$\overline{\text{MS}}$-OS quark mass relation with two mass scales

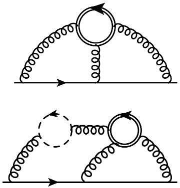
Radcor and LoopFest 2021, May 2021

Matthias Steinhauser | in collaboration with Matteo Fael, Fabian Lange, Kay Schönwald

TTP KIT

$\overline{\mathrm{MS}}$-OS quark mass relation with two mass scales

- Motivation
- Analytic 3-loop calculation
- A numerical approach
- Summary



$B \to X_c \ell \bar{\nu}$ to N^{3}LO

$$\Gamma(B \to X_c \ell \bar{\nu}) =$$

$$\Gamma_0 \left[X_0(m_c/m_b) + X_1(m_c/m_b) \frac{\alpha_s}{\pi} + X_2(m_c/m_b) \left(\frac{\alpha_s}{\pi}\right)^2 + X_3(m_c/m_b) \left(\frac{\alpha_s}{\pi}\right)^3 + \ldots \right]$$

- determination of $|V_{cb}|$
- $X_0 = 1 - 8\left(\frac{m_c}{m_b}\right)^2 - 24\left(\frac{m_c}{m_b}\right)^4 \log\left(\frac{m_c}{m_b}\right) + 8\left(\frac{m_c}{m_b}\right)^6 - \left(\frac{m_c}{m_b}\right)^8$
- X_3: [Fael,Schönwald,Steinhauser'20], talk by Kay Schönwald on Thursday (partial results: [Czakon,Dowling,Czarnecki'21])
- important scheme: m_b^{kin} and $\overline{m}_c$

Irreducible uncertainty of $m_t^{\rm pole}$

[Beneke,Nason,Marquard,Steinhauser'16] [Hoang,Lepenik,Preisser'16]

$$m_t^{\rm OS} = m_t^{\overline{\rm MS}}(\mu_m)\left(1+\sum_{n\geq 1} c_n\,\alpha_s^n(\mu)\right) \qquad c_n \underset{n\to\infty}{\longrightarrow} N\frac{\mu}{m(\mu_m)}\tilde{c}_n^{(\rm as)}$$

- IR renormalon predicts large-n behaviour: [Beneke,Braun'94; Beneke'94; Beneke'95]

$b = \beta_1/(2\beta_0)$

$$\tilde{c}_n^{(\rm as)} = (2\beta_0)^n\,\frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{s_1}{n+b}+\frac{s_2}{(n+b)(n+b-1)}+\cdots\right).$$

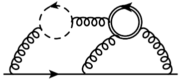
- exact 3- and 4-loop result + Borel summation ➪ N

[Marquard,Smirnov,Smirnov,Steinhauser,Wellmann'16]

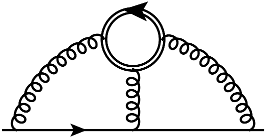
- typical loop momentum at $\mathcal{O}(\alpha_s^n)$: $m_t e^{-(n-1)}$ [Ball,Beneke,Braun'95]

$$\delta^{(5+)} m^{\rm OS} = 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \pm 0.009\,(\alpha_s) \pm 0.108\text{ (ambiguity) GeV}$$

- ± 0.108 GeV ➪ ± 0.71 GeV (no m_c, m_b effects)
- $\pm 0.030\,(m_{b,c})$ GeV from unknown light-mass effects at 4 loops



II. Analytic 3-loop 2-mass $\overline{\mathrm{MS}}$-OS



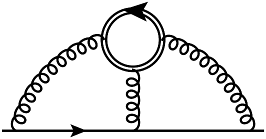
Canonical approach:

- IBP reduction; differential equations
- Fuchsian/cononical form [Henn'13,Lee'14,...]
- BCs
- (analytic) solution

Here:

- alternative approach to obtain solution in terms of iterated integrals
- no Fuchsian/cononical form
- `HarmonicSums` [Ablinger], `Sigma` [Schneider], `OreSys` [Gerhold'02]

