# Regge Limit Constraints for the Soft Anomalous Dimension at four loops

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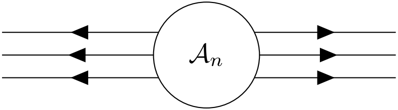
Falcioni,Milloy,Gardi,NM,Vernazza,2106.XXXXX

# Outline

- Introduction to the Soft Anomalous Dimension ($\Gamma_{\text{soft}}$)
- Three loops and the Regge Limit
- Four loop ansatz for the Soft Anomalous Dimension
- Signature separated expression $\Gamma_{\text{soft}}$ at four loop order NNLL
- Constraints for the kinematic functions

# Soft Anomalous dimension

It encodes the IR singularities of the amplitude




Amplitude factorisation

$$\mathcal{A}_n\left(\{p_i\},\epsilon,\alpha_s(\mu^2)\right) = Z_n\left(\{p_i\},\{\mathbf{T}_i\},\epsilon,\alpha_s(\mu_f^2)\right)\mathcal{H}_n\left(\{p_i\},\frac{\mu_f}{\mu},\epsilon,\alpha_s(\mu^2)\right)$$

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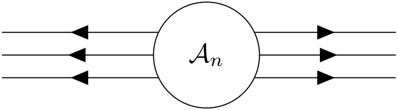


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Renormalisation group equation

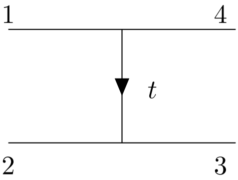
$$\frac{d}{d\ln\mu_f}Z_n\left(\{p_i\},\{\mathbf{T}_i\},\epsilon,\alpha_s(\mu_f)\right) = -Z_n\,\Gamma_n\left(\{p_i\},\{\mathbf{T}_i\},\mu_f,\alpha_s(\mu_f)\right)$$

$$Z_n = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu_f^2}\frac{d\lambda^2}{\lambda^2}\Gamma_n\left(\{p_i\},\{\mathbf{T}_i\},\lambda,\alpha_s(\lambda^2)\right)\right\}$$

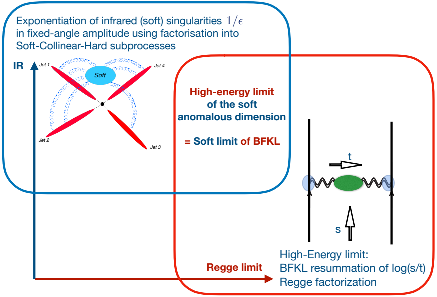
# Regge Limit: $2 \to 2$ scattering

Massless partons

Regge Limit: $s \gg -t$ with
$u \to -s$




$s = s_{12} = s_{34}$
$u = s_{13} = s_{24}$
$t = s_{14} = s_{23}$




[Gardi,Caron-Huot,
Reichel,Vernazza, 1912.10883]

# Amplitude Properties in the Regge Limit

- Signature: parity under $s \leftrightarrow u$
- Amplitude can be separated by signature

$$\mathcal{M}^{(\pm)}(s,t) = \mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t).$$

- Define the signature even log [Caron-Huot, Gardi, Vernazza 1701.05241]

$$L \equiv \log\left|\frac{s}{t}\right| - \frac{i\pi}{2} = \frac{1}{2}\left(\log\left(\frac{-s-i0}{-t}\right) + \log\left(\frac{-u-i0}{-t}\right)\right)$$

- Within the amplitude the coefficients of the signature even log are either even and imaginary or odd and real under $s \leftrightarrow u$.

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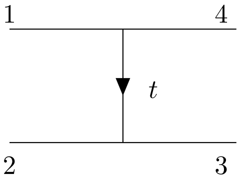
- Within the amplitude the coefficients of the signature even log are either even and imaginary or odd and real under $s \leftrightarrow u$.
- $\Gamma_{\text{soft}}$ multiplies a tree level amplitude which is odd and real $\implies$ $\Gamma_{\text{soft}}$ kinematic functions must then either be even and real or odd and imaginary.

# Colour flow notation




$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 \qquad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 \qquad \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3$$

$$\mathbf{T}_{s-u}^2 = \frac{1}{2}\left(\mathbf{T}_s^2 - \mathbf{T}_u^2\right)$$

Swapping $s \leftrightarrow u$ is equivalent to swapping $2 \leftrightarrow 3$ or $1 \leftrightarrow 4$ .

