Next-to-leading Power SCET in Higgs amplitudes induced by light quarks

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With Ze-long Liu, Bianka Mecaj, Matthias Neubert and Marvin Schnubel:

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Motivation

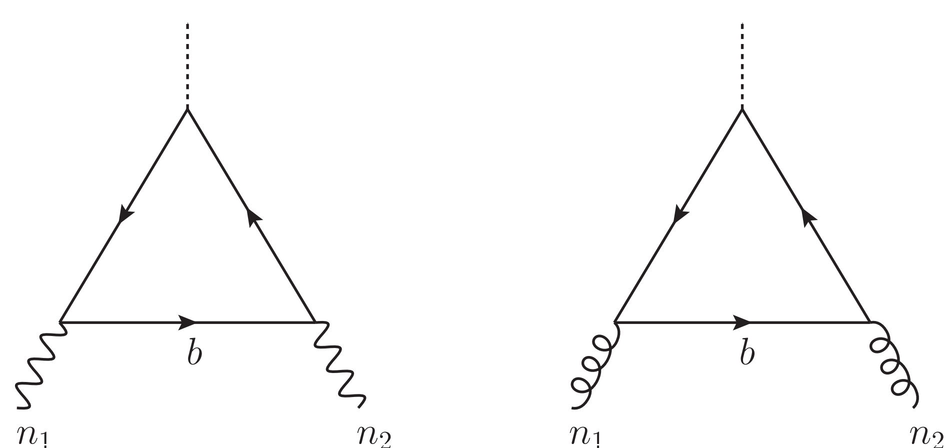
- The regarding Higgs amplitudes are power suppressed due to chirality flipping. But scale hierarchy $M_h^2 \gg m_{b(c)}^2$ gives Sudakov enhancement, which is relevant in precision study: $\sim 1.6\,\%$ for $H \to \gamma\gamma$ decay rate and $\sim 13\,\%$ for $gg \to H$ production rate.

- These are next-to-leading power (NLP) problems, and resummation has been studied using conventional QCD techniques. [Akhoury, Wang and Yakovlev, '98; Kotsky and Yakovlev, '01; Liu and Penin, '17, '18; Anastasiou and Penin, '20, '21]

- Recently NLP SCET has drawn a lot of attention [Beneke et al., Moult et al., 2016-2020; Liu, Neubert, '19, Wang, '19]. These processes are sufficiently complicated but simple enough (e.g., the operator basis is small) to investigate NLP SCET.

- Despite of some consensus of several generic features of NLP SCET (e.g., bare factorization), establishing a renormalized factorization and dealing with endpoint divergences is not fully understood yet.

- For these two processes, we can get **renormalized** factorization formulae and use "**plus-type subtraction**" to deal with endpoint divergences and push resummation **beyond** NLL in the end.



Brief Introduction to NLP SCET

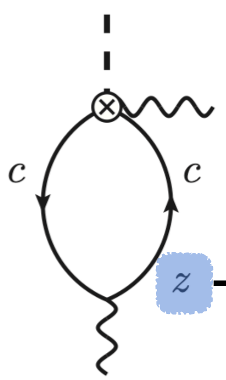
- In general, NLP effects start to be relevant in precision study at colliders. See talks by [Pal, van Beekveld, Penin, Mukherjee, Moch, Schnubel, Ajjath A H, Szafron...]
- LP Soft-Collinear Effective Theory (SCET) is very successful, e.g., [Ahrens, etc, '09; Becher, Neumann, '20; Stewart, Tackman, Waalewijn, '09; Ellis, etc, '10; Beneke and Kirilin, '12 ...]. How to apply NLP SCET at colliders?

1. NLP correction to measurement functions
2. NLP operators depending on processes
3. NLP SCET Lagrangian insertion(s)

?

NLP SCET Lagrangian has been known for a while [Beneke, Feldman, '02; Mount, Stewart, Vita, '19] , so it seems it should be straightforward. But NO.

$H_2(z) \otimes$

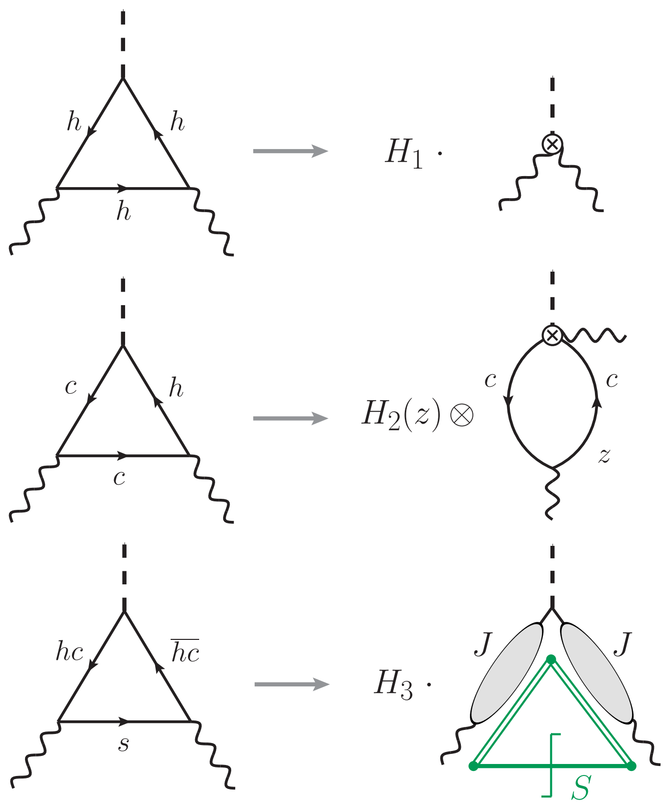


$$d\sigma \sim \sum_i H_i^{(0)} \otimes O_i^{(0)} \sim \sum_i H_i^{(0)} \otimes J_{i_1}^{(0)} \cdots \otimes J_{i_n}^{(0)} \otimes S_i^{(0)}$$

See Robert's talk

- Bare factorization is well understood for different processes, e.g., [DY: Beneke, etc. '16 - '20; Thrust: Mount, etc., '18-'20; $H\gamma\gamma$: Liu and Neubert, '19; Wang, '19]. The problem is when renormalizing endpoint divergences always occur.
- Without consistent renormalization, there is no way to perform resummation by standard RG evolution.

