On the Difference between the FOPT and CIPT Approach for Hadronic Tau Decays

André H. Hoang

University of Vienna

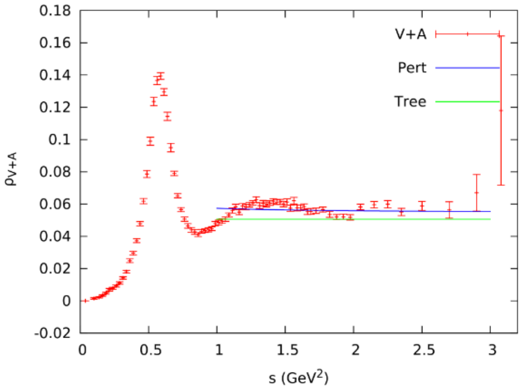
Based on arXiv:2008.00578
with Christoph Regner

Strong coupling from τ decays

ALEPH: τ hadronic width

(HFLAV 2019)

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\, \nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^- \overline{\nu}_e \nu_\tau(\gamma)]} = 3.6355 \pm 0.0081$$



4-loop: Gorishni etal., Surguladze etal. ‘91

5-loop: Baikov etal. ‘91

$$j^\nu_{v/av,jk} = \bar{q}_j \gamma^\mu (\gamma_5) q_k$$

$$(p^\mu p^\nu - g^{\mu\nu} p^2)\, \Pi(p^2) \equiv i \int dx\, e^{ipx}\, \langle \Omega |\, T\{ j^\mu_{v/av,jk}(x)\, j^\nu_{v/av,jk}(0)^\dagger \} \Omega \rangle$$

Adler function: $$\frac{1}{4\pi^2}\Big(1 + \hat{D}(s)\Big) \equiv -\, s\, \frac{\mathrm{d}\Pi(s)}{\mathrm{d}s}$$

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n, \qquad \text{CIPT}$$

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\!\left(\frac{-s}{s_0}\right) \qquad \text{FOPT}$$

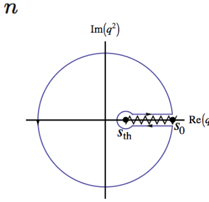
Theory: Operator product expansion ($s_0 = m_\tau^2$)

$$A_{W_i}(s_0) = \frac{N_c}{2} |V_{ud}|^2 \left[\delta^{\rm tree}_{W_i} + \delta^{(0)}_{W_i}(s_0) + \sum_{d\geq 2} \delta^{(d)}_{W_i}(s_0) + \delta^{\rm DV}_{W_i}(s_0) \right]$$

pQCD OPE Duality violation

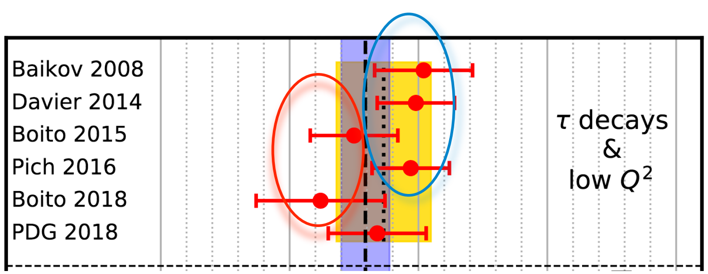
Contour-improved perturbation theory (CIPT):

$$\delta^{(0),\rm CIPT}_{W_i}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x) \left(\frac{\alpha_s(-x s_0)}{\pi}\right)^n$$



Fixed-order perturbation theory (FOPT):

$$\delta^{(0),\rm FOPT}_{W_i}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x) \ln^{k-1}(-x)$$



Outline

- Asymptotic series and renormalons
- FOPT and CIPT Borel representation do not agree
- Numerical Studies
- Implications for the OPE
- Conclusions

$$a(x) \equiv \frac{\beta_0\,\alpha_s(s)}{4\pi} = \frac{\beta_0\,\alpha_s(xs_0)}{4\pi}$$

$$a_0 \equiv \frac{\beta_0\,\alpha_s(s_0)}{4\pi}$$

Renormalon Calculus

Perturbative series in QCD are not convergent, but asymptotic. $\rightarrow$ $\hat{D}(s) \sim \sum_{n=1}^{\infty} n!\,\left(\frac{\alpha_s(-s)}{\pi}\right)^n$

Reminder of renormalon calculus: `t Hooft; David; Müller; Beneke; ...