MIs



- IBP reduction; differential equations
- decouple, factorize DEs
- BCs
- analytic solution

Iterated integrals:

$$J\left(\left\{a(\tau),\vec{b}(\tau)\right\},x\right)=\int_0^x \mathrm{d}\tau_1\; a(\tau_1)\; J\left(\left\{\vec{b}(\tau)\right\},\tau_1\right)$$

Here: $\frac{1}{\tau}$, $\frac{1}{1-\tau}$, $\frac{1}{1+\tau}$, $\frac{\sqrt{1-\tau^2}}{\tau}$, $\sqrt{1-\tau^2}$

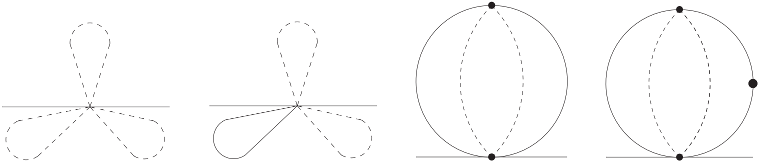
DEs

$$\vec{f}'(\epsilon, x) = \mathcal{M}(\epsilon, x) \cdot \vec{f}(\epsilon, x)$$

$x = m_2/m_1$
$\vec{f}$: vector with MIs

$$\mathcal{M}(\epsilon, x) = \begin{pmatrix} \times & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \times & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \times & \times & 0 & 0 & \dots \\ \times & 0 & \times & \times & 0 & 0 & \dots \\ 0 & \times & \times & \times & \times & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

DEs



$$\begin{pmatrix} f_1'(x) \\ f_2'(x) \\ f_3'(x) \\ f_4'(x) \end{pmatrix} = \begin{pmatrix} \frac{3(d-2)}{x} & 0 & 0 & 0 \\ 0 & \frac{2(d-2)}{x} & 0 & 0 \\ \frac{(d-2)^2}{2(d-3)x^3} & 0 & \frac{2(2d-5)}{x} & \frac{4}{x} \\ -\frac{3(d-2)^2}{8x^3(1-x^2)} & \frac{(2-d)^2}{4x(1-x^2)} & -\frac{(3-d)(3d-8)}{4x(1-x^2)} & \frac{1-(2d-5)x^2}{x(1-x^2)} \end{pmatrix} \cdot \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}$$

Solving differential equations

First order DE:

$$\left[\frac{\partial}{\partial x} - p(x)\right] f(x) = g(x)$$

Solution:

$$f(x) = f_p(x) + c_1 h(x)$$

$$h(x) = \exp\left(\int_0^x \mathrm{d}y p(y)\right) \qquad f_p(x) = h(x) \int_0^x \mathrm{d}y \frac{g(y)}{h(y)}$$

Solving differential equations

$$\left(\frac{\partial}{\partial x} - p_n(x)\right) \dots \left(\frac{\partial}{\partial x} - p_1(x)\right) f(x) = 0$$

is solved by

$$h_i(x) = \exp\left(\int_0^x \mathrm{d}y p_i(y)\right)$$

$$f_n(x) = h_1(x) \int_0^x \mathrm{d}x_1 \frac{h_2(x_1)}{h_1(x_1)} \int_0^{x_1} \mathrm{d}x_2 \frac{h_3(x_2)}{h_2(x_2)} \dots \int_0^{x_{n-2}} \mathrm{d}x_{n-1} \frac{h_n(x_{n-1})}{h_{n-1}(x_{n-1})}$$

$$f_p(x) = \sum_{i=1}^{n} f_i(x) \int_0^x \mathrm{d}y \frac{g(y) W_i(y)}{W(y)}$$

W: Wronskian determinant

The complete solution reads:

$$f(x) = f_p(x) + \sum_{i=1}^{n} c_i f_i(x)$$

Comments

- Solve order-by-order in ϵ
- decouple DE, factorizes DE
- homogenous equation does not depend on ϵ order
- BCs: $x \gg 1$ ($m_2 \gg m_1$)
- Pro: no need of special basis
- Con: integration in general harder
- Successfully applied in [Ablinger,Behring,Blümlein,De Freitas,von Manteuffel,Schneider'15; ... + Falcioni,Marquard,Rana'17; Ablinger,Blümlein,Marquard,Rana,Schneider'18]

