N3LO calculations for $2 \to 2$ processes using Simplified Differential Equations

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Based on: JHEP 02 (2021) 080 (arXiv:2010.06947 [hep-ph]) & ongoing work

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21/5/2021 RADCOR-LoopFest 2021, Florida (Virtual Conference)

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Massless 3-loop 2$\to$2 families with up to one external off-shell leg

HL-LHC and possible future upgrades/colliders will require high-precision theoretical predictions, which for 2 $\to$ 2 scatterings means reaching N3LO computations. These computations demand the calculation of 3-loop Feynman Integrals!

For all the external particles on-shell (relevant for di-jet or di-photon productions) there exist 9 (2) families of MIs, all of whom have been recently calculated in the literature

- *V. A. Smirnov, Phys. Lett.* ***B567*** *(2003) 193–199.*
- *J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP* ***07*** *(2013) 128.*
- *J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP* ***03*** *(2014) 088.*
- *J. Henn, B. Mistlberger, V. A. Smirnov and P. Wasser, JHEP* ***04****, 167 (2020).*

Keeping one external leg off-shell (Higgs−jet in gluon fusion production) we have 18 (3) families of MIs, of whom only one has been computed

- *S. Di Vita, P. Mastrolia, U. Schubert, and V. Yundin, JHEP* ***09****, 148 (2014)*
- *DC and N. Syrrakos, JHEP* ***02*** *(2021) 080.*

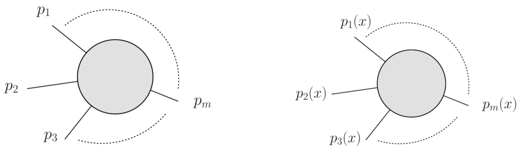
All these families together with the families with 2 external legs off-shell (di-boson productions) need to be calculated for future comparisons with the experiments.

First results for 3-loop 4-point Amplitude for the $q\bar{q} \to \gamma\gamma$ process in full-color QCD,
F. Caola, A. Von Manteuffel and L. Tancredi, Phys.Rev.Lett. ***126*** *(2021) 11, 112004!*
[Tancredi's talk]

Quick review of SDE

For any family of master integrals (MIs), **G**, one applies the following procedure [C. G. Papadopoulos, JHEP **07** (2014), 088]:

1) Parametrize the external momenta in terms of an dimensionless parameter, x, in such a way that captures the off-shellness of an external leg.



2) Take derivatives of the MIs with respect to x and create, using integration-by-parts identities (IBPs) a system of differential equations (DE) in one independent variable

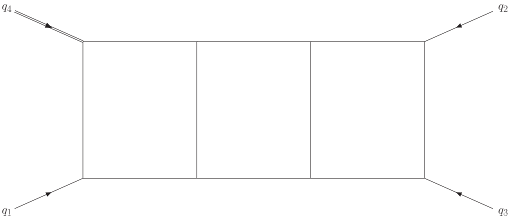
$$\partial_x \mathbf{G}(\{s_{ij}\}, x, \varepsilon) = \mathbf{H}(\{s_{ij}\}, x, \varepsilon)\mathbf{G}(\{s_{ij}\}, x, \varepsilon)$$

3) Find boundary conditions at $x \to 0$ and solve the differential equation.

The application of this method has plenty of advantages compared to the standard method of differential equations!

3-loop ladder-box with one external massive leg

We adopt the basis of universal transcendental MIs[1] (UT basis) and the notation for the kinematics from [S. Di Vita, et al, JHEP **09**, 148 (2014)], where this family was first studied



The external momenta can be expressed in Mandelstam variables

$$q_1^2 = q_2^2 = q_3^2 = 0\,, \quad q_4^2 = m^2\,, \quad q_2{\cdot}q_3 = s/2\,, \quad q_1{\cdot}q_3 = t/2\,, \quad q_1{\cdot}q_2 = (m^2{-}s{-}t)/2\,.$$

For this family we obtained a set of 83 master integrals in contrast with [S. Di Vita, et al, JHEP **09**, 148 (2014)], where a set of 85 was presented (we found via IBPs and analytic check of the solutions that $\mathcal{T}_7 = \mathcal{T}_8$ and $\mathcal{T}_{45} = \mathcal{T}_{46}$).

