals over elliptic curves [Adams, E.C., Weinzierl, '17] [Bogner, Schweitzer, Weinzierl, '17] [Broedel, Duhr, Dulat, Marzucca, Penante, Tancredi, '19] [Abreu, Becchetti, Duhr, Marzucca, '19].

Highlights

- We discuss a set of helicity amplitudes for top-quark pair production in the leading colour approximation.
- A compact helicity amplitudes obtained by sampling Feynman diagrams with finite field arithmetic [Peraro, '19].
- The helicity amplitudes contain complete information about top quark decays in the narrow width approximation.
- One 2-loop integral topology containing two elliptic curves previously unknown presented.
- Numerical evaluation of the amplitudes presented.

The setup

$$0 \to \bar{t}(p_1) + t(p_2) + g(p_3) + g(p_4)$$

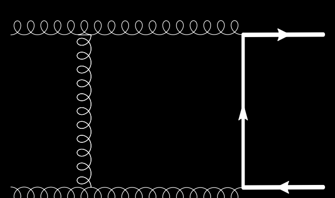
$$p_1^2 = p_2^2 = m_t^2, \qquad p_3^2 = p_4^3 = 0$$

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2,$$

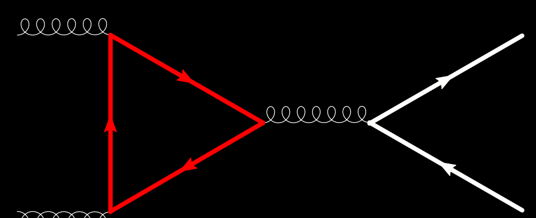
$$A^{(L)}(1_{\bar{t}}, 2_t, 3_g, 4_g) = n^L g_s^2 \left[(T^{a_3} T^{a_4})_{i_2}^{\bar{i}_1} A^{(L)}(1_{\bar{t}}, 2_t, 3_g, 4_g) + (3 \leftrightarrow 4) \right]$$

$$A^{(1)}(1_{\bar{t}}, 2_t, 3_g, 4_g) = N_c A^{(1),1} + N_l A^{(1),N_l} + N_h A^{(1),N_h},$$

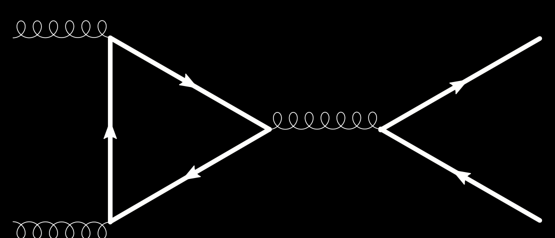
$$A^{(2)}(1_{\bar{t}}, 2_t, 3_g, 4_g) = N_c^2 A^{(2),1} + N_c N_l A^{(2),N_l} + N_c N_h A^{(2),N_h} + N_l^2 A^{(2),N_l^2} + N_l N_h A^{(2),N_l N_h} + N_h^2 A^{(2),N_h^2}$$



