Evaluating planar master integrals for Bhabha scattering

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in collaboration with Claude Duhr and Lorenzo Tancredi

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Two-loop Bhabha scattering in QED: four-point diagrams with all the external points on the mass shell, $p_i^2 = m^2$. Three variables, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, m^2.

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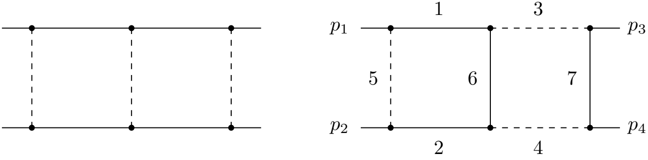

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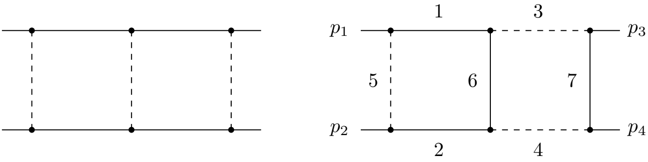

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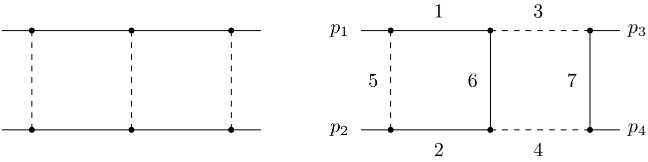

(a) (b)

Now: analytic evaluation of master integrals for graph (b).

Evaluating integrals for graph (a) with two different masses [M. Heller'21].

$$
\begin{aligned}
F_{a_1,a_2,\ldots,a_9} &= \int\int \frac{d^D k_1\, d^D k_2}{[-k_1^2+m^2]^{a_1}[-(k_1+p_1+p_2)^2+m^2]^{a_2}} \\
&\times \frac{[-(k_2+p_1)^2]^{a_8}\,[-(k_1-p_3)^2]^{a_9}}{[-k_2^2]^{a_3}[-(k_2+p_1+p_2)^2]^{a_4}[-(k_1+p_1)^2]^{a_5}} \\
&\times \frac{1}{[-(k_1-k_2)^2+m^2]^{a_6}[-(k_2-p_3)^2+m^2]^{a_7}}\,.
\end{aligned}
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\end{aligned}
$$

Solving IBP relations with KIRA or FIRE $\to$ 43 master integrals $g_1, \ldots, g_{43}$.

Solving differential equations

Differential equations

$$\partial_v g = A_v g \,,$$

$v = s, t, m^2$, $\partial_v = \frac{\partial}{\partial v}$ and matrices A_s, A_t, A_{m^2} are rational functions of s, t, m^2 and ϵ.

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Turn to an ϵ-basis [J. Henn'13], $g_i \to f_i$,

$$\partial_v f = \epsilon \bar{A}_v f$$

with $\bar{A}_v$ independent of ϵ.

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We use the strategy of
[T. Gehrmann, A. von Manteuffel, L. Tancredi & E. Weihs'14]

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Solution

$$f(s, t; \epsilon) = \mathbb{P}\mathrm{exp}\left[\epsilon \int_\gamma d\tilde{A}\right] f_0(\epsilon)$$

where $\mathbb{P}\mathrm{exp}$ is the path-ordered exponential and $f_0(\epsilon)$ is the initial condition related to the value of f at a specific point. The path γ connects the initial point (s_0, t_0) to the generic point (s, t).