Bose symmetry means that the symmetry properties of the colour structures are shared by the kinematic functions

# Soft Anomalous Dimension

[Becher, Neubert 2009, Gardi, Magnea 2009]
The dipole formula and its corrections given by $\Delta_n$.

$$\Gamma_n(\{p_i\},\{\mathbf{T}_i\},\mu,\alpha_s)) = \Gamma_n^{\text{dip.}}(\{p_i\},\{\mathbf{T}_i\},\mu,\alpha_s) + \Delta_n(\{\beta_{ijkl}\},\{\mathbf{T}_i\},\alpha_s) + \Delta_n(\{l_{ij}\},\frac{d_{RR_i}}{N_i},\alpha_s)$$

Momentum dependence

$$l_{ij} = \ln\frac{-s_{ij}}{\mu^2}$$

Conformal Invariant Cross Ratio

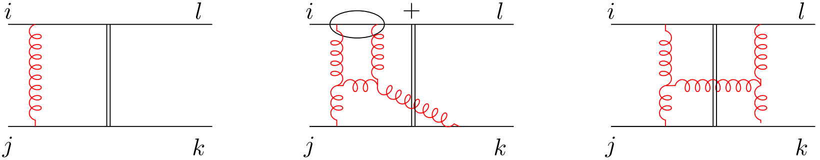
$$\beta_{ijkl} = \ln\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})} = l_{ij} + l_{kl} - l_{ik} - l_{jl}$$

Quartic Casimir starts appearing at four loops.

# Soft Anomalous Dimension at Three loops

Calculated explicitly in [Almelid,Duhr, Gardi 1507.0004]
Bootstrapped in [Almelid, Duhr, Gardi, McLeod, White 1706.10162], talks by R. Schabinger, N. Kidonakis, C. Dlapa

$$\Gamma_n(\{\beta_{ij}\},\mu,\alpha_s(\mu^2)) = -\sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\,\widehat{\gamma}_K(\alpha_s)\,l_{ij} + \sum_i^n \gamma_i(\alpha_s)$$
$$+ f(\alpha_s)\sum_{(i,j,k)} f^{ade}f^{bce}\{\mathbf{T}_i^a,\mathbf{T}_i^b\}\mathbf{T}_j^c\mathbf{T}_k^d$$
$$+ \sum_{(i,j,k,l)} f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\beta_{ijlk},\beta_{iklj};\alpha_s)$$



# Even Signature $\Gamma_{\text{soft}}$ in high energy limit

| | $L^0$ | $L^1$ | $L^2$ | $L^3$ | $L^4$ | $L^5$ |
|---|---|---|---|---|---|---|
| $\alpha_s^1$ | $\frac{1}{4}\widehat{\gamma}_K^{(1)} \ln \frac{-t}{\lambda^2} \sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(1)}$ | $\frac{1}{2}\widehat{\gamma}_K^{(1)} \mathbf{T}_t^2$ | | | | |
| $\alpha_s^2$ | $\frac{1}{4}\widehat{\gamma}_K^{(2)} \ln \frac{-t}{\lambda^2} \sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(2)}$ | $\frac{1}{2}\widehat{\gamma}_K^{(2)} \mathbf{T}_t^2$ | 0 | | | |
| $\alpha_s^3$ | $\frac{1}{4}\widehat{\gamma}_K^{(3)} \ln \frac{-t}{\lambda^2} \sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(3)} + \Delta^{(+,3,0)}$ | $\frac{1}{2}\widehat{\gamma}_K^{(3)} \mathbf{T}_t^2$ | 0 | 0 | | |
| $\alpha_s^4$ | | | $\Delta^{(+,4,2)}$ | 0 | 0 | |
| $\alpha_s^5$ | | | | | 0 | 0 |
| $\alpha_s^6$ | | | | | | 0 |

- Table from [Caron-Huot,Gardi,Reichel, Vernazza 1912.10883]
- $\Delta^{(+,3,0)}$ determined from the soft anomalous dimension at three loops [Almelid,Duhr, Gardi 1507.00047]
- With $\Delta^{(+,4,2)}$ calculated in [Falcioni,Gardi,Milloy,Vernazza,2012.00613]
- Yellow box at $\mathcal{O}(\alpha_s^4)$ is where the Casimir scaling is broken with the quartic Casimir.

