Two-Loop Five-Point One-Mass Integrals

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CERN

RADCOR/Loopfest, 17^{th} – 21^{st} May 2021

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based on [arXiv:2005.04195], [21xx.xxxxx]

Motivation

- Experimental precision will increase with more statistics.
- Need two-loop amplitudes to match with NNLO precision.
- Master integrals required.
- 5-point 1-mass relevant for W/Z/H-plus-two-jets production at the LHC.

[See Bayu's talk for $W + b\bar{b}$]

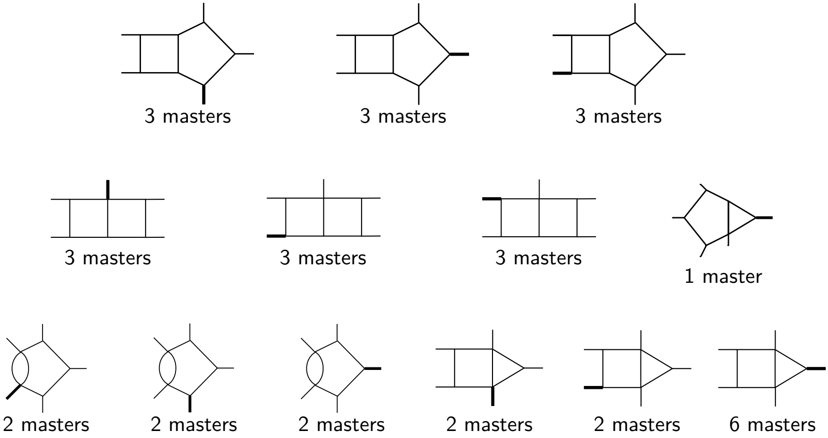
process	known	desired
⋮	⋮	⋮
$pp \to V + 2j$	$\mathrm{NLO}_{\mathrm{QCD}} + \mathrm{NLO}_{\mathrm{EW}}$ $\mathrm{NLO}_{\mathrm{EW}}$	$\mathrm{NNLO}_{\mathrm{QCD}}$
⋮	⋮	⋮
$pp \to H + 2j$	$\mathrm{NLO}_{\mathrm{HTL}} \otimes \mathrm{LO}_{\mathrm{QCD}}$ $\mathrm{N^3LO}^{(\mathrm{VBF}^*)}_{\mathrm{QCD}}$ incl. $\mathrm{NNLO}^{(\mathrm{VBF}^*)}_{\mathrm{QCD}}$ $\mathrm{NLO}^{(\mathrm{VBF})}_{\mathrm{EW}}$	$\mathrm{NNLO}_{\mathrm{HTL}} \otimes \mathrm{NLO}_{\mathrm{QCD}} + \mathrm{NLO}_{\mathrm{EW}}$ $\mathrm{NNLO}^{(\mathrm{VBF})}_{\mathrm{QCD}} + \mathrm{NLO}^{(\mathrm{VBF})}_{\mathrm{EW}}$
⋮	⋮	⋮

[Les Houches precision wishlist '19]

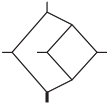
Status of Two-loop Five-point Integrals

- 5-point planar massless integrals [Gehrmann, Henn, lo Presti '15] [Papadopoulos, Tommasini, Wever '15].
- 5-point non-planar massless integrals [Abreu, BP, Zeng, '18], [Chicherin, Gehrmann, Henn, lo Presti, Mitev, Wasser '18], [Abreu, Dixon, Herrmann, BP, Zeng, '18], [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18], [Chicherin, Sotnikov '20].
- 5-point 1-mass planar family [Papadopoulos, Tommasini, Wever '15].
- [Papadopoulos, Wever '19] for 5-point 1-mass non-planar family.
- All planar 5-point 1-mass integrals. [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20], [Canko, Papadopoulos, Syrrakos '20], See Costas' talk.

