TWO-PARTON SCATTERING IN THE HIGH-ENERGY LIMIT: CLIMBING TWO- AND THREE-REGGEON LADDERS

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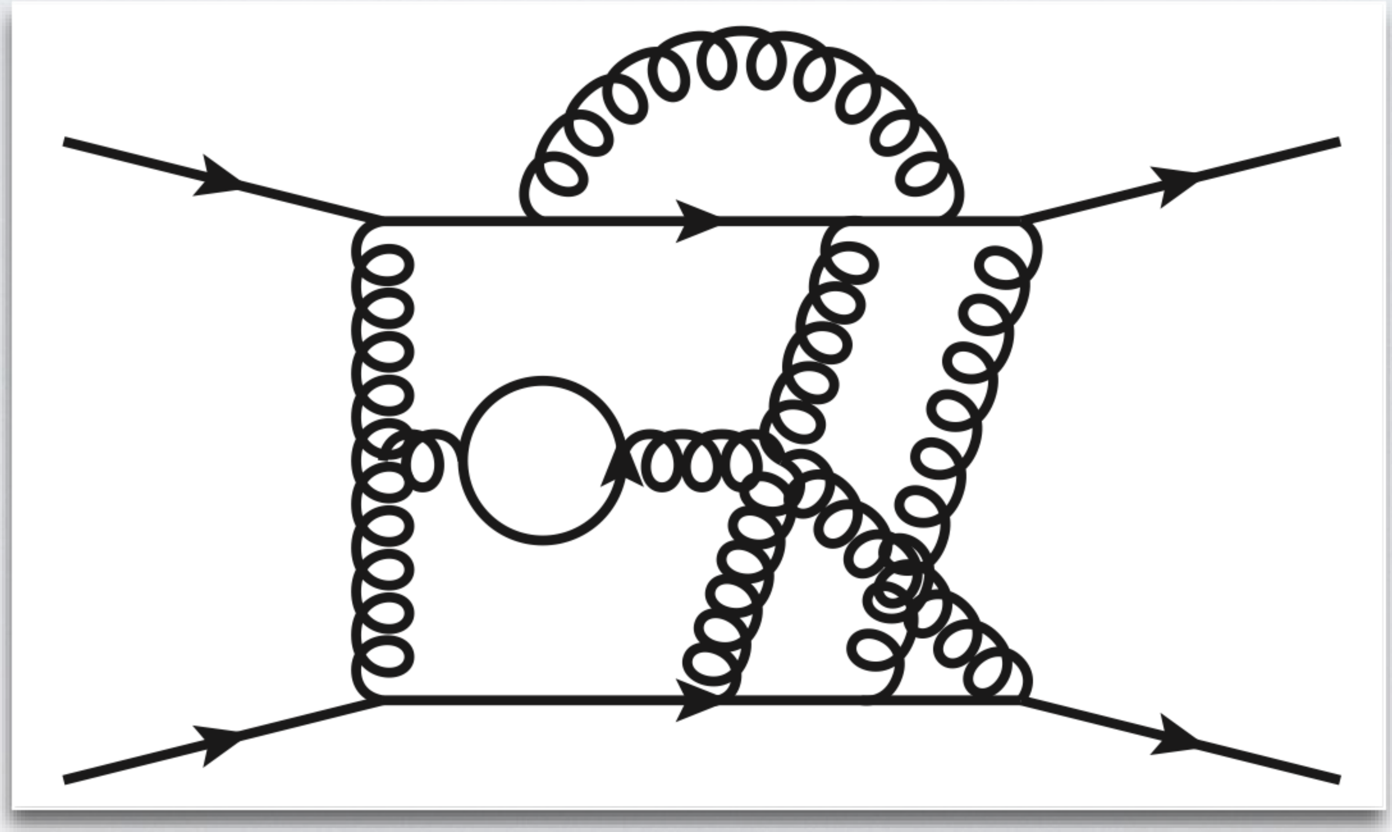
OUTLINE

- **Factorisation of amplitudes in the high-energy limit**
- **Scattering amplitudes by iterated solution of the BFKL equation**
- **The two-Reggeon cut: imaginary amplitude**
- **The three-Reggeon cut: real amplitude**

- *JHEP 1706 (2017) 016, [arXiv:1701.05241], with S. Caron-Huot and E. Gardi*
- *JHEP 1803 (2018) 098, [arXiv:1711.04850], with S. Caron-Huot, E. Gardi, and J. Reichel,*
- *JHEP 08 (2020) 116, [arXiv:2006.01267], with S. Caron-Huot, E. Gardi and J. Reichel*
- *[arXiv:2012.00613], with G. Falcioni, E. Gardi and C. Milloy*
- *and in preparation with G. Falcioni, E. Gardi N. Maher and C. Milloy*

FACTORISATION OF AMPLITUDES IN THE HIGH-ENERGY



HIGH-ENERGY LIMIT

- Very interesting theoretical problem:
  - Understand the high-energy QCD asymptotics in terms of Regge poles and cuts;
  - toy model for full amplitude, yet
    → retain rich dynamic in the 2D transverse plane,
    → non-trivial function spaces;
  - predict amplitudes and other observables in overlapping limits:
    → soft limit, infrared divergences.

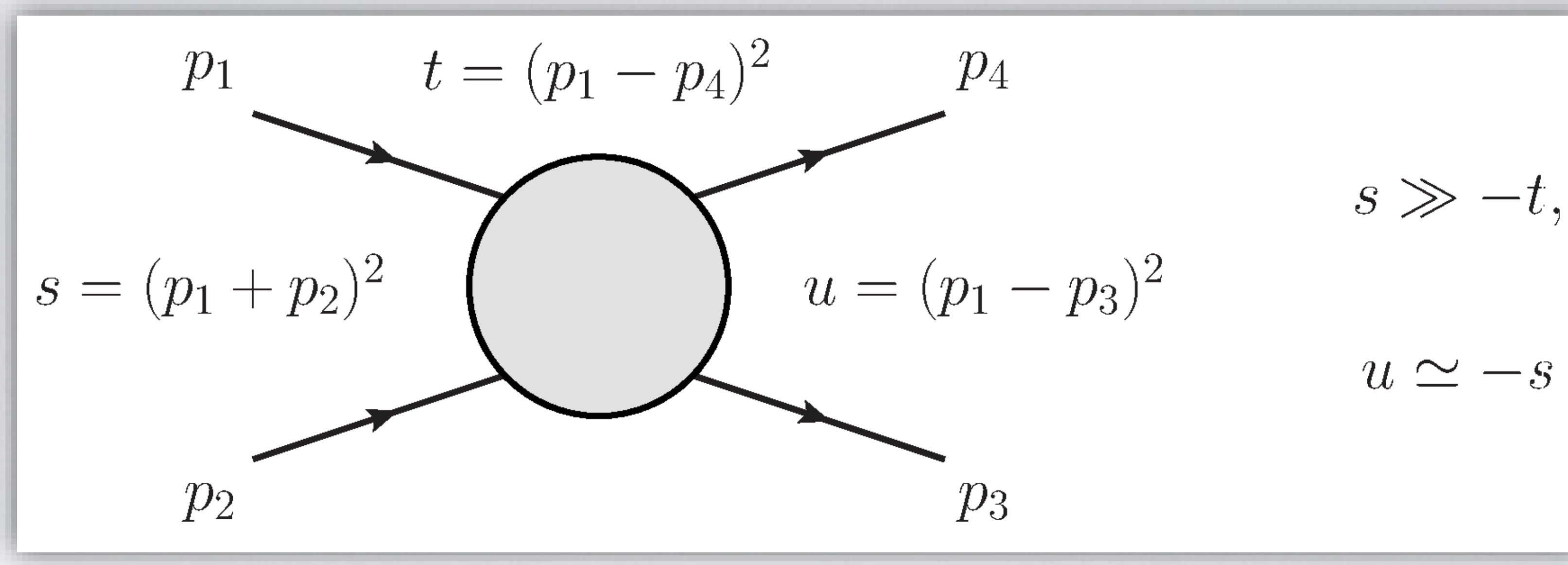
→ See talk by N. Maher

MRK in N=4 SYM:
Dixon, Pennington, Duhr, 2012;
Del Duca, Dixon, Pennington, Duhr, 2013;
Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2019

- Relevant for phenomenology at the LHC and future colliders:
  - perturbative phenomenology of forward scattering, e.g.
    → Deep inelastic scattering/saturation (small x = Regge, large Q^2 = perturbative),
    → Mueller-Navelet: pp → X+2jets, forward and backward.

See e.g. Andersen, Smillie, 2011; Andersen, Medley Smillie, 2016; Andersen, Hapola, Maier, Smillie, 2017; …

2→2 SCATTERING IN THE HIGH-ENERGY LIMIT



- Expansion in the strong coupling and in towers of (large) logarithms

$$\mathcal{M}_{ij\to ij} = \mathcal{M}^{(0)} + \frac{\alpha_s}{\pi}\log\frac{s}{-t}\,\mathcal{M}^{(1,1)} + \frac{\alpha_s}{\pi}\,\mathcal{M}^{(1,0)}$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \log^2\frac{s}{-t}\,\mathcal{M}^{(2,2)} + \left(\frac{\alpha_s}{\pi}\right)^2 \log\frac{s}{-t}\,\mathcal{M}^{(2,1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{M}^{(2,0)} + \dots$$

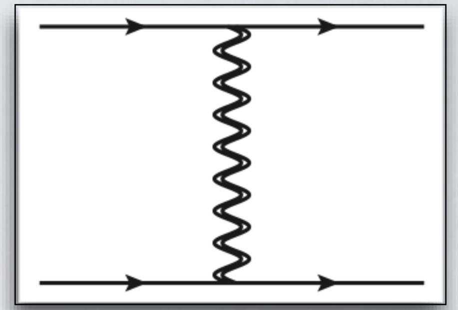
LL: $\frac{\alpha_s}{\pi}\log\frac{s}{-t}\,\mathcal{M}^{(1,1)}$, $\left(\frac{\alpha_s}{\pi}\right)^2 \log^2\frac{s}{-t}\,\mathcal{M}^{(2,2)}$

NLL: $\frac{\alpha_s}{\pi}\,\mathcal{M}^{(1,0)}$, $\left(\frac{\alpha_s}{\pi}\right)^2 \log\frac{s}{-t}\,\mathcal{M}^{(2,1)}$