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1}\left(\frac{\alpha_s(-s)}{\pi}\right)^n \Longrightarrow B[\hat{D}](u) \sim \sum_{n=1}^{\infty} \frac{c_{n,1}}{\Gamma(n)}\,u^{n-1} \Longrightarrow B[\hat{D}](u) \sim \underbrace{\frac{1}{(p-u)^\gamma} + \frac{1}{(\tilde{p}+u)^\gamma}}$$

Original series (asymptotic) — Borel function series (convergent) — Borel function

Borel transform — Analytic continuation

IR renormalon ambiguities associated to OPE corrections:

$$\hat{D}^{\rm OPE}(s) = \frac{1}{(-s)^2}\langle G^2\rangle + \sum_{p=3}^{\infty}\frac{1}{(-s)^p}\Big[C_0\langle \mathcal{O}_{2p,0}\rangle + C_1\langle \mathcal{O}_{2p,1}\rangle + \ldots\Big]\,.$$

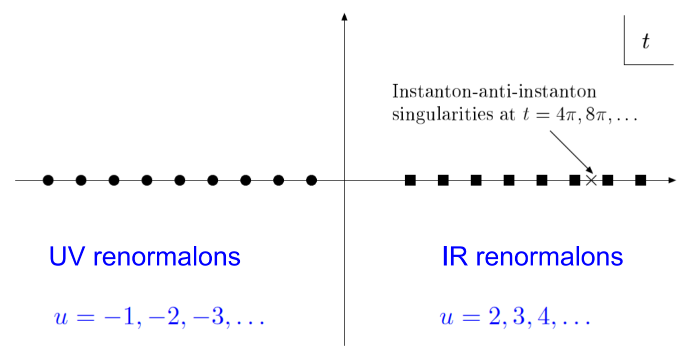
Borel representation and Borel sum:

(inverse Borel transform)

$$\hat{D}(s) = \int_0^\infty {\rm d}u\; B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0\alpha_s(-s)}}$$

Some regularization needed: PV prescription

(IR cutoff)



Beneke „Renormalons“

Borel Function Model Studies

- Apparent convergence of CIPT and FOPT series
- Discrepancy larger than suggested by individual series
- Motivated studies of Borel models for higher orders

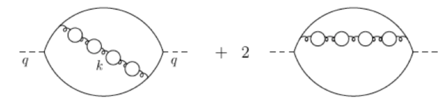
Beneke, Jamin 0806.3156
Jamin hep-ph/0509001
Caprini, Fischer 0906.5211
Descotes-Genon, Malaescu 1002.2968
Beneke, Jamin, Boito 1210.8038

$$B[\hat{D}]_{\rm model}(u) \;=\; B^{\rm IR}(u) + B^{\rm UV}(u) + B^{\rm ana}(u)$$

Types of IR renormalons singularities fixed by OPE.

Coefficients of IR renormalons cannot be fixed from first principles in full QCD $\longrightarrow$ model-dependence

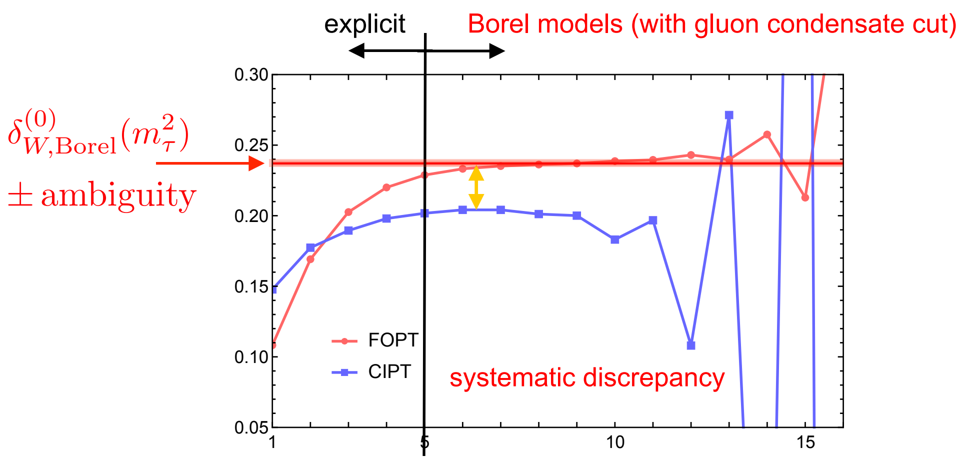
Exact results possible in „large-β_0 approximation“

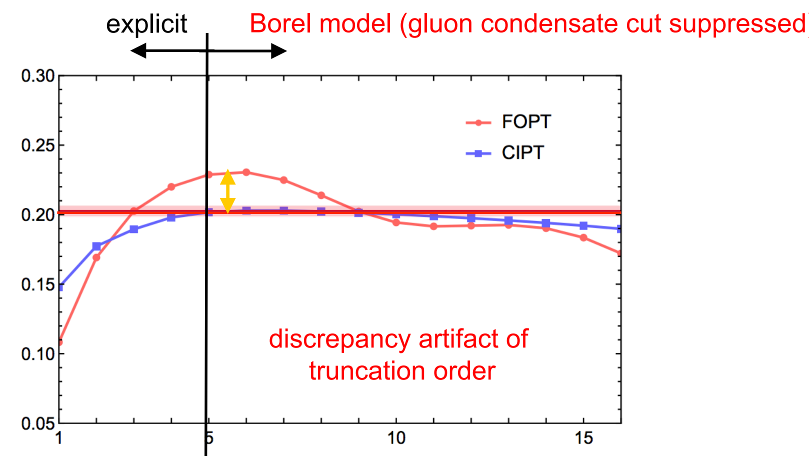


Borel Sum: $\delta^{(0)}_{W,\rm Borel}(s_0) = {\rm PV}\int_0^\infty {\rm d}u\;\frac{1}{2\pi i}\oint_{|x|=1}\frac{{\rm d}x}{x}\,W(x)\,B[\hat{D}](u)\,e^{-\frac{u}{a(-x)}}$

Discrepancy Systematic? Accidental?