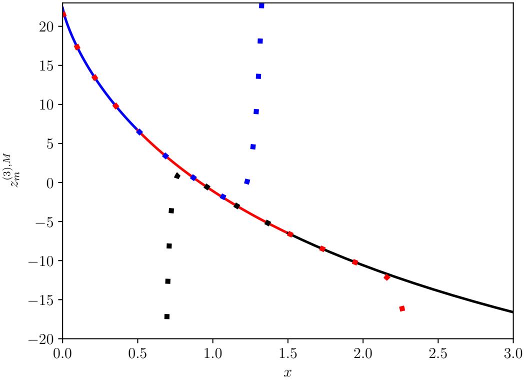
Example: $1/\epsilon^2$ term of f_3:

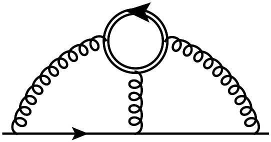
$$f_3''^{,(-2)} - \frac{2(3-4x^2)}{x(1-x^2)} f_3'^{,(-2)} + \frac{2(5-9x^2)}{x^2(1-x^2)} f_3^{(-2)} = \frac{3-26x^2+2x^4}{1-x^2} + \frac{4x^2(4-3x^2)}{1-x^2} H_0(x)$$

$\Leftrightarrow$

$$\left(\frac{\partial}{\partial x} - \frac{2(2-x^2)}{x(1-x^2)}\right)\left(\frac{\partial}{\partial x} - \frac{2(1-3x^2)}{x(1-x^2)}\right) f_3^{(-2)} = \frac{3-26x^2+2x^4}{1-x^2} + \frac{4x^2(4-3x^2)}{1-x^2} H_0(x)$$

Result: $z_m = \overline{m}_1 / m_1^{\mathrm{OS}}$





analytic expansions

$x \to 0$

$x \to 1$

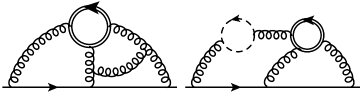
$1/x \to 0$

- $x = m_2/m_1$
- analytic result including $\mathcal{O}(\epsilon)$
- weight 5 (ϵ^0); weight 6 (ϵ^1)
- $x \to 0$ expansion + numerical evaluation: [Bekavac,Grozin,Seidel,Steinhauser'07]
- 2 loops: [Gray,Broadhurst,Grafe,Schilcher'90]

