arXiv:2101.00308[hep-ph]

Evaluation of two-loop EW box diagrams for $e^+e^- \to ZH$

Qian Song

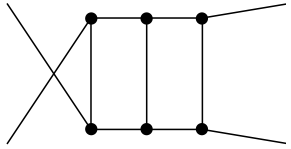
Ayres Freitas

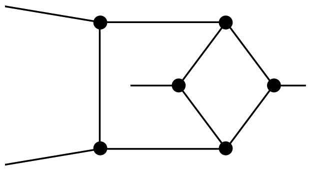
17 May, Monday

Content

- Introduction
- Evaluation method: planar box diagram

  non-planar box diagram
- Numerical result
- Summary



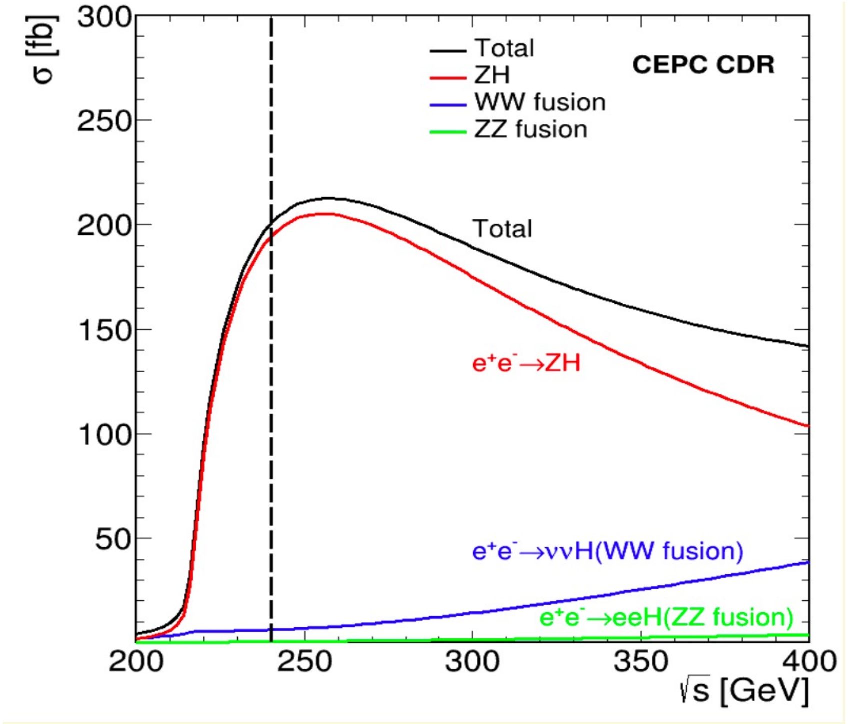


1. Introduction

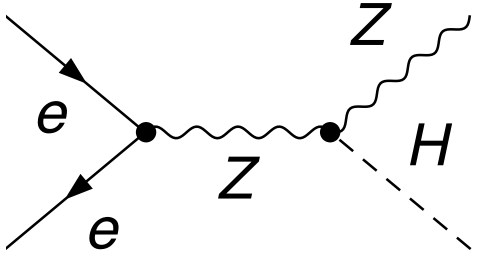
- Discovery of Higgs boson(2012,LHC): last fundamental particle in SM
- Experiments at the ATLAS and CMS: agrees with the result SM predicted
- Problems not solved: electroweak symmetry breaking, Higgs coupling to SM particles/DM, hierarchy problem... Require new physics beyond SM
- One promising way probing new physics: precision measurements of the properties of H
- LHC is difficult to reach very high precision due to complicated background

1. Introduction

- FCC-ee, CEPC, ILC: e+e- collider, large statistics, high luminosity, clean environment, measure H properties with very high precision ($\sqrt{s}$=240-250GeV)
- ILC: σ_{ZH}~1.2%, 250fb-1 (H. Baer et al. [arXiv:1306.6352 [hep-ph]])
- FCC-ee: σ_{ZH}~ 0.4%, 5ab-1(A.Abada et al[FCC Collaboration])
- CEPC: σ_{ZH}~ 0.5%, 5.6ab-1(arXiv:1811.10545)



1. Introduction



- LO on $\sigma(e^+e^- \to ZH)$:
  only consider s channel
  t,u channel amplitude is zero due to small Yukawa coupling
- NLO on $\sigma(e^+e^- \to ZH)$:
  unpolarized beam: 5-10%;
  (A. Denner et al,Phys. C 56, 261(1992))
  polarized beam: 10-20%;
  (S.Bondarenko, Phys. Rev. D 100,073002(2019))

$$\sigma(P_{e^-}, P_{e^+}) = \frac{1}{4}\sum_{\chi_1,\chi_2}(1+\chi_1 P_{e^+})(1+\chi_2 P_{e^-})\sigma_{\chi_1\chi_2},$$

TABLE I. Hard ($E_\gamma > 1$ GeV), Born, and one-loop cross sections in fb and relative corrections δ in % for the c.m. energy $\sqrt{s} = 250$ GeV and various polarization degrees of the initial particles.

