Decoupage at 5 loops

Alessandro Georgoudis
ENS Paris, CNRS

RADCOR-LoopFest 2021

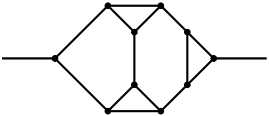
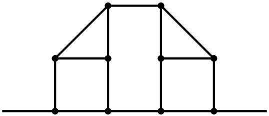
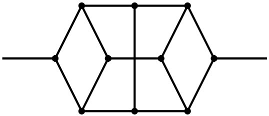
Work based on 2104.08272
with V. Goncalves, E. Panzer, R. Pereira, A. V. Smirnov and V. A Smirnov

Overview

We computed massless 5-loop p-integrals in $4 - 2\epsilon$ dimensions.

Overview

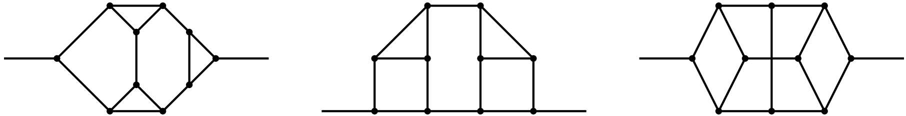
We computed massless 5-loop p-integrals in $4 - 2\epsilon$ dimensions.

they depend on a single external momenta p, which we choose as $p^2 = 1$, as $(p^2)^{-\omega}$ where $\omega = \sum a_i - \ell d/2$.

Overview

We computed massless 5-loop p-integrals in $4-2\epsilon$ dimensions.



they depend on a single external momenta p, which we choose as $p^2=1$, as $(p^2)^{-\omega}$ where $\omega=\sum a_i - \ell d/2$.
Then a general p-integral P can be written as

$$P(d) = (p^2)^{-\omega(P)} \sum_{n\in\mathbb{Z}} c_n(P)\varepsilon^j$$

where all the c_n for our case are rational numbers, $\zeta(i)$ or $\zeta(i,j)$ values.

Motivation

Motivation

- l-loop p-integrals can be used to compute l+1-loop counterterms.
- β-functions in different theories.
- They appear as boundary conditions in differential equations.
- Can be used for computing different physical quantities (structure constants, anomalous dimension, etc).

Motivation

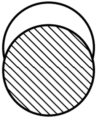
- l-loop p-integrals can be used to compute l+1-loop counterterms.
- β-functions in different theories.
- They appear as boundary conditions in differential equations.
- Can be used for computing different physical quantities (structure constants, anomalous dimension, etc).
- Cross-check that our results for $\hat{\zeta}(i)$ and $\hat{\zeta}(i,j)$ transformation hold for all p-integrals (not only position space) and compute the missing momentum space non-planar contribution.

How?

We want to use the Glue-and-Cut method to compute the p-integrals.

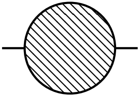
Glue-and-Cut

Let us consider a vacuum integrals with $\omega = 0$ and no other sub-divergences.



$$= \int \frac{\mathrm{d}^D p}{p^2 + m^2} \frac{P(\epsilon)}{(p^2)^{1+l\epsilon}} = \frac{c_0}{(l+1)\epsilon} + \mathrm{O}(\epsilon)$$

The value of the p-integral is then:



$$= \frac{c_0}{p^2} + \mathrm{O}(\epsilon))$$

By cutting different edges we can construct convergent p-integrals that have the same value at ϵ^0 order.

How?

We want to use the Glue-and-Cut method to compute the p-integrals.

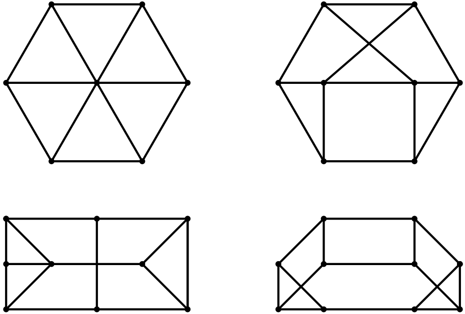
How?

We want to use the Glue-and-Cut method to compute the p-integrals.

- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.

Vacuum integrals

We need vacuum integrals with at least 12 propagators. This is the minimal number in 4 dimensions to have $\omega = 0$. Higher number of propagators can be generated by adding numerators.
For our computation we used convergent vacuum integrals with up to 2 numerators and 14 propagators.



How?

We want to use the Glue-and-Cut method to compute the p-integrals.

- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.

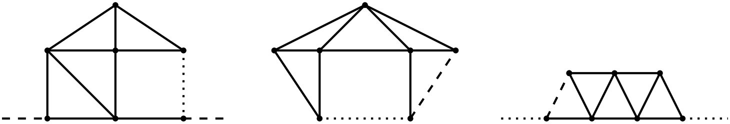
How?

We want to use the Glue-and-Cut method to compute the p-integrals.

- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.
- Construct all possible cuts. Map integrals to a specific p-integral family.

