The Kinetic Mass of Heavy Quarks at Three Loops

RADCOR-LoopFest 2021

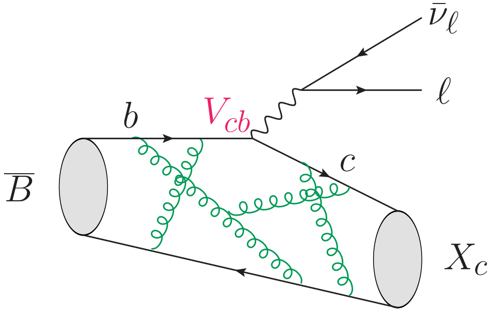
Matteo Fael | May 18, 2021

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS - KIT KARLSRUHE

based on Fael, Schönwald, Steinhauser, PRL 125 (2020) 052003, JHEP 10 (2020) 087

Inclusive semileptonic B-meson decays

- $|V_{cb}^{\mathrm{inc}}| = (42.19 \pm 0.78) \times 10^{-3}$
  HFLAV 2019
- Global fits in the **kinetic scheme**
  Bigi et al, PRD 56 (1997) 4017;
  Gambino, Schwanda, Phys.Rev.D 89 (2014) 1, 014022
  Alberti, Gambino, Healey, Nandi, Phys.Rev.Lett. 114 (2015) 6, 061802.
- Also 1S scheme
  Hoang, Ligeti, Manohar, Phys.Rev.Lett. 82 (1999) 277
  Bauer, Ligeti, Manohar, Trott, Phys.Rev.D 70 (2004) 094017



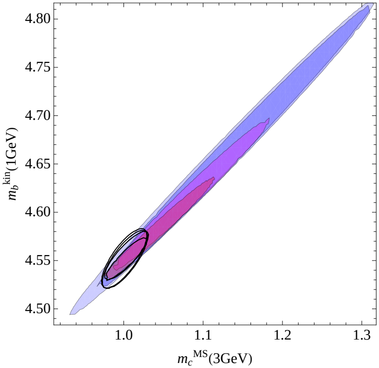
The heavy-quark expansion

$$\Gamma_{\text{sl}} = \mathcal{C}_0 + \mathcal{C}_{\mu_\pi} \frac{\mu_\pi^2}{m_b^2} + \mathcal{C}_{\mu_G} \frac{\mu_G^2}{m_b^2} + \mathcal{C}_{\rho_D} \frac{\rho_D^3}{m_b^3} + \mathcal{C}_{\rho_{LS}} \frac{\rho_{LS}^3}{m_b^3} + \ldots$$

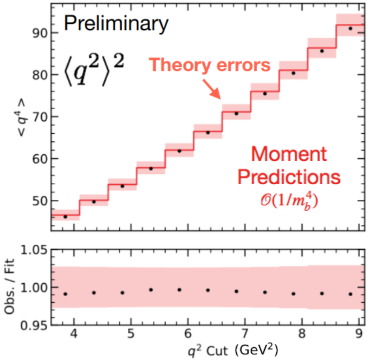
- Wilson coefficients $\mathcal{C}_i$ are calculable in pQCD
- Non-perturbative HQE parameters $\mu_\pi, \mu_G, \rho_D, \rho_{LS} \sim \langle B| \mathcal{O}_i |B\rangle$.
- Global fits determine $|V_{cb}|, m_b, m_c$ and the HQE parameters.

Why three-loop correction to m_b^{kin}?

- Precision predictions need short mass schemes for better convergence of α_s series.
- Precise conversion between $\overline{m}_b$ and m_b^{kin}.
- Improve the SM prediction by including α_s^3 corrections in pQCD.