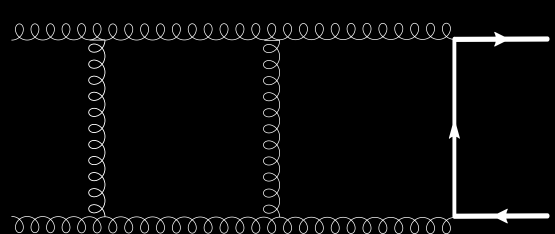
$A^{(1),1}$



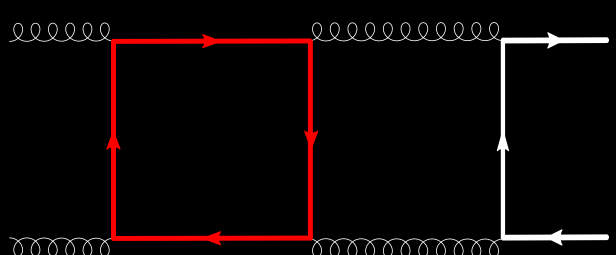
$A^{(1),N_l}$



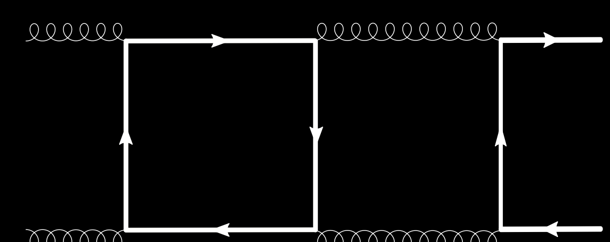
$A^{(1),N_h}$



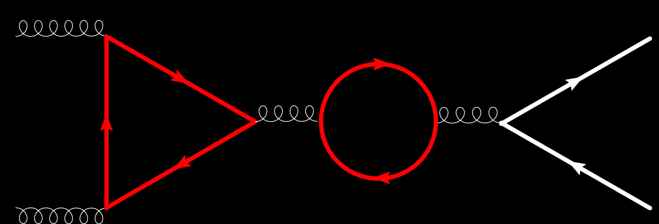
$A^{(2),1}$



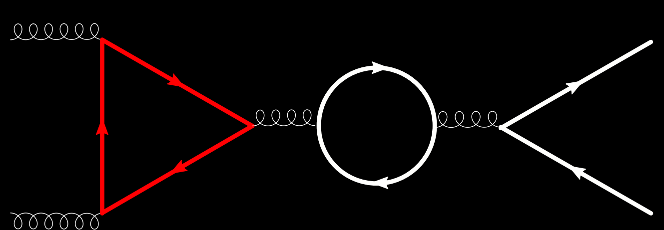
$A^{(2),N_l}$



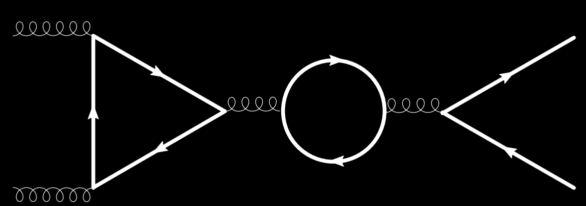
$A^{(2),N_h}$



$A^{(2),N_l^2}$



$A^{(2),N_l N_h}$



$A^{(2),N_h^2}$

$A^{(2),1},\ A^{(2),N_l},\ A^{(2),N_l^2},\ A^{(2),N_lN_h},\ A^{(2),N_h^2}$	$A^{(2),N_h}$
Solutions of all 1- and 2- loop MIs appearing in these can be expressed in terms of multiple polylogarithms [Bonciani, Ferroglia, Gehrmann, von Manteuffel, Studerus, '13] [Mastrolia, Passera, Primo, Schubert, '17].	For this 2-loop amplitude, MIs also involve elliptic generalisations & beyond [Adams, E.C., Weinzierl , '17,'18].

Amplitude Reduction

See also Bayu's talk

After colour ordering and helicity amplitude processing:

$$A^{(L),h} = \int \prod_{j=1}^{L} d^d k_j \frac{N_T^h}{\prod_{\alpha \in T} D_\alpha}$$

↓

Suitable for IBP:

$$A^{(L),h} = \sum_T \sum_i c_{T,i}^h G_{T,i}$$

↓

In terms of Master Integrals (MIs):

$$A^{(L),h} = \sum_k c_k^{\mathrm{IBP},h} \, \mathrm{MI}_k$$

↓

In terms of special functions:

$$A^{(L),h} = \sum_k \sum_{l=n(L)}^{0} \epsilon^l c_{kl}^h \, m_k + O(\epsilon)$$