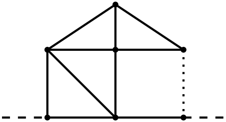
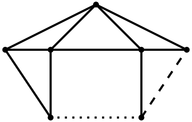
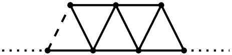
Cutting

We now need to cut all the possible vacuum integrals.

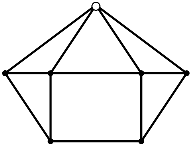
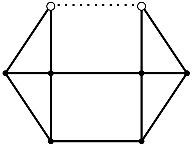
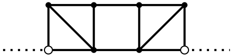


Cutting

We now need to cut all the possible vacuum integrals.

In order to produce p-integrals with a higher number of propagators we also have to blow up higher valence vertices.

We then need to map all the p-integrals into one of the 64 possible 3-valent maximal topologies.

How?

We want to use the Glue-and-Cut method to compute the p-integrals.

- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.
- Construct all possible cuts. Map integrals to a specific p-integral family.

How?

We want to use the Glue-and-Cut method to compute the p-integrals.

- Generate vacuum 6-loop graphs. Check for “convergence” of vacuum integrals, allowing also for numerators.
- Construct all possible cuts. Map integrals to a specific p-integral family.
- Apply constraints and equivalence of the different cuts of the same vacuum integral.

Constraints

How can we learn something from the equivalence of finite p-Integrals?

Constraints

How can we learn something from the equivalence of finite p-Integrals?
With integral by parts reductions!

- IBPs are integration of a total derivative:

$$0 = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \cdots \frac{\mathrm{d}^D \ell_L}{\mathrm{i}\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu}{D_1^{\nu_1} \cdots D_m^{\nu_m}} \,,$$

- They can be used to express a general integral as a combination of Master Integrals:

$$\mathsf{I} = \sum_{i=1}^{N} d_i \; \mathsf{I}_i \,,$$

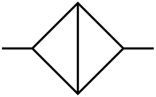
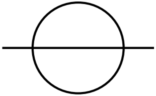
Constraints

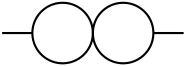
The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.

Constraints: Example

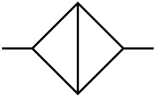
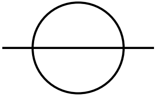
Reduction of two-loop propagator integral

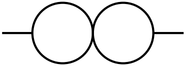
 $= \frac{2(3d-10)(8-3d)}{(d-4)^2}$ 

$- \frac{2(d-3)}{d-4}$ 

Constraints: Example

Reduction of two-loop propagator integral

 $= \frac{2(3d-10)(8-3d)}{(d-4)^2}$ 

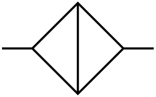
$- \frac{2(d-3)}{d-4}$ 

The spurious pole give a constrain on two epsilon orders. Convergence of the starting integral and the insertion of the value of the trivial integral $M_{\text{Dbubble}} = \frac{1}{\epsilon^2}$ gives:

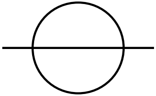
$$M_{\text{Sun}} = -\frac{1}{4\epsilon} - \frac{5}{8} - \frac{27}{16}\epsilon$$

Constraints: Example

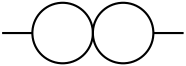
Reduction of two-loop propagator integral



$$= \frac{2(3d-10)(8-3d)}{(d-4)^2}$$



$$- \frac{2(d-3)}{d-4}$$



The spurious pole give a constrain on two epsilon orders. Convergence of the starting integral and the insertion of the value of the trivial integral $M_{\text{Dbubble}} = \frac{1}{\epsilon^2}$ gives:

$$M_{\text{Sun}} = -\frac{1}{4\epsilon} - \frac{5}{8} - \frac{27}{16}\epsilon$$

Same holds at higher loops

Constraints

The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
- Different cuts of the same vacuum integral are the same.

Constraints

The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
- Different cuts of the same vacuum integral are the same.

With this constraint we are able to relate all the coefficients of 5-loop master integrals, up to transcendental weight 9, to recursive 1-loop integrals and a product integral.

Recursive 1-loop

By recursively applying the bubble integration we can compute several integrals that appear at all loops.

$$\int \frac{\mathrm{d}^d \ell}{\pi^{d/2}} \frac{1}{\ell^{2a}(p-\ell)^{2b}} = (p^2)^{d/2-a-b} G(a,b)\,,$$

with G defined as

$$G(a,b) = \frac{\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}\,. \tag{1}$$

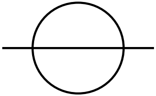
By recursively applying the bubble integration we can compute several integrals that appear at all loops.

$$\int \frac{\mathrm{d}^d \ell}{\pi^{d/2}} \frac{1}{\ell^{2a}(p-\ell)^{2b}} = (p^2)^{d/2-a-b} G(a,b)\,,$$

with G defined as

$$G(a,b) = \frac{\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}\,. \tag{1}$$

For example we can compute the sunset diagram at two loops:

 $= G(1,1)G(2-d/2,1)$

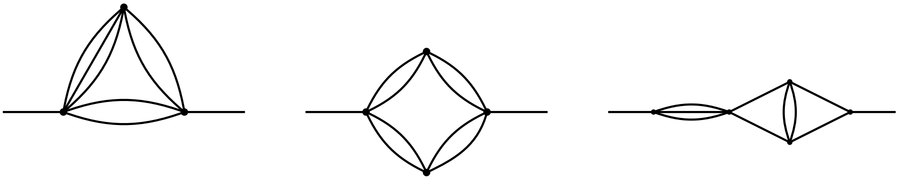
The same recursion can be done to compute the general watermelon diagram at each loop.