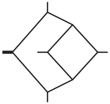
Planar 5-Point 1-Mass Masters [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]



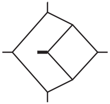
Hexabox 5-Point 1-Mass Masters [Abreu, Ita, BP, Tschernow (To appear)]



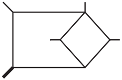
3 masters



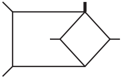
3 masters



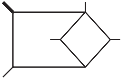
3 masters



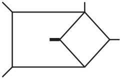
3 masters



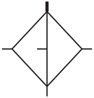
3 masters



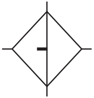
3 masters



6 masters



1 master



4 masters

Scattering Kinematics and Roots

- Five-point, one-mass.

$$p_1^2 \neq 0, \qquad p_i^2 = 0, \quad [i = 2\ldots,5].$$

- 6 independent Mandelstams.

$$\vec{s} = \{p_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}.$$

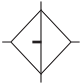
- Square roots of pentagon and three-mass triangle grams: $\sqrt{\Delta_5}$ and $\sqrt{\Delta_{3,i}}$.

$$\Delta_5 = -16\left[\epsilon_{\alpha,\beta,\gamma,\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta\right]^2.$$

$$\Delta_{3,1} = (p_1^2 - s_{23} - s_{45})^2 - 4 s_{23} s_{45}.$$

- New square roots $\sqrt{\Sigma_{5,i}}$, from leading singularities.

$$\Sigma_{5,1} = (s_{12}s_{15} - s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} - s_{15}s_{45})^2 - 4 s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15}).$$

LS [] $= \dfrac{1}{\sqrt{\Sigma_{5,1}}}.$

Strategy

1. Use (canonical) differential equations for each family.

$$\mathrm{d}\mathbf{I} = \mathbf{M}\,\mathbf{I}.$$

[Kotikov '91; Remiddi '97; Gehrmann '01, Remiddi; Henn '13]

Build efficiently with numerical tools/unitarity cuts.

2. Compute numerical boundary values using series expansion.

[Moriello '19]
[See also Martijn's talk]

Random Direction DE

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- Pure polylogarithmic integrals, $\Rightarrow$ simple connection $\mathbf{M}$.

  [Henn '13]

$$\mathbf{M} = \epsilon \sum_{\alpha} M_\alpha \, \mathrm{d} \log (W_\alpha).$$

- Alphabet $\mathcal{A}$ of letters W_α – algebraic kinematic functions.

$$W_\alpha \left(\vec{s}, \sqrt{\Delta_5}, \sqrt{\Delta_{3,i}}, \sqrt{\Sigma_{5,i}} \right) \in \mathcal{A}.$$

- DE matrices M_α. Entries in $\mathbb{Q}$.

$$M_{\alpha,i,j} \in \mathbb{Q}.$$

- Derivative in arbitrary direction.
- $\mathbf{C}(\epsilon, \vec{s})$ is single algebraic function of kinematics.

$$\vec{c} \cdot \frac{\partial}{\partial \vec{s}} \mathbf{I} = \mathbf{C}(\epsilon, \vec{s}) \, \mathbf{I}.$$

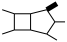
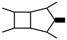
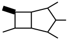
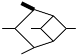
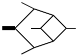
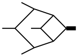
$$\mathbf{C}(\epsilon, \vec{s}) = \epsilon \sum_{\alpha} M_\alpha \, \vec{c} \cdot \frac{\partial}{\partial \vec{s}} \log (W_\alpha).$$

Counting Letters Numerically

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- $\dim(\mathcal{A}) = \#$ independent entries of $\mathbf{C}(\epsilon, \vec{s})$.
- Construct $\mathcal{C}_i(\epsilon, \vec{s})$, the vector of $\mathbf{C}(\epsilon, \vec{s})$ entries.
- $\dim(\mathcal{A})$ from rank of $\mathcal{C}$ evaluation matrix.