Gambino, Schwanda, Phys.Rev.D 89 (2014) 1



Belle coll. R. van Tonder, hep-ex/2105.08001
based on **MF, Mannel, Vos, JHEP 02 (2019) 177**

The heavy-quark expansion

	tree	α_s	α_s^2	α_s^3	
1	✓	✓	✓	**NEW**	Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77; Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015. **MF, Schönwald, Steinhauser, hep-ph:2011.13654**;
$1/m_b^2$	✓	✓	!		Alberti, Gambino, Nandi, JHEP 1401 (2014) 147; Mannel, Pivovarov, Rosenthal, PRD 92 (2015) 054025. Becher, Boos, Lunghi, JHEP 0712 (2007) 062.
$1/m_b^3$	✓	✓			Mannel, Pivovarov, PRD 100 (2019) 9.
$1/m_b^{4,5}$	✓				Dassinger, Mannel, Turczyk, JHEP 0703 (2007) 087, JHEP 1011 (2010) 109 **MF, Mannel, Vos, JHEP 02 (2019) 177, JHEP 12 (2019) 067.**
$\overline{m}_b - m_b^{\mathrm{kin}}$		✓	✓	**NEW**	Bigi et al, PRD 56 (1997) 4017; Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189. **MF, Steinhauser, Schönwald, PRL 125 (2020) 5; PRD 103 (2021) 014005**

The heavy-quark expansion

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K. Schönwald's talk on Thursday
see also M. Steinhauser and A. Czarnecki talks

	tree	α_s	α_s^2	α_s^3
1	✓	✓	✓	**NEW**
$1/m_b^2$	✓	✓	!	
$1/m_b^3$	✓	✓		
$1/m_b^{4,5}$	✓			
$\overline{m}_b - m_b^{\mathrm{kin}}$		✓	✓	**NEW**

Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77;
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A short distance mass for $B \to X_c \ell \bar{\nu}_\ell$

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\rm OS})^5}{192\pi^3} f(0.25) \left[1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3\right] + O\left(\frac{1}{m_b^2}\right)$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301; Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

- Mass scheme change: $m_b^{\rm OS} \to \tilde{m}_b \left(1 + c\,\frac{\alpha_s}{\pi}\right)$

$$\Gamma_{\rm sl} \propto (\tilde{m}_b)^n \left[1 + (nc + a_1)\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{n(n+1)}{2}c^2 + nc\,a_1 + a_2\right)\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right]$$

- Can we resum the power enhanced $(n\alpha_s)^k$ terms (with $n = 5$)?

Meson-quark mass relation

$$m_b = M_B - \bar{\Lambda} - \frac{\mu_\pi^2}{2m_b} + \dots$$

- $\bar{\Lambda}$: the B-meson binding energy.
- μ_π: the kinetic energy induced by the residual motion of the heavy quark.

The relevant parameter in Γ_{sl} is **m_b^5, not M_B^5**:

$$\Gamma_{\text{sl}} \simeq \frac{G_F^2 |V_{cb}|^5}{192\pi^3} (M_B - \bar{\Lambda})^5$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017

The kinetic mass

$$m_b^{\rm kin}(\mu) = m_b^{\rm OS} - [\overline{\Lambda}(\mu)]_{\rm pert} - \frac{[\mu_\pi^2(\mu)]_{\rm pert}}{2 m_b^{\rm kin}(\mu)} - \dots$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017.
see also: Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189;
Gambino, JHEP 09 (2011) 055;

- In pQCD, we can *peel off* the IR renormalon from the on-shell mass identifying:

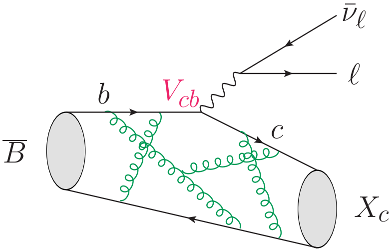
$$m_b(\mu) \to m_b^{\rm kin}(\mu) \qquad \overline{M}_B \to m_b^{\rm OS}$$

$$\overline{\Lambda}(\mu) \to [\overline{\Lambda}(\mu)]_{\rm pert} \qquad [\mu_\pi^2(\mu)] \to [\mu_\pi^2(\mu)]_{\rm pert}$$