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[1] [J. M. Henn, Phys. Rev. Lett. **110** (2013), 251601].

The class of Feynman Integrals (FI) describing this family can be expressed as follows

$$G_{a_1,\ldots,a_{15}}(\{q_j\},\varepsilon) = \int \left(\prod_{r=1}^{3} \frac{d^d l_r}{i\pi^{d/2}} \right) \frac{e^{3\varepsilon\gamma_E}}{D_1^{a_1}\ldots D_{15}^{a_{15}}} \quad \text{with} \quad d = 4-2\varepsilon\,,$$

where $D_{11},\ldots,D_{15}$ are propagators coming from irreducible-scalar-products (ISPs), thus obey $\{a_{11},a_{12},a_{13},a_{14},a_{15}\} \leq 0$, and the chosen parametrization for the propagators is

$$D_1 = l_1^2\,,\quad D_2 = l_2^2\,,\quad D_3 = l_3^2\,,\quad D_4 = (l_1-l_2)^2\,,\quad D_5 = (l_2-l_3)^2\,,$$
$$D_6 = (l_3+q_2)^2\,,\quad D_7 = (l_1+q_{23})^2\,,\quad D_8 = (l_2+q_{23})^2\,,\quad D_9 = (l_3+q_{23})^2\,,$$
$$D_{10} = (l_1+q_{123})^2\,,\quad D_{11} = (l_1+q_2)^2\,,\quad D_{12} = (l_2+q_2)^2\,,\quad D_{13} = (l_2+q_{123})^2\,,$$
$$D_{14} = (l_3+q_{123})^2\,,\quad \text{and}\quad D_{15} = (l_1-l_3)^2\,.$$

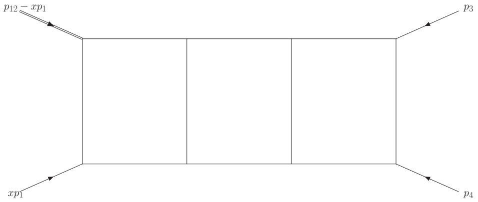
Moving to the SDE approach we choose the following parametrization

$$q_1 \to x p_1\,,\quad q_2 \to p_3\,,\quad q_3 \to -p_{123}\,,\quad q_4 \to p_{12} - x p_1 \quad \text{with} \quad p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0\,.$$

We express the Mandelstam variables and the external mass in terms of the parameter x and the new Mandelstam variables of the null momenta p_j

$$s = s_{12}\,,\quad t = x s_{23}\,,\quad m^2 = (1-x)s_{12}\,,$$

where $s_{12} = p_{12}^2$ and $s_{23} = p_{23}^2$.



After making the transformations ($l_1 \to k_1 - q_{23}$, $l_2 \to -k_2 - q_{23}$, $l_3 \to k_3 - q_{23}$) and applying the SDE approach to the propagators, they take the following form

$$D_1 = (k_1 + p_{12})^2\,, \quad D_2 = (k_2 - p_{12})^2\,, \quad D_3 = (k_3 + p_{12})^2\,, \quad D_4 = (k_1 + k_2)^2\,,$$
$$D_5 = (k_2 + k_3)^2\,, \quad D_6 = (k_3 + p_{123})^2\,, \quad D_7 = k_1^2\,, \quad D_8 = k_2^2\,, \quad D_9 = k_3^2\,,$$
$$D_{10} = (k_1 + xp_1)^2\,, \quad D_{11} = (k_1 + p_{123})^2\,, \quad D_{12} = (k_2 - p_{123})^2\,,$$
$$D_{13} = (k_2 - xp_1)^2\,, \quad D_{14} = (k_3 + xp_1)^2\,, \quad \text{and} \quad D_{15} = (k_1 - k_3)^2\,.$$

Having a UT basis we obtained a DE with respect to x, which is of canonical form

$$\partial_x \mathbf{g} = \varepsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g}$$

with $\mathbf{M}_i$ being purely numerical matrices and $l_i = \{0,\, 1,\, s_{12}/(s_{12} + s_{23}),\, -s_{12}/s_{23}\}$.

