Recent progress on two-loop massless pentabox integrals with one off-shell leg

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in collaboration with
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Feynman: 1949 to 2021, 72 years later

PHYSICAL REVIEW VOLUME 76, NUMBER 6 SEPTEMBER 15, 1949

Space-Time Approach to Quantum Electrodynamics

R. P. Feynman
Department of Physics, Cornell University, Ithaca, New York
(Received May 9, 1949)

In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.

The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation

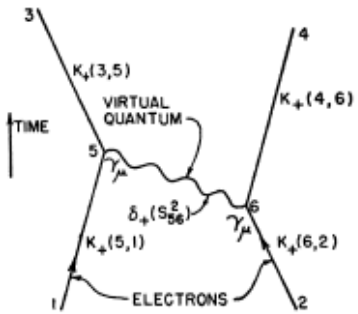
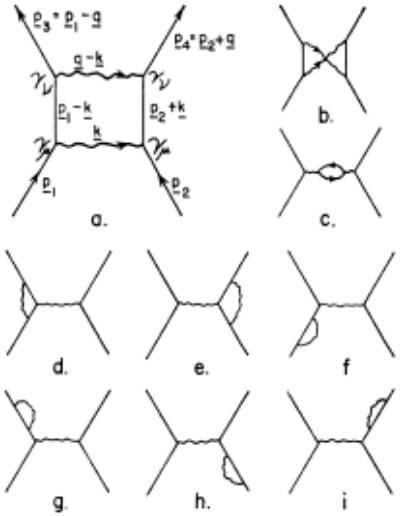


Fig. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.

D. More Complex Problems

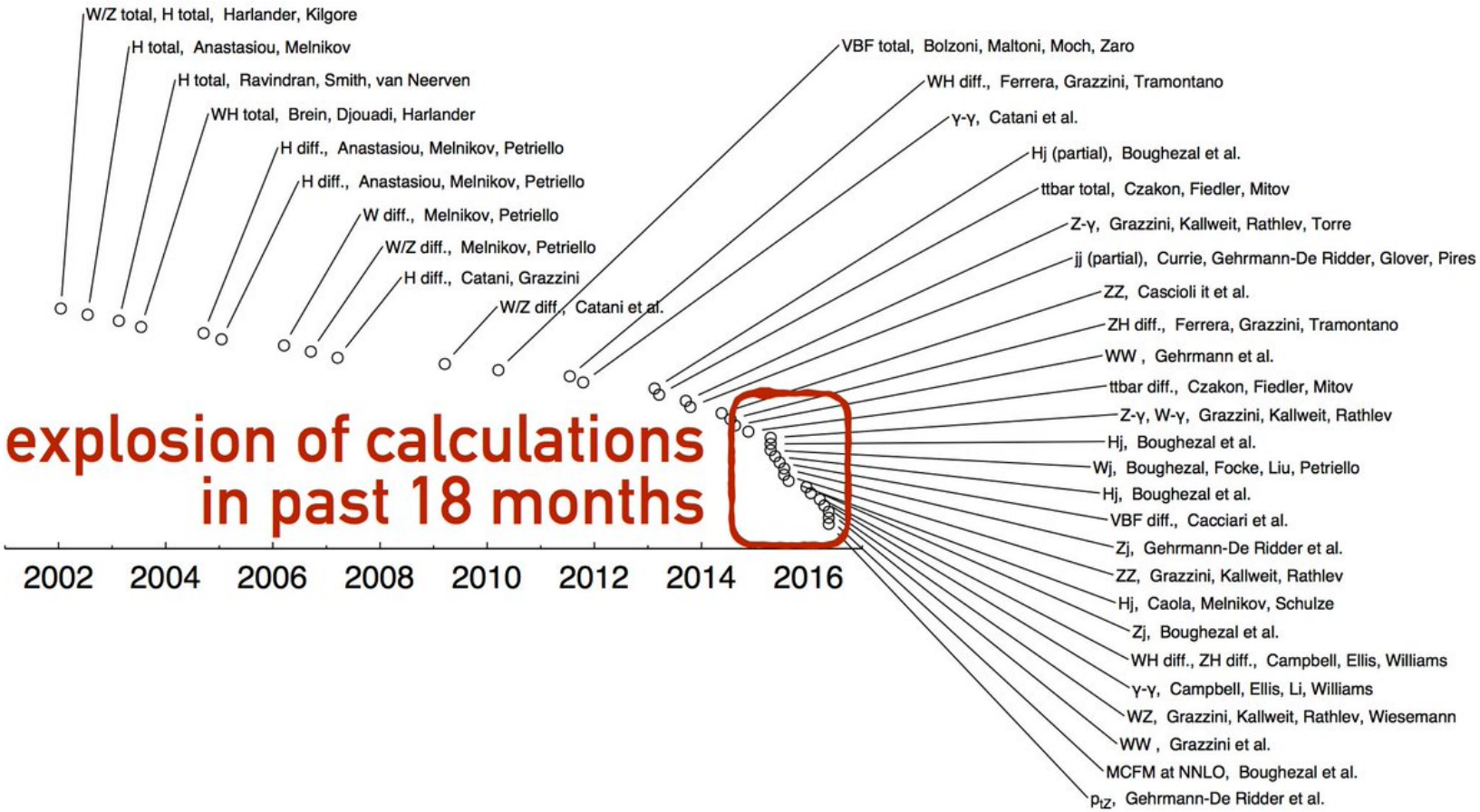
Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-