$$
\begin{aligned}
f_1 &= \epsilon^2 F_{2,0,0,0,0,2,0,0,0}\,,\\
f_2 &= -\epsilon^2 \frac{1}{2}\sqrt{-s}\sqrt{4m^2-s}F_{0,2,1,0,0,2,0,0,0}\\
&\quad -\epsilon^2\sqrt{-s}\sqrt{4m^2-s}F_{0,2,2,0,0,1,0,0,0}\,,\\
f_3 &= -\epsilon^2 sF_{0,2,1,0,0,2,0,0,0}\,,\\
f_4 &= -\frac{1}{2}\epsilon^2\sqrt{-t}\sqrt{4m^2-t}F_{0,0,0,0,1,2,2,0,0}\\
&\quad -\epsilon^2\sqrt{-t}\sqrt{4m^2-t}F_{0,0,0,0,2,1,2,0,0}\,,\\
f_5 &= -\epsilon^2 tF_{0,0,0,0,1,2,2,0,0}\,,\\
f_6 &= -\epsilon^2 m^2 F_{0,0,1,0,2,2,0,0,0}\\
f_7 &= -\epsilon^3\sqrt{-s}\sqrt{4m^2-s}F_{0,1,1,0,1,2,0,0,0}\,,\ldots
\end{aligned}
$$

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\begin{aligned}
r_s &= \sqrt{-s}\sqrt{4m^2 - s}, \quad r_t = \sqrt{-t}\sqrt{4m^2 - t}, \\
r_u &= \sqrt{-s - t}\sqrt{4m^2 - s - t}, \quad r_{st} = \sqrt{-s}\sqrt{4m^6 - s(m^2 - t)^2}\,.
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The square roots are chosen in such a way that that they are manifestly real at Euclidean values, $s, t < 0$.

The standard way to rationalize the first two square roots is to turn to dimensionless variables x and y

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x} \quad \frac{-t}{m^2} = \frac{(1-y)^2}{y}\,.$$

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The equations can be solved, first, in x, with results in terms of MPLs of x with the letters $\{0, -1, 1, -y, -1/y\}$.

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MPLs

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

$$G(\underbrace{0, \ldots, 0}_{n \text{ times}}; x) = \frac{1}{n!} \ln^n x$$

Then the equations with respect to y can be solved (after checking that the variable x disappears in them) in terms of MPLs of y with the letters $\{0, -1, 1\}$, i.e. harmonic polylogarithms [E. Remiddi & J. Vermaseren'99].

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and evaluating resulting parametric integrals
we obtain the following leading order asymptotic behaviour in this limit

$$
\begin{aligned}
f_1 &\sim 1 + \frac{\pi^2\epsilon^2}{6} - \frac{2\zeta(3)\epsilon^3}{3} + \frac{7\pi^4\epsilon^4}{360}\,, \\
f_6 &\sim -\frac{1}{4} - \frac{5\pi^2\epsilon^2}{24} - \frac{11\zeta(3)\epsilon^3}{6} - \frac{101}{480}\pi^4\epsilon^4\,, \\
f_9 &\sim -\frac{\pi^2\epsilon^2}{12} + \frac{1}{4}\epsilon^3\left(2\pi^2\log(2) - 7\zeta(3)\right) \\
&\quad + \frac{1}{180}\epsilon^4\left(13\pi^4 - 90\log^4(2) - 180\pi^2\log^2(2) - 2160\mathrm{Li}_4\left(\frac{1}{2}\right)\right)\,, \\
f_{18} &\sim \frac{1}{2}\epsilon^3\left(2\pi^2\log(2) - 3\zeta(3)\right) \\
&\quad + \frac{1}{20}\epsilon^4\left(7\pi^4 - 20\log^4(2) - 40\pi^2\log^2(2) - 480\mathrm{Li}_4\left(\frac{1}{2}\right)\right)\,, \\
f_{19} &\sim (-s)^{-\epsilon}\left(-1 + \frac{8\zeta(3)\epsilon^3}{3} + \frac{\pi^4\epsilon^4}{30}\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
f_{22} &\sim (-s)^{-\epsilon}\left(-\frac{1}{2}+\frac{4\zeta(3)\epsilon^3}{3}+\frac{\pi^4\epsilon^4}{60}\right) \\
&\quad +(-s)^{-2\epsilon}\left(\frac{1}{4}-\frac{\pi^2\epsilon^2}{24}-\frac{14\zeta(3)\epsilon^3}{3}-\frac{67}{480}\pi^4\epsilon^4\right)\,, \\
f_{23} &\sim (-s)^{-2\epsilon}\pi^2\left(\epsilon^2+2\epsilon^3\log(2)+2\epsilon^4\left(\pi^2+\log^2(2)\right)\right)\,, \\
f_{25} &\sim (-s)^{-\epsilon}\pi^2\left(-\epsilon^2-2\epsilon^3\log(2)-\frac{1}{2}\epsilon^4\left(\pi^2+4\log^2(2)\right)\right)
\end{aligned}
$$