Constraints

The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4 - 2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
- Different cuts of the same vacuum integral are the same.

With this constraint we are able to relate all the coefficients of 5-loop master integrals, up to transcendental weight 9, to recursive 1-loop integrals and a product integral.

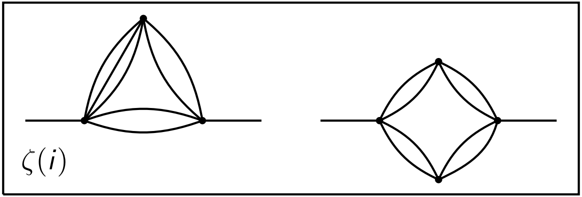


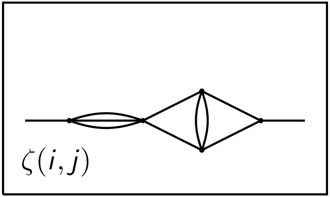
Constraints

The constraints are then:

- Degree of divergence of a l-loop 2-point integral does not exceed l. Important as IBP coefficients can have poles in $4-2\epsilon$.
- Cancellation of poles as p-integrals obtain through cutting are finite.
- Different cuts of the same vacuum integral are the same.

With this constraint we are able to relate all the coefficients of 5-loop master integrals, up to transcendental weight 9, to recursive 1-loop integrals and a product integral.





$\hat{\zeta}$ representation

There seem to exist a (ϵ dependent) redefinition of the ζ values such that the π dependent terms cancel for all the p-integrals.

$$
\begin{aligned}
\hat{\zeta}(3) &= \zeta(3) + \frac{3\varepsilon}{2}\zeta(4) - \frac{5\varepsilon^3}{2}\zeta(6) + \frac{21\varepsilon^5}{2}\zeta(8)\,, \\
\hat{\zeta}(5) &= \zeta(5) + \frac{5\varepsilon}{2}\zeta(6) - \frac{35\varepsilon^3}{4}\zeta(8)\,, \\
\hat{\zeta}(7) &= \zeta(7) + \frac{7\varepsilon}{2}\zeta(8)\,, \\
\hat{\zeta}(3,5) &= \zeta(3,5) - \frac{29}{12}\zeta(8) - \frac{15\varepsilon}{2}\zeta(4)\zeta(5), \\
\hat{\zeta}(9) &= \zeta(9)\,.
\end{aligned}
$$

$\hat{\zeta}$ representation

There seem to exist a (ϵ dependent) redefinition of the ζ values such that the π dependent terms cancel for all the p-integrals.

$$
\begin{aligned}
\hat{\zeta}(3) &= \zeta(3) + \frac{3\varepsilon}{2}\zeta(4) - \frac{5\varepsilon^3}{2}\zeta(6) + \frac{21\varepsilon^5}{2}\zeta(8)\,, \\
\hat{\zeta}(5) &= \zeta(5) + \frac{5\varepsilon}{2}\zeta(6) - \frac{35\varepsilon^3}{4}\zeta(8)\,, \\
\hat{\zeta}(7) &= \zeta(7) + \frac{7\varepsilon}{2}\zeta(8)\,, \\
\hat{\zeta}(3,5) &= \zeta(3,5) - \frac{29}{12}\zeta(8) - \frac{15\varepsilon}{2}\zeta(4)\zeta(5), \\
\hat{\zeta}(9) &= \zeta(9)\,.
\end{aligned}
$$

This type of relations, through the no-π theorem are related to some cancellations/relations of π terms in correlations functions, anomalous dimensions and β-functions.

Conclusions & Outlook

- Using the Glue-and-Cut method we were able to compute all the 5-loop 2 point master integral. Our result are up to transcendental weight $\zeta(9)$, which is the ϵ^0 order of the finite p-integrals.
- We have checked that our $\hat{\zeta}$ representation works also for momentum space p-integrals in the non-planar case.

Conclusions & Outlook

- Using the Glue-and-Cut method we were able to compute all the 5-loop 2 point master integral. Our result are up to transcendental weight $\zeta(9)$, which is the ϵ^0 order of the finite p-integrals.
- We have checked that our $\hat{\zeta}$ representation works also for momentum space p-integrals in the non-planar case.

- A major bottleneck are the IBP reductions. If one would like to push this to higher loops we need better tools.
- Would be interesting to apply the same method to different dimensions, for example $d = 3$.

Thank You!