NNLL: $\left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{M}^{(2,0)}$

2→2 SCATTERING IN THE HIGH-ENERGY LIMIT

- LL tower: one-Reggeon exchange in the t-channel
  (Regge pole in the complex angular momentum plane)



$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\frac{\alpha_s C_A}{\pi}\frac{r_\Gamma}{\epsilon}}$$

Regge, Gribov ~ 1960;
Lipatov; Fadin,Kuraev,Lipatov 1976

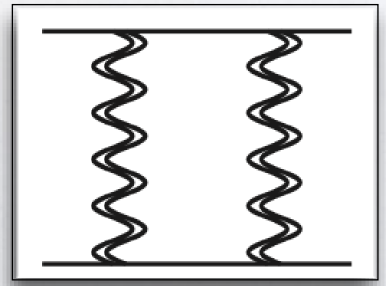
- LL amplitude

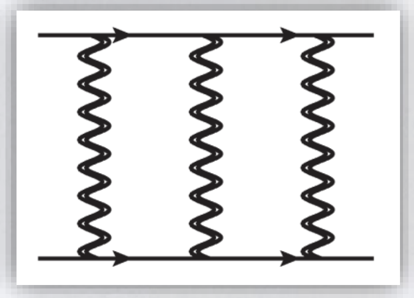
$$\mathcal{M}^{\rm LL} = e^{\frac{\alpha_s C_A L}{\pi}\frac{r_\Gamma}{\epsilon}}\mathcal{M}^{(0)}, \qquad r_\Gamma = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- Real amplitude at NLL: described by BFKL:

Fadin,Kuraev,Lipatov 1975-77; Balitsky,Lipatov 1978

- Beyond real NLL: compound states of multiple-Reggeon exchanges.





2➔2 SCATTERING IN THE HIGH-ENERGY LIMIT

- Multiple Reggeon exchange contribution in scattering amplitudes elusive, until recently.
- First evidence of violation of Regge-pole factorization in

  Del Duca, Glover 2001;

- Interplay with the infrared factorization theorem investigated in

  Del Duca, Duhr, Gardi, Magnea, White 2011; Del Duca, Falcioni, Magnea, LV, 2013, 2014;

- High-energy scattering via Wilson lines:

  Korchemskaya, Korchemsky, 1994,1996; Balitsky 1995; Babansky, Balitsky 2002;

- Two-parton scattering from rapidity evolution of Wilson lines

  Caron-Huot, 2013; Caron-Huot, Gardi, LV, 2017; Caron-Huot, Gardi, Reichel, LV, 2017, 2020; Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021.

  → This talk

- SCET-based formulation in

  Rothstein, Stewart 2016; Ridgway, Moult, Stewart, 2019, 2020.

- Calculation of multiple Reggeon exchanges within QCD also obtained in

  Fadin, Lipatov 2017; Fadin 2019, 2020.

2➔2 SCATTERING IN THE HIGH-ENERGY LIMIT

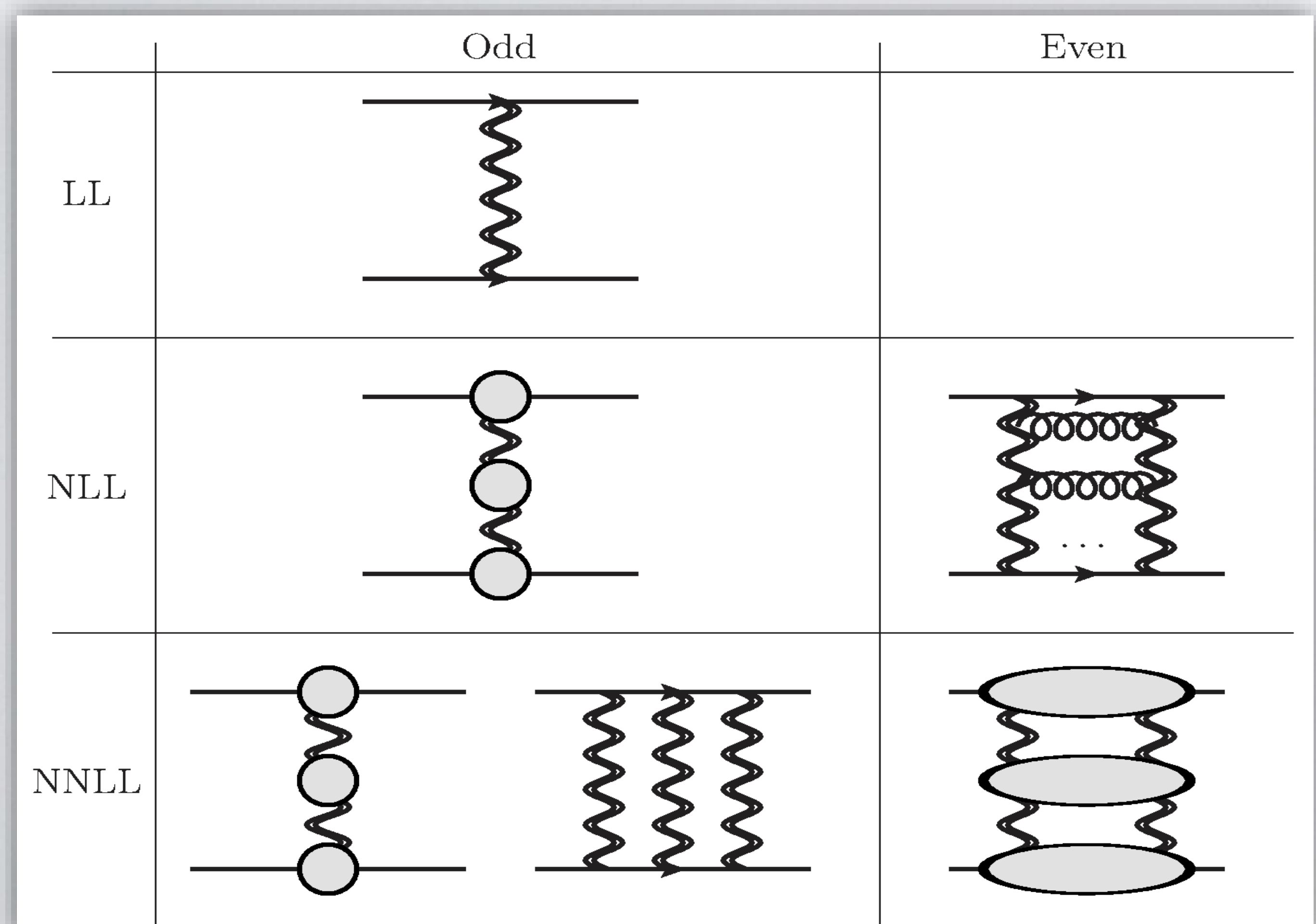
- Organizing principle: exploit symmetry under $s \leftrightarrow u$ exchange:
  → the amplitude decomposes into even (+) and odd (-) components under $s \leftrightarrow u$:

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

- Expand the amplitude in terms of the signature-even combination of logarithms:

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

  → $M^{(+)}$ imaginary with even number of Reggeons
  → $M^{(-)}$ real with odd number of Reggeons



Goals:

- Calculate multiple Reggeon exchanges to high-order in perturbation theory
- Understand the high-energy asymptotics of partonic amplitudes
- Investigate implications for IR divergences
- Do multiple Reggeon exchange exponentiate?

FROM BALITSKY-JIMWLK TO AMPLITUDES

- High-energy limit = forward scattering:

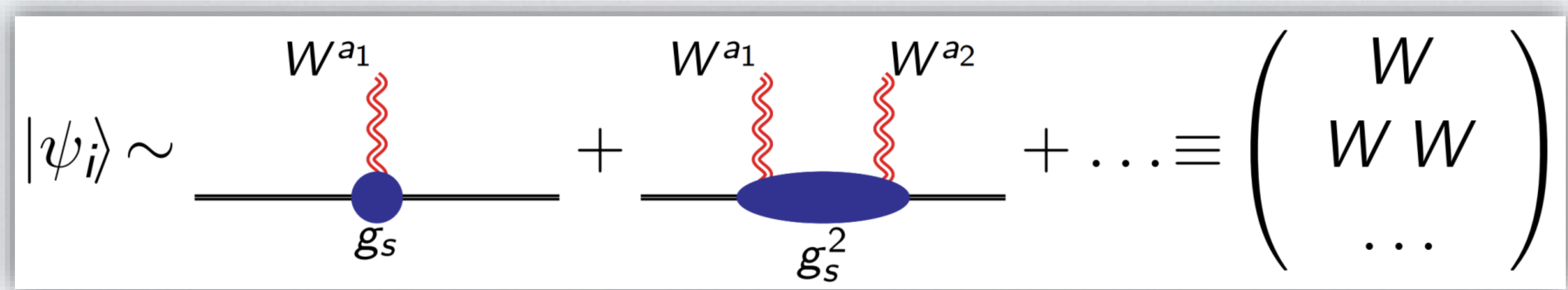
  → the projectile and target are described in terms of Wilson lines:

$$U^\eta(z_\perp) = \mathcal{P}\exp\left[ig_s\,\mathbf{T}^a\int_{-\infty}^{+\infty}dx^+\,A_+^a(x^+,x^-=0,z_\perp)\right] \equiv e^{ig_s\,\mathbf{T}^a\,W^a(z_\perp)}\,.$$

  - T^a group generator in parton representation
  - $\eta = L$ (implicit) cutoff

Korchemskaya, Korchemsky, 1994, 1996; Babansky, Balitsky, 2002, Caron-Huot, 2013

- Scattering states (target and projectile) are expanded in Reggeon fields W^a:



- Evolution in rapidity resums the high-energy log:

$$\frac{d}{dL}|\psi_i\rangle = -H|\psi_i\rangle.$$

Balitsky-JIMWLK Hamiltonian

Known at NLO:
Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016

- Scattering amplitude: expectation value of Wilson lines evolved to equal rapidity:

$$\frac{i}{2s}\frac{1}{Z_iZ_j}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-LH}|\psi_i\rangle.$$

(Z_i = collinear poles)

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

FROM BALITSKY-JIMWLK TO AMPLITUDES

- Structure of the leading-order Balitsky-JIMWLK equation:

$$H\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \dots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \dots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \dots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \dots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \dots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}.$$