The Small Velocity Sum Rules

- How to give an operative definition of $\overline{\Lambda}$ and μ_π^2?
- Moments of the excitation energy:

$$I_n(\vec{q}^2) = \int \mathrm{d}\omega \, \omega^n \frac{d\Gamma}{d\omega d\vec{q}^2}$$



with $\omega = E_{X_c} - M_D$ and $q = p_\ell + p_\nu$

The Small Velocity Sum Rules

Take the limit where the X_c's velocity is small: $|\vec{v}| = |\vec{q}/m_c| \ll 1$:

$$I_0(\vec{q}^{\,2}) = |\vec{q}\,| \frac{G_F^2 |V_{cb}|^2}{8\pi^3} (m_b - m_c)^2 + O\left(|\vec{v}|^2, \frac{\Lambda_{\mathrm{QCD}}}{m_b} \right)$$

$$I_1(\vec{q}^{\,2}) = I_0 \frac{\vec{v}^{\,2}}{2} \overline{\Lambda} + O\left(|\vec{v}|^3, \frac{\Lambda_{\mathrm{QCD}}^2}{m_b^2} \right)$$

$$I_2(\vec{q}\,) = I_0 \frac{\vec{v}^{\,2}}{3} \mu_\pi^2 + O\left(|\vec{v}|^3, \frac{\Lambda_{\mathrm{QCD}}^3}{m_b^3} \right)$$

The Small Velocity Sum Rules

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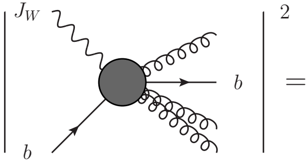
$$[\bar\Lambda(\mu)]_{\rm pert} = \lim_{\vec v\to 0}\ \lim_{m_b\to\infty}\ \frac{2}{\vec v^{\,2}}\ \frac{\int_0^\mu d\omega\, \omega\, W(\omega,\vec v)}{\int_0^\mu d\omega\, W(\omega,\vec v)}$$

$$[\mu_\pi^2(\mu)]_{\rm pert} = \lim_{\vec v\to 0}\ \lim_{m_b\to\infty}\ \frac{3}{\vec v^{\,2}}\ \frac{\int_0^\mu d\omega\, \omega^2\, W(\omega,\vec v)}{\int_0^\mu d\omega\, W(\omega,\vec v)}$$

- Consider only soft radiation $\Lambda_{\rm QCD} \ll \mu \ll m_b$



$= W(\omega, \vec v)$

The Small Velocity Sum Rules

$$[\overline{\Lambda}(\mu)]_{\text{pert}} = \lim_{\vec{v}\to 0} \lim_{m_b \to 0} \frac{2}{\vec{v}^{\,2}} \frac{\int_0^{\mu} d\omega\, \omega\, W(\omega, \vec{v})}{\int_0^{\mu} d\omega\, W(\omega, \vec{v})}$$

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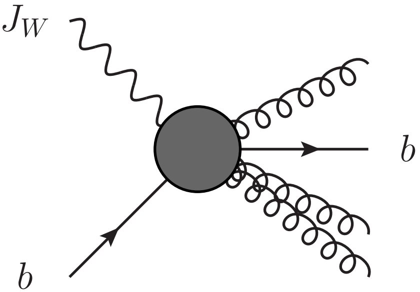
SV Limit $\leftrightarrow$ Threshold Limit

- Excite the heavy quark, but just a bit . . .