We solve the DE in a Laurent expansion of the MIs up to weight six, in the Euclidean region of the invariants, which is

$$0 < x < 1\,, \quad s_{12} < 0\,, \quad s_{12} < s_{23} < 0\,.$$

The solution can be written in the compact form

$$\begin{aligned}\mathbf{g} = {} & \varepsilon^0 \mathbf{b}_0^{(0)} + \varepsilon \left(\sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) + \varepsilon^2 \left(\sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) + \ldots \\ & + \varepsilon^6 \Bigg(\mathbf{b}_0^{(6)} + \sum \mathcal{G}_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)} \\ & + \sum \mathcal{G}_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} + \sum \mathcal{G}_{ijk} \mathbf{M}_i \mathbf{M}_j \mathrm{M}_k \mathbf{b}_0^{(3)} + \sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(5)} \Bigg)\,,\end{aligned}$$

where the matrices $\mathbf{b}_0^{(i)}$ are the boundary terms and $\mathcal{G}_i, \ldots, \mathcal{G}_{ijklmn}$ are Goncharov polylogarithms [A. B. Goncharov, Math. Res. Lett. **5** (1998), 497-516] of weight $1, \ldots, 6$, respectively, with argument x and letters from the set l_i.

Our results were numerically crossed-checked with the results from [S. Di Vita, et al, JHEP **09**, 148 (2014)] using `PolyLogTools` [C. Duhr and F. Dulat, JHEP **08** (2019), 135], and perfect agreement was found in all cases!

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Boundary conditions: known and zeroes

- Some integrals are known in close form and thus we can directly obtain boundary conditions for them

$$\{gb_1,\, gb_2,\, gb_3,\, gb_4,\, gb_5,\, gb_6,\, gb_7,\, gb_{17},\, gb_{18},\, gb_{19},\, gb_{44}\}\,.$$

- If a basis element (BE) has as an overall prefactor of x in such a power such as its leading regions contributing to its asymptotic limit $x \to 0$ (expansion-by-regions [M. Beneke and V. A. Smirnov, Nucl. Phys. B **522** (1998), 321-344]) are of the form $x^{\alpha+\beta\varepsilon}$ with $\alpha > 0$, then its boundary term should vanish

$$\{gb_{10},\, gb_{11},\, gb_{14},\, gb_{15},\, gb_{21},\, gb_{22},\, gb_{23},\, gb_{24},\, gb_{25},\, gb_{26},\, gb_{28},\\ gb_{31},\, gb_{37},\, gb_{38},\, gb_{45},\, gb_{46},\, gb_{47},\, gb_{48},\, gb_{50},\, gb_{53},\, gb_{55},\, gb_{58},\\ gb_{59},\, gb_{63},\, gb_{64},\, gb_{66},\, gb_{68},\, gb_{70},\, gb_{80},\, gb_{82},\, gb_{83}\} = 0\,.$$

Thus From 83 → 41 unknown boundaries!

Basis Element	**Asymptotic Limit of Master Integral $x \to 0$**
$g_{32} \equiv (s_{12} + s_{23}x)\varepsilon^5 F_{32}$	$F_{32} \equiv G_{1,0,0,1,1,2,0,1,0,1,0,0,0,0,0} \sim x^{-3\varepsilon},\, x^0$
$g_{41} \equiv (s_{12} + s_{23}x)\varepsilon^5 F_{41}$	$F_{41} \equiv G_{0,1,0,2,1,1,0,0,1,1,0,0,0,0,0} \sim x^{-3\varepsilon},\, x^0$
$g_{42} \equiv s_{12}s_{23}x\varepsilon^4 F_{42}$	$F_{42} \equiv G_{0,1,0,2,2,1,0,0,1,1,0,0,0,0,0} \sim x^{-1-3\varepsilon},\, x^{-3\varepsilon},\, x^0$
$g_{56} \equiv (s_{12} + s_{23}x)\varepsilon^6 F_{56}$	$F_{56} \equiv G_{1,1,0,1,1,1,0,0,1,1,0,0,0,0,0} \sim x^0$
$g_{71} \equiv s_{12}^2 s_{23}x\varepsilon^5 F_{71}$	$F_{71} \equiv G_{0,1,1,2,1,1,1,0,1,1,0,0,0,0,0} \sim x^{-1-3\varepsilon},\, x^{-3\varepsilon},\, x^0$
$g_{83} \equiv -s_{12}^3 x\varepsilon^6 F_{83}$	$F_{83} \equiv G_{1,1,1,1,1,1,1,1,1,1,0,-1,0,0,0} \sim x^{-3\varepsilon},\, x^0$