Bare Factorization: Plus-Type Subtraction and Emergence of Cutoff



$$\mathcal{M}_{\gamma\gamma}$$

$$||$$

$$\boxed{= H_1^{(0)}\langle O_1^{(0)}\rangle}$$

$$+$$

$$\boxed{= 2\int_0^1 dz H_2^{(0)}(z)\langle O_2^{(0)}(z)\rangle} \left(H_2^{(0)}(z) = \frac{\bar{H}_2(z)}{z(1-z)}\right)$$

$$+$$

$$\boxed{= H_3\langle O_3^{(0)}\rangle = H_3^{(0)}\int_0^\infty \frac{d\ell_+}{\ell_+}\int_0^\infty \frac{d\ell_-}{\ell_-} J^{(0)}\left(-M_h\ell_+\right) J^{(0)}\left(M_h\ell_-\right) S^{(0)}\left(\ell_+\ell_-\right)}$$

- Endpoint divergences occur when $z \to 0{,}1$ and $\ell_\pm \to \infty$.
- Some are regularized by DR, while others are rapidity divergences.
- Rapidity divergences are cancelled additively, not like LP SCET!

[Becher, Neubert, '10]
[Chiu, Jain, Neil, Rothstein, '11]

Cancellation of rapidity divergences indicates close relation between the two integrands in the endpoint region (next slide)

See Marvin's talk

"plus-type" subtraction

infinity bin

[Liu, Neubert, 1912.08818]

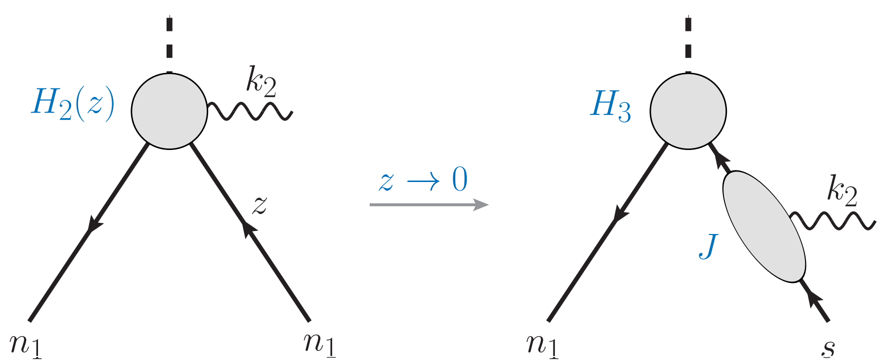
$$\mathcal{M}_{\gamma\gamma} = \left(H_1^{(0)} + \Delta H_1^{(0)}\right)\langle O_1^{(0)}\rangle + 2\int_0^1 dz\left[H_2^{(0)}(z)\langle O_2^{(0)}(z)\rangle - [\![H_2^{(0)}(z)]\!][\![\langle O_2^{(0)}(z)\rangle]\!] - [\![H_2^{(0)}(\bar{z})]\!][\![\langle O_2^{(0)}(\bar{z})\rangle]\!]\right]$$

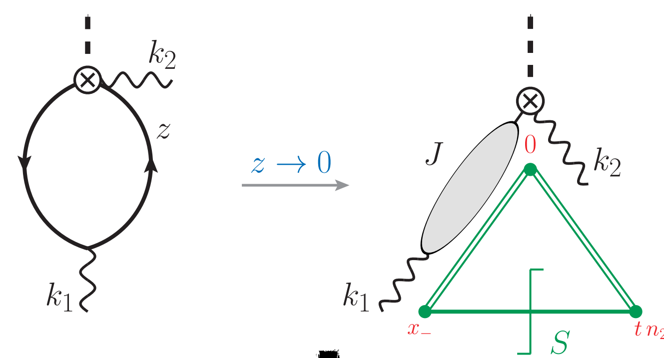
$$+ \lim_{\sigma\to -1} H_3^{(0)}\int_0^{\boxed{m_H}}\frac{d\ell_-}{\ell_-}\int_0^{\boxed{\sigma m_H}}\frac{d\ell_+}{\ell_+} J^{(0)}\left(m_H\ell_-\right) J^{(0)}\left(-m_H\ell_+\right) S^{(0)}\left(\ell_+\ell_-\right)\Big|_{\text{leading power}}$$

- $[\![f(z)]\!]$ means that one retains only the leading terms of the function $f(z)$.
- Cutoffs are emergent after adding back the subtraction and double counting is removed, which is $\Delta H_1^{(0)}$.
- Rapidity regulator is no longer needed due to plus-type subtraction.
- The factorization formula for $gg \to h$ to appear is very similar to its abelian cousin.

Re-factorization conditions

- Re-factorization conditions relate the integrands in the endpoint region.
- We proved them to all orders of α_s.





$$[\![H_2^{(0)}(z)]\!] = \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z} = -\frac{H_3^{(0)}}{z} J\left(zM_h^2\right)$$

$$[\![\langle\gamma\gamma|O_2^{(0)}(z)|h\rangle]\!] = -\frac{1}{2}\varepsilon_\perp^*\left(k_1\right)\cdot\varepsilon_\perp^*\left(k_2\right)\int_0^\infty \frac{d\ell_+}{\ell_+} J^{(0)}\left(-M_h\ell_+\right) S^{(0)}\left(zM_h\ell_+\right)$$

- These can also be used to obtain relations among renormalization factors, e.g., Z_J and $[\![Z_{22}]\!]$.
- They also ensure all order relations between "left-over" terms due to cutoffs when renormalizing operators.
- These conditions also hold in $gg \to h$ amplitude (to appear).
- Re-factorization could be generic to deal with endpoint divergences, including SCET1.