Quantifyable? Predictable?





CIPT vs. FOPT: Questions

We are not interested in which kinds of Borel models are more realistic!

Let us start from any Borel function model compatible with the OPE!

Questions we want to address:

- How can it happen that CIPT and FOPT “converge” to different values?
- Why does FOPT converge to the Borel sum, while CIPT does not for some Borel models.
- Is the Borel representation and Borel sum unique?
- Can one predict the CIPT-FOPT discrepancy for a given Borel model?
- Implications for α_s determinations?

Answers [our work]:

1) The CIPT and FOPT Borel representations are in general different.
2) The discrepancy between CIPT and FOPT can be computed for any given model.

3) OPE corrections for CIPT and FOPT do not agree !
4) OPE corrections for CIPT are not standard !

FOPT vs. CIPT Borel Representation

Renormalon calculus:

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n \quad \Longrightarrow \quad B[\hat{D}](u) \sim \sum_{n=1}^{\infty} \frac{c_{n,1}}{\Gamma(n)}\, u^{n-1} \quad \Longrightarrow \quad B[\hat{D}](u) \sim \frac{1}{(p-u)^\gamma} + \frac{1}{(\tilde{p}+u)^\gamma}$$

$$\Longrightarrow \qquad \hat{D}_{\rm Borel}(s) \;=\; \int_0^\infty \mathrm{d}u\; B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(-s)}}$$

FOPT approach (large-β_0): (more complicated in full QCD, outcome the same)

$$\delta^{(0),\rm FOPT}_{W_i}(s_0) \;=\; \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \underbrace{\sum_{k=1}^{n+1} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x)\, \ln^{k-1}(-x)}_{\text{coefficient}} \qquad \underbrace{B[\hat{D}](u)\, e^{-u\ln(-x)}}_{\text{Summed u-Taylor series}}\, e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}} \;=\; B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0\alpha_s(-x s_0)}}$$

$$\overbrace{e^{-u\ln(-x)}\, e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}}}^{e^{-\frac{4\pi u}{\beta_0\alpha_s(-x s_0)}}}$$

➡ previously known Borel representation = FOPT Borel representation

$$\delta^{(0),\rm FOPT}_{W_i,\rm Borel}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\; \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x)\, B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0\alpha_s(-x s_0)}}$$

FOPT vs. CIPT Borel Representation

Renormalon calculus:

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n \quad \Longrightarrow \quad B[\hat{D}](u) \sim \sum_{n=1}^{\infty} \frac{c_{n,1}}{\Gamma(n)} u^{n-1} \quad \Longrightarrow \quad B[\hat{D}](u) \sim \frac{1}{(p-u)^\gamma} + \frac{1}{(\tilde{p}+u)^\gamma}$$

$$\Longrightarrow \qquad \hat{D}_{\mathrm{Borel}}(s) = \int_0^\infty \mathrm{d}u \; B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(-s)}}$$

CIPT approach:

Complex-valued coupling is not the expansion parameter

$$\delta_{W_i}^{(0),\mathrm{CIPT}}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\pi}\right)^n = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \underbrace{c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)^n}_{\text{coefficient}},$$

➡ CIPT Borel representation: NEW !

$$\delta_{W_i,\mathrm{Borel}}^{(0),\mathrm{CIPT}}(s_0) = \int_0^\infty \mathrm{d}\bar{u} \, \frac{1}{2\pi i} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right) B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right) e^{-\frac{4\pi \bar{u}}{\beta_0 \alpha_s(s_0)}}$$

FOPT vs. CIPT Borel Representation

FOPT Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{1}{2\pi i}\oint\limits_{|x|=1}\frac{\mathrm{d}x}{x}\,W_i(x)\,B[\hat{D}](u)\,e^{-\frac{4\pi u}{\beta_0\alpha_s(-xs_0)}}$$

CIPT Borel representation

$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint\limits_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,W_i(x)\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right)e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