P_{e^-}	P_{e^+}	σ^{hard}, fb	σ^{Born}, fb	$\sigma^{\text{one-loop}}$, fb	δ, %
0	0	82.0(1)	225.59(1)	206.77(1)	−8.3(1)
−0.8	0	96.7(1)	266.05(1)	223.33(2)	−16.1(1)
−0.8	−0.6	46.3(1)	127.42(1)	111.67(2)	−12.4(1)
−0.8	0.6	147.1(1)	404.69(1)	334.99(1)	−17.2(1)

1. Introduction

- NNLO:(s= 240-250GeV)

EW+QCD:0.4-1.3% ($\alpha(0), \alpha(M_z), G_\mu$)

(Q.F.Sun, Phys.Rev.D 96,051301(2017))

EW+QCD:1.3% ($\overline{MS}, \alpha(M_z)$)

(Q.F.Sun, Phys.Rev.D 96,051301(2017))

$\sqrt{s}$	schemes	$\sigma_{\rm LO}$ (fb)	$\sigma_{\rm NLO}$ (fb)	$\sigma_{\rm NNLO}$ (fb)
240	$\alpha(0)$	223.14 ± 0.47	229.78 ± 0.77	$232.21^{+0.75+0.10}_{-0.75-0.21}$
	$\alpha(M_Z)$	252.03 ± 0.60	$228.36^{+0.82}_{-0.81}$	$231.28^{+0.80+0.12}_{-0.79-0.25}$
	G_μ	239.64 ± 0.06	$232.46^{+0.07}_{-0.07}$	$233.29^{+0.07+0.03}_{-0.06-0.07}$
250	$\alpha(0)$	223.12 ± 0.47	229.20 ± 0.77	$231.63^{+0.75+0.12}_{-0.75-0.21}$
	$\alpha(M_Z)$	252.01 ± 0.60	$227.67^{+0.82}_{-0.81}$	$230.58^{+0.80+0.14}_{-0.79-0.25}$
	G_μ	239.62 ± 0.06	231.82 ± 0.07	$232.65^{+0.07+0.04}_{-0.07-0.07}$

1. Introduction

- NNLO:(s= 240-250GeV)

EW+QCD:0.4-1.3% ($\alpha(0), \alpha(M_z), G_\mu$)

(Q.F.Sun, Phys.Rev.D 96,051301(2017))

EW+QCD:1.3% ($\overline{MS}, \alpha(M_z)$)

(Q.F.Sun, Phys.Rev.D 96,051301(2017))

$\sqrt{s}$	schemes	σ_{LO} (fb)	σ_{NLO} (fb)	σ_{NNLO} (fb)
	$\alpha(0)$	223.14 ± 0.47	229.78 ± 0.77	$232.21^{+0.75+0.10}_{-0.75-0.21}$
240	$\alpha(M_Z)$	252.03 ± 0.60	$228.36^{+0.82}_{-0.81}$	$231.28^{+0.80+0.12}_{-0.79-0.25}$
	G_μ	239.64 ± 0.06	$232.46^{+0.07}_{-0.07}$	$233.29^{+0.07+0.03}_{-0.06-0.07}$
	$\alpha(0)$	223.12 ± 0.47	229.20 ± 0.77	$231.63^{+0.75+0.12}_{-0.75-0.21}$
250	$\alpha(M_Z)$	252.01 ± 0.60	$227.67^{+0.82}_{-0.81}$	$230.58^{+0.80+0.14}_{-0.79-0.25}$
	G_μ	239.62 ± 0.06	231.82 ± 0.07	$232.65^{+0.07+0.04}_{-0.07-0.07}$

TABLE II. Total cross sections at various collider energies in the $\alpha(m_Z)$ scheme.