LO BFKL kernel

From LO B-JIMWLK

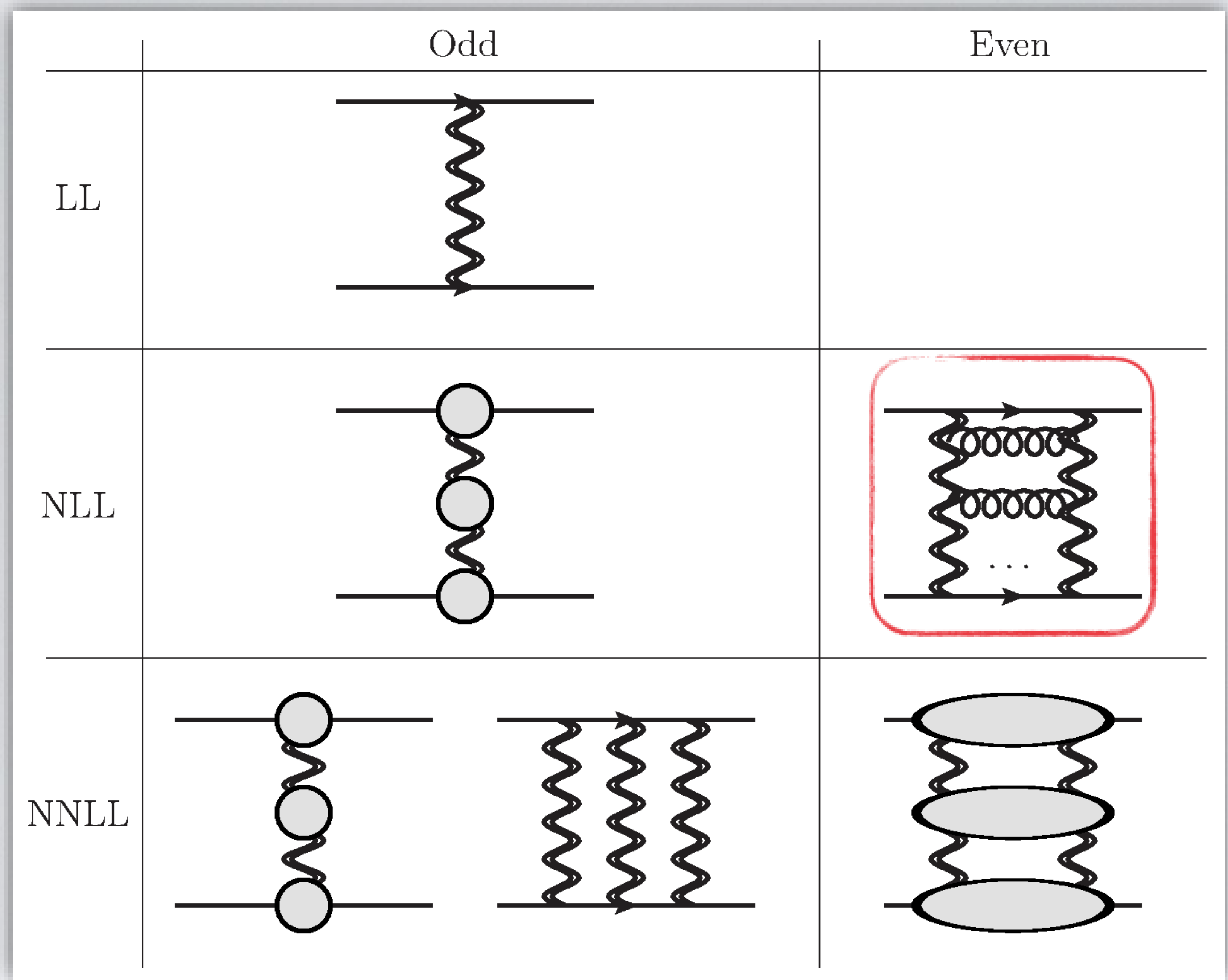
Terms in NNLO B-JIMWLK predicted by symmetry H = H^T

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

- At NLL we need *m* → *m* transition only → the LO BFKL kernel.
- At NNLL we need the *m* → *m+2* transition from the LO B-JIMWLK kernel.
- Define the reduced amplitude: subtract single-Reggeon exchange:

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} = \langle\psi_j|e^{-(H-H_{1\to1})L}|\psi_i\rangle \equiv \langle\psi_j|e^{-\hat{H}L}|\psi_i\rangle.$$

THE TWO-REGGEON CUT



THE TWO-REGGEON CUT

- The amplitude takes the form of an iterated integral over the BFKL kernel:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [{\rm D}k] \, \frac{p^2}{k^2(k-p)^2} \, \Omega^{(\ell-1)}(p,k) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)}, \qquad B_0 = e^{\epsilon\gamma_{\rm E}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- One rung = apply once the BFKL kernel on the "target averaged wave function":

$$\Omega^{(\ell-1)}(p,k) = \hat{H} \, \Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2) \, \hat{H}_{\rm i} + (C_A - \mathbf{T}_t^2) \, \hat{H}_{\rm m}$$

- "Integration" part:

Caron-Huot, Gardi, Reichel, LV, 2017

$$\hat{H}_{\rm i} \, \Psi(p,k) = \int [{\rm D}k'] \, f(p,k,k') \left[\Psi(p,k') - \Psi(p,k) \right],$$

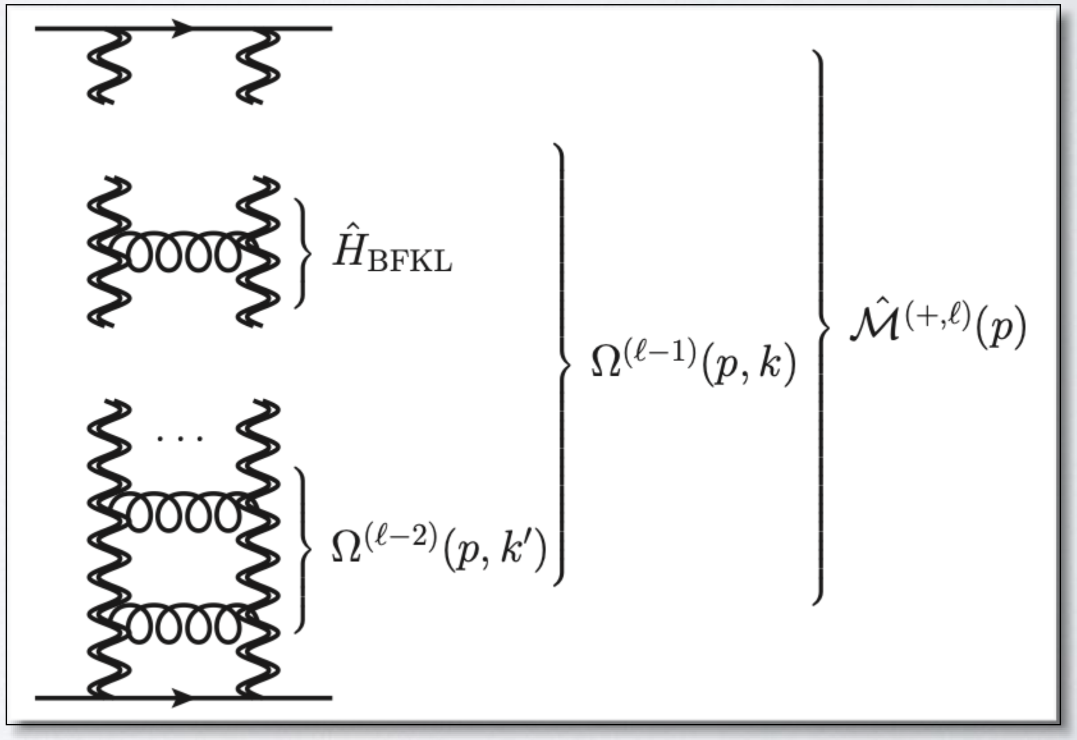
$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2}.$$

- "Multiplication" part:

$$\hat{H}_{\rm m} \Psi(p,k) = \frac{1}{2\epsilon} \left[2 - \left(\frac{p^2}{k^2} \right)^\epsilon - \left(\frac{p^2}{(p-k)^2} \right)^\epsilon \right] \Psi(p,k).$$

- Initial condition

$$\Omega^{(0)}(p,k) = 1.$$



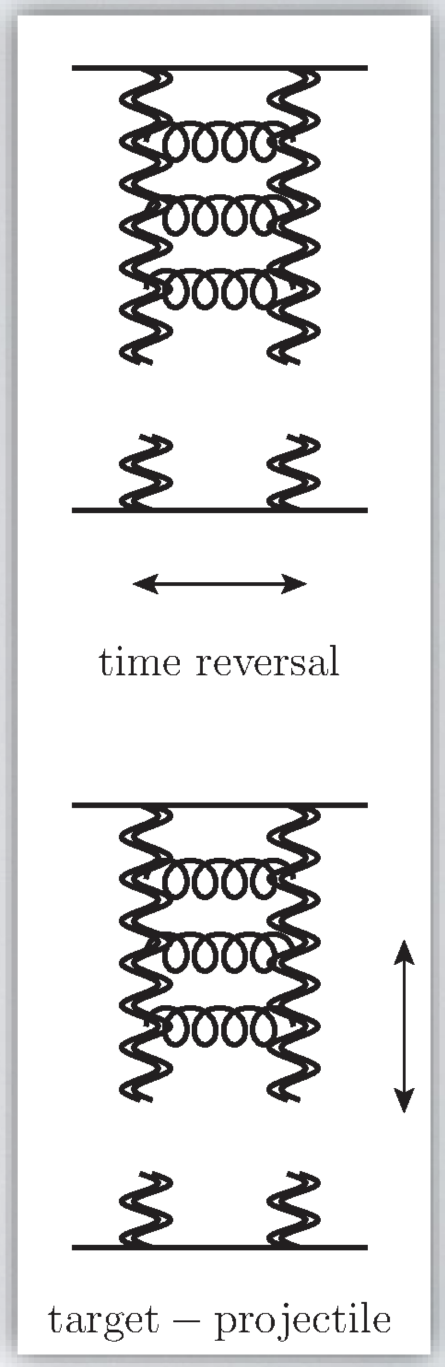
THE TWO-REGGEON CUT

- Exact solution in the adjoint channel: *Ω = 1*.
- For generic color representations in *d* dimension eigenfunctions are not known:
  → Iterative solution.
- General features:
  → target-projectile, time reversal and crossing symmetry;
  → outermost rungs are easy (multiplication);
  → first non-trivial integration at 4-loops:

Caron-Huot, 2013

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \frac{(B_0)^4}{4!}\Big\{(C_A - \mathbf{T}_t^2)^3\left(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) + C_A(C_A - \mathbf{T}_t^2)^2\left(-\frac{\zeta_3}{8\epsilon} - \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2)\right)\Big\}\mathbf{T}^2_{s-u}\, M^{(0)}.$$

- Integration in $d=2-2\epsilon$ involves Appell functions starting at 4 loops.
  → How to predict higher orders?