$$y = s - m_b^2 \simeq 2 m_b \omega \ll m_b^2$$

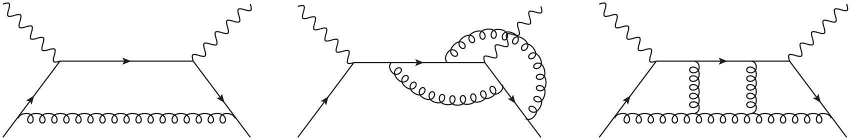


- SV Limit corresponds to **1 Particle Threshold limit**!
- Factorization:

$$W(\omega, \vec{v}) \simeq H \cdot U(\omega, \vec{v})$$

Ingredients for m_b^{kin} at $O(\alpha_s^3)$

- $W(\omega, \vec{v})$ up to $O(\alpha_s^3)$
- Imaginary part of forward scattering amplitudes:



$$W(\omega, \vec{v}) = W_{\mathrm{el}}(\vec{v}\,)\,\delta(\omega) + \frac{\vec{v}^2}{\omega} W_{\mathrm{real}}(\omega)\,\theta(\omega) + \mathcal{O}\left(v^4, \frac{\omega}{m_b}\right)$$

- Threshold expansion via *method of regions*: $y = s - m_b^2$.

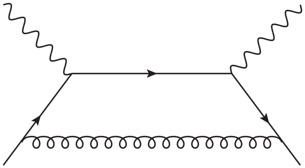
Beneke, Smirnov, NPB 522 (1998) 321; Smirnov Springer Tracts Mod. Phys. 250 (2010)

Expansion by Regions

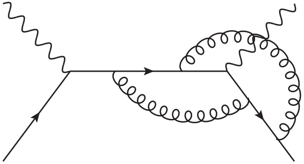
- For one heavy particle threshold, there are two regions:

  see also: Smirnov Springer Tracts Mod. Phys. 250 (2010)

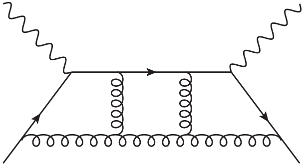
$$\text{hard (h): } k_i \sim m_b$$
$$\text{ultra-soft (us): } k_i \sim y/m_b$$



(u) (h)



(uu) (uh) (hh)



(uuu) (uuh) (uhh) (hhh)

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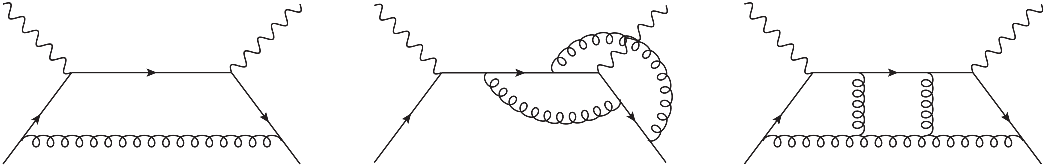

(u) ~~(h)~~ (uu) (uh) ~~(hh)~~ (uuu) (uuh) (uhh) ~~(hhh)~~

- **All hard** regions don't contribute: no imaginary part.
- After renormalization and decoupling $\alpha_s^{(n_l+n_h)} \to \alpha_s^{(n_l)}$, **only all ultra-soft part remains**

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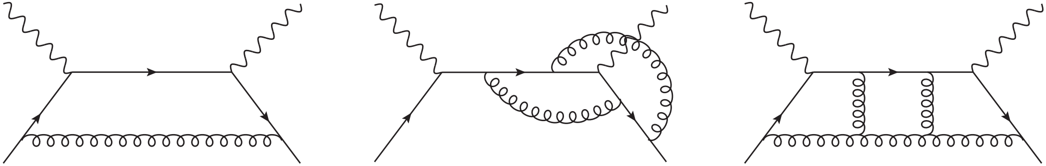

(u) ~~(h)~~ (uu) ~~(uh)~~ ~~(hh)~~ (uuu) ~~(uuh)~~ ~~(uhh)~~ ~~(hhh)~~

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For the statistics . . .

- 4, 66 and 1586 diagrams at one, two and three loops
- Partial fractioning and mapping between families implemented in `FORM` thanks to code `LIMIT`.