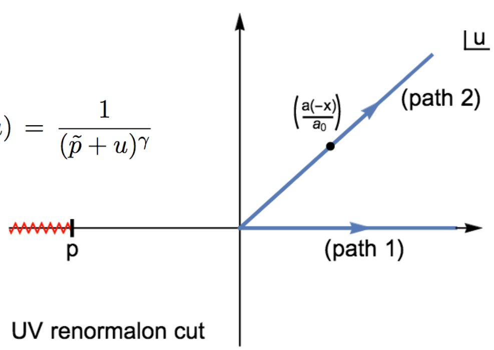
- Related through change of variables

$$u = \frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}$$

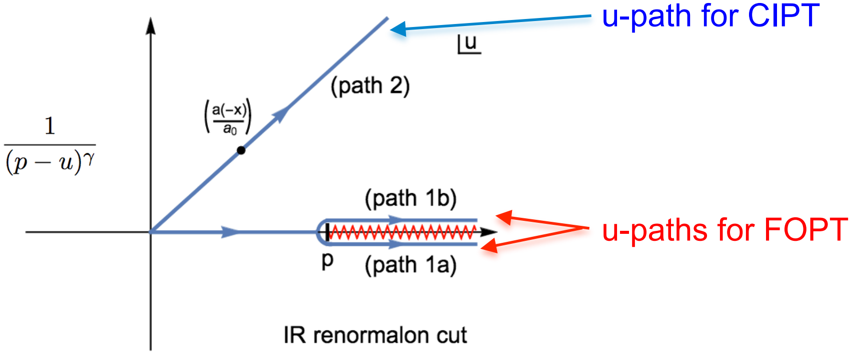
Complex number!

- Equivalent in perturbation theory (u-Taylor series)
- Different in presence of IR renormalon cuts

$$B^{\mathrm{UV}}_{\hat{D},-\tilde{p},\gamma}(u) = \frac{1}{(\tilde{p}+u)^\gamma}$$



$$B^{\mathrm{IR}}_{\hat{D},p,\gamma}(u) = \frac{1}{(p-u)^\gamma}$$



UV renormalons:

FOPT and CIPT Borel representations equivalent because closing up paths 1 and 2 does not contain cuts

IR renormalons: finite difference !

FOPT and CIPT Borel representations inequivalent

- FOPT: PV prescription needs to be imposed
- CIPT: automatically well-defined by complex-valued α_s
- Difference because closing paths 1a/1b and 2 always contains cuts

Asymptotic Separation

The difference between the CIPT and FOPT Borel representations can be computed analytically!

Generic IR renormalon contribution: $B^{\mathrm{IR}}_{\hat{D},p,\gamma}(u) = \dfrac{1}{(p-u)^{\gamma}} \quad \Longleftrightarrow \quad \langle \mathcal{O}_{2p} \rangle$

One can do u-integral first

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

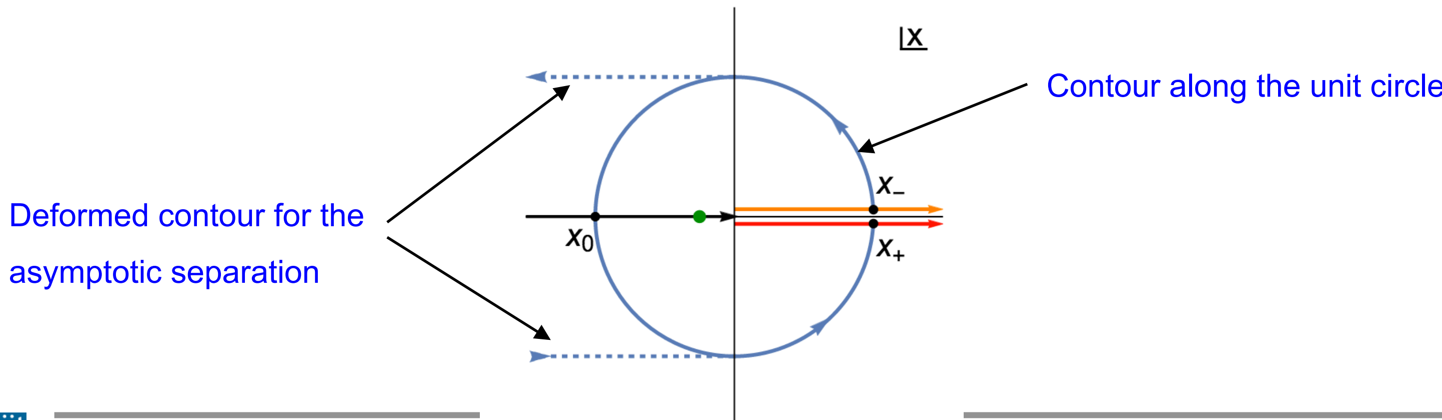
$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x}\, (-x)^m \,\mathrm{sig}[\mathrm{Im}[a(-x)]]\, (a(-x))^{1-\gamma}\, e^{-\frac{p}{a(-x)}} \,.$$

„Asymptotic Separation“

Cut along the negative real s-axis!

Power-suppressed $\sim \left(\dfrac{\Lambda^2_{\mathrm{QCD}}}{s}\right)^p$

Remaining contour integration must be deformed (to negative real infinity in the x-plane)



Asymptotic Separation

The difference between the CIPT and FOPT Borel representations can be computed analytically!