Towards higher precision: NNLO and beyond

NNLO Revolution

Gavin Salam → click to link



Missing generic approach

NNLO QCD: $pp \to \gamma\gamma\gamma + X$

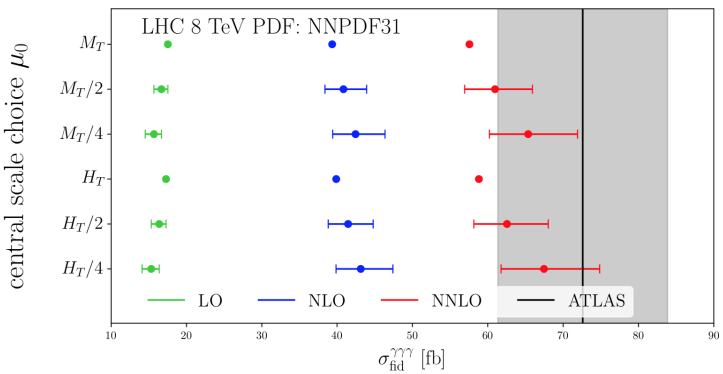


Figure 1. Predictions for the fiducial cross-section in LO (green), NLO (blue) and NNLO (red) QCD versus ATLAS data (black). Shown are predictions for six scale choices. The error bars on the theory predictions reflect scale variation only. For two of the scales only the central predictions are shown.

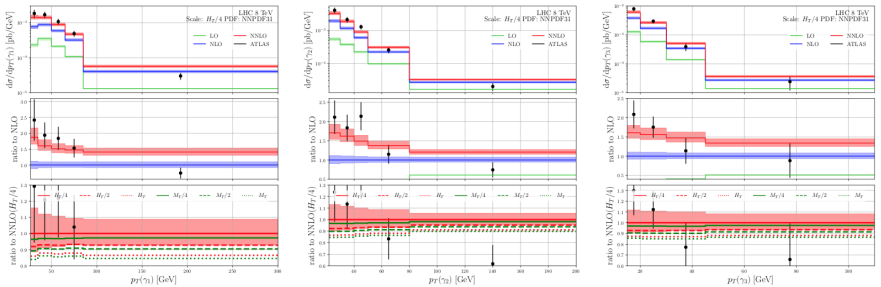


Figure 2. p_T distribution of the hardest photon γ_1 (left), γ_2 (center) and the softest one γ_3 (right). Top plot shows the absolute distribution at NNLO (red), NLO (blue) and LO (green) versus ATLAS data (black). Middle plot shows same distributions but normalized to the NLO. Bottom plot shows central NNLO predictions for 6 different scale choices (only the central scale is shown) with respect to the default choice $\mu_0 = H_T/4$. The bands represent the 7-point scale variations about the corresponding central scales.

H. A. Chawdhry, M. L. Czakon, A. Mitov and R. Poncelet, JHEP **2002** (2020) 057

NNLO QCD: $pp \to \gamma\gamma\gamma + X$

fiducial setup for $pp \to \gamma\gamma\gamma + X$; used in the ATLAS 8 TeV analysis of Ref. [37]
$p_{T,\gamma_1} \geq 27\,\text{GeV}, \quad p_{T,\gamma_2} \geq 22\,\text{GeV}, \quad p_{T,\gamma_3} \geq 15\,\text{GeV}, \quad 0 \leq \lvert\eta_\gamma\rvert \leq 1.37 \text{ or } 1.56 \leq \lvert\eta_\gamma\rvert \leq 2.37\,,$
$\Delta R_{\gamma\gamma} \geq 0.45, \quad m_{\gamma\gamma\gamma} \geq 50\,\text{GeV}, \quad$ Frixione isolation with $n = 1$, $\delta_0 = 0.4$, and $E_T^{\text{ref}} = 10\,\text{GeV}\,.$

Table 1: Definition of phase space cuts.

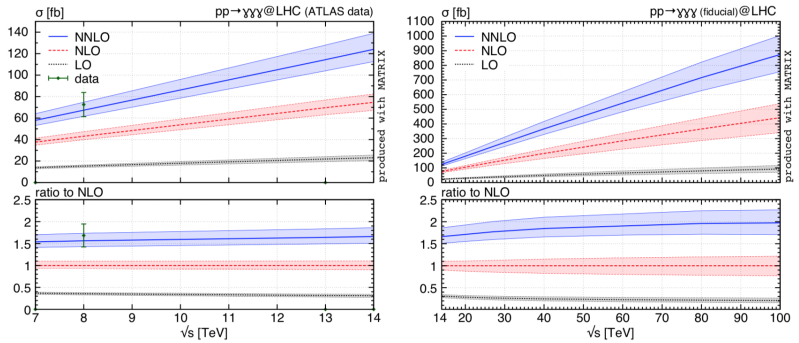


Figure 4: Fiducial cross sections for $pp \to \gamma\gamma\gamma + X$ as a function of the centre-of-mass energy at LO (black dotted), at NLO (red dashed), and at NNLO (blue, solid) The green data point at 8 TeV corresponds to the cross section measured by ATLAS in Ref. [37].