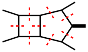
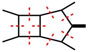
$$\mathrm{rank}\left[\mathcal{C}_i(\epsilon_0, \vec{s}^{(j)})\right] = \min[m, \dim(\mathcal{A})].$$

Family			
dim $(\mathcal{A})$	38	48	49
Family			
dim $(\mathcal{A})$	39	56	63

Alphabet From Cut DEs

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- Non-zero entries of $\mathbf{M}^{\text{cut}}$ match entries of $\mathbf{M}$.
- Cut DEs are easier to compute analytically.
- Use $\mathbf{C}$ to find most cut DE that spans the alphabet.
- Next-to-maximal-cut DE generates full alphabet, $\mathcal{A}$.

d ( $\vdots$) $= \mathbf{M}^{\text{cut}}$ ( $\vdots$)

$$\mathbf{M}^{\text{cut}}_{ij} = \mathbf{M}_{ij}, \quad \text{if} \quad \mathbf{M}^{\text{cut}}_{ij} \neq 0.$$

$$k : \dim(\mathcal{A}^{N^k-\max}) = \dim(\mathcal{A})$$

$$\mathcal{A} = \mathcal{A}^{N-\max}.$$

Differential Equation Matrices from Ansatz [Abreu, BP, Zeng '18]

- Canonical DE form very constrained ⇒ ansatz!
- M_α are free parameters.
- Use efficient evaluations and exact linear algebra over $\mathbb{F}_p$.
  [Schabinger, von Manteuffel '14; Peraro '16]
- Can reuse evaluations from dimension calculation.

$$\mathbf{C}(\epsilon, \vec{s}) = \epsilon \sum_\alpha M_\alpha \, \vec{c} \cdot \frac{\partial}{\partial \vec{s}} \log\left(W_\alpha\right).$$

$$\mathcal{W}_{\alpha k} = \vec{c} \cdot \left[\frac{\partial}{\partial \vec{s}} \log(W_\alpha) \right] \Big|_{\vec{s}=\vec{s}^{(k)}}.$$

$$M_\alpha = \sum_k \frac{1}{\epsilon_0} \mathcal{W}^{-1}_{\alpha k} \mathbf{C}(\epsilon_0, \vec{s}^{(k)}).$$

Penta/Hexabox Alphabet Organizing [Abreu, Ita, BP, Tschernow (To appear)]

- Close alphabet under (S_4) permutations of massless legs.

$$\mathcal{A} = \{W_1, \ldots, W_{204}\}.$$

- 204 letters W_α, organized as permutations of 21 generators.

$$p_2 \leftrightarrow p_5 \sim S[2,5].$$

- Some letters swap invariant.
- Galois group = root sign flips.

$$\sqrt{\Delta_5} \to -\sqrt{\Delta_5},$$
$$\sqrt{\Delta_{3,i}} \to -\sqrt{\Delta_{3,i}},$$
$$\sqrt{\Sigma_{5,i}} \to -\sqrt{\Sigma_{5,i}}.$$

- Useful object is tr_+.

$$\mathrm{tr}_+(i_1 \cdots i_n) = \mathrm{tr}\left(\left[\frac{1+\gamma_5}{2}\right] \not{p}_{i_1} \cdots \not{p}_{i_n}\right).$$

Galois Invariant Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ

Galois Invariant Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
p_1^2	Id
s_{12}	$S_4/S[3,4,5]$
s_{23}	$S_4/(S[2,3] \times S[4,5])$
$2\,p_1 \cdot p_2$	$S_4/S_3[3,4,5]$
$2\,p_2 \cdot (p_3 + p_4)$	$S_4/S_2[3,4]$

Galois Invariant Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
p_1^2	Id
s_{12}	$S_4/S[3,4,5]$
s_{23}	$S_4/(S[2,3]\times S[4,5])$
$2\,p_1\cdot p_2$	$S_4/S_3[3,4,5]$
$2\,p_2\cdot(p_3+p_4)$	$S_4/S_2[3,4]$
$\mathrm{tr}_+(1\,2\,1\,5)$	$S_4/(S_2[2,5]\times S_2[3,4])$
$\mathrm{tr}_+(1\,2\,1\,[4+5])$	$S_4/S_2[4,5]$
$\mathrm{tr}_+(1\,[2+3]\,4\,[2+3])$	$S_4/S_2[2,3]$
$\mathrm{tr}_+(1\,2\,[4+5]\,[2+3])$	$S_4/(S_2[3,4]\times S_2[2,5])$
$\mathrm{tr}_+(1\,2\,3\,4)-\mathrm{tr}_+(1\,2\,4\,5)$	S_4
$\mathrm{tr}_+(1\,2\,1\,[1+5]\,4\,[1+5])$	S_4