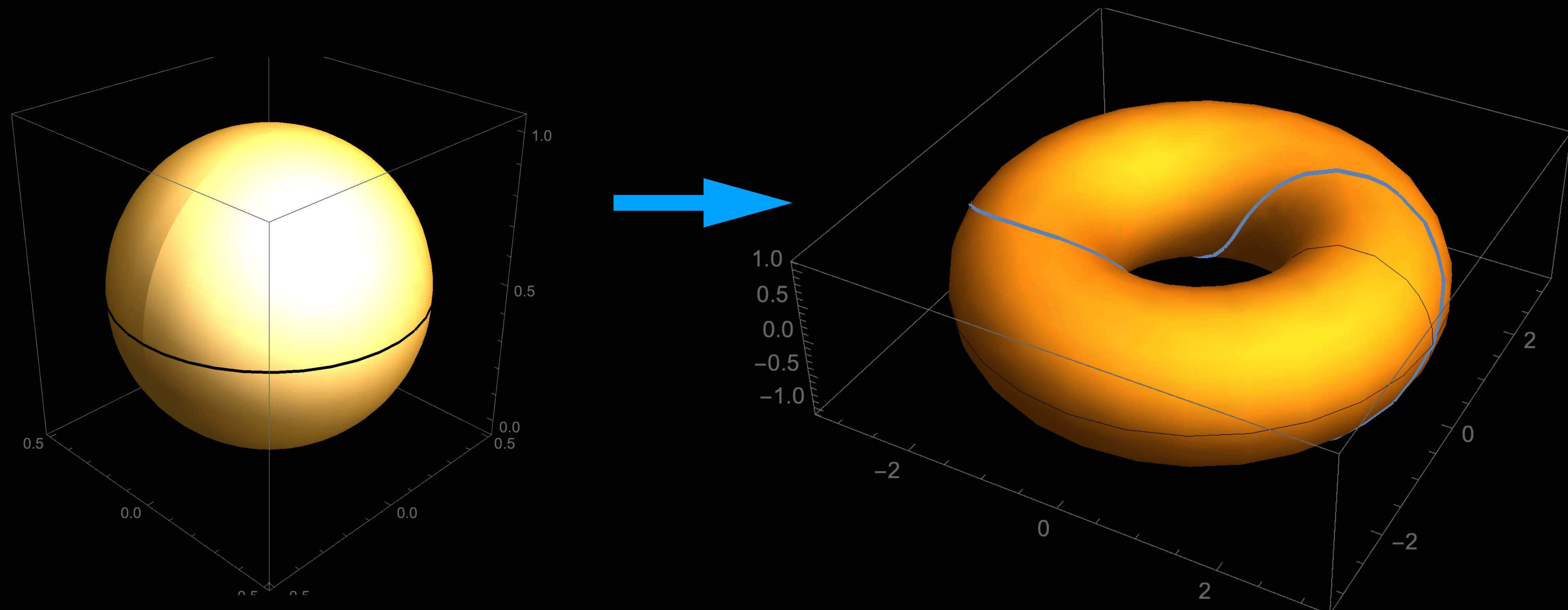
Iterated integrals

K. T. Chen, '77

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) ... \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k)$$

Multiple polylogarithms (MPLs) [Goncharov, '11]:

$$G(z_1, ..., z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$



$$E : \omega^2 - (z-z_1)(z-z_2)(z-z_3)(z-z_4) \; = \; 0$$

Vladimir's talk

Elliptic polylogarithms [Adams, Bogner, Weinzierl, '14,'15] [Bloch, Vanhove, '13] [Broedel, Duhr, Dulat, Tancredi, '17] [Brown, Levin,'11]

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 \ldots n_k \\ z_1 \ldots z_k \end{smallmatrix}; z; \tau\right) \; = \; \int_0^z dz^{'} g^{(n_1)}(z^{'} - z_1, \tau) \, \tilde{\Gamma}\left(\begin{smallmatrix} n_2 \ldots n_k \\ z_2 \ldots z_k \end{smallmatrix}; z; \tau\right)$$

Dependence of iterated integrals on elliptic curves enter through the appearance of elliptic periods in the integration kernels in $f_j(\lambda) d\lambda \; = \; \gamma^* \omega_j$.