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Boundary conditions: relations between boundaries

We define the resummation matrix at $x = 0$ through the Jordan-decomposition of $\mathbf{M}_0$

$$\mathbf{M}_0 = \mathbf{S}_0 \mathbf{D}_0 \mathbf{S}_0^{-1} \quad \longrightarrow \quad \mathbf{R}_0 = \mathbf{S}_0 e^{\varepsilon \mathbf{D}_0 \log(x)} \mathbf{S}_0^{-1} \,.$$

$\mathbf{R}_0$ correctly resumms the logarithms of x from the basis elements, meaning that we can write

$$\mathbf{g} = \mathbf{R}_0 \mathbf{g}_{\text{reg0}} \,,$$

where $\mathbf{g}_{\text{reg0}}$ is the regular part of the basis element at $x = 0$, via which are defined the asymptotic boundaries

$$\mathbf{g}_{\text{bound}} = \mathbf{g}_{\text{reg0}} \Big|_{x=0} \,.$$

Multiplying $\mathbf{R}_0$ from the right with $\mathbf{g}_{\text{bound}}$ and from the left with $\mathbf{T}^{-1}$ (transformation U.T. basis elements → MIs), we obtain the asymptotic limit at $x \to 0$ of the MI

$$\mathbf{F}_{x \to 0} = \mathbf{T}^{-1} \mathbf{R}_0 \mathbf{g}_{\text{bound}} \,.$$

This should be equal to the asymptotic limit found for the MI by expansion-by-regions (found by asy [B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012), 2139]). Thus by comparing the regions found by asy with that found by the resummation matrix method we obtain relations between different boundaries.

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Pure and Impure relations

1) We call *pure* the relations that contain only boundaries of UT basis elements. As an illustrated example we consider the master integral F_{71}:

i) Expansion-by-regions method yields for $x \to 0$: $x^{-1-3\varepsilon}$.

ii) The resummation matrix has produced two additional regions: $x^{-1-2\varepsilon}$ and x^{-1}.

iii) We proceed by setting the extra regions to zero since they are not predicted by `asy`.

From the second one, we obtain a relation which connects the boundary condition of g_{71} with the boundary condition of lower sector basis elements:

$$gb_{71} = \left(-12gb_2 + 4gb_{13} + 32gb_{16} + 48gb_{41} + 36gb_{42} - 45gb_{43}\right)/30 \,.$$

2) We call *impure* the relations between boundaries and asymptotic limits, which are obtained by equating the result of the `asy` with that of the resummation matrix. E.g.

$$gb_{41} = F_{41}^{\text{soft}} s_{12}\varepsilon^5 + gb_2/9 - gb_{13}/12 - 2gb_{16}/3 \,.$$

where F_{41}^{soft} is the $x^{-3\varepsilon}$ region of F_{41}.

As expected, in the *pure* relations between the boundaries the prefactors are just numbers $\longrightarrow$ Working perfectly even when a full analytic reduction is a bottleneck!!!

• By applying this method we obtain 28 *pure* relations and thus the problem of computing 41 boundaries is reduced to the calculation of the 13 asymptotic regions

$$\{F_8^{\text{hard}}, F_9^{\text{hard}}, F_{12}^{\text{hard}}, F_{13}^{\text{hard}}, F_{16}^{\text{hard}}, F_{20}^{\text{hard}}, F_{27}^{\text{hard}}, F_{29}^{\text{hard}}, F_{32}^{\text{soft}}, F_{39}^{\text{soft}}, F_{41}^{\text{soft}}, F_{51}^{\text{hard}}, F_{56}^{\text{hard}}\}$$

where with *hard* we denote the x^0 region and with *soft* the $x^{-3\varepsilon}$.