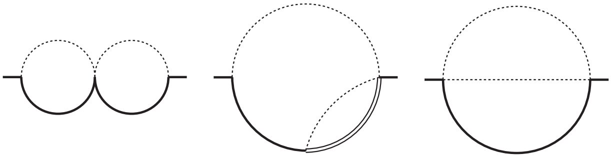
  Herren, PhD thesis, KIT, 2020

- `FIRE` and `LiteRed` reduction of integral families:

  Smirnov, Chuharev, hep-ph/1901.07808; Lee, hep-ph/1212.2685.

  - $\{2, 2\}$ in the $\{(uu), (uh)\}$ regions at two loops
  - $\{14, 4, 3\}$ in the $\{(uuu), (uuh), (uhh)\}$ regions at three loops
- **New master integrals:**
  - 3 in the (uu) region
  - 20 in the (uuu) region
- Heavy quark form factors up to $O(\alpha_s^2)$ (static limit)

  Lee, Smirnov, Smirnov, Steinhauser, JHEP 1805 (2018) 187; Blümlein, Marquard, Rana, PRD 99 (2019) 016013



I_1^{2l} I_2^{2l} I_3^{3l}

- The new master integrals contains linear-massive propagators:

$$I_2^{2l} = \int d^d k_1 d^d k_2 \frac{1}{k_1^2 (k_1 - k_2)^2 (2k_1 \cdot p - y)(2k_2 \cdot p)}$$

- Mellin-Barnes method

  `MB` package
  Czakon, Comput. Phys. Commun. 175 (2006) 559;
  Smirnov[2], EPJC 62 (2009) 445.

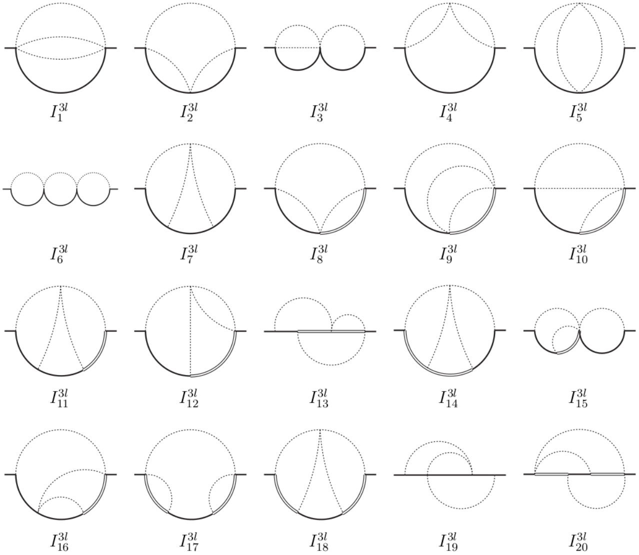
- PSLQ

- Analytic summation of residues

  HarmincSums
  www3.risc.jku.at/research/combinat/software/HarmonicSums/

- Differential equations in auxiliary variable

  Kotikov, PLB 254 (1991), 158
  Gehrmann, Remiddi, NPB 580 (2000) 485
  Henn, PRL 110 (2013), 251601.