# Odd Signature $\Gamma_{\text{soft}}$ in high energy limit

| | $L^0$ | $L^1$ | $L^2$ | $L^3$ | $L^4$ | $L^5$ |
|---|---|---|---|---|---|---|
| $\alpha_s^1$ | $\frac{1}{2}\widehat{\gamma}_K^{(1)} i\pi \mathbf{T}_{s-u}^2$ | 0 | | | | |
| $\alpha_s^2$ | $\frac{1}{2}\widehat{\gamma}_K^{(2)} i\pi \mathbf{T}_{s-u}^2$ | 0 | 0 | | | |
| $\alpha_s^3$ | $\frac{1}{2}\widehat{\gamma}_K^{(3)} i\pi \mathbf{T}_{s-u}^2 + \Delta^{(-,3,0)}$ | $\Delta^{(-,3,1)}$ | 0 | 0 | | |
| $\alpha_s^4$ | | | | $\Gamma_{\text{NLL}}^{(-,4)}$ | 0 | |
| $\alpha_s^5$ | | | | | $\Gamma_{\text{NLL}}^{(-,5)}$ | 0 |
| $\alpha_s^6$ | | | | | | $\Gamma_{\text{NLL}}^{(-,6)}$ |

- $\Delta^{(-,3,0)}$ and $\Delta^{(-,3,1)}$ determined from the $\Gamma_{\text{soft}}$ beyond-dipole correction [Almelid, Duhr, Gardi 1507.00047]
- The $\Gamma_{\text{NLL}}^{(-,\ell)}$ tower starting at four loops is known to all orders based on [Caron-Huot, Gardi,Reichel Vernazza, 1711.04850].

# Soft Anomalous Dimension at Three loops in the Regge limit

Red: from the ansatz   Blue: information from the Regge limit

$$\Gamma^{(3)}_{\rm NNLL} = \frac{1}{2}\widehat{\gamma}_K^{(3)}\mathbf{T}_t^2 L - \Big[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\Big]\mathcal{F}^{(-,3,1)} L + 2\Big[\mathbf{T}_{s-u}^2, [\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\Big]\mathcal{F}^{(+,3,1)} L$$

$$\Gamma^{(-,3)}_{\rm NNLL} = i\pi\zeta_3\Big[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\Big] L \qquad \Gamma^{(+,3)}_{\rm NNLL} = \frac{1}{2}\widehat{\gamma}_K^{(3)}\mathbf{T}_t^2 L$$

- Determined in [Caron-Huot, Gardi, Vernazza 1701.05241] using the exact calcuation at NNLL for the soft anomalous dimension in [Almelid, Duhr, Gardi 1507.00047].
- NLL and NNLL Regge limit results were crucial in bootstrapping the soft anomalous dimension at three loops in [Almelid, Duhr, Gardi, McLeod, White 1706.10162]

# Four loop Regge limit results

At four loops, NLL is at $L^3$ from [Caron-Huot, Gardi, Reichel, Vernazza 1711.04850]

$$\Gamma_{\rm NLL}^{(-,4)} = -i\pi\frac{\zeta_3}{24}\left[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\right]\mathbf{T}_t^2 L^3$$

$$\Gamma_{\rm NLL}^{(+,4)} = 0$$

From the odd amplitude at four loops, the NNLL result [Falcioni, Gardi,Milloy, Vernazza 2012.00613] is

$$\begin{aligned}\Gamma_{\rm NNLL}^{(+,4)} &= \zeta_2\zeta_3\Bigg(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24} + \frac{1}{4}\mathbf{T}_t^2[\mathbf{T}_t^2,(\mathbf{T}_{s-u}^2)^2] \\ &\quad + \frac{3}{4}[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2\Bigg)L^2 \\ &= \zeta_2\zeta_3\mathbf{C}^{(+,4,2)}L^2\end{aligned}$$

# Four loop n parton ansatz

The soft anomalous dimension at four loops from [Becher, Neubert 1908.11379] contains two new types of colour structures