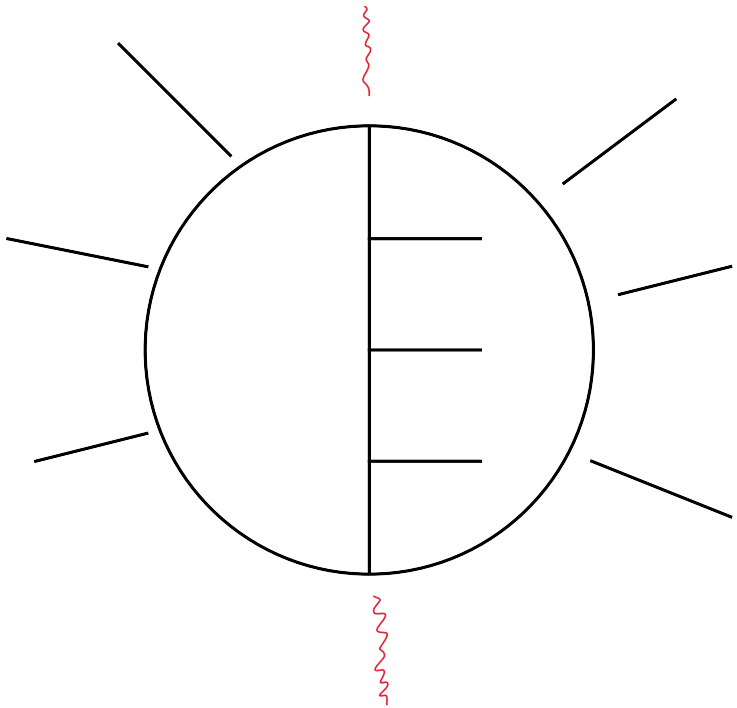
S. Kallweit, V. Sotnikov and M. Wiesemann, Phys. Lett. B **812** (2021) 136013

NNLO Revolution

- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, "Two-loop leading-colour QCD helicity amplitudes for two-photon plus jet production at the LHC," arXiv:2103.04319 [hep-ph].
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, "Two-loop leading-color helicity amplitudes for three-photon production at the LHC," arXiv:2012.13553 [hep-ph].
- S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov, "Leading-Color Two-Loop QCD Corrections for Three-Jet Production at Hadron Colliders," arXiv:2102.13609 [hep-ph].
- S. Abreu, B. Page, E. Pascual and V. Sotnikov, "Leading-Color Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders," JHEP **2101** (2021) 078
- S. Abreu *et al.*, "Caravel: A C++ Framework for the Computation of Multi-Loop Amplitudes with Numerical Unitarity," arXiv:2009.11957 [hep-ph].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, "Two-loop helicity amplitudes for diphoton plus jet production in full color," arXiv:2105.04585 [hep-ph].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, "Two-loop leading colour QCD corrections to $q\bar{q} \to \gamma\gamma g$ and $qg \to \gamma\gamma q$," JHEP **2104** (2021) 201
- S. Badger, H. B. Hartanto and S. Zoia, "Two-loop QCD corrections to $Wb\bar{b}$ production at hadron colliders," arXiv:2102.02516 [hep-ph].
- H. B. Hartanto, S. Badger, C. Brynnum-Hansen and T. Peraro, "A numerical evaluation of planar two-loop helicity amplitudes for a W-boson plus four partons," JHEP **1909** (2019) 119

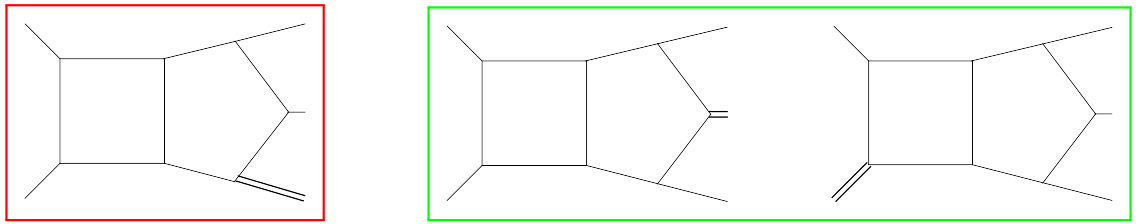
+ rational terms, IBP reduction, computation of Master Integrals, etc.