THE TWO-REGGEON CUT

- Observations:

  1) The wavefunction $\Omega(n)(p,k)$ is finite as $\epsilon \to 0$:

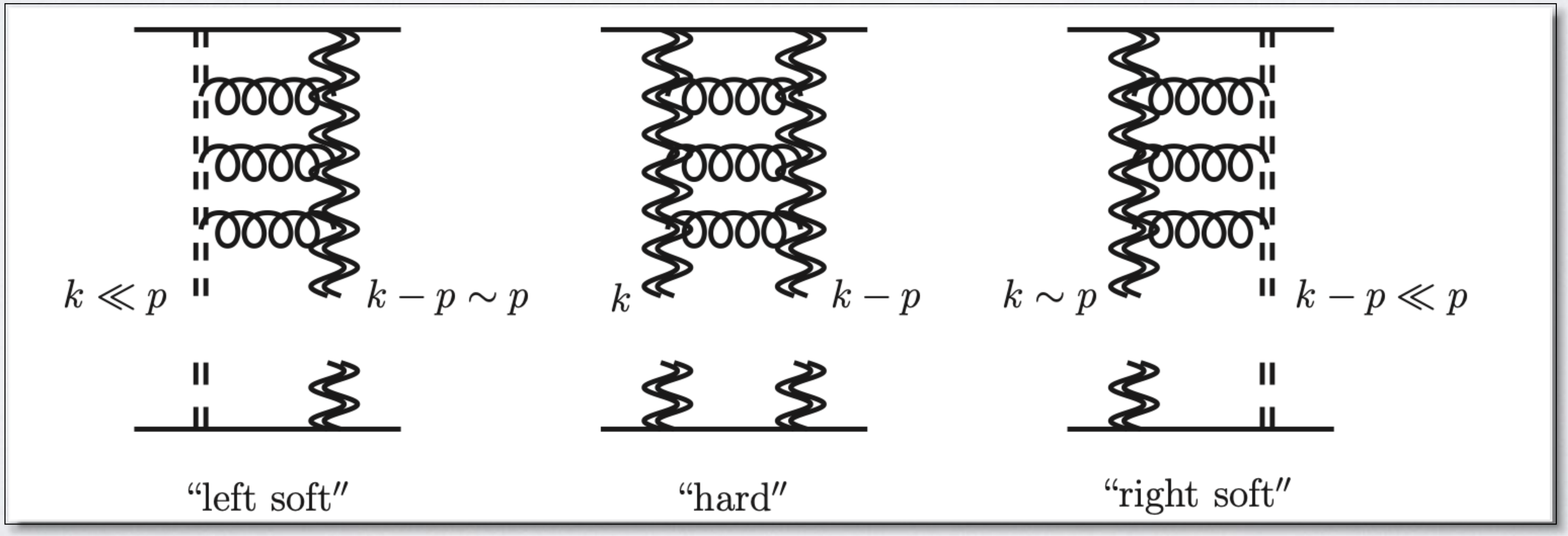
     → poles can only appear from the last integration.

  2) Evolution closes in the soft limit:

$$\int_{k\to 0} \Omega^{(\ell)}(p,k).$$

     → IR divergences occur only when a full rail goes soft!

     → Compute evolution in the (left) soft region and multiply by two.



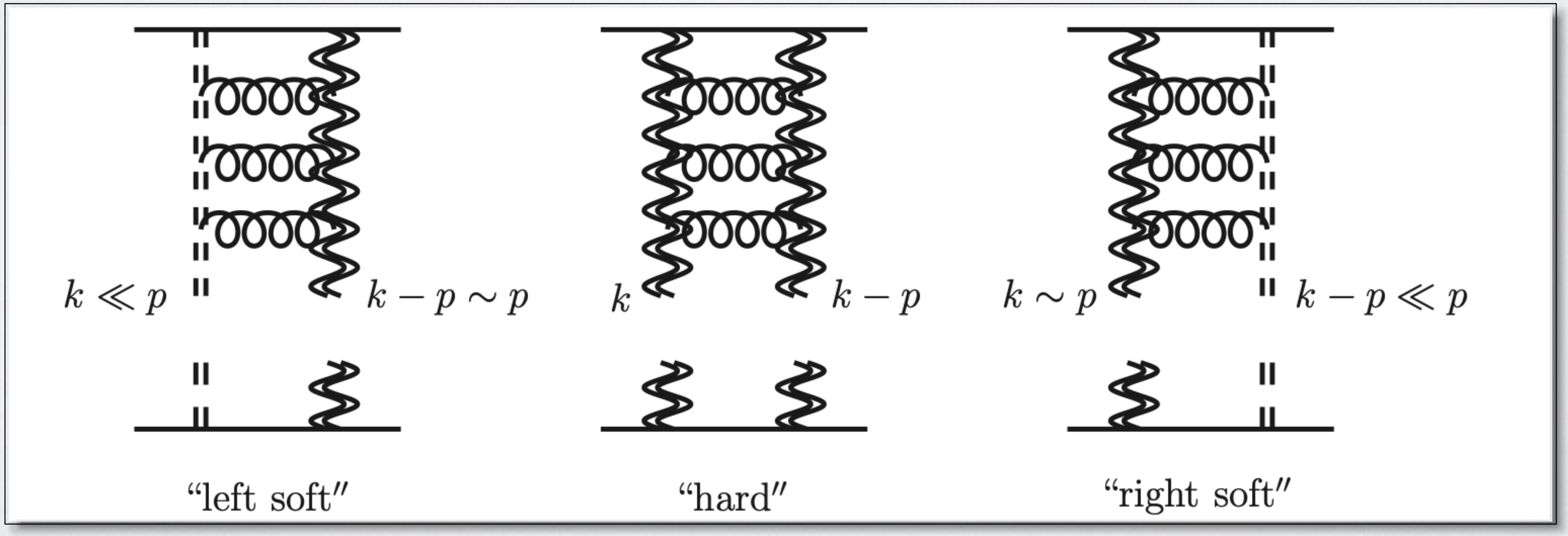
THE TWO-REGGEON CUT

- What about the finite part?
  - → Claim: the $\epsilon \to 0$ limit determined from evolution with $\epsilon = 0$.

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\left(\frac{s}{-t}\right) = -i\pi\Bigg\{ \underbrace{\int [Dk]\, \frac{p^2}{k^2(p-k)^2}\Omega_s(p,k)\ \mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\text{compute using soft limit of wavefunction in } D \text{ dimensions}} + \underbrace{\lim_{\epsilon\to 0}\int [Dk]\, \frac{p^2}{k^2(p-k)^2}\,\Omega^{(2d)}_h(p,k)\ \mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\text{compute in } D=2} \Bigg\}.$$

Caron-Huot, Gardi, Reichel, LV, 2020

Recall: the wavefunction is finite, singularities are generated upon the last integration for $k\to 0$.



TWO-REGGEON CUT: SOFT APPROXIMATION

Γ functions

- The soft function is polynomial in $(p^2/k^2)^\epsilon$:

$$\hat{H}_{\rm i}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon},$$

$$\hat{H}_{\rm m}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = \frac{1}{2\epsilon}\left[\left(\frac{p^2}{k^2}\right)^{n\epsilon} - \left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}\right].$$

Caron-Huot, Gardi, Reichel, LV, 2017

- Easy to compute to all orders: to $O(\epsilon^{-1})$ the amplitude reduces to a geometric series!

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL}|_s = i\pi\,\frac{1}{(2\epsilon)^\ell}\,\frac{B_0^\ell(\epsilon)}{\ell!}\,\left(1 - R(\epsilon)\frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1}(C_A - \mathbf{T}_t^2)^{\ell-1}\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

where

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

TWO-REGGEON CUT: D=2

- Translate the action of the BFKL kernel into a set of differential equations:

$$z\frac{d}{dz}\left[\hat{H}_{2\mathrm{d,i}}\Psi(z,\bar{z})\right] = \hat{H}_{2\mathrm{d,i}}\left[z\frac{d}{dz}\Psi(z,\bar{z})\right].$$

- The full algorithm requires to take care of contact terms,

$$\partial_z\partial_{\bar{z}}\,\log(z\bar{z}) = \pi\,\delta^2(z),$$

and to considering the action of (1-z)d/dz as well.