Renormalization ($h \to \gamma\gamma$ & $gg \to h$)

- ☑ There are operator mixings when renormalizing them.

- ☑ The renormalization for the non-abelian case is slightly different, since the amplitude itself is not IR safe.

  Extra divergences can be accounted for by a global renormalization: [Becher and Neubert, '09]

$$\mathcal{M}_{gg}(\mu) = Z_{gg}^{-1}(\mu)\mathcal{M}_{gg}^{(0)}, \quad \text{with} \quad Z_{gg}^{-1} = 1 + \frac{\alpha_s(\mu)}{4\pi}\left[\frac{2C_A}{\epsilon^2} + \frac{-2C_A \ln(-M_h^2/\mu^2) + \beta_0}{\epsilon}\right] + \mathcal{O}(\alpha_s^2)$$

- ☑ This global renormalization factor changes the renormalization factors for the operators,

  and therefore the anomalous dimensions. Here are the Z factors of the soft function S at NLO as an example:

Lange-Neubert kernel

$$Z_S^{gg}(w,w';\mu) = \delta(w-w') + \frac{\alpha_s(\mu)}{4\pi}\left\{\left[(C_F - C_A)\left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{\omega}{\mu^2}\right) - \frac{3C_F - \beta_0}{\epsilon}\right]\delta(w-w') - \frac{4(C_F - C_A/2)}{\epsilon} w\Gamma(w,w')\right\}$$

$$Z_S^{\gamma\gamma}(w,w';\mu) = \delta(w-w') + \frac{\alpha_s(\mu)}{4\pi}\left\{\left[C_F\left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{\omega}{\mu^2}\right) - \frac{3C_F}{\epsilon}\right]\delta(w-w') - \frac{4C_F}{\epsilon} w\Gamma(w,w')\right\}$$

- ☑ The structure of cusp term is also observed in [Mount, Stewart, Vitra Zhu, '09; Beneke, Garny, Jaskiewicz, Szafron, Vernazza' Wang, 20]

- ☑ We derived the non-abelian renormalization factors, not only from RG invariance, but also

  using the method in [Bodwin, et al., 2101.04872].

Renormalized Factorization: Plus-Type Subtraction and Cutoff

$$\mathcal{M}_{\gamma\gamma} = \underbrace{H_1(\mu)\,\langle O_1(\mu)\rangle}_{T_1(\mu)} + \underbrace{2\int_0^1 dz \Big[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - [\![H_2(z,\mu)]\!][\![\langle O_2(z,\mu)\rangle]\!] - [\![H_2(\bar z,\mu)]\!][\![\langle O_2(\bar z,\mu)\rangle]\!]\Big]}_{T_2(\mu)}$$

$$+ \underbrace{\lim_{\sigma\to -1} H_3(\mu) \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J\left(M_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right)}_{T_3(\mu)}\Big|_{\text{leading power}}$$

- This master formula is free of any divergences for $H \to \gamma\gamma$. Its non-abelian cousin is similar, but needs Z_{gg}^{-1}.

- To establish such a renormalized formula is not so straightforward:
  - With cutoffs in the convolution, exchanging integration limits doesn't commute with renormalization, e.g.,

$$S(w,\mu) = \int_0^{\boxed{\infty}} dw'\, Z_S\left(w,w';\mu\right) S^{(0)}\left(w'\right) \quad \text{v.s} \quad \int_0^{\boxed{\sigma M_h^2}} \frac{dw'}{w'} S^{(0)}(w') \times \cdots$$

- After exchanging the integration limits when expressing everything in terms of renormalized ones, there are some "left-over" terms (mismatch). We proved to all orders that the sum of these terms is purely **hard**, and it can be absorbed into H_1. The same procedure also applies to $gg \to H$.

$$H_1(\mu) = \Big(H_1^{(0)} + \underbrace{\Delta H_1^{(0)}}_{\text{infinity bin}} - \underbrace{\delta H_1^{(0)}}_{\text{left-over}}\Big) Z_{11}^{-1} + 2\int_0^1 dz \underbrace{\Big[H_2^{(0)}(z) Z_{21}^{-1}(z) - [\![H_2^{(0)}(z)]\!][\![Z_{21}^{-1}(z)]\!] - [\![H_2^{(0)}(\bar z)]\!][\![Z_{21}^{-1}(\bar z)]\!]\Big]}_{\text{mixing}}$$

Renormalization Group Equations for Operators

$$\frac{d}{d\ln\mu}\left\langle O_1(\mu)\right\rangle = -\gamma_{11}\left\langle O_1(\mu)\right\rangle$$

$$\frac{d}{d\ln\mu}\left\langle O_2(z,\mu)\right\rangle = -\int_0^1 dz'\gamma_{22}\left(z,z'\right)\left\langle O_2\left(z',\mu\right)\right\rangle - \gamma_{21}(z)\left\langle O_1(\mu)\right\rangle$$

$$\frac{d}{d\ln\mu}[\![\left\langle O_2(z,\mu)\right\rangle]\!] = -\int_0^1 dz'\,[\![\gamma_{22}\left(z,z'\right)]\!][\![\left\langle O_2\left(z',\mu\right)\right\rangle]\!] - [\![\gamma_{21}(z)]\!]\left\langle O_1(\mu)\right\rangle$$

$$\frac{d}{d\ln\mu}J\left(p^2,\mu\right) = -\int_0^\infty dx\,\gamma_J\left(p^2,xp^2\right)J\left(xp^2,\mu\right)$$

$$\frac{d}{d\ln\mu}S(w,\mu) = -\int_0^\infty dx\,\gamma_S(w,w/x)S(w/x,\mu)$$

- There are non-local kernels (Brodsky-Lepage kernel and Lange-Neubert kernel) in all the above except $\left\langle O_1(\mu)\right\rangle$.
- Two-loop solutions to $J(p^2,\mu)$ and $S(\omega,\mu)$ are obtained in [2003.03393] and [2005.03013].
- The solutions to $\langle O_2(z,\mu)\rangle$ and its endpoint region version will be given in our on-coming paper.
- The anomalous dimensions for the non-abelian case are different but the structure is similar.
- We evolve the operators up to the hard scale μ_h, such that resummation is obtained by evolution of operators.