Two-loop graph

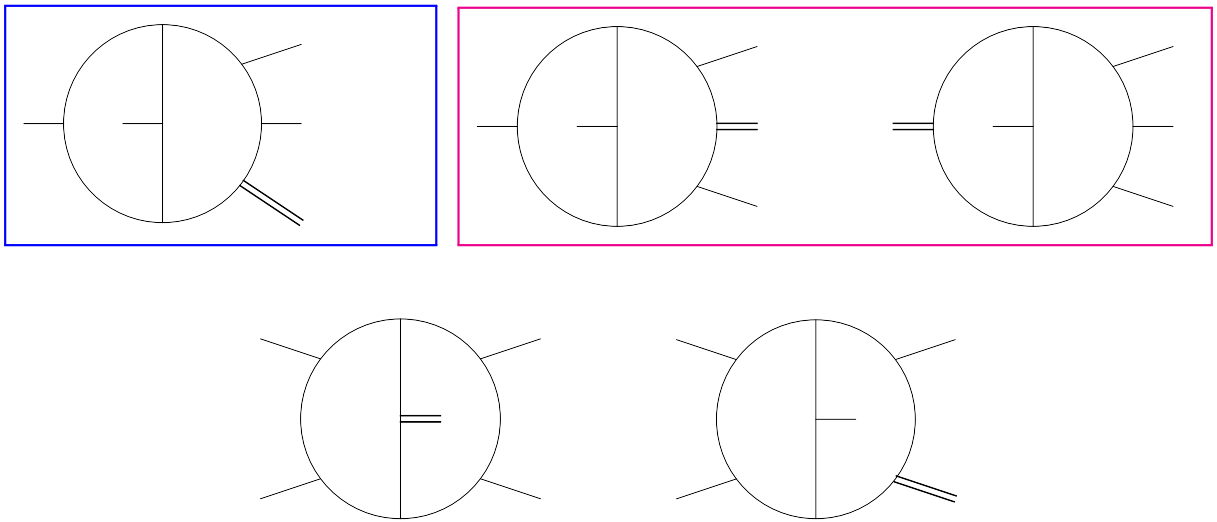


5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].
C. G. Papadopoulos and C. Wever, JHEP **2002** (2020) 112
S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199
[talk by Ben Page]

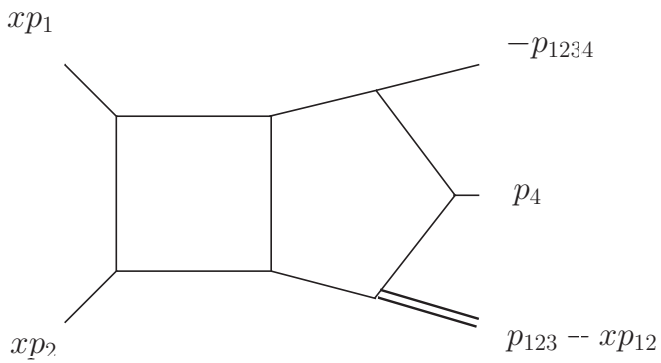


The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

5BOX - ONE LEG OFF-SHELL: P1



$$q_1 \to p_{123} - xp_{12},\ q_2 \to p_4,\ q_3 \to -p_{1234},\ q_4 \to xp_1$$

SDE parametrisation: n off-shell legs $\to n-1$ off-shell legs + the x variable.

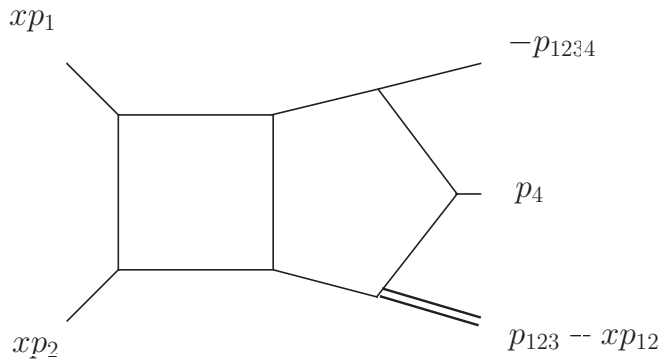
C. G. Papadopoulos, "Simplified differential equations approach for Master Integrals," JHEP **1407** (2014) 088

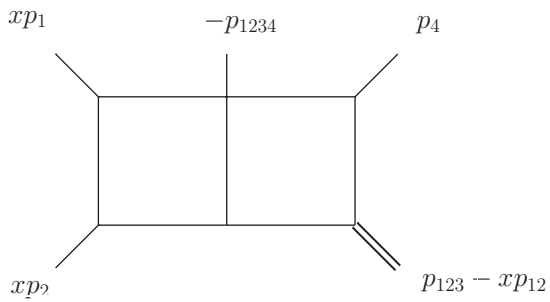
[talk by N. Syrrakos]

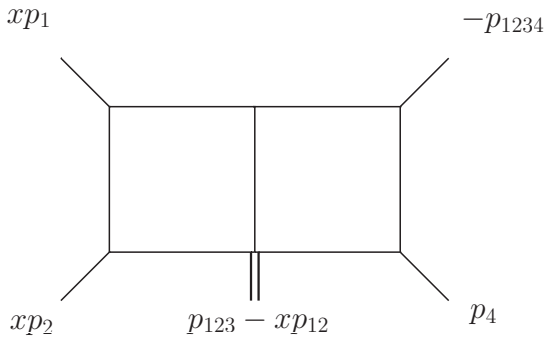
- $p_i,\ i = 1 \ldots 5$, satisfy $\sum_1^5 p_i = 0$, with $p_i^2 = 0,\ i = 1 \ldots 5,\ p_{i\ldots j} := p_i + \ldots + p_j$. The set of independent invariants: $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$.