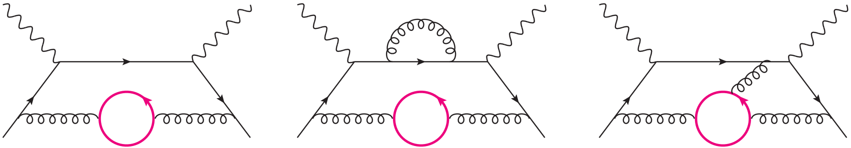
The kinetic mass

$$
\begin{aligned}
\frac{m^{\mathrm{kin}}}{m^{\mathrm{OS}}} &= 1 - \frac{\alpha_s^{(n_l)}}{\pi} C_F \left(\frac{4}{3}\frac{\mu}{m^{\mathrm{OS}}} + \frac{1}{2}\frac{\mu^2}{(m^{\mathrm{OS}})^2} \right) + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^2 C_F \left\{ \frac{\mu}{m^{\mathrm{OS}}} \left[C_A \left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9} l_\mu \right) + n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \right. \\
&\left. + \frac{\mu^2}{(m^{\mathrm{OS}})^2} \left[C_A \left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12} l_\mu \right) + n_l T_F \left(\frac{13}{18} - \frac{1}{3} l_\mu \right) \right] \right\} + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^3 C_F \left\{ \frac{\mu}{m^{\mathrm{OS}}} \left[C_A^2 \left(-\frac{130867}{1944} \right. \right. \right. \\
&+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27} \right) l_\mu - \frac{121}{27} l_\mu^2 \Big) + C_A n_l T_F \left(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3 \right. \\
&+ \left(-\frac{1654}{81} + \frac{8\pi^2}{27} \right) l_\mu + \frac{88}{27} l_\mu^2 \Big) + C_F n_l T_F \left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81} l_\mu - \frac{16}{27} l_\mu^2 \right) \Big] \\
&+ \frac{\mu^2}{(m^{\mathrm{OS}})^2} \left[C_A^2 \left(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36} \right) l_\mu - \frac{121}{72} l_\mu^2 \right) + C_A n_l T_F \left(\frac{13699}{1296} - \frac{23\pi^2}{54} \right. \right. \\
&- \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9} \right) l_\mu + \frac{11}{9} l_\mu^2 \Big) + C_F n_l T_F \left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27} l_\mu - \frac{2}{9} l_\mu^2 \right) \Big] \Big\}, \quad (4)
\end{aligned}
$$

MF, Schönwald, Steinhauser, PRL 125 (2020) 052003

- The mass relation is written in terms of $\alpha_s^{(n_l)}$.
- n_l = number of **massless quarks**, $l_\mu = \log(2\mu/\mu_s)$.

Charm mass effects: $0 < m_c$, $m_c \neq m_b$



- Charm mass effects to the $\overline{\mathrm{MS}}$-on-shell mass relation known up to $O(\alpha_s^3)$.
  Davydychev, Grozin, PRD 59 (1999) 054024; Broadhurst, Gray, Schlicher, Z. Phys. C 52 (1991) 111;
  Bekavac, Grozin, Seidel, Steinhauser, JHEP 10 (2007) 006; **MF, Schönwald, Steinhauser, JHEP 10 (2020) 087.**
- No charm mass effects known for the kinetic-on-shell mass relation.
- We assume $|y| \ll m_c^2, m_b^2$, i.e. no cut through a charm loop

The kinetic-$\overline{\rm MS}$ mass relation for bottom

- Input values:
  $\alpha_s^{(5)}(M_Z) = 0.1179$ $\quad \overline{m}_c(3\text{ GeV}) = 0.993\text{ GeV}$ $\quad \overline{m}_b(\overline{m}_b) = 4.163\text{ GeV}$
- The charm quark wants to be treated as ***heavy***.

scheme	$\alpha_s^{(n_r)}$	m_c in $\overline{\rm MS}$-OS	m_c in kin-OS	
(A)	3	✓	-	$m_b^{\rm kin}(1\text{ GeV}) = 4163 + 248 + 80 + 30 = 4520\text{ MeV}$
(B)	4	✓	✓	$4163 + 259 + 78 + 26 = 4526\text{ MeV}$
(C)	4	✓	✗	$4163 + 259 + 84 + 41 = 4547\text{ MeV}$
(D)	3	✗	✗	$4163 + 248 + 81 + 30 = 4521\text{ MeV}$