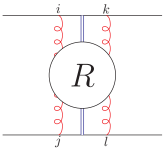
$$\begin{aligned}
\Gamma_n(\{\beta_{ij}\},\mu,\alpha_s(\mu^2)) = &-\sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\,\widehat{\gamma}_K(\alpha_s)\, l_{ij} + \sum_i^n \gamma_i(\alpha_s) \\
&+ f(\alpha_s)\sum_{(i,j,k)} f^{ade}f^{bce}\{\mathbf{T}_i^a,\mathbf{T}_i^b\}\mathbf{T}_j^c\mathbf{T}_k^d \\
&+ \sum_{(i,j,k,l)} f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\beta_{ijlk},\beta_{iklj};\alpha_s) \\
&+ \Gamma_n(\{\beta_{ij}\},\mu,\alpha_s(\mu^2))|_{\text{Quartic Casimir}} \\
&+ \Gamma_n(\{\beta_{ij}\},\mu,\alpha_s(\mu^2))|_{\text{Five generators}}
\end{aligned}$$

# Four loop n parton ansatz: Quartic Casimir

$$\Gamma_n(\{\beta_{ij}\}, \alpha_s(\mu^2)) = -\sum_R g^R(\alpha_s)\left[\sum_{(i,j)} \left(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij}\right) l_{ij} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk}\, l_{ij}\right] + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, \mathcal{G}^R(\beta_{ijlk}, \beta_{iklj}; \alpha_s)$$

Completely symmetric colour structure

$$\mathcal{D}^R_{ijkl} = \frac{1}{6}\sum_{\sigma\in S_3} \mathrm{Tr}_R\left(T^a T^{\sigma(b)} T^{\sigma(c)} T^{\sigma(d)}\right) \mathbf{T}^a_i \mathbf{T}^b_j \mathbf{T}^c_k \mathbf{T}^d_l$$

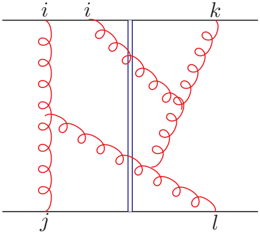



Quartic Casimir : $\frac{d_{RR_i}}{N_i} = \mathcal{D}^R_{iiii}$

# Four loop n parton ansatz: Five generators

$$\Gamma_n(\{\beta_{ij}\}, \mu, \alpha_s(\mu^2)) = \sum_{(i,j,k,l)} \mathcal{T}_{ijkli}\, \mathcal{H}_1(\beta_{ijlk}, \beta_{iklj}; \alpha_s) + \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, \mathcal{H}_2(\beta_{ijkl}, \beta_{ijmk}, \beta_{ikmj}, \beta_{jiml}, \beta_{jlmi}; \alpha_s)$$

$$\mathcal{T}_{ijklm} = i f^{adf} f^{bcg} f^{efg} \left( \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \mathbf{T}_m^e \right)_+$$



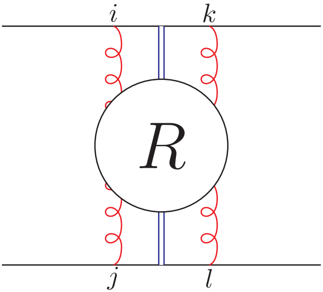
# Four loop order NNLL accuracy

$$\Gamma^{(4)}_{\mathrm{NNLL}}(\{\beta_{ij}\},\mu) = \sum_{(1,2,3,4)} f^{ade} f^{bce} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \, \mathcal{F}_R^{(4)}(\beta_{ijlk}, \beta_{iklj})$$
$$+ \sum_R \sum_{(1,2,3,4)} \mathcal{D}_{ijkl}^R \mathcal{G}_R^{(4)}(\beta_{ijlk}, \beta_{iklj})$$
$$+ \sum_{(1,2,3,4)} \mathcal{T}_{ijkli} \mathcal{H}_1^{(4)}(\beta_{ijlk}, \beta_{iklj}).$$

- $H_1$ and $F$ are antisymmetric under the swapping of their two arguments, unlike $G$ which is fully symmetric
- Group even and odd terms under $2 \leftrightarrow 3$ or $1 \leftrightarrow 4$ which makes $s \leftrightarrow u$ parity manifest.