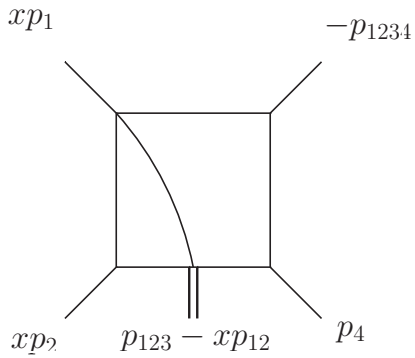
$$q_1^2 = (1-x)(S_{45} - S_{12}x),\ s_{12} = (S_{34} - S_{12}(1-x))\,x,\ s_{23} = S_{45},\ s_{34} = S_{51}x,$$
$$s_{45} = S_{12}x^2,\ s_{15} = S_{45} + (S_{23} - S_{45})x$$

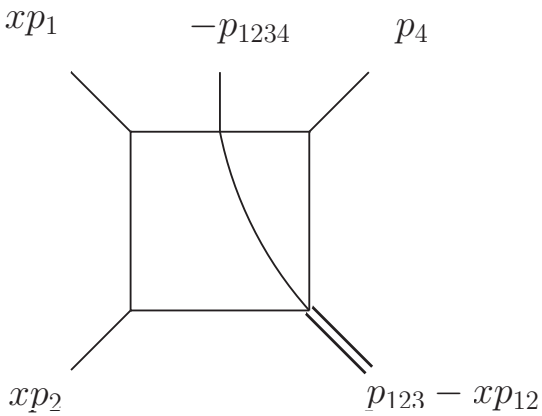
5box - one leg off-shell: P1

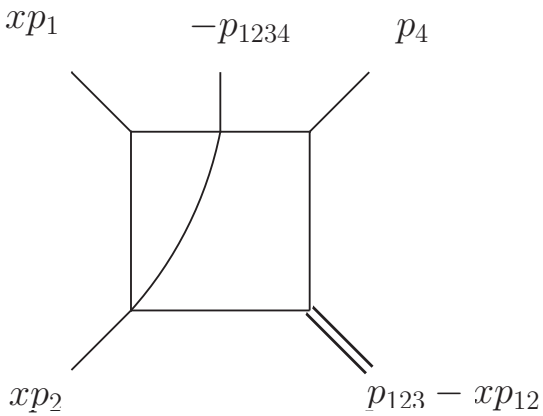


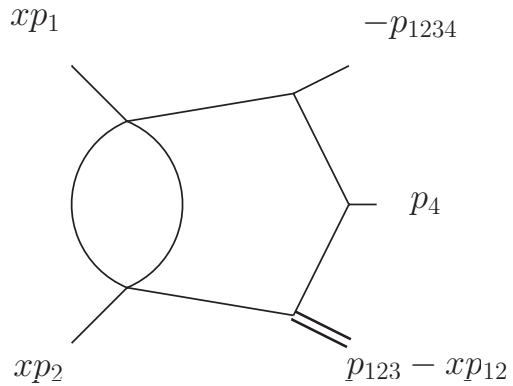












4-point up to two legs off-shell

J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1405** (2014) 090
F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1409** (2014) 043
C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072

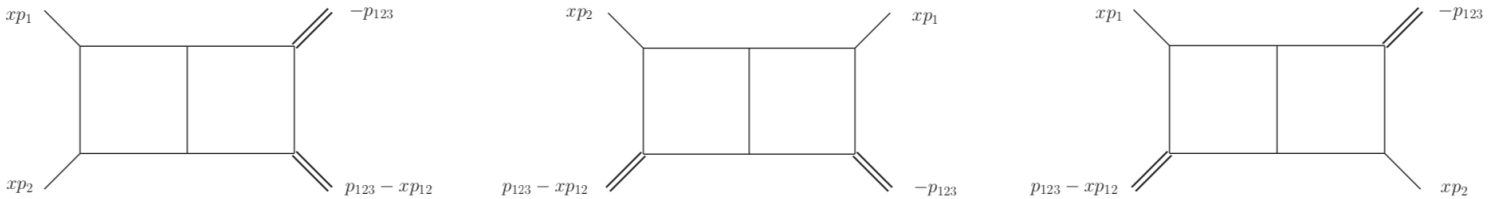


Figure 3. The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

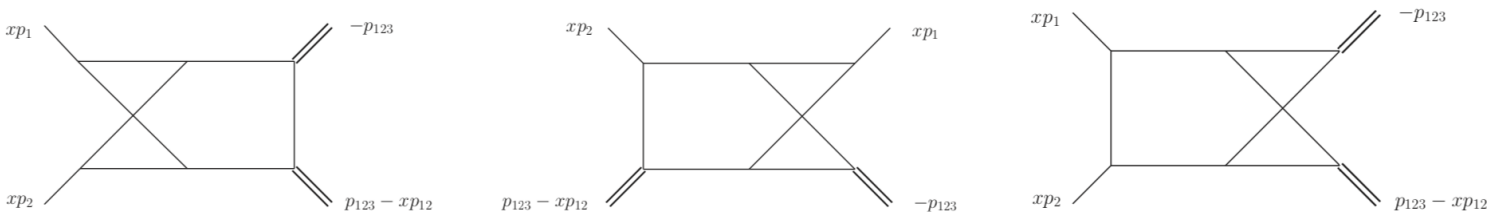


Figure 4. The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

As well as planar and nonplanar double box with one off-shell leg expressed in UT basis.