and $f_i \sim 0$, i.e. $f_i = o(s,t)$ for all the other elements.

For example,

$$
\begin{aligned}
f_{42} = \ldots + \varepsilon^4 \Big(& -\pi^2 G(-1;y)G(0,x) + \frac{1}{2}\pi^2 G(0;y)G(0,x) - \frac{1}{3}\pi^2 G(1;y)G(0,x) - 36G(-1,-1,0;y)G(0,x) \\
& + 24G(-1,0,0;y)G(0,x) - 12G(-1,1,0;y)G(0,x) + 24G(0,-1,0;y)G(0,x) - 10G(0,0,0;y)G(0,x) \\
& + 8G(0,1,0;y)G(0,x) - 12G(1,-1,0;y)G(0,x) + 8G(1,0,0;y)G(0,x) - 4G(1,1,0;y)G(0,x) \\
& + 11\zeta(3)G(0,x) - \frac{4}{3}\pi^2 G(-1,x)G(0;y) + 2\pi^2 G(-1;y)G(-1/y;x) - \frac{1}{6}\pi^2 G(0;y)G(-1/y;x) \\
& - 2\pi^2 G(-1;y)G(-y,x) + \frac{3}{2}\pi^2 G(0;y)G(-y,x) - \frac{1}{3}\pi^2 G(-1,0,x) \\
& - 12G(-1,0,x)G(-1,0;y) - 4\pi^2 G(-1,0;y) + \pi^2 G(-1,-1/y;x) - \pi^2 G(-1,-y,x) \\
& - 2\pi^2 G(0,-1;y) + 8G(-1,0,x)G(0,0;y) + 2G(-1,-1/y;x)G(0,0;y) \\
& - 2G(-1,-y,x)G(0,0;y) + \frac{7}{2}\pi^2 G(0,0;y) - 4G(-1,0,x)G(1,0;y) - \frac{4}{3}\pi^2 G(1,0;y) \\
& + \pi^2 G(-1/y,-1;x) + 6G(-1,0;y)G(-1/y,0;x) - 4G(0,0;y)G(-1/y,0;x) + 2G(1,0;y)G(-1/y,0;x) \\
& - \frac{1}{6}\pi^2 G(-1/y,0;x) - G(0,0;y)G(-1/y,-1/y;x) - \frac{1}{2}\pi^2 G(-1/y,-1/y;x) + G(0,0;y)G(-1/y,-y;x) + \ldots \Big)
\end{aligned}
$$

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It can be rationalized by the following further change of variables $x \to w$:

$$x = \frac{2\left((1-w)\left(y^2-y+1\right)^2 - 2y^2\right)}{\left(1-w^2\right)\left(y^2-y+1\right)^2}.$$

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The equations are solved, first, in w and then in y. The results are written in terms of $G(\ldots, w)$ and $G(\ldots, y)$.

The letters in $G(\ldots, w)$ and $G(\ldots, y)$ are cumbersome and the result is rather complicated, the contributions of weight 4 take $\sim$ 60mb. Still we obtain an answer to the question about the class of functions: these are MPLs, with the exception of f_{14}.

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Evaluating the weight 4 results with `GiNaC` [C. W. Bauer, A. Frink & R. Kreckel'00; J. Vollinga & S. Weinzierl'04] meets certain problems connected with timing and stability, so that such results become impractical.