Large Logarithms at 3-loop

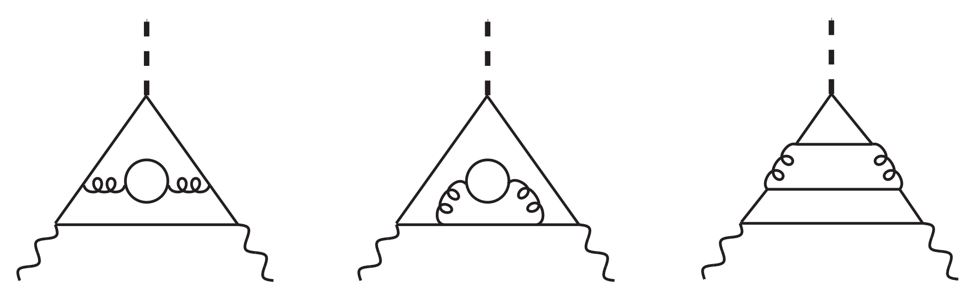
- The renormalized factorization formulae for $H \to \gamma\gamma$ and $gg \to H$ can reproduce 2-loop fixed order results.
- Using the RGEs up to 2-loop ($H \to \gamma\gamma$), we can even analytically reproduce 3-loop amplitude for $\alpha_s^2 L^{6,5,4,3}$, in perfect agreement with [Czakon, Niggetiedt, '20].

$$L = \ln \frac{-m_H^2 - i0}{m_b^2}$$

$$\begin{aligned}\mathcal{M}_b^{\gamma\gamma} =& \frac{N_c \alpha_b}{\pi} \frac{m_b^2}{v} \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2) \Bigg\{ \frac{L^2}{2} - 2 \\ &+ \frac{C_F \alpha_s(\hat{\mu}_h)}{4\pi} \left[-\frac{L^4}{12} - L^3 - \frac{2\pi^2}{3} L^2 + \left(12 + \frac{2\pi^2}{3} + 16\zeta_3 \right) L - 20 + 4\zeta_3 - \frac{\pi^4}{5} \right] \\ &+ C_F \left(\frac{\alpha_s(\hat{\mu}_h)}{4\pi} \right)^2 \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} - \frac{\beta_0}{30} \right) L^5 + d_4^{\mathrm{OS}} L^4 + d_3^{\mathrm{OS}} L^3 + \dots \right] \Bigg\}\end{aligned}$$

$$0.01975L^6 - 0.31111L^5 - 8.74342L^4 - 68.6182L^3$$

- d_4^{OS} and d_3^{OS} are numbers in terms of zeta values and color factors. The large logarithms correspond to the diagrams in full theory.



- Note that there is hierarchy between the coefficients of the logarithms. Although L is large, subleading logarithm is not smaller than the leading one at all, not to mention L^4 and L^3. This means LL and NLL resummation would not be sufficient.

Resummation: RG improved LO

- The previous slide shows you that we should treat equally leading logarithms, subleading logarithms, etc.

  In other words, we resum the large logarithms in RG improved perturbation theory, keeping the counting in exponential.

$$T_i \sim \exp\left[L^2 \cdot g_0^i(\alpha_s L^2) + g_1^i(\alpha_s L^2) + \alpha_s \cdot g_2^i(\alpha_s L^2) + \ldots\right]$$

- It turns out that NLL resummation only receives contribution from resummed T_3. However, RG improved LO resummation also needs resummed T_2. The RG improved LO T_3 for $H \to \gamma\gamma$ reads:

$$T_{3,\mathrm{LO}}^{\mathrm{RGi}} = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu_h)}{\sqrt{2}} m_b(\mu_h) \int_0^{m_H} \frac{d\ell_-}{\ell_-} \int_0^{\sigma m_H} \frac{d\ell_+}{\ell_+} e^{2S_\Gamma(\mu_s,\mu_h) - 2S_\Gamma(\mu_-,\mu_h) - 2S_\Gamma(\mu_+,\mu_h) + a_{\gamma_s} + a_{\gamma_m} + 2\gamma_E\left(2a_\Gamma^s - a_\Gamma^- - a_\Gamma^+\right)}$$
$$\left(\frac{\sigma m_H \ell_-}{\mu_-^2}\right)^{a_\Gamma^-} \left(\frac{m_H \ell_+}{\mu_+^2}\right)^{a_\Gamma^+} \left(\frac{\ell_- \ell_+}{\mu_s^2}\right)^{-a_\Gamma^s} \frac{\Gamma(1-a_\Gamma^-)}{\Gamma(1+a_\Gamma^-)} \frac{\Gamma(1-a_\Gamma^+)}{\Gamma(1+a_\Gamma^+)} \underbrace{G_{4,4}^{2,2}\left(\begin{matrix} 1,0,1,1 \\ 1+a_\Gamma^s, 1+a_\Gamma^s, a_\Gamma^s, a_\Gamma^s \end{matrix} \middle| \frac{\ell_-\ell_+}{m_b^2} \right)}_{\text{from resummed soft function}}$$

Meijer-G

from resummed soft function

$$S_\Gamma(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\mathrm{cusp}}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}, \quad a_V(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_V(\alpha)}{\beta(\alpha)}$$

- For RG improved LO resummation, we use two loop Γ_{cusp} in S_Γ, one loop γ_V in a_Γ^i, a_{γ_s} and a_{γ_m}.