Canonical form for DEs

$$d\vec{I} = A\vec{I}, \quad dA - A \wedge A = 0$$

$$\vec{J} = U\vec{I}, \quad d\vec{J}(x,\epsilon) = \epsilon\,(d\tilde{A})\,\vec{J}(x.\epsilon)$$

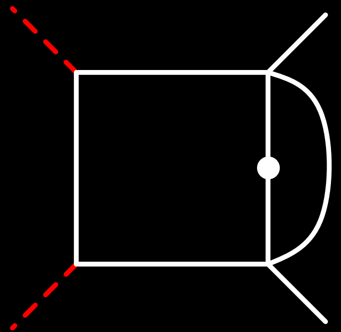
$$\tilde{A} = \sum_k A_k\ \log\alpha_k(x)$$

Find a basis that brings DEs to canonical form [Henn, '13].

For algebraic cases (involving roots), transformation to a canonical form may involve: algebraic functions in kinematic variables, period of the elliptic curve and their derivatives [Adams, Weinzierl, '18].

For example,

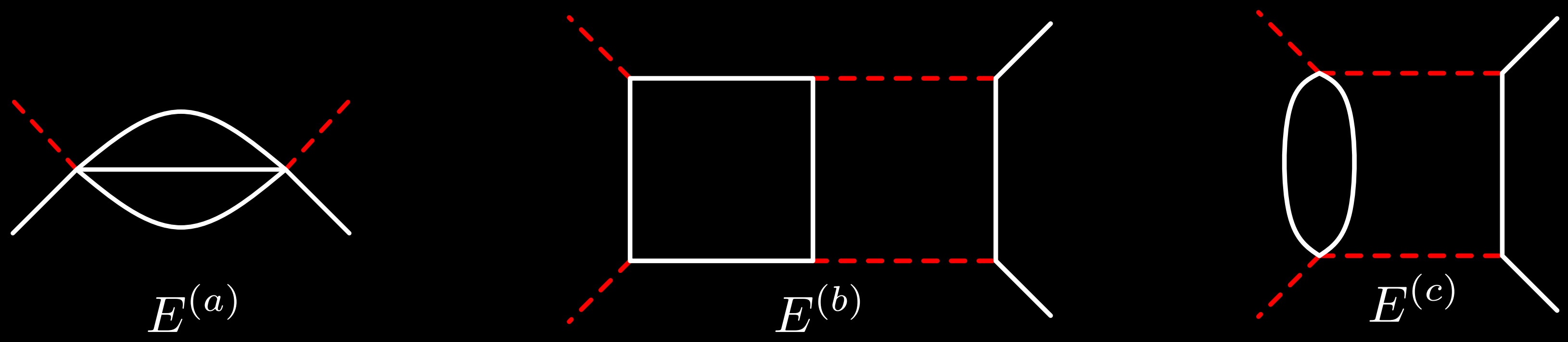
$$J \;=\; \epsilon^3\,\frac{(1-x)^2}{x}\,\frac{\pi}{\psi}$$



$\psi \propto K[x]$

Top loop integrals

MIs of topbox solved using DEs & expressed as iterated integrals in [Adams, E.C., Weinzierl, '18].