For these complicated elements, we prefer to apply the recently developed code `DiffExp` to evaluate Feynman integrals numerically using differential equations [M. Hidding'20; talk at this session].

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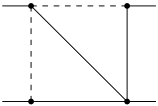
With a canonical basis, the code works much better.

Elliptic sector

$f_{14} \equiv \epsilon^4 \bar{f} = -\epsilon^4 \sqrt{-s-t}\sqrt{4m^2-s-t}$ times



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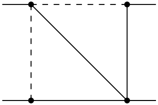

The differential equation equations give

$$\frac{\partial}{\partial x}\bar{f}(x,y) = \frac{1}{(x-1)x\sqrt{(x+y)(xy+1)\left(x^2y+xy^2-4xy+x+y\right)}}$$
$$\times\left[(x-1)G(0,x)\left(2\left(3x^2y+x(y-1)^2+y\right)G(0,0,y)+\pi^2\left(x^2-1\right)y\right)\right.$$
$$-(x+1)\left(2G(0,y)\left(x\left(y^2-1\right)G(0,0,x)+(x-1)^2y\left(G\left(-\frac{1}{y},0,x\right)-G(-y,0,x)\right)\right)\right.$$
$$-2(x-1)^2y\left(-G\left(-\frac{1}{y},0,0,x\right)-G(-y,0,0,x)+2G(0,0,0,x)-2G(1,0,0,x)\right.$$
$$\left.+G(0,0,0,y)-2G(1,0,0,y)-\zeta(3)\right)+(x-1)^2y\left(2G(0,0,y)+\pi^2\right)G\left(-\frac{1}{y},x\right)$$
$$\left.\left.+(x-1)^2y\left(2\;G(0,0,y)+\pi^2\right)G(-y,x)\right)\right]\;.$$

$$
\begin{aligned}
\frac{\partial}{\partial x}\bar{f}(x,y) = {} & \frac{1}{(x-1)x\sqrt{(x+y)(xy+1)\left(x^2y+xy^2-4xy+x+y\right)}} \\
& \times\Big[(x-1)G(0,x)\left(2\left(3x^2y+x(y-1)^2+y\right)G(0,0,y)+\pi^2\left(x^2-1\right)y\right) \\
& -(x+1)\left(2G(0,y)\left(x\left(y^2-1\right)G(0,0,x)+(x-1)^2y\left(G\left(-\frac{1}{y},0,x\right)-G(-y,0,x)\right)\right)\right. \\
& -2(x-1)^2y\left(-G\left(-\frac{1}{y},0,0,x\right)-G(-y,0,0,x)+2G(0,0,0,x)-2G(1,0,0,x)\right. \\
& \left.+G(0,0,0,y)-2G(1,0,0,y)-\zeta(3)\right)+(x-1)^2y\left(2G(0,0,y)+\pi^2\right)G\left(-\frac{1}{y},x\right) \\
& \left.+(x-1)^2y\left(2\ G(0,0,y)+\pi^2\right)G(-y,x)\right)\Big]\ .
\end{aligned}
$$

The function $\bar{f}(x,y)$ is symmetrical, $\bar{f}(y,x)=\bar{f}(x,y)$.

The differential equation is solved on a path which consists of two straight-line segments: the straight line from the point $(1, 1)$ (where the function $= 0$) to the point $(1, y), 0 \leq y \leq 1$, The differential equation is solved on a path which consists of two straight-line segments: the straight line from the point $(1, 1)$ (where the function $= 0$) to the point $(1, y), 0 \leq y \leq 1$, and from $(1, y)$ to the general (x, y) in the Euclidean region $0 < x < 1, 0 < y < 1$.

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The square root $\sqrt{(x+y)(xy+1)\left(x^2y+xy^2-4xy+x+y\right)}$ cannot be rationalized [M. Besier, D. van Straten & S. Weinzierl'18] so that, maybe, it is not possible to arrive at a result in terms of MPLs.