# Key Steps for comparing $\Gamma_{\text{soft}}$ in the Regge limit to the results

- Analyse the matter content present in the amplitude to deduce what representation would appear at NNLL accuracy in the soft anomalous dimension




- In the Regge limit, the kinematic functions have an expansion in the signature even logarithm for example

$$P|_{\text{regge limit}} = P^{(0)} + P^{(1)} L + P^{(2)} L^2 + P^{(3)} L^3$$

# Signature odd $\Gamma_{\text{soft}}$ at NLL at four loops

From BFKL resummation in[Caron-Huot, Reichel, Gardi, Vernazza 2017]

$$\Gamma_{\text{NLL}}^{(-,4)} = -i\pi \frac{\zeta_3}{24} \left[ \mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right] \mathbf{T}_t^2 L^3$$

Equating to the Regge limit of [Becher, Neubert 2019]

$$\begin{aligned}
\Gamma_{\text{NLL}}^{(-,4)} = {} & \mathcal{F}_A^{(-,4,3)} \left[ \mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right] L^3 \\
& + \frac{1}{8} \left[ \mathbf{T}_t^2, \left[ \mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right] \right] \left( -\tilde{\mathcal{H}}_1^{(-,4,3)} + 2\mathcal{H}_1^{(-,4,3)} \right) L^3 \\
& + \frac{1}{2} \left[ \mathbf{T}_{s-u}^2, [\mathbf{T}_{s-u}^2, [\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]] \right] \mathcal{H}_1^{(-,4,3)} L^3
\end{aligned}$$

Using $\mathbf{T}_t^2 \mathcal{M}_{\text{tree}} = C_A \mathcal{M}_{\text{tree}}$

# Constraints for signature odd $\Gamma_{\text{soft}}$ at NLL at four loops

At $L^3$ order,

$$\mathcal{F}_A^{(-,4)} = C_A i\pi \frac{\zeta_3}{24} L^3 + O(L^2) \qquad \mathcal{F}_F^{(-,4)} = 0L^3 + O(L^2)$$

$$\mathcal{H}_1^{(-,4)} = 0L^3 + O(L^2) \qquad \tilde{\mathcal{H}}_1^{(-,4)} = 0L^3 + O(L^2)$$

- At $L^2$ order, the amplitude is currently unknown.
- Consistent with work in [Vladimirov 1707.07606] to have no $H$ sector (terms with five generators)

# Signature even $\Gamma_{\mathrm{soft}}$ at NNLL at four loops

From the reggeon ladders in [Falcioni, Gardi, Milloy,Vernazza 2020]

$$\Gamma^{(+,4)}|_{\text{Regge Limit}} = 0L^3 + \zeta_2\zeta_3 \mathbf{C}_{\Delta}^{(+,4,2)} L^2 + \mathcal{O}(L)$$

Expanding the functions in $\Gamma^{(+,4)}_{\mathrm{NNLL}}$ in the signature even log and equating to

$$\begin{aligned}
\Gamma^{(+,4,2)} &= 2\mathbf{C}_{\Delta}^{(+,4,2)} \mathcal{G}_A^{(+,4,2)} \\
&+ \left[\mathbf{T}_{s-u}^2, [\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\right] \left\{ 2\mathcal{F}_A^{(+,4,2)} + C_A \left( \frac{1}{2}\mathcal{G}_A^{(+,4,2)} + \mathcal{H}_1^{(+,4,2)} \right) \right\} \\
&- \frac{1}{4}\left( 3\mathbf{T}_{s-u}^2 [\mathbf{T}_{s-u}^2, \mathbf{T}_t^2] \mathbf{T}_t^2 + \mathbf{T}_{s-u}^2 \mathbf{T}_t^2 [\mathbf{T}_{s-u}^2, \mathbf{T}_t^2] \right) \mathcal{H}_1^{(+,4,2)}
\end{aligned}$$

# Constraints for signature even $\Gamma_{\text{soft}}$ at NNLL at four loops

At $L^3$ order, all symmetric functions are zero in the Regge limit.