Galois Invariant Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
p_1^2	Id
s_{12}	$S_4/S[3,4,5]$
s_{23}	$S_4/(S[2,3] \times S[4,5])$
$2\,p_1 \cdot p_2$	$S_4/S_3[3,4,5]$
$2\,p_2 \cdot (p_3 + p_4)$	$S_4/S_2[3,4]$
$\mathrm{tr}_+(1\,2\,1\,5)$	$S_4/(S_2[2,5] \times S_2[3,4])$
$\mathrm{tr}_+(1\,2\,1\,[4+5])$	$S_4/S_2[4,5]$
$\mathrm{tr}_+(1\,[2+3]\,4\,[2+3])$	$S_4/S_2[2,3]$
$\mathrm{tr}_+(1\,2\,[4+5]\,[2+3])$	$S_4/(S_2[3,4] \times S_2[2,5])$
$\mathrm{tr}_+(1\,2\,3\,4) - \mathrm{tr}_+(1\,2\,4\,5)$	S_4
$\mathrm{tr}_+(1\,2\,1\,[1+5]\,4\,[1+5])$	S_4

Galois Non-Trivial Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ

Galois Non-Trivial Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
$\frac{s_{12}+s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}+s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{s_{12}-s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}-s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{\mathrm{tr}_+(1\,2\,3\,4)}{\mathrm{tr}_-(1\,2\,3\,4)}$	$\mathcal{S}$ $(\lvert\mathcal{S}\rvert = 8)$
$\frac{\mathrm{tr}_+(1\,5\,3\,[1+2])}{\mathrm{tr}_-(1\,5\,3\,[1+2])}$	S_4
$\frac{a+\sqrt{\Sigma_{5,1}}}{a-\sqrt{\Sigma_{5,1}}}$	S_4

$$a = s_{12}s_{23} + s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15} - s_{12}s_{15}$$

Galois Non-Trivial Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
$\frac{s_{12}+s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}+s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{s_{12}-s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}-s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{\mathrm{tr}_+(1\,2\,3\,4)}{\mathrm{tr}_-(1\,2\,3\,4)}$	$\mathcal{S} \quad (\lvert\mathcal{S}\rvert = 8)$
$\frac{\mathrm{tr}_+(1\,5\,3\,[1+2])}{\mathrm{tr}_-(1\,5\,3\,[1+2])}$	S_4
$\frac{a+\sqrt{\Sigma_{5,1}}}{a-\sqrt{\Sigma_{5,1}}}$	S_4
$\frac{\Omega^{--}\Omega^{++}}{\Omega^{-+}\Omega^{+-}}$	$S_4/(S_2[2,3]\times S_2[4,5]\times S_2[s_{23},s_{45}])$
$\frac{\tilde\Omega^{--}\tilde\Omega^{++}}{\tilde\Omega^{-+}\tilde\Omega^{+-}}$	$S_4/(S_2[3,4]\times S_2[2,5])$

$$a = s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45}+s_{45}s_{15}-s_{12}s_{15}$$

$$\Omega^{\pm\pm} = s_{12}s_{15}-s_{12}s_{23}-s_{45}s_{15} \pm s_{34}\sqrt{\Delta_{3,1}} \pm \sqrt{\Delta_5},$$

$$\tilde\Omega^{\pm\pm} = p_1^2 s_{34} \pm \sqrt{\Delta_5} \pm \sqrt{\Sigma_{5,1}}$$