MF, Schönwald, Steinhauser, Phys.Rev.D 103 (2021) 014005

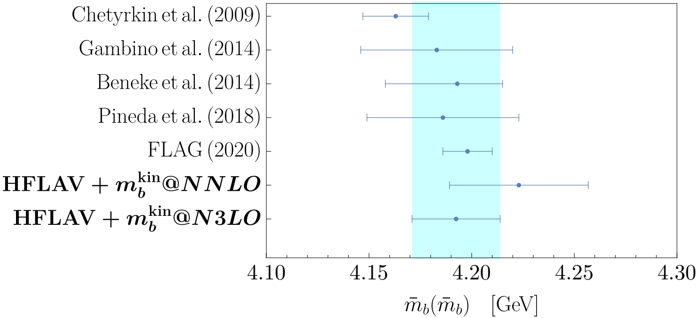
Scheme conversion uncertainty

- Half of α_s^3 correction

  $\delta m_b^{\text{kin}} \simeq 15$ MeV

- Large-β_0 term at α_s^4

  $\delta m_b^{\text{kin}} \simeq 8$ MeV



Compared with:

- scheme conversion uncertainty at two-loops: $\delta m_b^{\text{kin}} = 30$ MeV Gambino, JHEP 09 (2011) 055
- $m_b^{\text{kin}}(1\ \text{GeV}) = 4554 \pm 18$ MeV from $B \to X_c \ell \bar{\nu}_\ell$ global fit HFLAV 2019

Conclusions

- We computed the $O(\alpha_s^3)$ relation between the kinetic and the on-shell mass.
- We studied finite charm mass effects in m_b^{kin}: the charm *wants* to be heavy!
- Scheme conversion uncertainty is reduced by a factor of 2.
- Mass formula is necessary to improve $B \to X_c \ell \bar{\nu}$ prediction in the SM.
- Our results are crucial for future extractions of $|V_{cb}|$ and m_b from Belle-II data.

Spare

Let's include radiative corrections . . .

$$I_n(\vec{q}^2) = \int_{|\vec{q}|}^{q_0^{\mathrm{max}}} dq_0\, \omega^n \frac{d\Gamma_{\mathrm{tree}}}{dq_0\, d\vec{q}^2} + \int_{q_0^{\mathrm{max}}-\mu}^{q_0^{\mathrm{max}}} dq_0\, \omega^n \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2} + \int_{|\vec{q}|}^{q_0^{\mathrm{max}}-\mu} dq_0\, \omega^n \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2}$$

- We introduce a **Wilsonian cutoff** μ, with $\Lambda_{\mathrm{QCD}} \ll \mu \ll M_B$, to separate gluons with
  - $\omega < \mu$ that belong to the non-perturbative regime,
  - $\omega > \mu$ that can be described in pQCD.

Let's include radiative corrections . . .

$$I_1(\vec{q}^2) = \underbrace{\int_{|\vec{q}|}^{q_0^{\max}} dq_0\, \omega\, \frac{d\Gamma_{\text{tree}}}{dq_0\, d\vec{q}^2} + \int_{q_0^{\max}-\mu}^{q_0^{\max}} dq_0\, \omega\, \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2}}_{\text{use this to define in the SV limit } \overline{\Lambda}(\mu)} + \int_{|\vec{q}|}^{q_0^{\max}-\mu} dq_0\, \omega\, \frac{d\Gamma_{\alpha_s}}{dq_0\, d\vec{q}^2}$$

- We introduce a **Wilsonian cutoff** μ, with $\Lambda_{\mathrm{QCD}} \ll \mu \ll M_B$, to separate gluons with
  - $\omega < \mu$ that belong to the non-perturbative regime,
  - $\omega > \mu$ that can be described in pQCD.

m_b^{kin}(1 GeV) from $\overline{m}_b(\overline{m}_b)$

in MeV

scheme	$\alpha_s^{(n_r)}$	m_c in $\overline{\text{MS}}$-OS	m_c in kin-OS	
(A)	3	✓	–	$4163 + 248 + (81 + 7_{\Delta_{m_c}} + 12_{dec} - 20_{n_c}) + (30 + 14_{\Delta_{m_c}} + 16_{dec} - 30_{n_c} - 1_{n_c \times dec} + 0.4_{\Delta_{m_c} \times dec}) = 4163 + 248 + 80 + 30 = 4520$
(B)	4	✓	✓	$4163 + 259 + (88 + 7_{\Delta_{m_c}} + 5_{\Delta_{m_c}^{\text{kin}}} - 22_{n_c}) + (34 + 16_{\Delta_{m_c}} + 10_{\Delta_{m_c}^{\text{kin}}} - 34_{n_c}) = 4163 + 259 + 78 + 26 = 4526$