Z_m^{OS}, ϵ^0

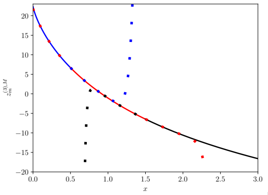
```
... cR^2*I2R*nm*(((120*x + 20*Pi^2*x - 384*x^3 - 64*Pi^2*x^3 + 408*x^5 +
     68*Pi^2*x^5 - 144*x^7 - 24*Pi^2*x^7 + 27*Pi^3*Sqrt[1 - x]*Sqrt[1 + x] + 18*Pi^3*Sqrt[1 - x]*x^4*Sqrt[1 + x])*
    GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL]}, x])/(27*Sqrt[1 - x]*Sqrt[1 + x]) -
  (4*(6 + Pi^2)*(3 + 2*x^4)*GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL]}, x]^2)/9 +
  ((8 + Pi^2)*(2 + 3*x^4)*GL[{(Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL}, x]^2)/
   6 + (16*Sqrt[1 - x]*(-1 + x)*x*(1 + x)^(3/2)*(-5 + 6*x^2)*
    GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL], VarGL^(-1)}, x])/9 +
  (8*Sqrt[1 - x]*(-1 + x)*x*(1 + x)^(3/2)*(-5 + 6*x^2)*
    GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL], (1 - VarGL)^(-1), VarGL^(-1)}, x])/
   9 + (16*Sqrt[1 - x]*(-1 + x)*x*(1 + x)^(3/2)*(-5 + 6*x^2)*
    GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL], VarGL^(-1), VarGL^(-1)}, x])/9 -
  (8*Sqrt[1 - x]*(-1 + x)*x*(1 + x)^(3/2)*(-5 + 6*x^2)*
    GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL], (1 + VarGL)^(-1), VarGL^(-1)}, x])/
   9 - (32*(3 + 2*x^4)*GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL],
      Sqrt[1 - VarGL]*Sqrt[1 + VarGL], VarGL^(-1)}, x])/3 -
  (4*Sqrt[1 - x]*(-1 + x)*(1 + x)^(3/2)*(5 + 9*x^2)*
    GL[{(Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL, (1 - VarGL)^(-1),
      VarGL^(-1)}, x])/9 + (4*Sqrt[1 - x]*(-1 + x)*(1 + x)^(3/2)*(5 + 9*x^2)*
    GL[{(Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL, (1 + VarGL)^(-1),  VarGL^(-1)}, x])/9 -
  (16*(3 + 2*x^4)*GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL],
      Sqrt[1 - VarGL]*Sqrt[1 + VarGL], (1 - VarGL)^(-1), VarGL^(-1)}, x])/3 -
  (32*(3 + 2*x^4)*GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL],
      Sqrt[1 - VarGL]*Sqrt[1 + VarGL], VarGL^(-1), VarGL^(-1)}, x])/3 +
  (16*(3 + 2*x^4)*GL[{Sqrt[1 - VarGL]*Sqrt[1 + VarGL],
      Sqrt[1 - VarGL]*Sqrt[1 + VarGL], (1 + VarGL)^(-1), VarGL^(-1)}, x])/3 +
  (4*(2 + 3*x^4)*GL[{(Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL,
      (Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL, (1 - VarGL)^(-1), VarGL^(-1)},
     x])/3 - (4*(2 + 3*x^4)*GL[{(Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL,
      (Sqrt[1 - VarGL]*Sqrt[1 + VarGL])/VarGL, (1 + VarGL)^(-1), VarGL^(-1)},
     x])/3 + (5*(1 + x)^2*(1 - x + x^2)*H[-1, x]^2*H[0, x])/72 -
  (5*x^4*H[0, x]^3)/4 + ((-1 + x)*(1 + x)*(1 + x^2)*H[0, -1, x]^2)/3 -
  ((-1 + x)^2*(-1303 - 3760*x - 3703*x^2 - 4800*x^3 - 1824*x^4 + 1152*x^5 +
     576*x^6)*H[0, 1, x])/432 + ((-1 + x)*(1 + x)*(1 + x^2)*H[0, 1, x]^2)/3 +
  H[0, -1, x]*(((1 + x)*(-1303 - 144*Pi^2 + 2457*x + 144*Pi^2*x + 57*x^2 -
```

III. 4-loop integrals with 2 mass scales

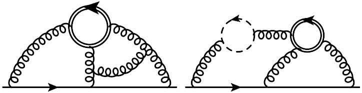


- elliptic sectors
- complicated analytic expression
- iterated integrals of high weight
- numerical evaluation tedious; can be time consuming

➪ Idea: (numerical) expansion around kinematic limits (e.g.: $x \to 0, 1, \infty$)

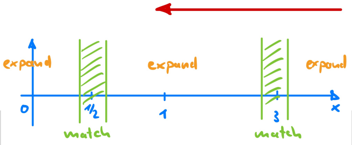


III. 4-loop integrals with 2 mass scales

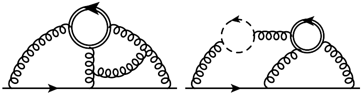


Algorithm

1. Reduction to master integrals; establish differential equations (DE).
2. Compute BCs
3. (Power-log) ansatz for MIs; expand DEs for $x \gg 1$; use BCs to fix constants
4. (Power-log) ansatz for MIs; expand DEs for $x \approx 1$
5. Numerical matching for $x = x_1$
6. Repeat procedure for expansion around $x = 0$ and $x = 1$