Galois Non-Trivial Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
$\frac{s_{12}+s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}+s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{s_{12}-s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}-s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{\mathrm{tr}_+(1\,2\,3\,4)}{\mathrm{tr}_-(1\,2\,3\,4)}$	$\mathcal{S}$ $(\lvert\mathcal{S}\rvert = 8)$
$\frac{\mathrm{tr}_+(1\,5\,3\,[1+2])}{\mathrm{tr}_-(1\,5\,3\,[1+2])}$	S_4
$\frac{a+\sqrt{\Sigma_{5,1}}}{a-\sqrt{\Sigma_{5,1}}}$	S_4
$\frac{\Omega^{--}\Omega^{++}}{\Omega^{-+}\Omega^{+-}}$	$S_4/(S_2[2,3]\times S_2[4,5]\times S_2[s_{23},s_{45}])$
$\frac{\tilde\Omega^{--}\tilde\Omega^{++}}{\tilde\Omega^{-+}\tilde\Omega^{+-}}$	$S_4/(S_2[3,4]\times S_2[2,5])$
$\sqrt{\Delta_{3,1}}$	$S_4/(S_2[2,3]\times S_2[4,5]\times S_2[s_{23},s_{45}])$
$\sqrt{\Delta_5}$	Id
$\sqrt{\Sigma_{5,1}}$	$S_4/(S_2[3,4]\times S_2[2,5])$

$$a = s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45} + s_{45}s_{15}-s_{12}s_{15}$$

$$\Omega^{\pm\pm} = s_{12}s_{15}-s_{12}s_{23}-s_{45}s_{15} \pm s_{34}\sqrt{\Delta_{3,1}} \pm \sqrt{\Delta_5},$$

$$\tilde\Omega^{\pm\pm} = p_1^2 s_{34} \pm \sqrt{\Delta_5} \pm \sqrt{\Sigma_{5,1}}$$

Galois Non-Trivial Letters

[Abreu, Ita, BP, Tschernow (To appear)]

Generator W	Permutations σ
$\frac{s_{12}+s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}+s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{s_{12}-s_{13}+\sqrt{\Delta_{3,1}}}{s_{12}-s_{13}-\sqrt{\Delta_{3,1}}}$	$S_4/(S_2[2,3]\times S_2[4,5])$
$\frac{\mathrm{tr}_+(1\,2\,3\,4)}{\mathrm{tr}_-(1\,2\,3\,4)}$	$\mathcal{S} \quad (\lvert\mathcal{S}\rvert = 8)$
$\frac{\mathrm{tr}_+(1\,5\,3\,[1{+}2])}{\mathrm{tr}_-(1\,5\,3\,[1{+}2])}$	S_4
$\frac{a+\sqrt{\Sigma_{5,1}}}{a-\sqrt{\Sigma_{5,1}}}$	S_4
$\frac{\Omega^{--}\Omega^{++}}{\Omega^{-+}\Omega^{+-}}$	$S_4/(S_2[2,3]\times S_2[4,5]\times S_2[s_{23},s_{45}])$
$\frac{\tilde\Omega^{--}\tilde\Omega^{++}}{\tilde\Omega^{-+}\tilde\Omega^{+-}}$	$S_4/(S_2[3,4]\times S_2[2,5])$
$\sqrt{\Delta_{3,1}}$	$S_4/(S_2[2,3]\times S_2[4,5]\times S_2[s_{23},s_{45}])$
$\sqrt{\Delta_5}$	Id
$\sqrt{\Sigma_{5,1}}$	$S_4/(S_2[3,4]\times S_2[2,5])$

$$a = s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45}+s_{45}s_{15}-s_{12}s_{15}$$

$$\Omega^{\pm\pm} = s_{12}s_{15}-s_{12}s_{23}-s_{45}s_{15}\pm s_{34}\sqrt{\Delta_{3,1}}\pm\sqrt{\Delta_5},$$

$$\tilde\Omega^{\pm\pm} = p_1^2 s_{34}\pm\sqrt{\Delta_5}\pm\sqrt{\Sigma_{5,1}}$$

Strategy for Boundary Conditions

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- Potential branch points of integrals at $p_1^2 = 0$, $s_{i,j} = 0$.
- DE has other, spurious poles.
- Regularity constrains $\mathbf{I}^{(n)}$. [Gaiotto, Maldacena, Sever, Vieira '11]
- Put DE on 1-d path.
- Once and for all, compute 128-digit boundary condition, using series expansion.