Resummation: RG improved LO

- Contribution from T_2, which originates from mixing with $O_1(\mu)$, is smaller than T_3. However, it is hard to obtain an analytic expression like the case of T_3 due to the subtraction nature.

$$T_2 = 2\int_0^1 dz\Big[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - [\![H_2(z,\mu)]\!][\![\langle O_2(z,\mu)\rangle]\!] - [\![H_2(\bar{z},\mu)]\!][\![\langle O_2(\bar{z},\mu)\rangle]\!]\Big]$$

- We can numerically evaluate it. Nevertheless let me show you analytically how subtraction kills endpoint divergences. To see this explicit, we go to Gegenbauer space:

$$(H_2 \otimes \langle O_2\rangle)_{\rm div} \propto \sum_{m=0}^{\infty} \frac{4m+3}{(2m+1)(2m+2)}\frac{1}{\beta_0 + 2C_F\,(2H_{2m+1}-3)}$$
$$2\big([\![H_2]\!] \otimes [\![\langle O_2\rangle]\!]\big)_{\rm div} \propto \sum_{m=0}^{\infty} \frac{4m+3}{(2m+1)(2m+2)}\frac{1}{\beta_0 + 2C_F\,(2H_{2m+1}-3-1/((2m+1)(2m+2)))}$$

- The logic applies to the non-abelian case too.

NLL from RG improved LO resummation

- Although we argue that one should resum in RG improved perturbation theory, it is helpful to extract NLL results from RG improved LO results for academic purpose:

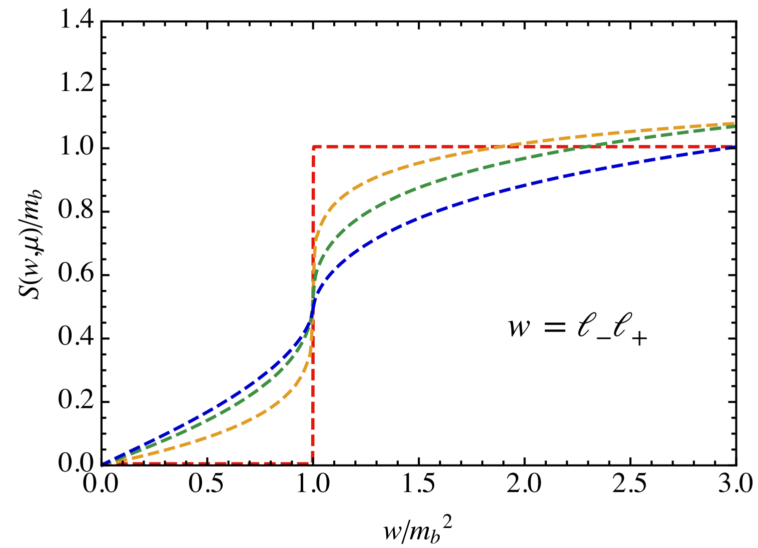
$$\mathcal{M}_{\gamma\gamma}^{\rm NLL} \propto \frac{L^2}{2}\sum_{n=0}^{\infty}(-\rho_\gamma)^n \frac{2\Gamma(n+1)}{\Gamma(2n+3)}\left[1+\frac{3\rho_\gamma}{2L}\frac{2n+1}{2n+3}-\frac{\beta_0}{C_F}\frac{\rho_\gamma^2}{4L}\frac{(n+1)^2}{(2n+3)(2n+5)}\right] \qquad \rho_\gamma = \frac{C_F\alpha_s(\mu_h)L^2}{2\pi}$$

$$\mathcal{M}_{gg}^{\rm NLL}(\hat{\mu}_h) \propto \frac{L^2}{2}\sum_{n=0}^{\infty}(-\rho_g)^n \frac{2\Gamma(n+1)}{\Gamma(2n+3)}\left[1+\frac{C_F}{C_F-C_A}\frac{3\rho_g}{2L}\frac{2n+1}{2n+3}-\frac{\beta_0}{C_F-C_A}\frac{\rho_g^2}{4L}\frac{(n+1)^2}{(2n+3)(2n+5)}\right] \qquad \rho_g = \frac{(C_F-C_A)\alpha_s(\mu_h)L^2}{2\pi}$$

- At NLL, abelian case is recovered by $C_F - C_A \to C_F$ from the non-abelian case (**in general not true beyond NLL**).
- Abelian LL agree with [Akhoury et al, '01] but NLL **not,** and non-abelian NLL agrees with the latest version of [Anastasiou and Penin, '20].

How to obtain NLL from RG improved LO

☑ $H \to \gamma\gamma$:



$$L_s = \ln \frac{\mu_h^2}{\ell_- \ell_+}$$
$$L_- = \ln \frac{\mu_h^2}{m_H \ell_-}$$
$$L_+ = \ln \frac{\mu_h^2}{-m_H \ell_+}$$

$$\exp\left[-\frac{C_F \alpha_s(\mu_h)}{4\pi}\left(L_s^2 - L_-^2 - L_+^2 + \frac{\alpha_s(\mu_h)}{4\pi}\frac{\beta_0}{3}\left(L_s^3 - L_-^3 - L_+^3\right)\right) - 6L_s\right]$$

☑ $gg \to H$ is involved due to the color factors of local kernel and non-kernel in the jet and soft function are **different**.

Resummed jet and soft function are different from the abelian case. For example, the jet function has the following change:

$$\frac{\Gamma\left(1 - a_{\Gamma_c}\right)}{\Gamma\left(1 + a_{\Gamma_c}\right)} \longrightarrow \left(\frac{\Gamma\left(1 - a_{\Gamma_c}\right)}{\Gamma\left(1 + a_{\Gamma_c}\right)}\right)^{\frac{C_F - C_A/2}{C_F - C_A}} \qquad \Gamma_c = C_F - C_A, \quad \left.\frac{C_F - C_A/2}{C_F - C_A}\right|_{C_A \to 0} = 1$$

☑ Soft function is the same story. After expanding the new soft function to step function and also the exponential, everything else follows in the same way as in the abelian case. The integrand now reads:

$$\exp\left[-\left(C_F - C_A\right)\frac{\alpha_s\left(\mu_h\right)}{4\pi}\left(L_s^2 - L_-^2 - L_+^2 + \frac{\alpha_s\left(\mu_h\right)}{4\pi}\frac{\beta_0}{3}\left(L_s^3 - L_-^3 - L_+^3\right)\right) + \frac{\beta_0 - 6C_F}{C_F - C_A}L_s\right]$$