• We calculated the *hard* limits with the use of the method of expansion-by-regions in the momentum space (significantly easier in SDE) and IBP reduction $\to$ We found that the *hard* limits are equal to some of the known MI.

• The soft limits were calculated using standard expansion-by-region approach, meaning computing their Feynman-parameter representation provided by `asy`. In order to facilitate the integrations we used a technique of integrating out bubble subintegrals (inspired by J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP **07** (2013) 128.), using

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)^{a_1}((k+p)^2)^{a_2}} = \frac{\Gamma(a-d/2)\Gamma(d/2-a_1)\Gamma(d/2-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(d-a)} (p^2)^{d/2-a}$$

where $a = a_1 + a_2$. In our cases $a_1 = 2$ and $a_2 = 1$ and after integrating out the bubble subintegral we arrive at a two-loop integral with one index shifted from 1 to $1+\varepsilon$.

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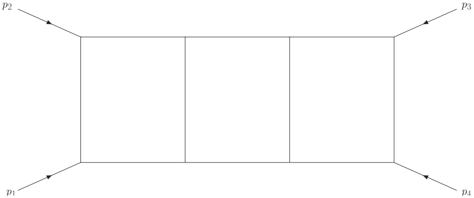
$x \to 1$ *limit: Massless problem*

The $x \to 1$ limit yields the solution for a canonical basis of the massless ladder-box:



The chosen normalisation of the FI is

$$G_{a_1,\ldots,a_{15}}\left(\{p_j\},\varepsilon\right) = \left(-s_{12}\right)^{3\varepsilon} \int \left(\prod_{l=1}^{3} \frac{d^d k_l}{i\pi^{d/2}}\right) \frac{e^{3\varepsilon\gamma_E}}{D_1^{a_1}\ldots D_{15}^{a_{15}}} \quad \text{with} \quad d = 4-2\varepsilon$$

and the propagators being

$$\begin{aligned}
&D_1 = (k_1+p_{12})^2, \quad D_2 = (k_2-p_{12})^2, \quad D_3 = (k_3+p_{12})^2, \quad D_4 = (k_1+k_2)^2, \\
&D_5 = (k_2+k_3)^2, \quad D_6 = (k_3+p_{123})^2, \quad D_7 = k_1^2, \quad D_8 = k_2^2, \quad D_9 = k_3^2, \\
&D_{10} = (k_1+p_1)^2, \quad D_{11} = (k_1+p_{123})^2, \quad D_{12} = (k_2-p_{123})^2, \\
&D_{13} = (k_2-p_1)^2, \quad D_{14} = (k_3+p_1)^2, \quad \text{and} \quad D_{15} = (k_1-k_3)^2.
\end{aligned}$$

We compared our results numerically with pySecDec [S. Borowka et al, Comput. Phys. Commun. **222** (2018), 313-326] and perfect agreement was found in all cases!

Procedure for taking the $x \to 1$

Briefly the procedure for taking the $x \to 1$ limit is:

1) Rewrite the solution as an expansion in $\log(1-x)$:

$$\mathbf{g} = \sum_{n\geq 0} \epsilon^n \sum_{i=0}^{n} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i(1-x)$$

2) Define the regular part of $\mathbf{g}$ at $x = 1$ and from it the truncated part:

$$\mathbf{g}_{reg} = \sum \epsilon^n \mathbf{c}_0^{(n)} \quad \text{and} \quad \mathbf{g}_{trunc} = \mathbf{g}_{reg}\Big|_{x=1}$$

3) Define the resummation matrix $\mathbf{R}_1$ and from it the purely numerical matrix $\mathbf{R}_{10}$:

$$\mathbf{R}_1 = e^{\epsilon \mathbf{M}_1 \log(1-x)} = \mathbf{S}_1 e^{\epsilon \mathbf{D}_1 \log(1-x)} \mathbf{S}_1^{-1} \quad \text{and} \quad \mathbf{R}_1 \xrightarrow{(1-x)^{a_i \epsilon} \to 0} \mathbf{R}_{10}$$