At $L^2$ order

$$\mathcal{F}_A^{(+,4)} = -C_A \frac{\zeta_2\zeta_3}{8} L^2 + O(L) \qquad \mathcal{F}_F^{(+,4)} = 0L^2 + O(L)$$

$$\mathcal{G}_A^{(+,4)} = \frac{\zeta_2\zeta_3}{2} L^2 + O(L) \qquad \mathcal{G}_F^{(+,4)} = 0L^2 + O(L)$$

$$\mathcal{H}_1^{(+,4)} = 0L^2 + O(L)$$

# Summary of constraints

Constraints for the kinematic functions in the $\Gamma_{\text{soft}}$ for $2 \to 2$ scattering of massless partons at four loop order in the Regge Limit separated by signature

Signature even

$$\mathcal{F}_F^{(+,4)} = 0L^2 + O(L)$$
$$\mathcal{F}_A^{(+,4)} = -C_A \frac{\zeta_2\zeta_3}{8} L^2 + O(L)$$
$$\mathcal{G}_A^{(+,4)} = \frac{\zeta_2\zeta_3}{2} L^2 + O(L)$$
$$\mathcal{G}_F^{(+,4)} = 0L^2 + O(L)$$
$$\mathcal{H}_1^{(+,4)} = 0L^2 + O(L)$$

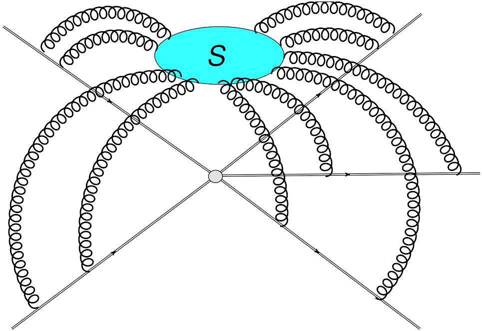
Signature odd

$$\mathcal{F}_F^{(-,4)} = 0L^3 + O(L^2)$$
$$\mathcal{F}_A^{(-,4)} = -C_A i\pi \frac{\zeta_3}{24} L^3 + O(L^2)$$
$$\mathcal{H}_1^{(-,4)} = 0L^3 + O(L^2)$$
$$\mathcal{H}_1^{(-,4)} = 0L^3 + O(L^2)$$

# Conclusion and outlook

- Expressed the four loop order NNLL terms of the soft anomalous dimension for massless partons with $s \leftrightarrow u$ signature explicit.
- Provided constraints for the kinematic functions at four loops from the NLL even amplitude and NNLL odd amplitude in the Regge limit
- The constraints will be used in bootstrapping the soft anomalous dimension at four loops

# Extra Slides




- Massless partons ansatz $n = 4$ parton 4 loop order
- $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}_1$ terms with more detail

# Massless n=4 parton ansatz four loop order

$$\Gamma_4^{(4)} = -\sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \widehat{\gamma}_K^{(4)} \, l_{ij} + \sum_{i}^{4} \gamma_i^{(4)}$$

1 loop

$$+ f^{(4,R)} \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl} \, \mathcal{F}_R^{(4)}(\beta_{ijlk}, \beta_{iklj})$$

3 loop

$$- \sum_R g^{(4,R)} \left[ \sum_{(i,j)} (\mathcal{D}_{iijj}^R + 2\mathcal{D}_{iiij}^R) \, l_{ij} + \sum_{(i,j,k)} \mathcal{D}_{ijkk}^R \, l_{ij} \right]$$
$$+ \sum_R \sum_{(i,j,k,l)} \mathcal{D}_{ijkl}^R \, \mathcal{G}_R^4 \beta_{ijlk}, \beta_{iklj}; \alpha_s) + \sum_{(i,j,k,l)} \mathcal{T}_{ijkli} \, \mathcal{H}_1^{(4)}(\beta_{ijlk}, \beta_{iklj}) \quad (1)$$

# Quartic Casimir and the cusp anomalous dimension

$$\Gamma^{i}_{\text{cusp}}(\alpha_s) = \widehat{\gamma}_K(\alpha_s) C_i + 2 \sum_R g^R(\alpha_s) \frac{d_{RR_i}}{N_i}$$

- Multiplies $l_{ij}$ so at four loop order only starts to contribute at $\mathrm{N^3LL}$
- Four loop cusp anomalous dimension was computed analytically for QCD and $\mathcal{N}=4$ super Yang-Mills in [Henn, Korchemsky,Mistlberger 1911.10174,von Manteuffel, Panzer,Schabinger 2002.04617]
- Governs the renormalisation of null Wilson lines [Korchemsky, Korchemskaya, 1992]
- IR divergences of scattering amplitudes and form factors [Sterman 1981; Korchemsky, Radyushkin 1992; Korchemskaya, Korchemsky, 1996; Dixon, Magnea, Sterman 2008, Becher, Neubert, 2009, . . . ]