$$E^{(a)}: w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4\right)$$

$$E^{(b)}: w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)$$

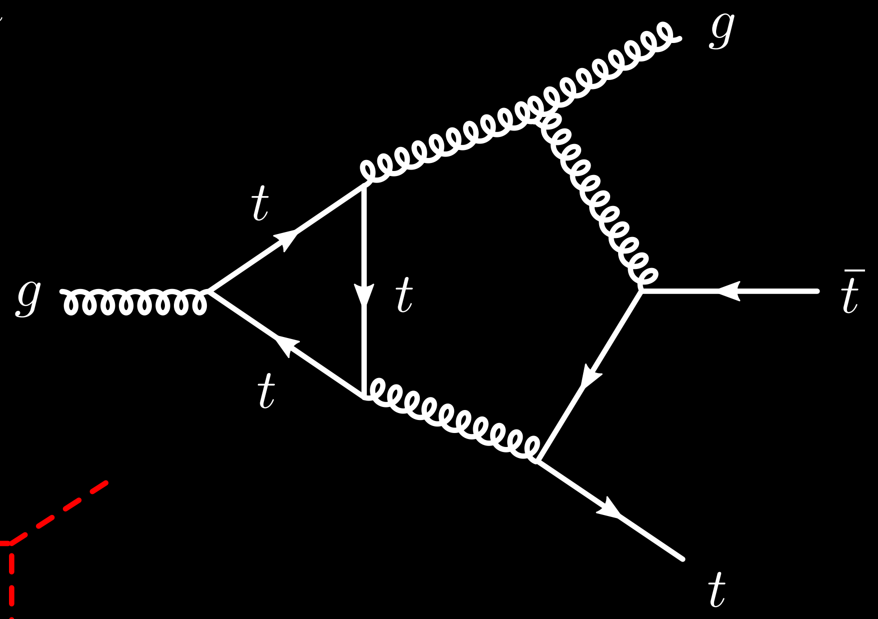
$$E^{(c)}: w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+\frac{2m^2\left(s+4t\right)}{\left(s-4m^2\right)}z+\frac{sm^2\left(m^2-4t\right)-4m^2t^2}{s-4m^2}\right)$$

DE system simplifies for $t = m^2$ as well as for $s = \infty$.

- For $t = m^2$ MIs expressible in terms of MPLs.
- For $s = \infty$ MIs expressible in terms of iterated integrals of kernels of sunrise.

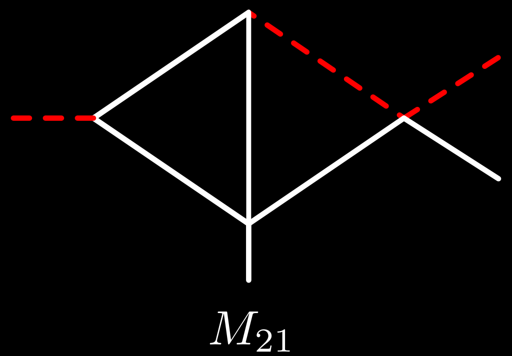
New integrals

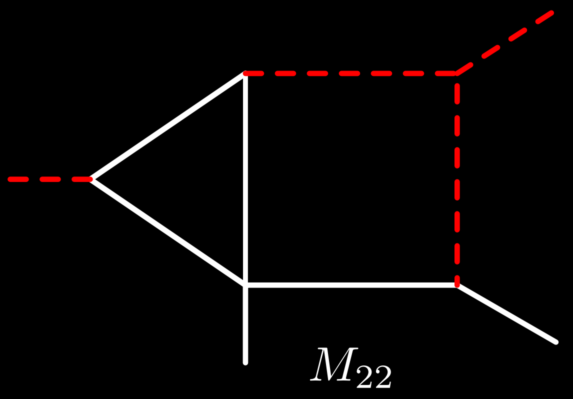
Feynman diagram contributing to $A^{(2),N_h}$



leads to new MIs.

Two integrals previously missing:





Canonical DEs for these MIs:

$$J^{21} = -\epsilon^4 (1-y) M_{21},$$

$$J^{22} = -\epsilon^4 \frac{(1-x)^2(1-y)}{x} M_{22}.$$

This family is also associated to elliptic curves due to sub-sectors from topbox.

Results

All four orders of J_{21} :

$$J_{21}^{(0)} = 0, \qquad J_{21}^{(1)} = 0,$$
$$J_{21}^{(2)} = 0, \qquad J_{21}^{(3)} = 0,$$
$$J_{21}^{(4)} = -\frac{\pi^4}{60} + (G(0,y) - 2G(1,y))\zeta_3 + G(0,0,0,1,y) - 2G(1,0,0,1,y) + \frac{\pi^2}{36} I_\gamma(g_0, f_3; \lambda) + \frac{1}{18} I_\gamma(g_0, f_3, \eta_0^{(a)}, f_3; \lambda),$$