Conclusion and take-home message

- ☑ We derived the **renormalized factorization** formula in the "**plus-type subtraction**" scheme to get rid of endpoint divergences.
- ☑ The framework can be applied to other SCET2 problems, e.g., NLP qT resummation.
- ☑ Subtraction scheme is successful to deal with endpoint divergences. Re-factorization conditions play a key role here, and we believe they are also important in other cases.
- ☑ Our prediction is in perfect agreement with QCD three-loop calculations.
- ☑ It is the first NLP resummation which can be done to RG improved LO.
- ☑ As NLL, non-abelian and abelian seem to be the same under the replacement $C_F - C_A \to C_F$. But in general it is **not** true beyond.

Thank you for your attention!

Backup: Bare results

$$
\begin{aligned}
H_1^{(0)} =& \frac{y_{b,0}}{\sqrt{2}} \frac{N_c \alpha_{b,0}}{\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} (1-3\epsilon) \frac{2\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(3-2\epsilon)} \\
& \times \left\{ 1 - \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{\Gamma(2-3\epsilon)} \right. \\
& \times \left[\frac{2(1-\epsilon)\left(3 - 12\epsilon + 9\epsilon^2 - 2\epsilon^3\right)}{1-3\epsilon} + \frac{8}{1-2\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(2-\epsilon)\Gamma(2-3\epsilon)}{\Gamma(1+2\epsilon)\Gamma^3(1-2\epsilon)} \right. \\
& \left. \left. - \frac{4\left(3 - 18\epsilon + 28\epsilon^2 - 10\epsilon^3 - 4\epsilon^4\right)}{1-3\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(1+\epsilon)\Gamma(2-2\epsilon)} \right] \right\} \\
H_2^{(0)}(z) =& \frac{y_{b,0}}{\sqrt{2}} \left\{ \frac{1}{z} + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \right. \\
& \left. \times \left[\frac{2 - 4\epsilon - \epsilon^2}{z^{1+\epsilon}} - \frac{2(1-\epsilon)^2}{z} - 2\left(1 - 2\epsilon - \epsilon^2\right) \frac{1 - z^{-\epsilon}}{1-z} \right] \right\} + (z \to 1-z) \\
H_3^{(0)} =& \frac{y_{b,0}}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} 2(1-\epsilon)^2 \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \right]
\end{aligned}
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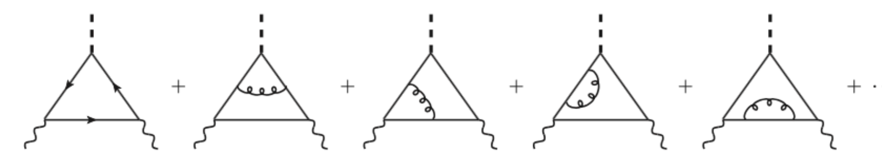
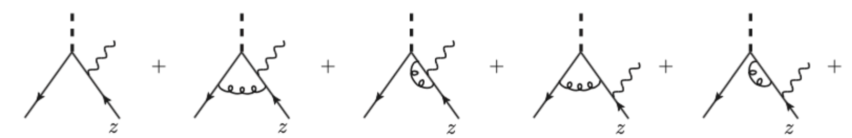
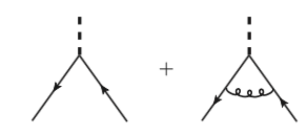
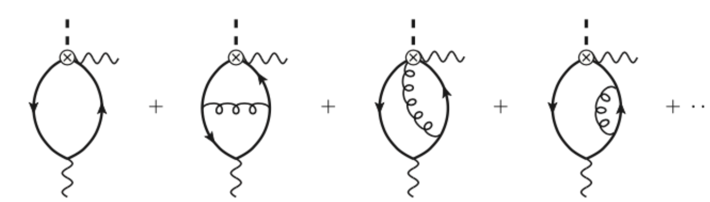
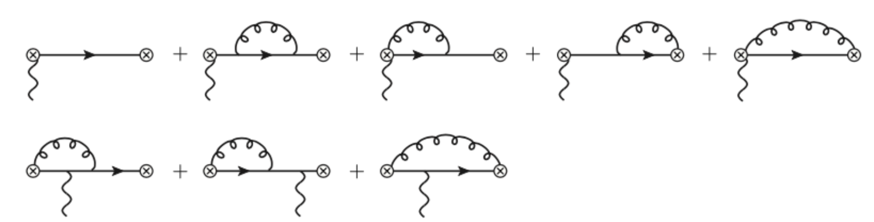


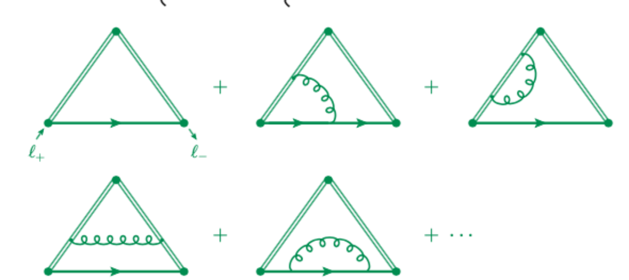




$$
\begin{aligned}
\langle O_1^{(0)} \rangle =& m_{b,0} g_\perp^{\mu\nu} \\
\langle O_2^{(0)}(z) \rangle =& \frac{N_c \alpha_{b,0}}{2\pi} m_{b,0} g_\perp^{\mu\nu} \left[e^{\epsilon \gamma_E} \Gamma(\epsilon) \left(m_{b,0}^2\right)^{-\epsilon} + \frac{C_F \alpha_{s,0}}{4\pi} \left(m_{b,0}^2\right)^{-2\epsilon} \left[K(z) + K(1-z)\right] \right] \\
J^{(0)}\left(p^2\right) =& 1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-p^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \left(2 - 4\epsilon - \epsilon^2\right) \\
S^{(0)}(w) =& -\frac{N_c \alpha_{b,0}}{\pi} m_{b,0} \left[S_a^{(0)}(w) \theta\left(w - m_{b,0}^2\right) + S_b^{(0)}(w) \theta\left(m_{b,0}^2 - w\right) \right]
\end{aligned}
$$