# F term

$$\Gamma_{4,\mathcal{F}}(\{\beta_{ij}\}, \mu, \alpha_s(\mu^2)) = \sum_{(i,j,k,l)} f^{ade} f^{bce} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \mathcal{F}(\beta_{ijlk}, \beta_{iklj}; \alpha_s),$$

$$\mathcal{F}^{(+)}(\{\beta_{ij}\}, \alpha_s) \equiv \frac{1}{2} \left(\mathcal{F}(\beta_{1324}, \beta_{1423}; \alpha_s) + \mathcal{F}(\beta_{1234}, \beta_{1432}; \alpha_s)\right),$$

$$\mathcal{F}^{(-)}(\{\beta_{ij}\}, \alpha_s) \equiv \frac{1}{2} \left(\mathcal{F}(\beta_{1234}, \beta_{1432}; \alpha_s - \mathcal{F}(\beta_{1324}, \beta_{1423}; \alpha_s)\right) + \mathcal{F}(\beta_{1243}, \beta_{1342}; \alpha_s).$$

# F term

Symmetry under $2 \leftrightarrow 3$ which is $s \leftrightarrow u$

$$\Gamma_{4,\mathcal{F}}(\{\beta_{ij}\},\mu,\alpha_s) = 8\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\left[\left(f^{abe}f^{cde}+f^{ace}f^{bde}\right)\mathcal{F}^{(+)}(\{\beta_{ij}\},\alpha_s) + f^{ade}f^{bce}\mathcal{F}^{(-)}(\{\beta_{ij}\},\alpha_s)\right].$$

$$\Gamma_{4,\mathcal{F}}^{(4)}(\{\beta_{ij}\},\mu) = -\mathcal{F}_R^{(-,4)}\left[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\right]+2\mathcal{F}_R^{(+,4)}\left[\mathbf{T}_{s-u}^2,[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\right], \quad (2)$$

# Fully Symmetric quartic contribution (G term)

$$\Gamma_{4,\mathcal{G}}(\{\beta_{ij}\},\mu,\alpha_s(\mu^2)) = \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\,\mathcal{G}_R(\beta_{ijlk},\beta_{iklj};\alpha_s), \qquad (3)$$

with

$$\mathcal{D}^R_{ijkl} = \frac{1}{6}\sum_{\sigma\in S_3} \mathrm{Tr}_R\left(T^a T^{\sigma(b)} T^{\sigma(c)} T^{\sigma(d)}\right) \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d. \qquad (4)$$

The odd amplitude at NNLL only has gluonic contributions so for the signature even part of the soft anomalous dimension only the adjoint part is required.

$$\Gamma^A_{4,\mathcal{G}}(\{\beta_{ij}\},\mu,\alpha_s(\mu^2)) = 24\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\Big[\mathrm{Tr}_A\big(T^a T^b T^c T^d\big) + \frac{\mathbf{C}_A}{6}\left(f^{abe}f^{cde} - f^{ade}f^{bce}\right)\Big]\mathcal{G}^A(\{\beta_{ij}\}\alpha_s). \qquad (5)$$

# Fully Symmetric quartic contribution (G term)

Commutator identity

$$f^{abc}\mathbf{T}^c = -i[\mathbf{T}^a, \mathbf{T}^b]$$

$$\begin{aligned}
f^{abe}f^{cde}\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d &= -if^{cde}[\mathbf{T}_1^b, \mathbf{T}_1^e]\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \\
&= -[\mathbf{T}_1\cdot\mathbf{T}_2, \mathbf{T}_1^e][\mathbf{T}_3^d, \mathbf{T}_3^e]\mathbf{T}_4^d \\
&= -[\mathbf{T}_1\cdot\mathbf{T}_2, \mathbf{T}_1^e][\mathbf{T}_3\cdot\mathbf{T}_4, \mathbf{T}_3^e] \\
&= -\Big[\mathbf{T}_1\cdot\mathbf{T}_2, [\mathbf{T}_3\cdot\mathbf{T}_4, \mathbf{T}_1\cdot\mathbf{T}_3]\Big] \\
&= \frac{1}{16}\left(-\Big[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\Big] + 2\Big[\mathbf{T}_{s-u}^2, [\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\Big]\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}\Big[F^aF^bF^cF^d\Big]\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d &= \frac{1}{2}\Big[\big[\mathbf{T}_1^b, \mathbf{T}_1^\gamma\big], \mathbf{T}_1^\alpha\Big]\mathbf{T}_2^b\Big[\big[\mathbf{T}_3^d, \mathbf{T}_3^\alpha\big], \mathbf{T}_3^\gamma\Big]\mathbf{T}_4^d \\
&+ \frac{1}{2}\mathbf{T}_1^a\Big[\mathbf{T}_2^\gamma, \big[\mathbf{T}_2^\alpha, \mathbf{T}_2^a\big]\Big]\mathbf{T}_3^c\Big[\mathbf{T}_4^\alpha, \big[\mathbf{T}_4^\gamma, \mathbf{T}_4^c\big]\Big]
\end{aligned}$$