Sunrise kernels

The kernels are given by: $g_0 = dy\,\frac{y+3}{y(1-y)}, \quad f_3 = \frac{(3dy\;\psi_1^a)}{\pi}, \quad \eta_0^{(a)} = -\frac{(2dy\;\pi^2)}{{\psi_1^a}^2(-9+y)(-1+y)y}$

Results for J_{22} given by:

$$J_{22}^{(0)} = 0, \qquad J_{22}^{(1)} = 0,$$
$$J_{22}^{(2)} = -\frac{1}{2} G(0,0,x),$$
$$J_{22}^{(3)} = G(1,y)G(0,0,x) + 3G(0,-1,0,x) - \frac{3}{2} G(0,0,0,x) + \frac{\pi^2}{4} G(0,x) + \frac{9}{2}\zeta_3.$$

Topbox kernels

$$J_{22}^{(4)} = I_\gamma(\ldots, \eta^{\frac{b}{a}}, \ldots) + \ldots$$

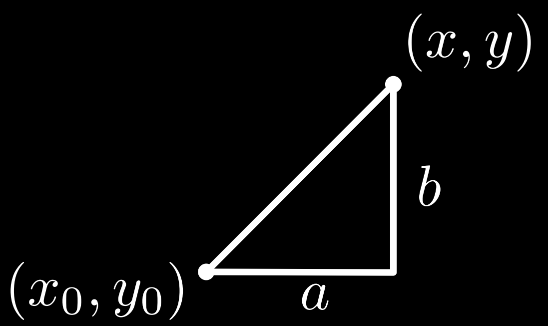
$$\eta^{\frac{a}{b}} = f(x,y)\frac{\psi_1^{(b)}}{\psi_1^{(a)}}dx + g(x,y)\frac{\psi_1^{(b)}}{\psi_1^{(a)}}dy$$

Numerical evaluation

Series expand the kernels around some points to compute iterated integrations.

Use the properties of iterated integrals.

Path decomposition formula,



$$\int_{a\,b} I(\omega_1 ... \omega_n; \lambda) = \sum_{i=0}^{n} \int_b I(\omega_1 ... \omega_i; \lambda) \int_a I(\omega_{i+1} ... \omega_n; \lambda),$$

where 0-fold integrals are defined by $I_\gamma(;\lambda) = 1.$

Need to make sure singularities of the kernels taken into account.

Properties of integrals often make evident which path to choose.

(see also Martijn's talk)

Euclidean region

For Euclidean region, $x = \frac{7}{120}, \qquad y = \frac{10}{11}$ $\qquad \frac{s}{m_t^2} = -\frac{(1-x)^2}{x}, \qquad \frac{t}{m_t^2} = y$

we may choose $a \ := \ dy = 0, \qquad b \ := dx = 0$

Series expand the kernels along b.

Use path decomposition formula iteratively for points far away.

We check the results in this region by comparing the numerical results for the squared matrix element against an independent computation.