Generic IR renormalon contribution: $$B^{\rm IR}_{\hat{D},p,\gamma}(u) = \frac{1}{(p-u)^\gamma} \quad \Longleftrightarrow \quad \langle \mathcal{O}_{2p} \rangle$$

One can do u-integral first

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\rm CIPT}_{\{(-x)^m,p,\gamma\},\rm Borel}(s_0) - \delta^{(0),\rm FOPT}_{\{(-x)^m,p,\gamma\},\rm Borel}(s_0)$$

$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{{\rm d}x}{x} \, (-x)^m \, \underset{\uparrow}{{\rm sig}}[{\rm Im}[a(-x)]] \, (a(-x))^{1-\gamma} \, \underset{\uparrow}{e^{-\frac{p}{a(-x)}}} \, .$$

„Asymptotic Separation“

Cut along the negative real s-axis!

Power-suppressed $\sim \left(\frac{\Lambda^2_{\rm QCD}}{s}\right)^p$

Properties:

- Renormalization scheme invariant
- Much larger than canonical FOPT Borel sum ambiguity estimate if the Borel model has a sizeable gluon condensate cut
- Fully analytic results

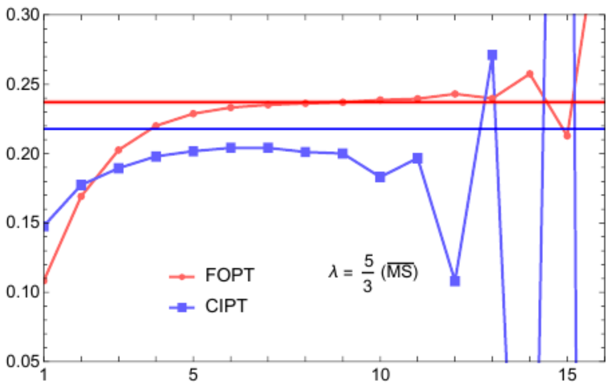
Numerical Tests

Full QCD: Tau decay rate R_τ

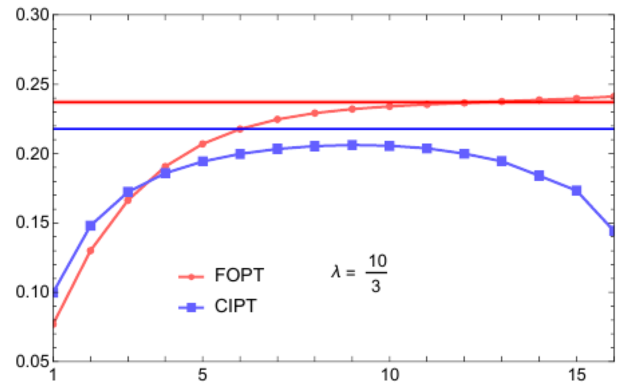
(Beneke/Jamin Borel Model, with gluon cond. cut)

$$W_\tau(x) = (1-x)^3(1+x) = 1 - 2x + 2x^3 - x^4$$

- Updated to 5-loop precision



(a) $\delta^{(0)}_{W_\tau}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{\overline{\mathrm{MS}}}$, full β-function



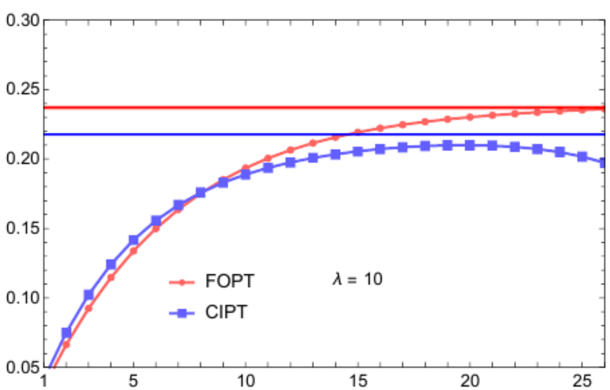
(b) $\delta^{(0)}_{W_\tau}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{(5/3)}$, full β-function

width of line

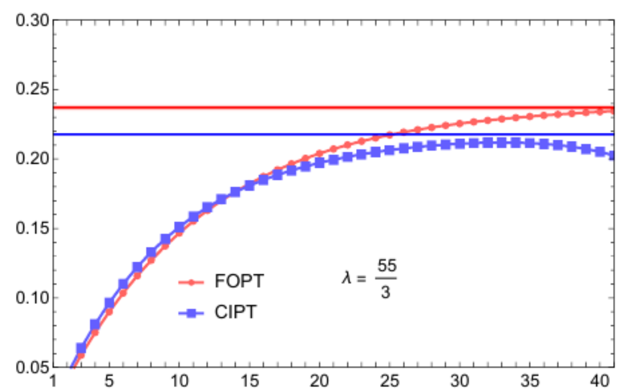
= FOPT Borel sum ambiguity

= renormalon ambiguity used

in previous literature



(c) $\delta^{(0)}_{W_\tau}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{(25/3)}$, full β-function



(d) $\delta^{(0)}_{W_\tau}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{(50/3)}$, full β-function

Agreement of CIPT series behavior with CIPT Borel sum can depend on the scheme.

Better agreement in schemes where $\alpha_s(m_\tau)$ is small.