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Let us apply elliptic MPLs (eMPLs)
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Use the variable $\bar{x} = 1 - x$. Here is the result

$$
\begin{aligned}
&2\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & \frac{1}{y}+1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & y+1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + \left(-3\log^2(y)-\pi^2\right)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & 1\end{smallmatrix};\bar{x},\vec{a}\right)\\
&+\left(\log^2(y)+\pi^2\right)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & \frac{1}{y}+1\end{smallmatrix};\bar{x},\vec{a}\right) + \left(\log^2(y)+\pi^2\right)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & y+1\end{smallmatrix};\bar{x},\vec{a}\right)\\
&+2\log(y)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1\\ \infty & \frac{1}{y}+1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) - 2\log(y)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1\\ \infty & y+1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & \frac{1}{y}+1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right)\\
&+2\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & y+1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + \left(\log^2(y)-\pi^2\right)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1\\ 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + \left(\log^2(y)+\pi^2\right)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1\\ 1 & \frac{1}{y}+1\end{smallmatrix};\bar{x},\vec{a}\right)\\
&+\left(\log^2(y)+\pi^2\right)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1\\ 1 & y+1\end{smallmatrix};\bar{x},\vec{a}\right) + 4\log(y)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1\\ 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + 2\log(y)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1\\ 1 & \frac{1}{y}+1 & 1\end{smallmatrix};\bar{x},\vec{a}\right)\\
&-2\log(y)\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1\\ 1 & y+1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + 4\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ \infty & 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right)\\
&+4\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{smallmatrix};\bar{x},\vec{a}\right) + (-4\mathrm{Li}_3(-y)-4\mathrm{Li}_3(y)+4\mathrm{Li}_2(-y)\log(y)\\
&+4\mathrm{Li}_2(y)\log(y)-\frac{2}{3}\log^3(y)+2\log(1-y)\log^2(y)+2\log(y+1)\log^2(y)-\pi^2\log(y)\\
&+2\pi^2\log(y+1)-2\zeta(3)\Big)\,\mathcal{E}_4\left(\begin{smallmatrix}-1\\ \infty\end{smallmatrix};\bar{x},\vec{a}\right) + (-4\mathrm{Li}_3(-y)-4\mathrm{Li}_3(y)+4\mathrm{Li}_2(-y)\log(y)\\
&+4\mathrm{Li}_2(y)\log(y)-\frac{2}{3}\log^3(y)+2\log(1-y)\log^2(y)+2\log(y+1)\log^2(y)-\pi^2\log(y)\\
&+2\pi^2\log(y+1)-2\zeta(3)\Big)\,\mathcal{E}_4\left(\begin{smallmatrix}-1\\ 1\end{smallmatrix};\bar{x},\vec{a}\right) - 12\mathrm{Li}_4(-y)-12\mathrm{Li}_4(y)-2\mathrm{Li}_2(y)\log^2(y)\\
&-2\mathrm{Li}_2(-y)\left(\log^2(y)+\pi^2\right)+8\mathrm{Li}_3(-y)\log(y)+8\mathrm{Li}_3(y)\log(y)-2\zeta(3)\log(y)\\
&-\frac{1}{6}\log^4(y)-\frac{1}{2}\pi^2\log^2(y)-\frac{3\pi^4}{20}
\end{aligned}
$$

eMPLs

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

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The set of eMPLs in our case is associated with the elliptic curve $z^2 = P_n(x, y)$, where P_n is a polynomial of degree $n = 3$ or 4. Here
$P_4(x, y) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$ with
$a_1 = y + 1, a_2 = (y - 1)\left(\sqrt{y^2 - 6y + 1} + y - 1\right)/(2y)$,
$a_3 = (y - 1)\left(-\sqrt{y^2 - 6y + 1} + y - 1\right)/(2y), a_4 = 1/y + 1$

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MPLs are partial cases of eMPLs:

$$\mathcal{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = G(c_1, \ldots, c_k; x)$$

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- All the master integrals but one are expressed in terms of MPLs.
- We have derived a compact result for one master integral in terms of eMPLs.