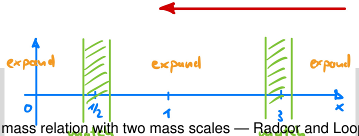


III. 4-loop integrals with 2 mass scales



Algorithm

1. Reduction to master integrals; establish differential equations (DE).
   Kira [Klappert,Lange,Maierhöfer,Usovitsch'20], FireFly [Klappert,Klein,Lange'20]
2. Compute BCs $x = m_2/m_1$; $x \gg 1$
3. (Power-log) ansatz for MIs; expand DEs for $x \gg 1$; use BCs to fix constants only even powers; 50 expansion terms
4. (Power-log) ansatz for MIs; expand DEs for $x \approx 1$ simple Taylor expansion
5. Numerical matching for $x = x_1$ $x_1 = 3$; 50 expansion terms
6. Repeat procedure for expansion around $x = 0$ and $x = 1$ matching for $x_0 = 1/2$



Other approaches ...

...based on expansions, numerical approaches, etc.:

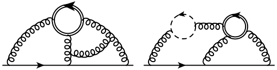
[Laporta'01; ...; Liu,Ma,Wang'17; Blümlein,Schneider'17]

[Lee,Smirnov,Smirnov'17]: `DESS.m`

Differences: Our approach ...

- ...does not require special form of differential equation
- ...provides (numerical) expansion around kinematic limits
  ➪ approximation in whole kinematic range
- ...can reproduce log-divergent behaviour
- ...is applied to physical quantity

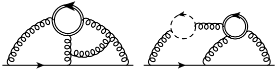
z_m



[Fael,Lange,Schönwald,Steinhauser]

- $\overline{m}_1 = z_m(\mu) m_1^{\rm OS}$
- $$z_m = C_F T^3 n_m n_l^2 z_m^{FMLL} + C_F T^3 n_m n_l n_h z_m^{FMLH} + C_F T^3 n_m n_h^2 z_m^{FMHH} + C_F T^3 n_m^2 n_l z_m^{FMML} + C_F T^3 n_m^2 n_h z_m^{FMMH} + C_F T^3 n_m^3 z_m^{FMMM} + C_F^2 T^2 n_m n_l z_m^{FFML} + C_F C_A T^2 n_m n_l z_m^{FAML}$$
  $+$ 8 further colour structures involving n_m
  $+$ 23 further colour structures without n_m
- reduction to MIs: `Kira` [Klappert,Lange,Maierhöfer,Usovitsch'20] and `FireFly` [Klappert,Klein,Lange'20]
- $>$ 300 4-loop 2-scale MIs
- analytic results (for all 8 colour structures) ➪ cross check
- $\approx$ 10 digits for $0 \leq x < \infty$
- $x = 0$ ➪ agreement with
  [Lee,Marquard,Smirnpv,Smirnov,Steinhauser'13; Marquard,Smirnov,Smirnov,Steinhauser,Wellmann'16]
- $x = 1$: new analytic results; agreement with numeric results from
  [Marquard,Smirnov,Smirnov,Steinhauser,Wellmann'16]

z_m



[Fael,Lange,Schönwald,Steinhauser]