5BOX - ONE LEG OFF-SHELL: P1-3

$$G^{P_1}_{a_1\cdots a_{11}} := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+q_1)^{2a_2}(k_1+q_{12})^{2a_3}(k_1+q_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2+q_{123})^{2a_6}(k_2+q_{1234})^{2a_7}(k_1-k_2)^{2a_8}(k_1+q_{1234})^{2a_9}(k_2+q_1)^{2a_{10}}(k_2+q_{12})^{2a_{11}}},$$

$$G^{P_2}_{a_1\cdots a_{11}} := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1-q_{1234})^{2a_2}(k_1-q_{234})^{2a_3}(k_1-q_{34})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-q_{34})^{2a_6}(k_2-q_4)^{2a_7}(k_1-k_2)^{2a_8}(k_2-q_{1234})^{2a_9}(k_2-q_{234})^{2a_{10}}(k_1-q_4)^{2a_{11}}},$$

$$G^{P_3}_{a_1\cdots a_{11}} := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+q_2)^{2a_2}(k_1+q_{23})^{2a_3}(k_1+q_{234})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2+q_{234})^{2a_6}(k_2-q_1)^{2a_7}(k_1-k_2)^{2a_8}(k_1-q_1)^{2a_9}(k_2+q_2)^{2a_{10}}(k_2+q_{23})^{2a_{11}}},$$

where $q_{i\ldots j} := q_i + \ldots + q_j$.

5box - one leg off-shell: P1

J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601

S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log(W_a)\, \tilde{M}_a \vec{g}$$

- Also from direct differentiation of MI wrt to x. Just g in terms of FI.

$$\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \vec{g}$$

- ℓ_b, are independent of x, some depending only on the reduced invariants, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. M_b are independent of the invariants.
- number of letters smaller than in AIMPTZ representation
- Main contribution for us from AIMPTZ: the canonical basis (+ numerics)

5box - one leg off-shell: P1

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$$d\vec{g} = \epsilon \sum_a d\log(W_a)\, \tilde{M}_a \vec{g}$$

$$\frac{d\log(W_a)}{dx}$$

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5BOX - ONE LEG OFF-SHELL: P1-3

$$\frac{d\mathbf{g}}{dx} = \sum_a \frac{1}{x-\ell_a}\mathbf{M}_a\mathbf{g}$$

$$\begin{aligned}
\mathbf{g} = {} & \epsilon^0\mathbf{b}_0^{(0)} + \epsilon\left(\sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
& + \epsilon^2\left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \\
& + \epsilon^3\left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)}\right) \\
& + \epsilon^4\left(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(1)}\right. \\
& \left. + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)}\right) + \ldots
\end{aligned}$$

$$\mathcal{G}_{ab\ldots} := \mathcal{G}(\ell_a, \ell_b, \ldots; x)$$

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. `Results.txt`

5box - one leg off-shell: Boundary conditions

- starting from the full equation

$$\frac{d\vec{g}}{dx} = \epsilon \frac{1}{x} M_0 \vec{g} + \mathcal{O}(x^0)$$

- using all letters W_a, with the solution ($\mathbf{b} := \sum_{i=0}^{4} \epsilon^i \mathbf{b}_0^{(i)}$)

$$\mathbf{g}_0 = \mathbf{S} e^{\epsilon \log(x) \mathbf{D}} \mathbf{S}^{-1} \mathbf{b}$$

- $\mathbf{S}$ and $\mathbf{D}$ are obtained through Jordan decomposition of the $\mathbf{M}_0$
- Resummed: $\mathbf{R}_0 = \mathbf{S} e^{\epsilon \log(x) \mathbf{D}} \mathbf{S}^{-1}$
- What we know about:

$$\mathbf{R}_0 = \sum_i x^{n_i \varepsilon} \mathbf{R}_{0i} + \sum_j \varepsilon x^{n_j \varepsilon} \log(x) \mathbf{R}_{0j0}$$

5box - one leg off-shell: Boundary conditions

- IBP reduction in terms of Master Integrals

$$\mathbf{g} = \mathbf{TG}.$$

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. `Masters.m`

- Expansion by regions. [no logarithmic terms]

$$G_i \underset{x\to 0}{=} \sum_j x^{b_j + a_j \varepsilon}\, G_i^{(j)}$$

- Linear equations:

$$\mathbf{g}_0 := \mathbf{R}_0\mathbf{b} = \lim_{x\to 0} \mathbf{TG}\Big|_{O\left(x^{0+a_j\varepsilon}\right)}$$

- Matrix $\mathbf{T}$ is horrible-looking depending on x, ε and S_{ij}. But

$$\mathbf{R}_0\mathbf{b} \to \varepsilon, x, \mathit{Rationals} \otimes \mathit{polyLogs} \qquad G_i^{(j)} \to \mathit{Simple}\left[S_{ij}\right] \otimes \mathit{polyLogs}$$

so we can afford IBP reduction with only x, ε symbolic: i.e. `FIRE6` or `Kira2`.

5box - one leg off-shell: Boundary conditions

- No regions in the top-sector are needed.
- To obtain expressions for regions, $G_i^{(j)}$, in Feynman parameter space, we use `FIESTA` `asyexpand`, for $x \to 0$ limit (SDE).
- In most cases integration is straightforward and the resulting ${}_2F_1$ hypergeometric functions are expanded with `HypExp`.
- In few cases we use Mellin-Barnes techniques using the `MB`, `MBSums` and `XSummer` along with the in-house (A. Kardos) package `Gsuite`.
- Boundary terms only depends on 12 Goncharov

$$G\left[0,1,-\frac{S12-S34}{S51}\right], G\left[1,-\frac{S12-S34}{S51}\right], G\left[0,0,1,-\frac{S12-S34}{S51}\right], G\left[0,1,1,-\frac{S12-S34}{S51}\right],$$
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$$G\left[1,1,0,1,-\frac{S12-S34}{S51}\right]$$

- and 4 Logarithms $\{\text{Log}[-S12], \text{Log}[-S45], \text{Log}[S12-S34], \text{Log}[-S51]\}$.