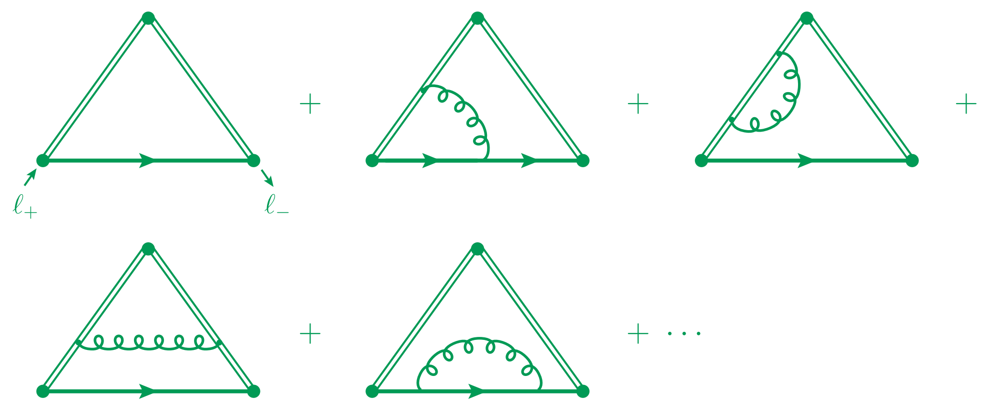






Backup: Renormalized quantities: Soft Quark Soft Function

- The soft quark soft function is the most complex.
- The diagrams for $H \to \gamma\gamma$ and $gg \to H$ are basically the same, taking color factors into account.



$$\boxed{L_w = \ln\left(w/\mu^2\right), \ \hat{w} = w/m_b^2}$$

$$S^\gamma(w,\mu) = -\frac{N_c \alpha_b}{\pi} m_b(\mu) \left[S_a^\gamma(w,\mu)\theta\left(w - m_b^2\right) + S_b^\gamma(w,\mu)\theta\left(m_b^2 - w\right)\right]$$

$$S_a^\gamma(w,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + g(\hat{w})\right]$$

$$S_b^\gamma(w,\mu) = \frac{C_F\alpha_s}{\pi}\ln(1-\hat{w})\left[L_m + \ln(1-\hat{w})\right]$$

$$S^g(w,\mu) = -\frac{T_F\delta_{AB}\alpha_s}{\pi} m_b(\mu)\left[S_a^g(w,\mu)\theta\left(w - m_b^2\right) + S_b^g(w,\mu)\theta\left(m_b^2 - w\right)\right]$$

$$S_a^g(w,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + g(\hat{w})\right] + \frac{C_A\alpha_s}{4\pi}\left[L_w^2 - \frac{\pi^2}{6} + h(\hat{w})\right]$$

$$S_b^g(w,\mu) = \left(C_F - \frac{C_A}{2}\right)\frac{\alpha_s}{\pi}\ln(1-\hat{w})\left[L_m + \ln(1-\hat{w})\right]$$

Backup: Renormalization ($h \to \gamma\gamma$)

$$\{O_1,\ O_2(z),\ [\![O_2(z)]\!]\} \qquad O_i(\mu) = Z_{ij} \otimes O_j^{(0)} \qquad \boldsymbol{Z} = \begin{pmatrix} Z_{11} & 0 & 0 \\ Z_{21} & Z_{22} & 0 \\ [\![Z_{21}]\!] & 0 & [\![Z_{22}]\!] \end{pmatrix}.$$

- Renormalization of O_1 is trivial, which is just the quark mass renormalization
- The diagonal Z_{22} can be understood by noticing that the coloured fields in O_2 have the same structure as in leading-twist LCDA of a transversely polarized vector meson: Brodsky-Lepage kernel
- Z_{22} is not enough to absorb all the UV divergence in O_2. The remaining can be absorbed by the mixing with O_1, which is just Z_{21}. Since the final states are photons, the mixing is natural
- The renormalization of $[\![O_2(z)]\!]$ can be obtained by the limiting behaviour of that of O_2

$$J^{(0)} \otimes J^{(0)} \otimes S^{(0)} = O_3^{(0)} = \mathrm{T}\left\{h\bar{\xi}_{n_1}\xi_{n_2}, i\int d^D x \mathcal{L}^{(1/2)}_{q\xi_{n_1}}(x), i\int d^D y \mathcal{L}^{(1/2)}_{\xi_{n_2}q}(y)\right\} + \text{h.c.}$$

- NLP SCET Lagrangian doesn't need renormalization, so the renormalization of $O_3^{(0)}$ comes from that of the scalar current $J_S = h\bar{\xi}_{n_1}\xi_{n_2}$, which is known to three loops:

$$\int_0^\infty d\ell_- \int_0^\infty d\ell_+ Z_J(\ell'_-, \ell_-) Z_J(\ell'_+, \ell_+) Z_S(\ell_-\ell_+, \omega) = Z_{33}\delta(\omega - \ell'_-\ell'_+)$$

- Z_J is related to $[\![Z_{22}]\!]$ by re-factorization formula and we prove that it can also be obtained from first principle
- Z_S can be obtained from the above relation and recently confirmed by Bodwin et al. first principle calculation at NLO