- The 2D wavefunction is expressed as function of SVHPLs, e.g.

$$\Omega^{(1)}_{2\mathrm{d}} = \frac{1}{2}C_2\left(\mathcal{L}_0 + 2\mathcal{L}_1\right)$$

$$\Omega^{(2)}_{2\mathrm{d}} = \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right).$$

where $C_1=2C_A-T_t^2$, $C_2 = C_A-T_t^2$ and, e.g.,

$$\mathcal{L}_{0,1}(z,\bar{z}) = H_0(z)H_1(\bar{z}) + H_{0,1}(z) + H_{1,0}(\bar{z}).$$

Brown, 2004, 2013, Schnetz, 2013

Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2019, …

TWO-REGGEON CUT: D=2

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\left(\frac{s}{-t}\right) = -i\pi\Bigg\{ \underbrace{\int [Dk]\, \frac{p^2}{k^2(p-k)^2} \Omega_s(p,k)\; \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\text{compute using soft limit of wavefunction in } D \text{ dimensions}} + \underbrace{\lim_{\epsilon\to 0} \int [Dk]\, \frac{p^2}{k^2(p-k)^2}\, \Omega^{(\rm 2d)}_h(p,k)\; \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\text{compute in } D=2} \Bigg\}.$$

- Two methods to perform the last integration and sum consistently soft and hard region.

$$\hat{\mathcal{M}}^{(1)}|_{\epsilon^0} = 0, \qquad \hat{\mathcal{M}}^{(2)}|_{\epsilon^0} = 0,$$

$$\hat{\mathcal{M}}^{(3)}|_{\epsilon^0} = -i\pi \frac{(B_0)^3}{2!}\left[C_2^2\left(-\frac{11}{4}\zeta_3\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(4)}|_{\epsilon^0} = -i\pi \frac{(B_0)^4}{3!}\left[C_1 C_2^2\left(-\frac{3}{16}\zeta_4\right) + C_2^3\left(\frac{3}{16}\zeta_4\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(5)}|_{\epsilon^0} = -i\pi \frac{(B_0)^5}{4!}\left[C_2^4\left(-\frac{717}{16}\zeta_5\right) + C_1 C_2^3\left(\frac{333}{16}\zeta_5\right) + C_1^2 C_2^2\left(-\frac{5}{2}\zeta_5\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(6)}|_{\epsilon^0} = -i\pi \frac{(B_0)^6}{5!}\left[C_2^5\left(-\frac{2879}{32}\zeta_3^2 + \frac{5}{32}\zeta_6\right) + C_1 C_2^4\left(\frac{2637}{32}\zeta_3^2 - \frac{5}{32}\zeta_6\right) + C_1^2 C_2^3\left(-\frac{399}{16}\zeta_3^2\right) + C_1^3 C_2^2\left(\frac{39}{16}\zeta_3^2\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\ldots$$

Caron-Huot, Gardi, Reichel, LV, 2020

REGGE VS INFRARED FACTORISATION

- Applications: 1) test (and predict) the analytic structure of infrared divergences.
- The infrared divergences of amplitudes are controlled by a renormalization group equation:

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

where $\boldsymbol{Z}_n$ is given as a path-ordered exponential of the soft-anomalous dimension:

Becher, Neubert, 2009; Gardi, Magnea, 2009

$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons is an operator in color space given by

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

- Given M_n as calculated in the high-energy limit, use IR factorisation to extract the soft anomalous dimension.

→ ***See talk by N. Maher***

TWO-REGGEON CUT: NUMBER THEORY

- Applications: 2) number theory.

$$\hat{\mathcal{M}}_{\rm h}^{(11)} = \frac{i\pi}{8!}\Bigg\{C_2^2 C_A^8\Big(-\frac{44253\,g_{533}}{5120} - \frac{652795\zeta_3^2\zeta_5}{2048} - \frac{818318 27\zeta_{11}}{327680}\Big)$$
$$+ C_2^3 C_A^7\Big(\frac{510873\,g_{533}}{5120} + \frac{10645591\zeta_3^2\zeta_5}{2048} + \frac{14761239427\zeta_{11}}{1966080}\Big)$$
$$+ \ldots + C_2^8 C_A^2\Big(-\frac{2158233\,g_{533}}{5120} - \frac{852453151\zeta_3^2\zeta_5}{2048} - \frac{1295244371839\zeta_{11}}{655360}\Big)$$
$$+ C_2^9 C_A\Big(\frac{6979863\,g_{533}}{5120} + \frac{2225183081\zeta_3^2\zeta_5}{2048} + \frac{741771390019\zeta_{11}}{655360}\Big)$$
$$+ C_2^{10}\Big(\frac{1094181\,g_{533}}{2560} + \frac{2638860059\zeta_3^2\zeta_5}{1024} + \frac{4498262900131\zeta_{11}}{655360}\Big)\Bigg\}.$$

Brown, 2004, 2013; Schnetz, 2013

- Hard regions: only odd ζ_n, consistent with 2D wavefunction made of SVHPLs.
- Finite (hard) amplitude contains g_{533} at 11 loops:

Caron-Huot, Gardi, Reichel, LV, 2020

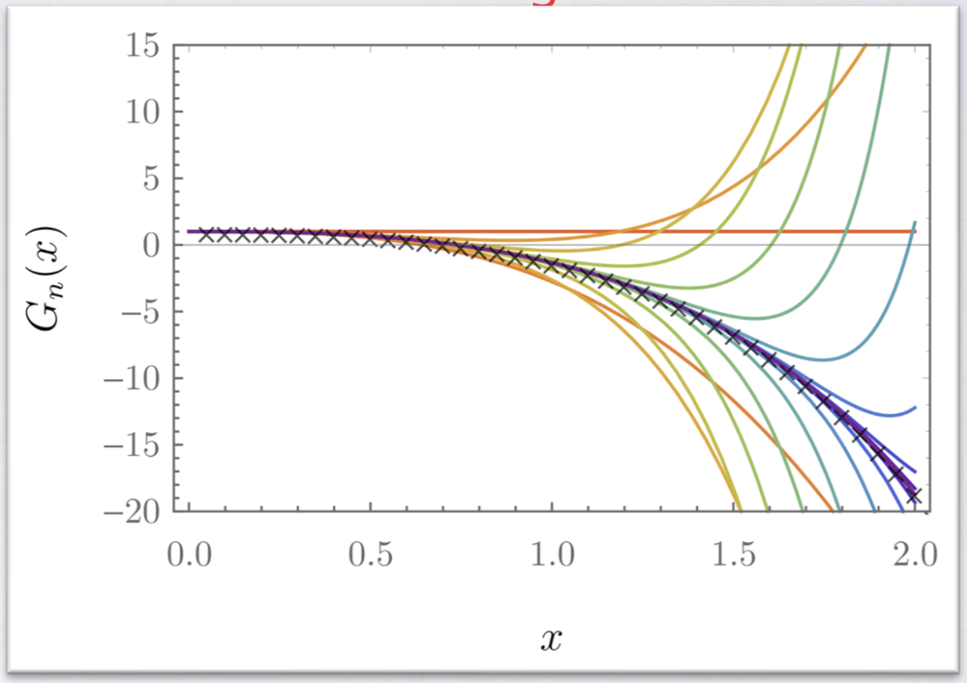
$$g_{5,3,3} = -\frac{4}{7}\zeta_2^3\zeta_5 + \frac{6}{5}\zeta_2^2\zeta_7 + 45\zeta_2\zeta_9 + \zeta_{5,3,3}\,.$$

→ no exponentiation in terms of **Γ** functions.

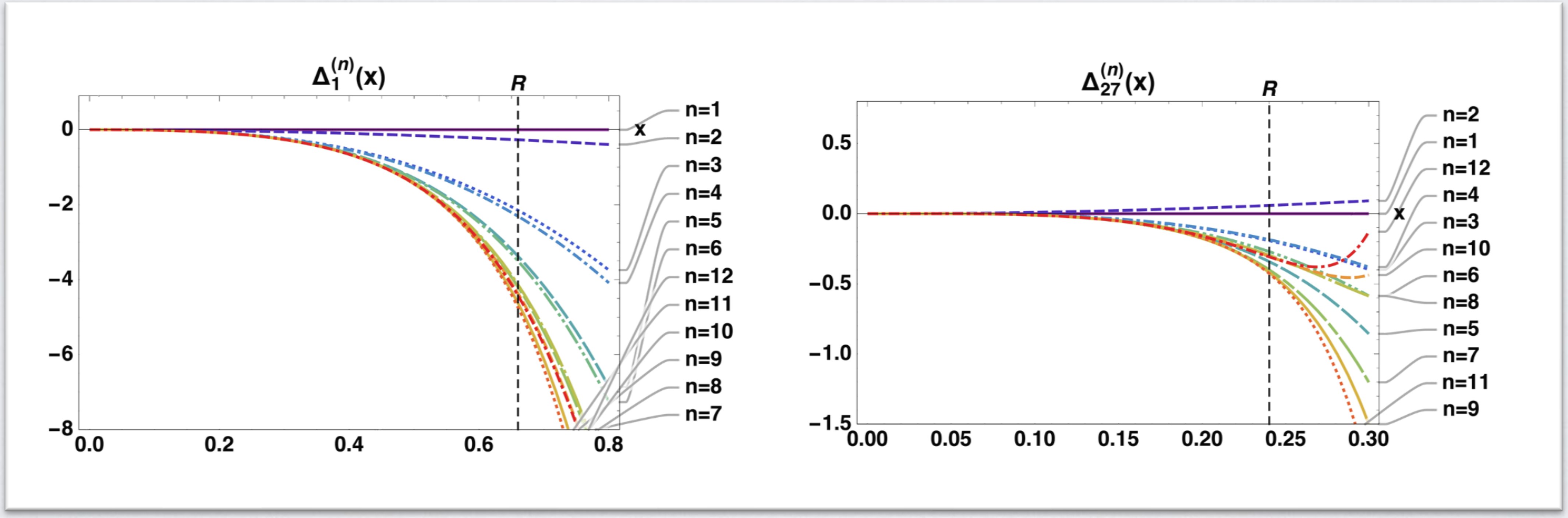
TWO-REGGEON CUT: NUMERICAL STUDIES

- Applications: 3) numerical studies.
- The soft anomalous dimension has an infinite radius of convergence: entire function, free of singularities for any finite $x=\alpha_s/\pi\, L$.