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. `Boundaries.m`

5box - one leg off-shell: Kinematical Regions

- Euclidean region:

$$\left\{\mathrm{S12} \to -2, \mathrm{S23} \to -3, \mathrm{S34} \to -5, \mathrm{S45} \to -7, \mathrm{S51} \to -11, x \to \frac{1}{4}\right\}$$

no letter l in the region $[0, x]$, all boundary terms real. [very fast `GiNaC`]

Family	W=1	W=2	W=3	W=4
P_1 (g_{72})	17 (14)	116 (95)	690 (551)	2740 (2066)
P_2 (g_{73})	25 (14)	170 (140)	1330 (1061)	4950 (3734)
P_3 (g_{84})	22 (12)	132 (90)	1196 (692)	4566 (2488)

Table: Number of GP entering in the solution, as explained in the text.

- with timings, running the `GiNaC` Interactive Shell `ginsh`, given by 1.9, 3.3, and 2 seconds for P_1, P_2 and P_3 respectively and for a precision of 32 significant digits
- A very different canonical basis, several elements start at ϵ^4.

5box - one leg off-shell: Kinematical Regions

- One-scale integrals - closed form

$$
\begin{aligned}
(-s_{34})^{-\epsilon} &= (-S_{51})^{-\epsilon} x^{-\epsilon} \\
(-s_{45})^{-\epsilon} &= (-S_{12})^{-\epsilon} x^{-2\epsilon} \\
(-s_{15})^{-\epsilon} &= (-S_{45})^{-\epsilon} \left(1 - \frac{S_{45} - S_{23}}{S_{45}} x\right)^{-\epsilon} \\
(-p_{1s})^{-\epsilon} &= (1-x)^{-\epsilon} (-S_{45})^{-\epsilon} \left(1 - \frac{S_{12}}{S_{45}} x\right)^{-\epsilon} \\
(-s_{12})^{-\epsilon} &= x^{-\epsilon} (S_{12} - S_{34})^{-\epsilon} \left(1 - \frac{S_{12}}{S_{12} - S_{34}} x\right)^{-\epsilon},
\end{aligned}
$$

- One-scale integrals - expanded form

$$
\begin{aligned}
&\text{Log}[-\text{p1s} - i\delta] \to G[1, x] + G\left[\frac{\text{S45}}{\text{S12}}, x\right] + \text{Log}[-\text{S45}], \\
&\text{Log}[-\text{s34} - i\delta] \to \text{Log}[-\text{S51}] + \text{Log}[x], \\
&\text{Log}[-\text{s12} - i\delta] \to G\left[\frac{\text{S12} - \text{S34}}{\text{S12}}, x\right] + \text{Log}[\text{S12} - \text{S34}] + \text{Log}[x], \\
&\text{Log}[-\text{s45} - i\delta] \to \text{Log}[-\text{S12}] + 2\,\text{Log}[x], \\
&\text{Log}[-\text{s15} - i\delta] \to G\left[\frac{\text{S45}}{-\text{S23} + \text{S45}}, x\right] + \text{Log}[-\text{S45}]
\end{aligned}
$$

5box - one leg off-shell: Kinematical Regions

- In general many letters will be now in $[0, x]$. This has two consequences:
  1. Need to fix infinitesimal imaginary part of $\frac{l_i}{x}$
  2. Increasing CPU time in `GiNaC`.
- Since the $\mathcal{F}$ polynomial maintains the sign of the $i0$ prescription of Feynman propagators with all original invariants assuming $s_{ij}(p_{1s}) \to s_{ij}(p_{1s}) + i\delta$, we determine the corresponding infinitesimal imaginary part of $\frac{l_i}{x}$ from

$$p_{1s} + i\delta = (1-x)(S_{45} - S_{12}x),\ s_{12} + i\delta = (S_{34} - S_{12}(1-x))\,x,$$
$$s_{23} + i\delta = S_{45},\ s_{34} + i\delta = S_{51}x,$$
$$s_{45} + i\delta = S_{12}x^2,\ s_{15} + i\delta = S_{45} + (S_{23} - S_{45})x$$

  with $S_{ij} \to S_{ij} + i\delta\eta_{ij}$, $x \to x + i\delta\eta_x$,
- Building a Fibration Basis using for instance `PolyLogTools`.

5box - one leg off-shell: Validation

- All regions of AIMPTZ checked @precision
- One-loop pentagon at order $\mathcal{O}(\varepsilon^4)$ [any order, analytic]

  N. Syrrakos, "Pentagon integrals to arbitrary order in the dimensional regulator," arXiv:2012.10635 [hep-ph].

  [talk by N. Syrrakos]
- Taken the limit $x = 1$ in all families to obtain the result for on-shell planar 5box

  SDE is not only capable to produce analytic results for off-shell MI but it can also give, almost for free, the on-shell MI.

  [talk by D. Canko]
- Evaluating phase-space points for $pp \to W^+ j_1 j_2$ generated by `HELAC-PHEGAS`, i.e. arbitrary floating points.