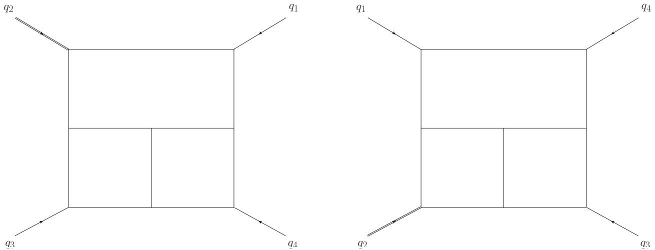
4) Find the $x \to 1$ limit by acting $\mathbf{R}_{10}$ to $\mathbf{g}_{trunc}$:

$$\mathbf{g}_{x\to 1} = \mathbf{R}_{10} \mathbf{g}_{trunc}$$

5) Reduce the number of the basis elements to the number of the MI of the massless problem using the property $\mathbf{R}_{10}^2 = \mathbf{R}_{10} \Rightarrow \mathbf{R}_{10}\mathbf{g}_{x\to 1} = \mathbf{g}_{x\to 1}$ and/or IBPs.

4-point 3-loop planar families with 1 off-shell leg

To complete the set of all planar families one needs to solve the two tennis-courts:



- The first (lets call F2) contains 117 MI of whom 59 are new!
- The second (lets call F3) contains 166 MI of whom 32 are new!
- For fast evaluations, analytical solutions will be needed for the 3 physical regions

1) $m^2 > 0, \quad s \geq m^2, \quad t \leq 0, \quad u \leq 0$

2) $m^2 > 0, \quad s \leq 0, \quad t \geq m^2, \quad u \leq 0$

3) $m^2 > 0, \quad s \leq 0, \quad t \leq 0, \quad u \geq m^2$

Focusing on the F2, we have constructed an UT basis using the following methods:

- *One-loop (boxes, triangles, etc) and two-loop (UT integrals from Double-box families) building blocks [P. Wasser, MSc thesis (2016)].*
- `DlogBasis` *combined with the SDE parametrization, to find integrands of d-log form [J. Henn, et al, JHEP **04**, 167 (2020)].*
- *Magnus exponential [M. Argeri, et al, JHEP **1403** (2014) 082].*
- `Fuchsia.cpp` *[https://github.com/magv/fuchsia.cpp].*

Intermediate checks of the UTness of the chosen BEs were made by semi-numerical derivations of the DE.

An analytic reduction through `FIRE6` [A. Smirnov, et al, Comput.Phys.Commun. **247** (2020)] was possible in a personal laptop (`i7, 8-core, 16GB RAM`) using SDE approach which produced 104 integrals for reduction in order to derive the DE, while was not possible using the standard approach which produced 1096 integrals.

Currently working on the computation of the boundaries of F2 and the determination of a UT Basis for F3, using the methods described herein.

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Thank you!

This research work was supported by the HFRI (Hellenic Foundation for Research and Innovation) under the HFRI PhD Fellowship grant (Fellowship Number: 554).

H.F.R.I.
Hellenic Foundation for Research & Innovation

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DlogBasis combined with SDE parametrization

DlogBasis depends on the spinor helicity parametrization, which can not be applied when we deal with massive external momenta. When one deals with such problems the standard way to proceed is the decomposition of the external massive momentum in terms of two (arbitrary) massless momenta [J. Henn, et al, JHEP **04**, 167 (2020)], or the use of Baikov representation [J. Henn, et al,JHEP **04** (2020) 018] & [C. Dlapa et al, 2103.04638 [hep-th]].

Another possible way of proceeding is the use of the SDE notation for the propagators where by definition the external momenta that appear in it for 1-mass problems are massless[2] and thus the spinor helicity parametrization can be applied. Thus while the command

`SetParametrization[SpinorHelicityParametrization[`$\{l_1, l_2, l_3\}, \{a, b, c\}, \{q_1, q_2, q_3\}$`]]`

doen't work when someone uses the standard notation for the propagators, it works when one uses the SDE notation for the propagators

`SetParametrization[SpinorHelicityParametrization[`$\{k_1, k_2, k_3\}, \{a, b, c\}, \{p_1, p_2, p_3\}$`]]`

[2] the same approach can be used for 2-mass problems introducing an extra y parameter beyond x