Caron-Huot, Gardi, Reichel, LV, 2017, 2020

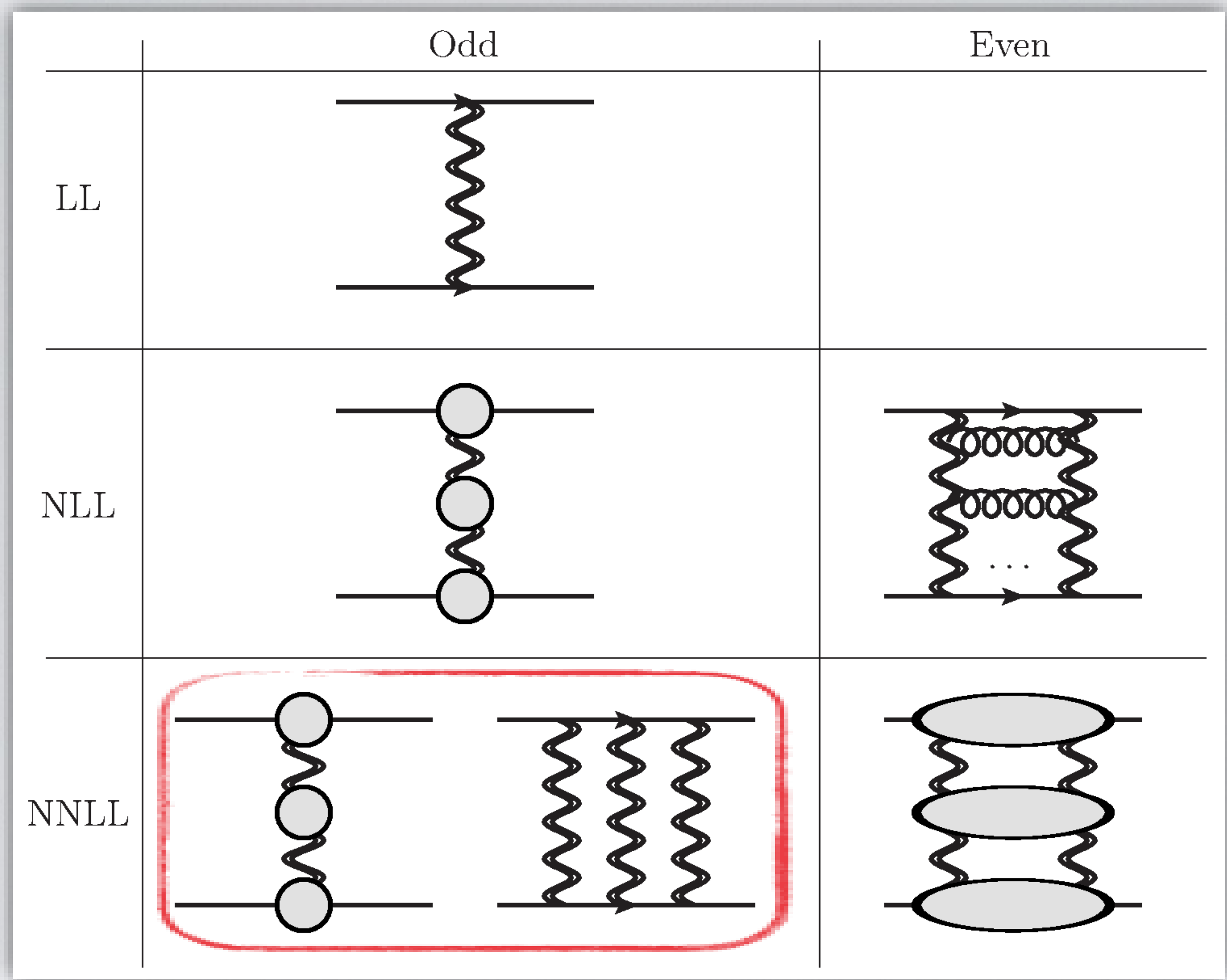


- The finite amplitude is an alternating series, whose coefficients grows geometrically:



- Finite radius of convergence in $\alpha_s/\pi\, L$ *that* stabilises to $|R| \simeq 0.66$ for singlet, $|R| \simeq 0.24$ for 27 representation, by means of a Padé approximant (pole at $-|R|$).

THE THREE-REGGEON CUT



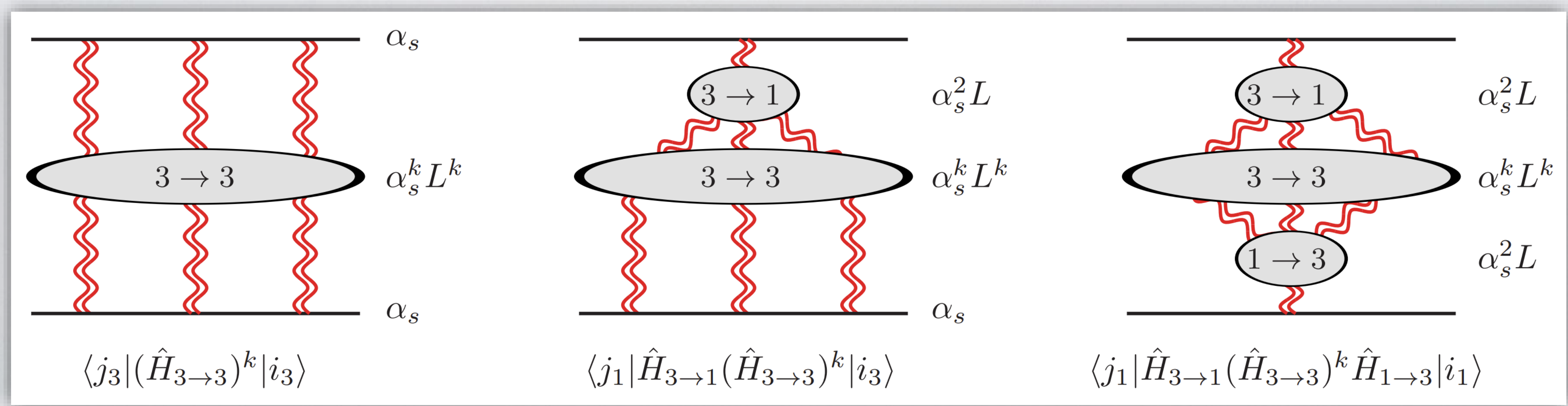
THE THREE-REGGEON CUT

- To all orders the amplitude takes the form

Falcioni, Gardi, Milloy, LV, 2020

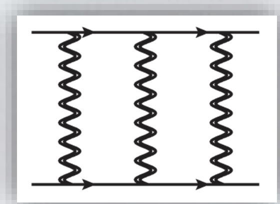
$$\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ r_\Gamma^2 \pi^2 \Bigg[\sum_{k=0}^{\infty} \frac{(-X)^k}{k!} \langle j_3 | \hat{H}^k_{3\to 3} | i_3 \rangle + \sum_{k=1}^{\infty} \frac{(-X)^k}{k!} \left[\langle j_1 | \hat{H}_{3\to 1} \hat{H}^{k-1}_{3\to 3} | i_3 \rangle + \langle j_3 | \hat{H}^{k-1}_{3\to 3} \hat{H}_{1\to 3} | i_1 \rangle \right] + \sum_{k=2}^{\infty} \frac{(-X)^k}{k!} \langle j_1 | \hat{H}_{3\to 1} \hat{H}^{k-2}_{3\to 3} \hat{H}_{1\to 3} | i_1 \rangle \Bigg]^{\mathrm{LO}} + \langle j_1 | i_1 \rangle^{\mathrm{NNLO}} \Bigg\} .$$

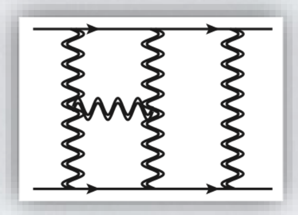
- Graphically:

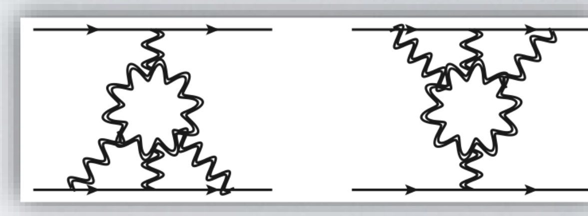


THE THREE-REGGEON CUT

- Two and three loops:



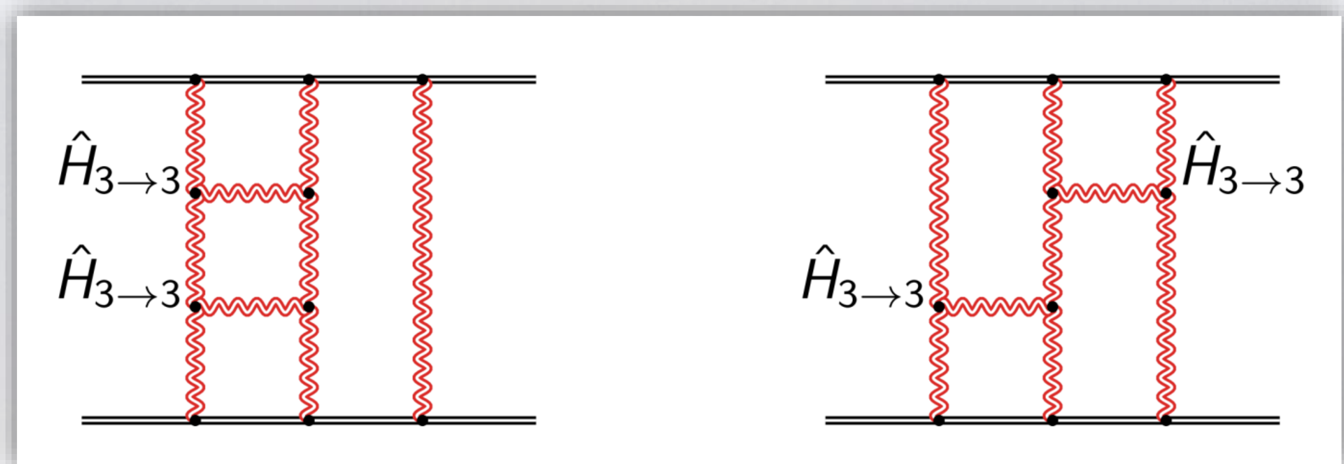




- At four loops one needs to take into account:

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-,4,2)} = \frac{r_\Gamma^4\pi^2}{2}\Big[\langle j_1|\hat{H}_{3\to1}\hat{H}_{1\to3}|i_1\rangle + \langle j_3|\hat{H}^2_{3\to3}|i_3\rangle + \langle j_1|\hat{H}_{3\to1}\hat{H}_{3\to3}|i_3\rangle + \langle j_3|\hat{H}_{3\to3}\hat{H}_{1\to3}|i_1\rangle\Big].$$

- 1) *3 → 3* transition: two independent contributions



- All integrals are massless 4-loops propagators: Γ functions or computed with FORCER.
- Problem: factorize the color structure in universal operators acting on the tree level amplitude, with

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2, \qquad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4, \qquad \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3.$$