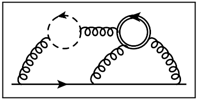
- $\overline{m}_1 = z_m(\mu) m_1^{\mathrm{OS}}$
- $$\begin{aligned} z_m = {} & C_F T^3 n_m n_l^2 z_m^{FMLL} + C_F T^3 n_m n_l n_h z_m^{FMLH} + C_F T^3 n_m n_h^2 z_m^{FMHH} + C_F T^3 n_m^2 n_l z_m^{FMML} \\ & + C_F T^3 n_m^2 n_h z_m^{FMMH} + C_F T^3 n_m^3 z_m^{FMMM} + C_F^2 T^2 n_m n_l z_m^{FFML} + C_F C_A T^2 n_m n_l z_m^{FAML} \\ & + 8 \text{ further colour structures involving } n_m \\ & + 23 \text{ further colour structures without } n_m \end{aligned}$$
- reduction to MIs: `Kira` [Klappert,Lange,Maierhöfer,Usovitsch'20] and `FireFly` [Klappert,Klein,Lange'20]
- $>$ 300 4-loop 2-scale MIs
- analytic results (for all 8 colour structures) ➪ cross check

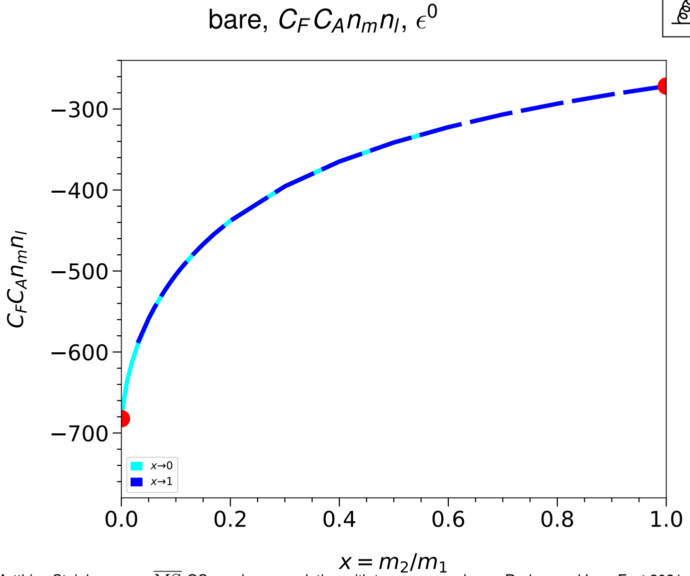
$$\begin{aligned} z_m^{FAML}(x=1) = {} & -\frac{6250177}{248832} + \frac{89\pi^6}{3780} + \frac{25\log^4(2)}{18} - \frac{\log^5(2)}{45} + \pi^4\left(\frac{5633}{25920} + \frac{49\log(2)}{1080} + \frac{\log^2(2)}{12}\right) + \frac{100}{3}\mathrm{Li}_4(1/2) + \frac{8}{3}\mathrm{Li}_5(1/2) \\ & + \frac{4777\zeta(3)}{288} - \frac{11\zeta(3)^2}{16} + \frac{61\zeta(5)}{12} + \pi^2\left[\frac{15649}{15552} + \frac{16}{9}C - \frac{535\log(2)}{54} + \frac{245\log^2(2)}{54} + \frac{\log^3(2)}{27} - \frac{\log^4(2)}{12} - 2\mathrm{Li}_4(1/2) \right. \\ & \left. + \left(-\frac{47}{24} - \frac{7\log(2)}{4}\right)\zeta(3)\right] \end{aligned}$$

$C = \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.915966\ldots$

Examples

[Fael,Lange,Schönwald,Steinhauser]