Asymptotic separation provides quantitative description of CIPT-FOPT discrepancy for any given model !

Numerical Tests

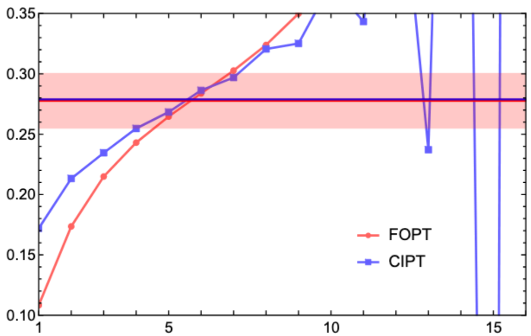
Spectral function moments with small asymptotic separation

(Beneke/Jamin Borel Model)

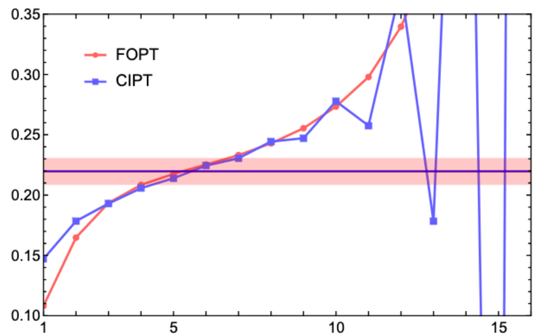
vanishing asymptotic separation from gluon condensate renormalon in large-β_0

$$W_c(x) = (1-x)^2(1+cx+x^2)$$

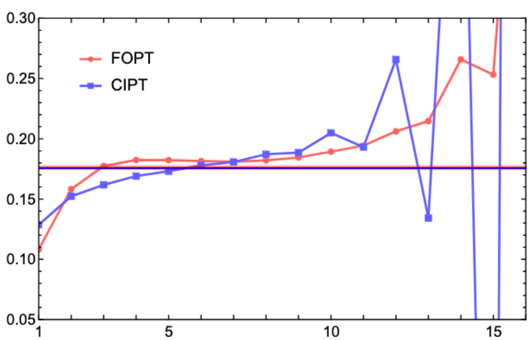
Spectral function moments with small CIPT-FOPT discrepancy can be designed.



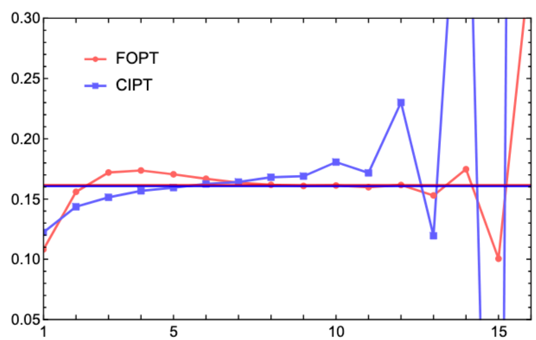
(a) $\delta^{(0)}_{W_{c=-1}}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{\overline{\mathrm{MS}}}$, full β-function



(b) $\delta^{(0)}_{W_{c=0}}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{\overline{\mathrm{MS}}}$, full β-function



(c) $\delta^{(0)}_{W_{c=0.75}}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{\overline{\mathrm{MS}}}$, full β-function

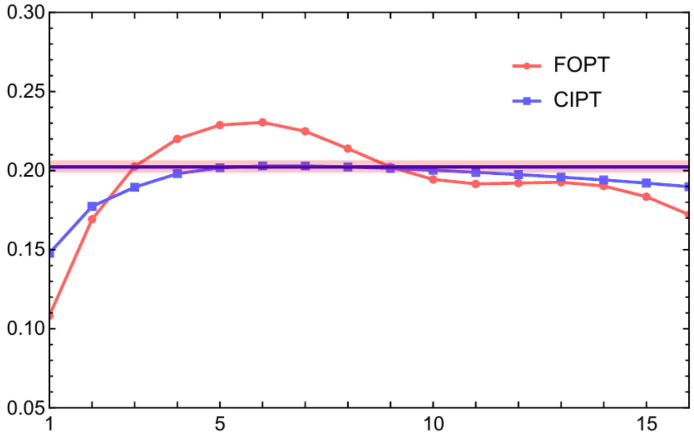


(d) $\delta^{(0)}_{W_{c=1}}(m_\tau^2)$, $B_{\hat{D},\mathrm{mr}}$, $\alpha_s^{\overline{\mathrm{MS}}}$, full β-function

Numerical Tests

Model with strongly suppressed gluon condesate cut

$$W_\tau(x) = (1-x)^3(1+x) = 1 - 2x + 2x^3 - x^4$$



(b) $\delta^{(0)}_{W_\tau}(m_\tau^2)$, $B_{\hat{D},p=3}$, $\alpha_s^{\overline{\rm MS}}$, full β-function