$m_b^{\text{kin}}(1\text{ GeV})$ from $\overline{m}_b(\overline{m}_b)$

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in MeV

	$\alpha_s^{(n_r)}$	m_c in $\overline{\text{MS}}$-OS	m_c in kin-OS	
(C)	4	✓	✗	$$\begin{aligned} &4163 + 259 + (99 + 7_{\Delta_{m_c}} - 22_{n_c}) \\ &+ (59 + 16_{\Delta_{m_c}} - 34_{n_c}) \\ &= 4163 + 259 + 84 + 41 = 4547 \end{aligned}$$
(D)	3	✗	✗	$4163 + 248 + 81 + 30 = 4521$

Charm quark mass

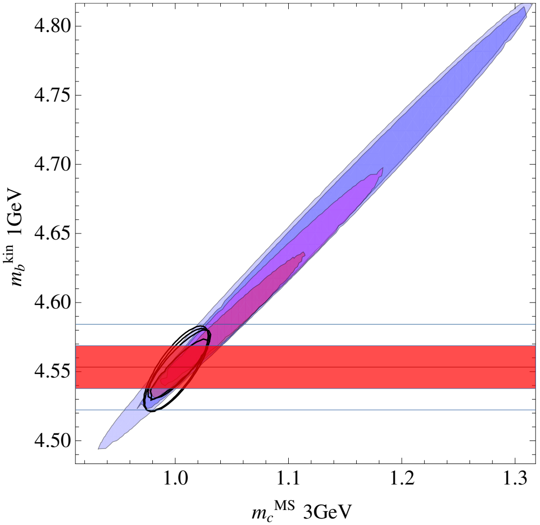
- $m_c^{\mathrm{kin}}(0.5\ \mathrm{GeV})$:

$$\begin{aligned} m_c^{\mathrm{kin}}(0.5\ \mathrm{GeV}) &= \ \ 993 + 191 + 100 + 52\ \mathrm{MeV} = 1336\ \mathrm{MeV}\,, \\ m_c^{\mathrm{kin}}(0.5\ \mathrm{GeV}) &= 1099 + 163 + \ \ 76 + 34\ \mathrm{MeV} = 1372\ \mathrm{MeV}\,, \\ m_c^{\mathrm{kin}}(0.5\ \mathrm{GeV}) &= 1279 + \ \ 84 + \ \ 30 + 11\ \mathrm{MeV} = 1404\ \mathrm{MeV}\,. \end{aligned}$$

- $m_c^{\mathrm{kin}}(1\ \mathrm{GeV})$:

$$\begin{aligned} m_c^{\mathrm{kin}}(1\ \mathrm{GeV}) &= \ \ 993 + 83 + 35 + 14\ \mathrm{MeV} = 1125\ \mathrm{MeV}\,, \\ m_c^{\mathrm{kin}}(1\ \mathrm{GeV}) &= 1099 + 37 + \ \ 2 - \ \ 3\ \mathrm{MeV} = 1135\ \mathrm{MeV}\,, \\ m_c^{\mathrm{kin}}(1\ \mathrm{GeV}) &= 1279 - 73 - 61 - 17\ \mathrm{MeV} = 1128\ \mathrm{MeV}\,, \end{aligned}$$

where from top to bottom $\mu_s = 3\ \mathrm{GeV}, 2\ \mathrm{GeV}$ and $\overline{m}_c$ for $\overline{m}_c(\mu_s)$ and $\alpha_s^{(3)}(\mu_s)$.



Original plot from

Gambino, Schwanda, PRD 89 (2014) 014022

Horizontal error bands superimposed by MF