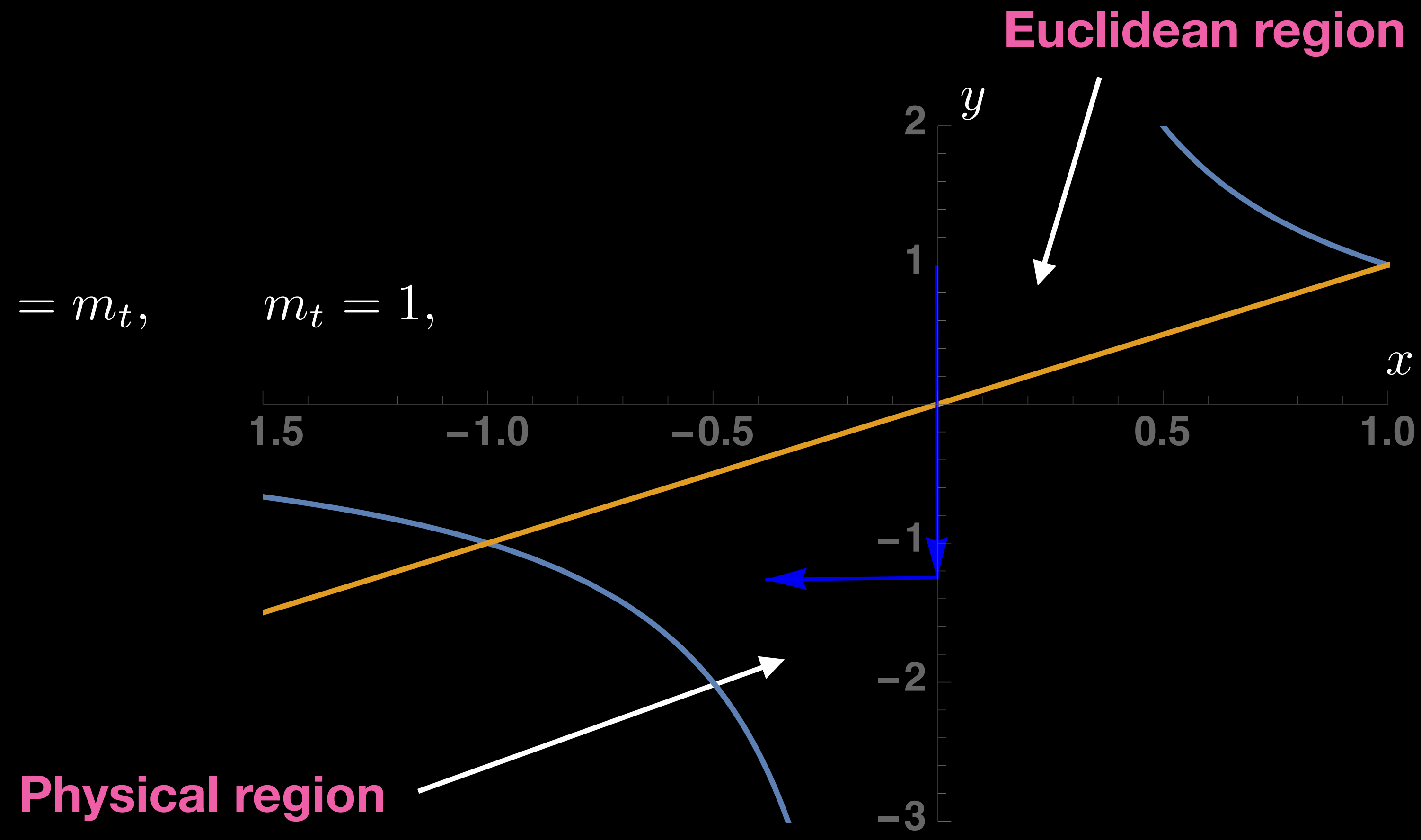
Physical region

Analytic continuation of integrals around physical branch points needed.

Use multiple (1-dimensional) path segments and use series expansion of integrands on each.

We evaluate at the point

$$\frac{s}{m_t^2} = 5, \qquad \frac{t}{m_t^2} = -\frac{5}{4}, \qquad \mu = m_t, \qquad m_t = 1,$$



take the path shown, using the path decomposition formula recursively.

Results for the physical region

- We were able to analytically continue all the integrals associated with elliptic curves a and b to the physical region.
- For the MIs containing curve c, we use FIESTA and pySecDec.
- We compared the finite remainder of the squared matrix element at the physical point against [Bärnreuther, Czakon, Fiedler, '13] and found good agreement.

Lessons

- The choice of number of path segments and their sizes is important.
- Analytic continuation of integrals depending on multiple elliptic curves not straightforward.
- Numerical evaluation in other regions needs automation.
- Stable and efficient evaluation over the whole physical scattering regions still needs work.

Summary & Outlook

- First analytic results of 2-loop $t\bar{t}gg$ amplitude with top quark loops.
- New integrals containing elliptic curves found and evaluated.
- Explored an approach for a direct numerical evaluation of iterated integrals with highly nontrivial kernels.
- Complications involved in analytic continuation of integrals with multiple elliptic curves studied.
- Construction of a nicer basis of transcendental function to remove some observed redundancy is under examination.

Thanks!