5box - one leg off-shell: Outlook

- Non-planar families
  - We have completed the first hexa-box family, N1.
  - Preliminary check against AIMPTZ group results @lowaccuracy: at least 13 digits.
  - N2 and N3 families UT bases from AIMPTZ: analytic results expected soon.
  - Complete all 5-point families: penta-pentagon N4 and N5.
  - Obtain all 5-point families with on-shell legs, $x \longrightarrow 1$. [new wrt pentagon functions?]
  - MI: 4-point with up to 2 off shell legs and 5-point with up to one off-shell leg.
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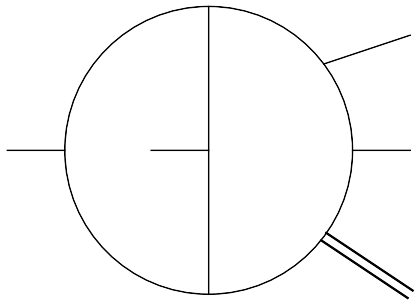

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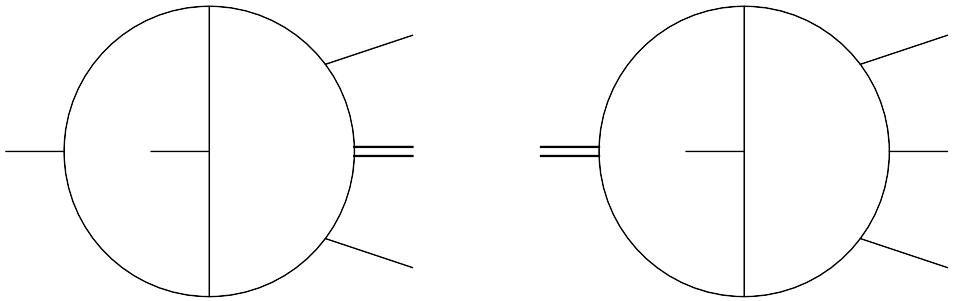

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Acknowledging

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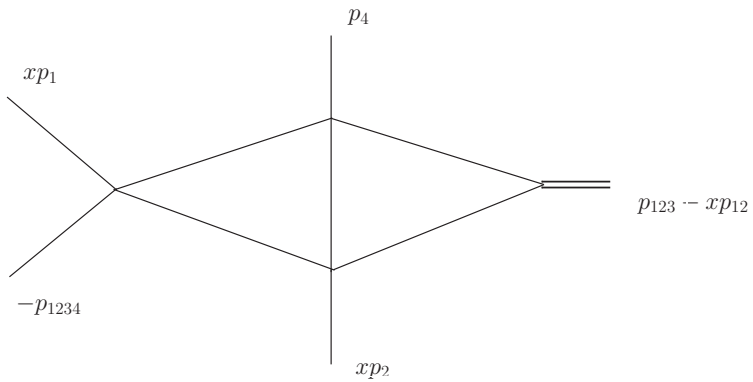
Thank you for your attention !

Backup slides

5BOX - ONE LEG OFF-SHELL: P2-3



The two-loop diagram representing the decoupling basis element.

- Basis element 46 for P2 (53 for P3) known from double box P23 family; starts at $\mathcal{O}(\epsilon^4)$. [decoupling]

$$q_1 \to P_{123} - yP_{12},\ q_2 \to yP_1,\ q_3 \to P_4,\ q_4 \to -P_{1234},\ q_5 \to yP_2$$

$$q_1 \to p_{123} - xp_{12},\ q_2 \to p_4,\ q_3 \to -p_{1234},\ q_4 \to xp_1,\ q_5 \to xp_2$$

$$q_1^2 = (1-y)(S'_{45} - S'_{12}y),\ s_{12} = S'_{45} - (S_{12} + S'_{23})y,\ s_{23} = \left(S'_{34} - S_{12}(1-y)\right)y,$$
$$s_{34} = S'_{45},\ s_{45} = -(S'_{12} - S'_{34} + S'_{51})y,\ s_{15} = S'_{45} + S'_{23}y$$

$$q_1^2 = (1-x)(S_{45} - S_{12}x),\ s_{12} = (S_{34} - S_{12}(1-x))\,x,\ s_{23} = S_{45},\ s_{34} = S_{51}x,$$
$$s_{45} = S_{12}x^2,\ s_{15} = S_{45} + (S_{23} - S_{45})x$$

5BOX - ONE LEG OFF-SHELL: P2-3

$$d\vec{g} = \left[\sum_b d\log\left(x-\ell_b\right) M_b + \sum_c d\log\left(y-\ell_c\right) \bar{M}_c + d\log\left(W_{58}\left(x,y\right)\right) \tilde{M}_{58}\right] \vec{g}$$

- all letters W_a, except W_{58}, are linear functions only of x or y.
- M matrices have zeroes in the row and the column corresponding to the basis element 46 for P2 (53 for P3).
- $\bar{M}$ matrices have non-zero matrix elements only in the row and the column corresponding to the basis element 46 for P2 (53 for P3).
- $\tilde{M}$ matrix have non-zero matrix elements only in the column corresponding to the basis element 46 for P2 (53 for P3).

$$\frac{d\vec{g}'}{dx} = \sum_a \frac{1}{x-\ell_a} M_a \vec{g}'$$