Asymptotic separation ≈ FOPT Borel sum ambiguity

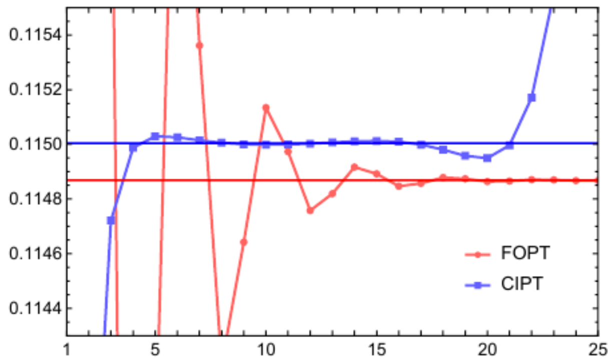
If the Borel function a suppressed gluon condensate cut, the CIPT-FOPT discrepancy is an artifact of the truncation order and may be reconciled by higher order corrections.

[Beneke, Jamin ‘2012: such models not plausible]

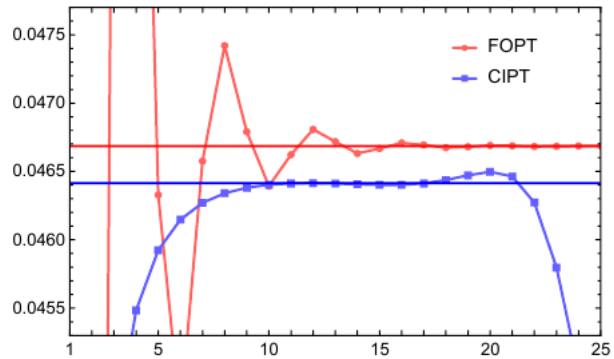
→ Asymptotic separation only relevant phenomenologically if the Borel function of the Adler function has a sizeable gluon condensate cut.

Numerical Tests

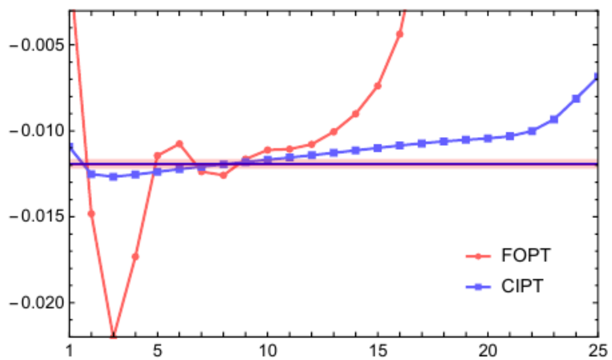
Single renormalon models (large-b0): $$B(u) = \frac{1}{(2-u)} \quad \Longleftrightarrow \quad \langle \alpha_s G^{\mu\nu} G_{\mu\nu} \rangle$$



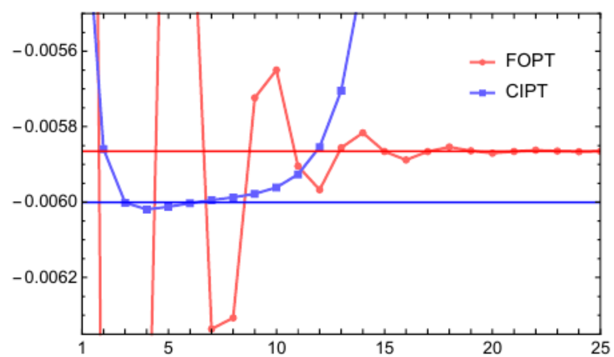
(a) Simple pole, p=2, $W(x) = 1$, large-β_0



(b) Simple pole, p=2, $W(x) = (-x)$, large-β_0



(c) Simple pole, p=2, $W(x) = (-x)^2$, large-β_0



(d) Simple pole, p=2, $W(x) = (-x)^4$, large-β_0

Excellent description of the asymptotic behavior of the CIPT series using the CIPT Borel representation.

Convergence behavior strongly depending on the power of the weight function.

Intriguing observation:

For moments with $W(x) = x^{m \neq 2}$

FOPT convergent series!

Gluon cond. corr. vanishes

CIPT series divergent!

(Apparently unnoticed in the literature)

→ CIPT expansion not compatible with standard OPE ?

Implications

What does the asymptotic separation mean?

- FOPT Borel representation: PV prescription needs to be imposed
- CIPT Borel representation: automatically well-defined by complex-valued α_s

Prescriptions represents different types of IR regularizations/cutoffs

➡ FOPT and CIPT do not have the same OPE corrections!

Asymptotic separation quantifies the difference of these OPE corrections.