$$\vec{s}(t) = \vec{s}_b + (\vec{s}_e - \vec{s}_b)\, t\,, \quad t \in [0,1]\,.$$

$$\frac{d}{dt}\mathbf{I}(t,\epsilon) = \epsilon \mathbf{A}(t)\mathbf{I}(t,\epsilon).$$

$$\mathbf{I}(\epsilon) = \sum_{n=0}^{\infty} \mathbf{I}^{(n)}\, \epsilon^n.$$

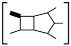
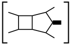
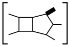
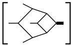
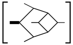
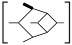
$$\mathbf{A}(t) = \frac{1}{t-t_k}\mathbf{A}_{-2,k} + \mathcal{O}[(t-t_k)^0]$$

$$\boxed{\mathbf{A}_{-2,k}\left[\mathbf{I}^{(n-1)}(t_k)\right] = 0\,.}$$

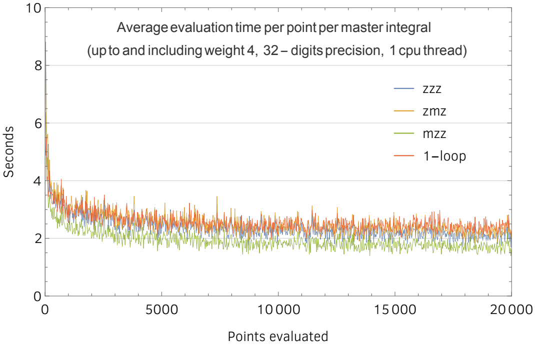
High Precision Evaluation [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20/To appear]

- Can evolve to any region maintaining high precision.
- E.g. weight-4 128-digit at $\vec{s}_{\text{ph-1}} = \left(\frac{137}{50}, -\frac{22}{5}, \frac{241}{25}, -\frac{377}{100}, \frac{13}{50}, \frac{249}{50}\right)$.

[DiffExp]

$\Big[$  $\Big]^{(4)}_{3}$	Re	$+11.90852968084159332956737844434123\ldots$
	Im	$-143.8383823509733651355372899165828\ldots$
$\Big[$  $\Big]^{(4)}_{3}$	Re	$+44.16216574473530086723311855418285\ldots$
	Im	$-46.21874613385033996994440307755667\ldots$
$\Big[$  $\Big]^{(4)}_{3}$	Re	$+29.80276365179310881202389321759335\ldots$
	Im	$+273.8662784626651511391329522557241\ldots$
$\Big[$  $\Big]^{(4)}_{1}$	Re	$+0.196005898233033010683344809972819\ldots$
	Im	$-0.379706717274129286445770827639486\ldots$
$\Big[$  $\Big]^{(4)}_{1}$	Re	$-0.308803846088998937983338935724276\ldots$
	Im	$+0.646842335546650495163249938099425\ldots$
$\Big[$  $\Big]^{(4)}_{1}$	Re	$+0.410978253386789374756950888794838\ldots$
	Im	$-0.527369133359713739013647088111916\ldots$

Pentabox Over Physical Phase Space [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

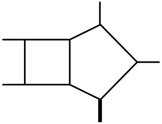
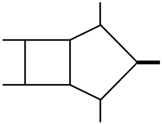
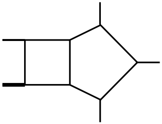


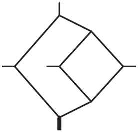
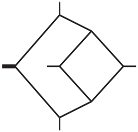
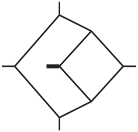
- Asymptotically more efficient strategy over phase space.
- Pick $\vec{s}_b$ from previous evaluations near to $\vec{s}_e$.
- Over time need fewer segments.

Conclusions

- Master integrals for leading-colour W-plus-two-jets production are now known.
- All one-mass non-planar hexabox master integrals known.
- Canonical differential equations can be efficiently constructed using finite field methods combined with unitarity cuts.
- Generalized series solutions to differential equations are an efficient, generic method for evaluating Feynman integrals.