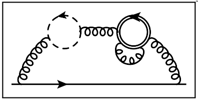


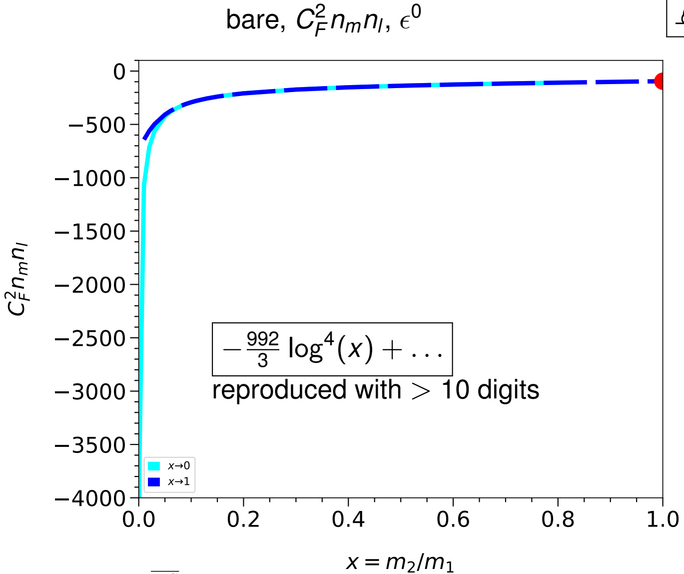
shown:
numeric expansion

perfect agreement
with analytic result

Examples

[Fael,Lange,Schönwald,Steinhauser]



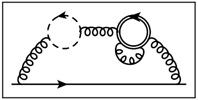


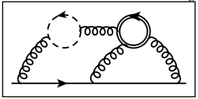
shown:
numeric expansion

perfect agreement
with analytic result

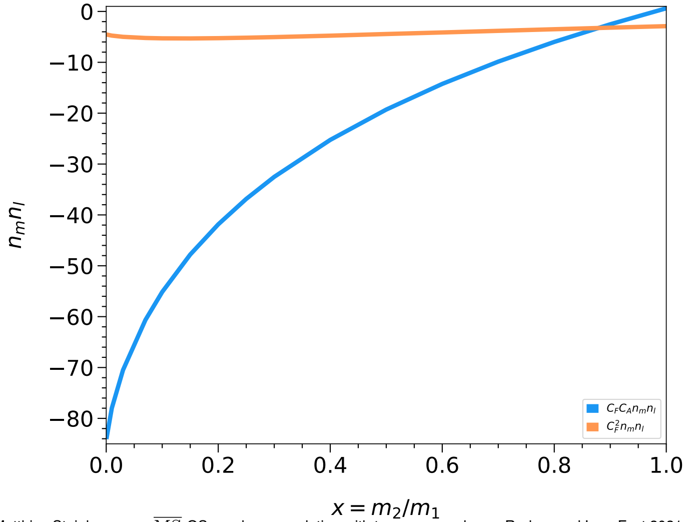
Examples

[Fael,Lange,Schönwald,Steinhauser]





renormalized, $C_F C_A n_m n_l$ and $C_F^2 n_m n_l$

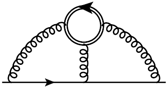


shown:
numeric expansion

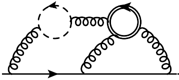
perfect agreement
with analytic result

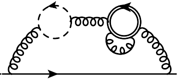
Summary

KIT
Karlsruhe Institute of Technology



- Analytic 3-loop 2-scale calculation
- Method for computation of multi-loop 2-scale integrals
  – (analytic) BCs
  – no special demands on differential equations





- $\overline{\mathrm{MS}}$-OS at 4 loops with $m_2 \neq 0$
  – analytic calculation for 8 colour factors
  – $\approx$ 10 digits agreement with numerical approximation
- other applications . . .


LOOPS AND LEGS IN QUANTUM FIELD THEORY
16th Workshop on Elementary Particle Physics,
Ettal, Germany, April 25 - 30, 2022
Programme committee:
Z.Bern UCLA
J. Blümlein DESY
R. Bonciani Sapienza
J. Fujimoto KEK
T. Gehrmann U Zurich
E.W.N. Glover Durham
J. Gracey Liverpool
W. Hollik MPI Munich
D. Kreimer HU Berlin
M. Misiak Warsaw
S. Moch Hamburg
L. Reina FSU
G. Rodrigo Valencia
C. Schneider Linz
M. Steinhauser KIT
Y. Schröder Bio-Bio
D. Stöckinger Dresden
J.A.M. Vermaseren Nikhef
Organizing committee:
P. Marquard
M. Steinhauser
www.desy.de/LL2022
supported by
. Photo: © M. Fend