➡ Difference must already exist at the level of the Adler function

FOPT and CIPT expansion of the Adler function

$$\hat{D}^{\rm CIPT}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

$$\hat{D}^{\rm FOPT}(s) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n} k\, c_{n,k}\, \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

⟺

$$\hat{D}^{\rm CIPT}_{\rm Borel}(s) = \int_0^\infty {\rm d}u\, \left(\frac{\alpha_s(-s)}{\alpha_s(|s|)}\right) B[\hat{D}]\left(\frac{\alpha_s(-s)}{\alpha_s(|s|)}\bar{u}\right) e^{-\frac{4\pi u}{\beta_0 \alpha_s(|s|)}}$$

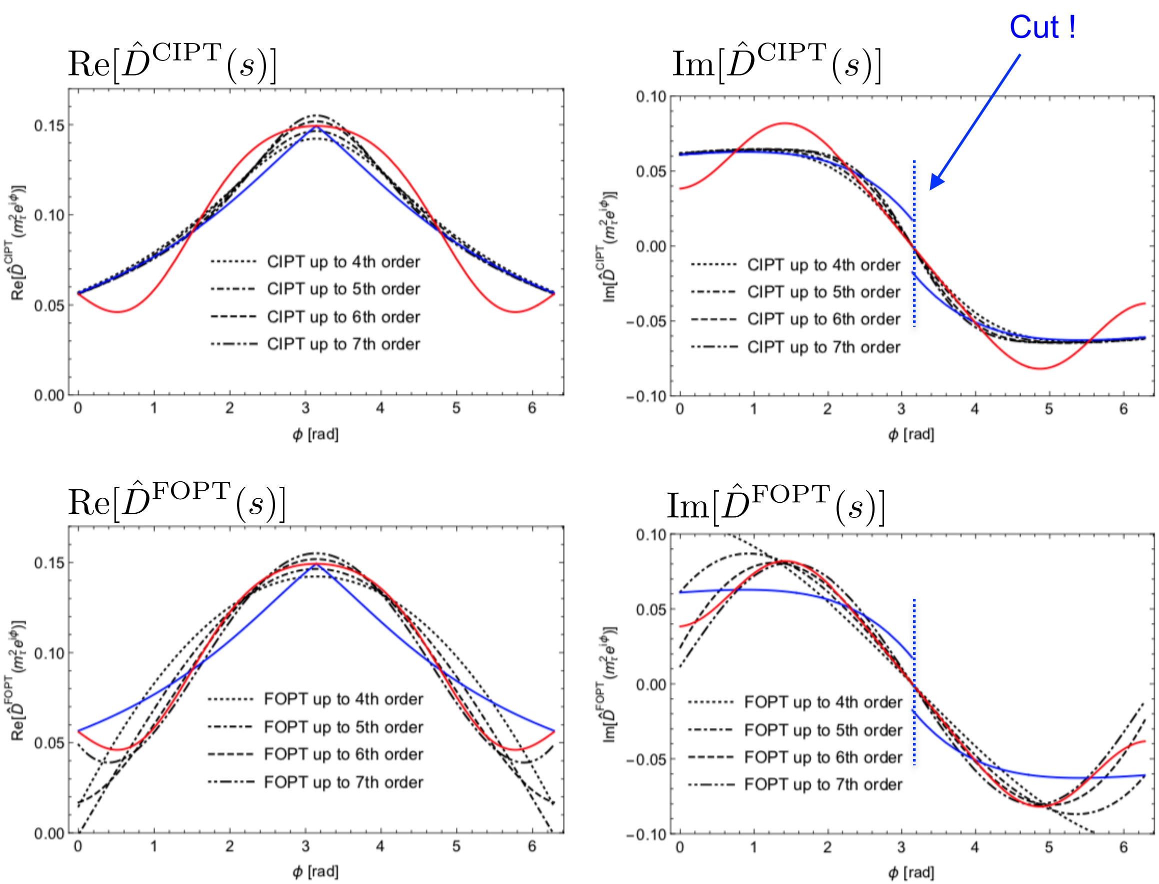
$$\hat{D}^{\rm FOPT}_{\rm Borel}(s) = {\rm PV}\int_0^\infty {\rm d}u\, B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(-s)}}$$

CIPT: expansion in $\alpha_s(-s)$

FOPT: expansion in $\alpha_s(|s|)$

FOPT and CIPT for the Adler Function

(Beneke/Jamin Borel Model)



FOPT Borel sum

CIPT Borel sum

CIPT-FOPT difference exists already for the Adler function

CIPT: expansion in $\alpha_s(-s)$

FOPT: expansion in $\alpha_s(|s|)$

CIPT expansion agrees with CIPT Borel sum

FOPT expansion agrees with FOPT Borel sum

CIPT Borel sum has cut along the negative real s-axis for any Borel model !

CIPT expansion appears not compatible with standard OPE in general !

(Standard OPE corrections cannot correct the unphysical cut)

Summary

- The use of FOPT and CIPT for the spectral function moments implies a different treatment of IR momenta.

- FOPT and CIPT Borel representations and their Borel sums differ
  $\longrightarrow$ "asymptotic separation" computable

- Discrepancy between FOPT and CIPT described well by asymptotic separation if 5-loop series is already asymptotic (~ gluon condensate renormalon large).

- Asymptotic separation can reconcile 5-loop CIPT-FOPT discrepancy if the Alder function Borel function has a large gluon condensate cut.

- CIPT Borel representation (and thus also CIPT) not compatible with standard OPE approach: difference to standard OPE = asymptotic separation