One-Mass Penta/Hexa-box Integral Families

Top Topology			
# Masters	74	75	86

Top Topology			
# Masters	86	86	135

Local Solutions for Canonical Differential Equation

- Expand around t_k and integrate. Locally valid.

$$\mathbf{M}(t) = \epsilon \mathbf{A}(t).$$

- If t_k is singular point, $\mathbf{A}(t)$ expansion has square roots/poles.

$$\frac{d}{dt}\mathbf{I}^{(0)}(t) = 0,$$
$$\frac{d}{dt}\mathbf{I}^{(n)}(t) = \mathbf{A}(t)\mathbf{I}^{(n-1)}(t).$$

- Integration promotes poles to logarithms.

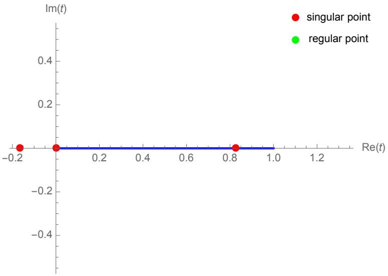
$$\mathbf{A}(t) = \sum_{i=-2}^{\infty} \mathbf{A}_{i,k}(t - t_k)^{\frac{i}{2}}.$$

- In practice we work with truncated series.

$$\mathbf{I}_k^{(n)}(t) = \mathbf{c}_{k,(n)}^{(0,0)} + \sum_{j_1,j_2} \mathbf{c}_{k,(n)}^{(j_1,j_2)} (t - t_k)^{\frac{j_1}{2}} \log(t - t_k)^{j_2}.$$

Choosing Expansion Points and Matching

- Convergence of $\mathbf{I}_k$ limited by nearest singular point of DE.
- As truncating, take radius r_k as $\frac{1}{2}$ radius of convergence.
- First expand around relevant singular points.
- Fill in remainder by bisection.
- Continuity fixes integration constants $\mathbf{c}^{(0,0)}_{k,(n)}$ of each $\mathbf{I}_k$.



$$\mathbf{I}^{(n)}_k(t_k - r_k) = \mathbf{I}^{(n)}_{k-1}(t_{k-1} + r_{k-1}).$$

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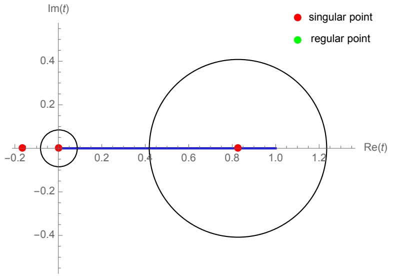

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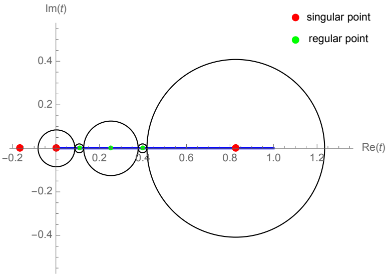

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Asymptotic Timings

	Family	MI's	time per MI (s)	total time (s)	truncation order
32 digits	zzz	86	2.08	179	$70 < n_k < 140$
	zmz	75	2.24	168	
	mzz	74	1.69	125	
	1-loop	13	2.38	31	
16 digits	zzz	86	1.10	94	$40 < n_k < 80$
	zmz	75	1.15	86	
	mzz	74	0.88	65	
	1-loop	13	1.69	22	

Square Roots in Finite Fields

- Finite fields are not algebraically closed.
- Letters are algebraic over mandelstams.
- Pick $\vec{s}$ where Δ_3, Δ_3^{nc}, and tr_5^2 are perfect squares.
- Surprisingly easy as $\sim \frac{1}{2}$ of $\mathbb{F}_p$ are perfect squares.
- Picking randomly gives $\frac{1}{8}$ chance of perfect square.

$$\nexists \; x : \quad x^2 - 3 = 0 \pmod{7}.$$

$$0^2, 1^2, 2^2, \ldots, \left(\frac{p-1}{2}\right)^2.$$