THE THREE-REGGEON CUT

$$\langle j_3|\hat{H}^2_{3\to3}|i_3\rangle = \frac{1}{144}\left[\frac{\mathbf{C}^{(4,-4)}_{33}}{\epsilon^4} + \frac{2f_\epsilon}{\epsilon}\mathbf{C}^{(4,-1)}_{33} + \mathcal{O}(\epsilon)\right]\langle j_1|i_1\rangle,$$ with

$$\mathbf{C}^{(4,-4)}_{33} = -6\left(6C_A^2 - 17C_A\mathbf{T}_t^2 + 6(\mathbf{T}_t^2)^2\right)\mathbf{C}^{(2)}_{33} - \frac{3}{4}\mathbf{T}^2_{s-u}(\mathbf{T}_t^2)^2\mathbf{T}^2_{s-u} + \frac{25}{144}C_A^4 + \frac{1}{3}\frac{d_{AA}}{N_A} - 3C_A(d_i+d_j),$$

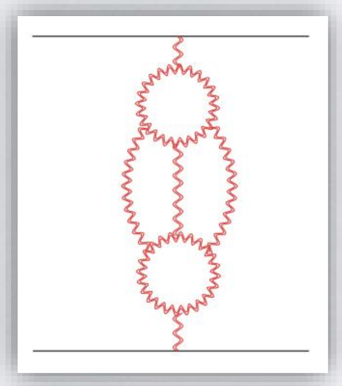
$$\mathbf{C}^{(4,-1)}_{33} = -18\left(300C_A^2 - 521C_A\mathbf{T}_t^2 + 220(\mathbf{T}_t^2)^2\right)\mathbf{C}^{(2)}_{33} - 101\mathbf{C}^{(4,-4)}_{33}.$$

- $f_\epsilon = \zeta_3 + \frac{3}{2}\epsilon\zeta_4 + \mathcal{O}(\epsilon^2)$ appears in every term at NNLL;
- Color operators $\mathbf{T}_t^2$ and $\mathbf{T}_{s-u}^2$ acting on $M^{(+)}$;
- Contribution of quartic Casimir.

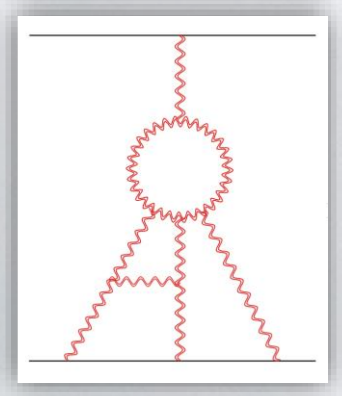
(Similar relations observed in *Baikov, Chetyrkin, 2018*)

- To all orders, terms with a single Reggeon are $\propto M^{(0)}$:

Falcioni, Gardi, Milloy, LV, 2020



$$\langle j_1|\hat{H}_{3\to1}\hat{H}_{1\to3}|i_1\rangle = \frac{1}{432}\left[-\left(\frac{C_A^4}{12} + \frac{d_{AA}}{N_A}\right)\frac{1}{\epsilon^4} + \left(\frac{101}{6}C_A^4 + 220\frac{d_{AA}}{N_A}\right)\frac{f_\epsilon}{\epsilon} + \mathcal{O}(\epsilon)\right]\langle j_1|i_1\rangle\,,$$



$$\langle j_1|\hat{H}_{3\to1}\hat{H}_{3\to3}|i_3\rangle = \frac{C_A d_i}{144}\left[\frac{1}{\epsilon^4} - 208\frac{f_\epsilon}{\epsilon} + \mathcal{O}(\epsilon)\right]\langle j_1|i_1\rangle\,.$$

THE THREE-REGGEON CUT

- The NNLL amplitude at four loops reads

Falcioni, Gardi, Milloy, LV, 2020

$$\hat{\mathcal{M}}^{(-,4,2)} = \frac{r_\Gamma^4 \pi^2}{144}\left[C_{\mathcal{M}}^{(-4)}\frac{1}{\epsilon^4} + C_{\mathcal{M}}^{(-1)}\frac{f_\epsilon}{\epsilon} + \mathcal{O}(\epsilon)\right]\mathcal{M}^{\rm Tree}, \quad \text{with}$$

$$C_{\mathcal{M}}^{(-4)} = \frac{1}{2}\mathbf{C}_{33}^{(4,-4)} - \frac{C_A^4}{72} - \frac{1}{6}\frac{d_{AA}}{N_A} + \frac{1}{2}C_A(d_i+d_j), \quad C_{\mathcal{M}}^{(-1)} = \mathbf{C}_{33}^{(4,-1)} + \frac{101 C_A^4}{36} + \frac{110}{3}\frac{d_{AA}}{N_A} - 104 C_A(d_i+d_j)\,.$$

The result holds in every gauge theory.

- Applications: 1) extract infrared divergences. → ***See talk by N. Maher***
- Applications: 2) finite terms:

$$\mathcal{H}^{(-,4,2)} = \left\{\frac{C_A^2}{2}\left(\hat{\alpha}_g^{(2,0)}\right)^2 + \frac{3}{16}\zeta_4\zeta_2 C_{\mathbf{\Delta}}^{(4)}\right\}\mathcal{M}^{\rm tree}, \quad \mathbf{C}_{\Delta}^{(4,2)} = \frac{1}{4}\mathbf{T}_t^2[\mathbf{T}_t^2,(\mathbf{T}_{s-u}^2)^2] + \frac{3}{4}[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 + \frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\,.$$

→ We can calculate it both in QCD and N=4 SYM!

- QCD:

$$\mathcal{H}_{\rm QCD}^{(-,4,2)} = \left\{C_A^2 T_F^2 n_f^2 \frac{49}{1458} + C_A^3 T_F n_f\left(\frac{7\zeta_3}{216} - \frac{707}{2916}\right) + C_A^4\left(\frac{\zeta_3^2}{128} - \frac{101\zeta_3}{864} + \frac{10201}{23328}\right) + \frac{3}{16}\zeta_4\zeta_2 C_{\mathbf{\Delta}}^{(4)}\right\}\mathcal{M}^{\rm tree}.$$

- N=4 SYM (from QCD according to maximum trascendentality):

$$\mathcal{H}_{\mathcal{N}=4}^{(-,4,2)} = \left\{\frac{C_A^4}{128}\zeta_3^2 + \frac{3}{16}\zeta_4\zeta_2 C_{\mathbf{\Delta}}^{(4)}\right\}\mathcal{M}^{\rm tree}.$$

Matches the large Nc limit (first term)

New non-planar term, proportional to Δ(4,2) (second term)

CONCLUSION

- Modern approach to high-energy scattering via Wilson lines:
  - $\rightarrow$ new theoretical control up to NNLL.
  - $\rightarrow$ $2 \rightarrow 2$ amplitudes obtained by iteration of the Balitsky-JIMWLK Hamiltonian.
- Imaginary part at NLL obtained to all orders in the strong coupling:
  - $\rightarrow$ Extracted the soft anomalous dimension to all orders;
  - $\rightarrow$ Numerical studies on the convergence of the perturbative expansion.
- Real part at NNLL obtained up to four loops:
  - $\rightarrow$ Extracted the corresponding term of the soft anomalous dimension;
  - $\rightarrow$ Real part of the $2 \rightarrow 2$ amplitude in QCD and N=4 SYM at four loops.

EXTRA SLIDES

TWO-REGGEON CUT: D=2

- Introduce complex variables

$$\frac{k}{p} = \frac{z}{z-1}, \qquad \frac{k'}{p} = \frac{w}{w-1}.$$

Caron-Huot, Gardi, Reichel, LV, 2020

- BFKL kernel in D=2:

$$\hat{H}_{\rm 2d} = (2C_A - \mathbf{T}_t^2)\hat{H}_{\rm 2d,i} + (C_A - \mathbf{T}_t^2)\hat{H}_{\rm 2d,m}$$

- “Integration” part:

$$\hat{H}_{\rm 2d,i} = \frac{1}{4\pi}\int d^2w\, K(w,\bar{w},z,\bar{z})\Big[\Psi(w,\bar{w}) - \Psi(z,\bar{z})\Big],$$

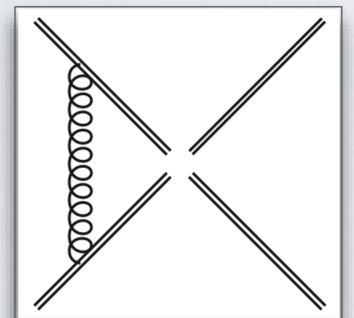
$$K(w,\bar{w},z,\bar{z}) = \frac{1}{\bar{w}(z-w)} + \frac{2}{(z-w)(\bar{z}-\bar{w})} + \frac{1}{w(\bar{z}-\bar{w})}.$$

- “Multiplication” part:

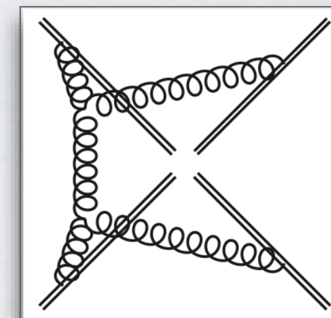
$$\hat{H}_{\rm 2d,m} = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right]\Psi(z,\bar{z}).$$

REGGE VS INFRARED FACTORISATION

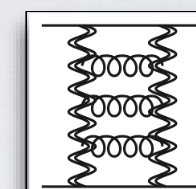
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

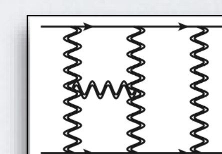


“dipole formula”



“quadrupole correction”





- Early studies of constraints from soft-collinear factorisation, collinear limits, and the high-energy limit in Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012;
- First evidence of “beyond dipole” contribution at four loops in Caron-Huot, 2013;
- Calculated a three loops in Almelid, Duhr, Gardi, 2015, 2016;
- Confirmed, in 2 ➔ 2 scattering in N=4 SYM in Henn, Mistlberger, 2016;
- Confirmed, in the high energy limit, in Caron-Huot, Gardi, LV, 2017;
- Re-derived based on a bootstrap approach in Almelid, Duhr, Gardi, McLeod, White, 2017.