# Fully Symmetric quartic contribution (G term)

$$\begin{aligned}\Gamma_{4,\mathcal{G}}(\{\beta_{ij}\},\mu,\alpha_s(\mu^2)) &= \left(2\left(\frac{d_{AA}}{N_A}-\frac{C_A^4}{24}\right)-\frac{1}{2}\mathbf{T}_t^2[(\mathbf{T}_{s-u}^2)^2,\mathbf{T}_t^2]\right.\\ &\left.+\frac{3}{2}[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2+\frac{\mathbf{C}_A}{2}\left[\mathbf{T}_{s-u}^2,[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\right]\right)\\ &\times\mathcal{G}^A(\{\beta_{ij}\};\alpha_s)\\ &=\left(2C^{(+,4,2)}-\frac{\mathbf{C}_A}{2}\left[\mathbf{T}_{s-u}^2,[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\right]\right)\\ &\times\mathcal{G}^A(\{\beta_{ij}\};\alpha_s).\end{aligned}$$

# Five generators: H1 term for 4 partons

$$\Gamma_{\mathcal{H}_1}(\{\beta_{ij}\},\mu,\alpha_s(\mu^2)) = \sum_{(1,2,3,4)} if^{adg} f^{bch} f^{egh} \{\mathbf{T}_i^a, \mathbf{T}_i^e\} \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \times \mathcal{H}_1(\beta_{ijlk}, \beta_{iklj}; \alpha_s)$$

$$\mathcal{T}_{ijkli} = f^{adg} f^{bch} f^{egh} \mathbf{T}_i^a \mathbf{T}_i^e \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d + \frac{\mathbf{C}_A}{4} f^{ade} f^{bce} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d$$

$$\mathcal{H}_1^{(+)}(\{\beta_{ij}\},\alpha_s) = \frac{1}{2}\Big(H_1(\beta_{1324},\beta_{1423};\alpha_s) + H_1(\beta_{1234},\beta_{1432};\alpha_s)\Big)$$

$$\mathcal{H}_1^{(-)}(\{\beta_{ij}\},\alpha_s) = \frac{1}{2}\Big(H_1(\beta_{1324},\beta_{1423};\alpha_s) - H_1(\beta_{1234},\beta_{1432};\alpha_s)\Big)$$

$$\tilde{\mathcal{H}}_1^{(-)}(\{\beta_{ij}\},\alpha_s) = H_1(\beta_{1243},\beta_{1342};\alpha_s).$$

# Five generators: H1 term for 4 partons

$$
\begin{aligned}
\Gamma_{\mathcal{H}_1}(\{\beta_{ij}\},\alpha_s(\mu^2)) = {} & \frac{\mathcal{H}_1^{(+)}(\{\beta_{ij}\},\alpha_s)}{4}\Bigg(4C_A\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big] \\
& - 3C_A\mathbf{T}^2_{s-u}[\mathbf{T}^2_{s-u},\mathbf{T}^2_t] - \mathbf{T}^2_t[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\mathbf{T}^2_{s-u}\Bigg) \\
& + \tilde{\mathcal{H}}_1^{(-)}(\{\beta_{ij}\},\alpha_s)\left(\frac{1}{4}\Big[\mathbf{T}^2_t,\big[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\big]\Big]\right) \\
& + \mathcal{H}_1^{(-)}(\{\beta_{ij}\},\alpha_s)\Bigg(\frac{1}{2}\Big[\mathbf{T}^2_{s-u},\big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\big]\Big] \\
& - \frac{1}{8}\Big[\mathbf{T}^2_t,\big[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\big]\Big]\Bigg)
\end{aligned}
$$