Backup: Renormalization Group Equations for Hard Functions

$$\frac{d}{d\ln\mu}H_1(\mu) = \gamma_{11}H_1(\mu) + \underbrace{D_{\text{cut}}\,(\mu)}_{\text{left-over}} + 2\int_0^1 dz\Big[H_2(z,\mu)\gamma_{21}(z) - \underbrace{[\![H_2(z,\mu)]\!][\![\gamma_{21}(z)]\!] - [\![H_2(\bar{z},\mu)]\!][\![\gamma_{21}(\bar{z})]\!]}_{\text{mixing}}\Big]$$

$$\frac{d}{d\ln\mu}H_2(z,\mu) = \int_0^1 dz' H_2\left(z',\mu\right)\gamma_{22}\left(z',z\right)$$

$$\frac{d}{d\ln\mu}H_3(\mu) = \gamma_{33}H_3(\mu)$$

- Due to the cutoffs, those "left-over" terms will give the inhomogeneous contribution in the end:

$$D_{\text{cut}}(\mu) = -\frac{N_c\alpha_b}{\pi}\frac{y_b(\mu)}{\sqrt{2}}\left[\frac{C_F\alpha_s}{4\pi}16\zeta_3 + \left(\frac{\alpha_s}{4\pi}\right)^2 d_{\text{cut},2} + \mathcal{O}\left(\alpha_s^3\right)\right] \ni \alpha_b\left(\alpha_s L_h\right)^n$$

- The RGE for $H_1(\mu)$ is **not** Sudakov type due to $D_{\text{cut}}(\mu)$, which makes it difficult to be solved.

  However we can just set the scale $\mu = \mu_h$ and evolve the scales of the operators up.

- A similar pattern holds for the non-abelian case.

Backup: H_1 mismatch and amplitudes

$$\delta H_1^{(0)} + \delta' H_1^{(0)} = H_3(\mu) \int_{M_h}^{\infty} \frac{d\rho_-}{\rho_-} \int_{\sigma M_h}^{\infty} \frac{d\rho_+}{\rho_+} \frac{S^{(\epsilon)}\left(\rho_+\rho_-, \mu\right)}{m_{b,0}} J^{(\epsilon)}\left(M_h\rho_-, \mu\right) J^{(\epsilon)}\left(-M_h\rho_+, \mu\right)\Big|_{\text{leading power}}$$

$$- H_3^{(0)} \int_{M_h}^{\infty} \frac{d\rho_-}{\rho_-} \int_{\sigma M_h}^{\infty} \frac{d\rho_+}{\rho_+} \frac{S^{(0)}\left(\rho_+\rho_-\right)}{m_{b,0}} J^{(0)}\left(M_h\rho_-\right) J^{(0)}\left(-M_h\rho_+\right)\Big|_{\text{leading power}}$$

$$+ 4\left[\int_0^1 dz \int_0^\infty dz' - \int_0^\infty dz \int_0^1 dz'\right] \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z} [\![Z_{22}^{-1}\left(z, z'\right)]\!][\![Z_{21}\left(z'\right)]\!],$$

$$T_1 \propto -2 + \frac{C_F\alpha_s}{4\pi}\left[-\frac{\pi^2}{3}L_h^2 + \left(12 + 8\zeta_3\right)L_h + \cdots\right], \quad T_2 \propto \frac{C_F\alpha_s}{4\pi}\left[\frac{2\pi^2}{3}L_h L_m - \frac{\pi^2}{3}L_m^2 + \cdots\right],$$

$$T_3 \propto \frac{L^2}{2} + \frac{C_F\alpha_s}{4\pi}\left[-\frac{L^4}{12} - L^3 + \frac{1}{2}\left(-3L_m + 4 - 4\zeta_2\right)L^2 + 4\left(\zeta_2 + 2\zeta_3\right)L - 8\zeta_3 L_m + \cdots\right]$$

$$\boxed{L = \ln\frac{-m_H^2 - i0}{m_b^2}}$$

$$\boxed{L_h = \ln\frac{-m_H^2 - i0}{\mu^2}}$$

$$\boxed{L_m = \ln\frac{m_b^2}{\mu^2}}$$

Backup: Anomalous Dimensions

$$\gamma_J^{\gamma\gamma}\left(p^2, xp^2\right) = \frac{\alpha_s}{\pi}\left[C_F \ln\frac{-p^2}{\mu^2}\delta(1-x) + C_F\Gamma(1,x)\right] + C_F\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\theta(1-x)}{1-x}h(x) + \mathcal{O}\left(\alpha_s^3\right)$$

$$\gamma_S^{\gamma\gamma}\left(w, w'; \mu\right) = -\frac{\alpha_s}{\pi}\left[C_F\left(\ln\frac{w}{\mu^2} + \frac{3}{2}\right)\delta\left(w - w'\right) + 2C_F w\Gamma\left(w, w'\right)\right] - 2C_F\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{w\theta\left(w'-w\right)}{w'\left(w'-w\right)}h\left(\frac{w}{w'}\right) + \mathcal{O}\left(\alpha_s^3\right)$$

$$\gamma_J^{gg}\left(p^2, xp^2\right) = \frac{\alpha_s}{\pi}\left[\left(C_F - C_A\right)\ln\frac{-p^2}{\mu^2}\delta(1-x) + \left(C_F - \frac{C_A}{2}\right)\Gamma(1,x)\right] + \mathcal{O}\left(\alpha_s^2\right)$$

$$\gamma_S^{gg}\left(w, w'\right) = -\frac{\alpha_s}{\pi}\left\{\left[\left(C_F - C_A\right)\ln\frac{w}{\mu^2} + \frac{3C_F - \beta_0}{2}\right]\delta\left(w - w'\right) + 2\left(C_F - \frac{C_A}{2}\right)w\Gamma\left(w, w'\right)\right\} + \mathcal{O}\left(\alpha_s^2\right)$$

$$h(x) = \ln x\left[\beta_0 + 2C_F\left(\ln x - \frac{1+x}{x}\ln(1-x) - \frac{3}{2}\right)\right]$$

- This two loop non-local kernel was first derived in [Braun, Ji, Manashov, '19] for LCDA;
  And was confirmed by a direct calculation of two loop radiative photon jet function in [Liu, Neubert, '20].
- The two loop version for the gluon case is ongoing…