TWO-REGGEON CUT: IR SINGULARITIES

- Expand the soft anomalous dimension in the high-energy logarithm:

$$\mathbf{\Gamma}\left(\alpha_s(\lambda)\right) = \mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s(\lambda), L\right) + \ldots .$$

- At LL gluon Reggeization fixes $\mathbf{\Gamma}_{\rm LL}$ from gluon trajectory:

$$\mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda)\right) = \frac{\alpha_s(\lambda)}{\pi}\frac{\gamma_K^{(1)}}{2} L\,\mathbf{T}_t^2 = \frac{\alpha_s(\lambda)}{\pi} L\,\mathbf{T}_t^2 .$$

- At NLL

$$\mathbf{\Gamma}_{\rm NLL} = \mathbf{\Gamma}^{(+)}_{\rm NLL} + \mathbf{\Gamma}^{(-)}_{\rm NLL},$$

- with

$$\mathbf{\Gamma}^{(+)}_{\rm NLL} = \frac{\alpha_s(\lambda)}{\pi}\sum_{i=1}^{2}\left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right) + \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\,\mathbf{T}_t^2,$$

$$\mathbf{\Gamma}^{(-)}_{\rm NLL} = i\pi\frac{\alpha_s(\lambda)}{\pi}\mathbf{T}_{s-u}^2 + O(\alpha_s^4 L^3).$$

Del Duca, Duhr, Gardi, Magnea, White, 2011

TWO-REGGEON CUT: IR SINGULARITIES

- Derive an infrared-factorised representation of the reduced amplitude:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \exp\left\{ -\frac{\alpha_s(\mu)}{\pi}\frac{B_0(\epsilon)}{2\epsilon}\, L\mathbf{T}_t^2\right\} \Big[\mathbf{Z}^{(-)}_{\rm NLL}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right) \mathcal{H}^{(-)}_{\rm LL}\left(\{p_i\},\mu,\alpha_s(\mu)\right)$$

$$+\, \mathbf{Z}^{(+)}_{\rm LL}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right) \mathcal{H}^{(+)}_{\rm NLL}\left(\{p_i\},\mu,\alpha_s(\mu)\right)\Big],$$

No poles

- By matching we get the soft anomalous dimension to all orders:

$$\mathbf{\Gamma}^{(-,\ell)}_{\rm NLL} = \frac{i\pi}{(\ell-1)!}\left(1 - R\left(\frac{x}{2}(C_A-\mathbf{T}_t^2)\right)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\Bigg|_{x^{\ell-1}} \mathbf{T}^2_{s-u}\,,$$

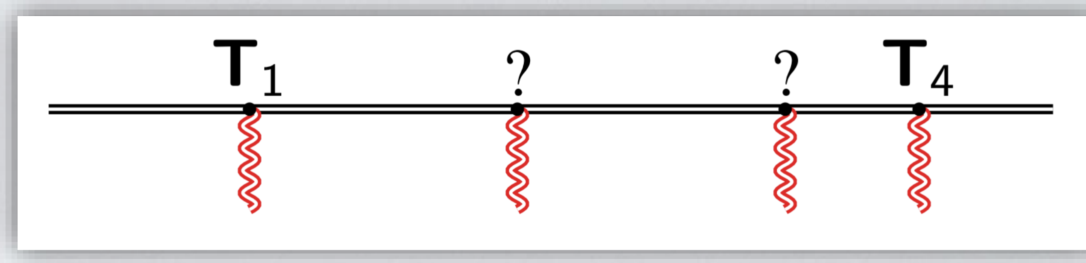
with

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2+10\zeta_6\right)\epsilon^6 + \ldots\,.$$

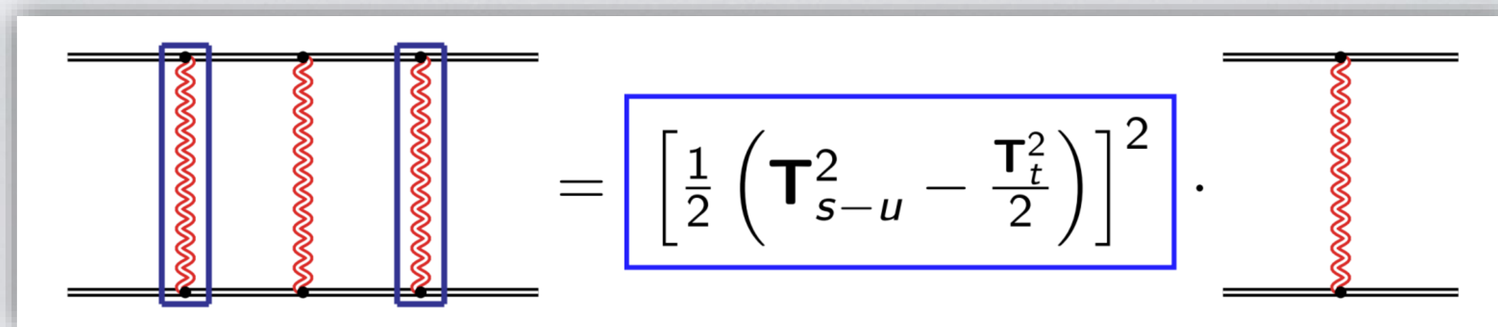
Caron-Huot, Gardi, Reichel, LV, 2017

THE THREE-REGGEON CUT

- Outmost generators clearly associated with external particles

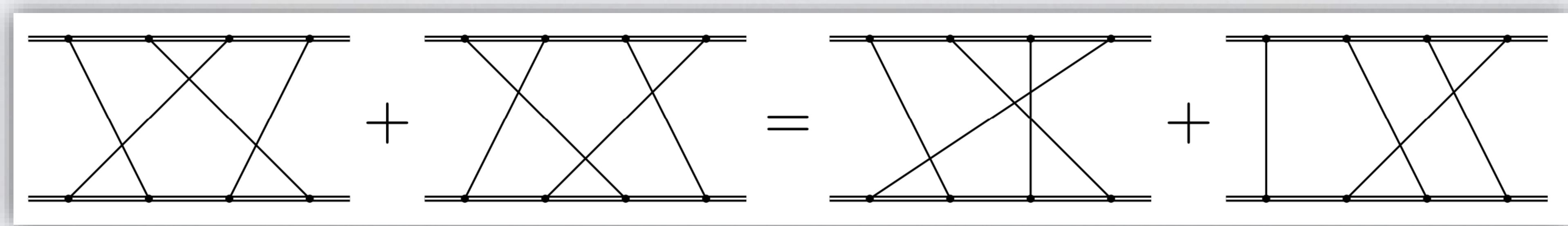


- At lowest order there is no ambiguity



with $\mathbf{T}^2_{s-u} = (\mathbf{T}^2_s - \mathbf{T}^2_u)/2.$

- Starting at three loops one has entangled contributions, for which identities such as



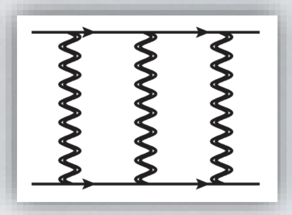
are needed.

THE THREE-REGGEON CUT

- At two loops one has

Caron-Huot, Gardi, LV, 2017

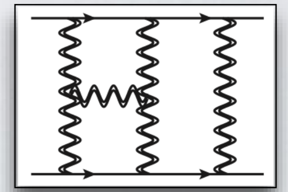
$$\langle j_1|i_1\rangle^{\rm NNLO} = \left(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)}\right) \langle j_1|i_1\rangle,$$



and

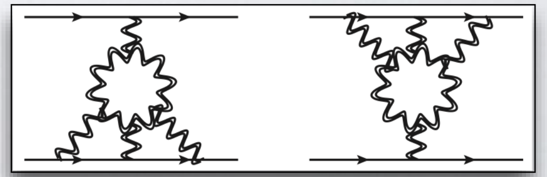
$$\langle j_3|i_3\rangle = -72\left(\frac{1}{\epsilon^2} - 6\epsilon f_\epsilon\right) \mathbf{C}_{33}^{(2)} \langle j_1|i_1\rangle, \quad f_\epsilon \equiv \zeta_3 + \frac{3}{2}\epsilon\zeta_4 + \mathcal{O}(\epsilon^2), \quad \mathbf{C}_{33}^{(2)} \equiv \frac{1}{24}\left((\mathbf{T}_{s-u}^2)^2 - \frac{C_A^2}{12}\right).$$

- At three loops



$$\langle j_3|\hat{H}_{3\to 3}|i_3\rangle = \left[\frac{1}{\epsilon^3}\left(C_A - \frac{5}{6}\mathbf{T}_t^2\right) + 2f_\epsilon\left(C_A - \frac{41}{6}\mathbf{T}_t^2\right) + \mathcal{O}(\epsilon^2)\right] \mathbf{C}_{33}^{(2)} \langle j_1|i_1\rangle\,,$$

and



$$\langle j_1|\hat{H}_{3\to 1}|i_3\rangle = \frac{1}{36}\left(\frac{1}{\epsilon^3} - 70 f_\epsilon + \mathcal{O}(\epsilon^2)\right) d_i \,\langle j_1|i_1\rangle,$$

$$d_i \equiv \frac{d_{AR_i}}{N_{R_i}}\frac{1}{C_{R_i}}, \qquad d_{AR_i} \equiv \frac{1}{6}\sum_{\sigma\in\mathcal{S}_3} {\rm tr}\left(F^a F^b F^c F^d\right) {\rm tr}\left(\mathbf{T}^a \mathbf{T}^{\sigma(b)} \mathbf{T}^{\sigma(c)} \mathbf{T}^{\sigma(d)}\right).$$

THE THREE-REGGEON CUT

Falcioni, Gardi, Milloy, LV, 2020

- Applications: 1) extract infrared divergences: from

$$\mathcal{H} = \tilde{\mathbf{Z}}^{-1} e^{-H_{11} L} \hat{\mathcal{M}}, \qquad \mathbf{Z} = \mathbf{P} \exp \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \mathbf{\Gamma} \right\},$$

- We get

$$\mathrm{Re}\left[\mathbf{\Delta}^{(4,2)}\right] = \frac{\zeta_2 \zeta_3}{4} \bigg(-\frac{3}{8} \mathbf{T}_{s-u}^2 \mathbf{T}_t^2 [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] - \frac{9}{8} [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \mathbf{T}_t^2 \mathbf{T}_{s-u}^2 + \frac{3}{8} [(\mathbf{T}_t^2)^2, (\mathbf{T}_{s-u}^2)^2] + \frac{d_{AA}}{N_A} - \frac{C_A^4}{24} \bigg).$$

- Manifestly non-planar (planar terms cancel in $\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)$). New quartic Casimir.

→ ***See talk by N. Maher***