$\sqrt{s}$ (GeV)	σ_{LO} (fb)	σ_{NLO} (fb)	σ_{NNLO} (fb)	σ_{NNLO}^{exp} (fb)
240	252.0	228.6	231.5	231.5
250	252.0	227.9	230.8	230.8
300	190.0	170.7	172.9	172.9
350	135.6	122.5	124.2	124.0
500	60.12	54.03	54.42	54.81

1. Introduction

- EW+QCD:0.4-1.3% ($\alpha(0), \alpha(M_z), G_\mu$) (Q.F.Sun, Phys.Rev.D 96,051301(2017))

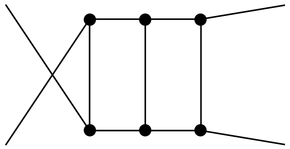
  EW+QCD:1.3% ($\overline{MS}, \alpha(M_z)$) (Q.F.Sun, Phys.Rev.D 96,051301(2017))
- EW+EW: ~1% (arXiv:1906.05379) (25377 diagrams(arXiv:2102.15213))

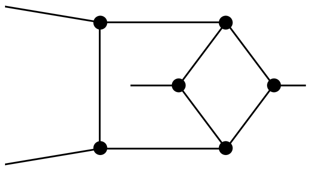
  challenging type: 2250 diagrams with 7 denominators, 4 independent mass scale, 2 independent energy scale

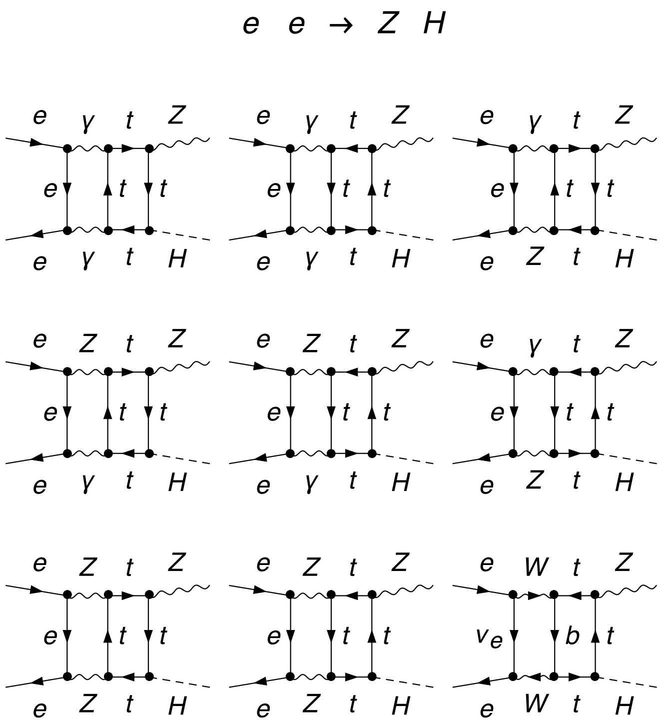
  diagrams with closed fermion loop dominant due to large top-quark Yukawa coupling and large number of fermions in SM

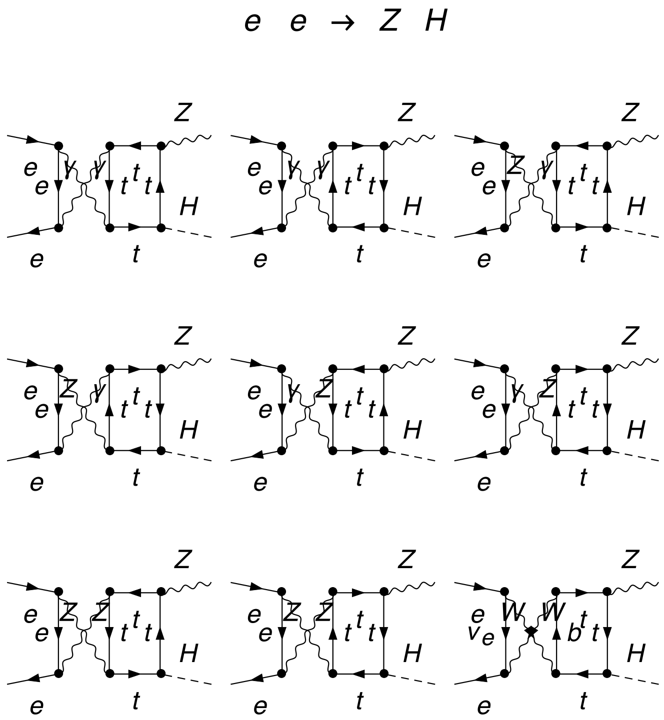
  → planar & Non-planar diagrams with closed top-quark loop (18+9)

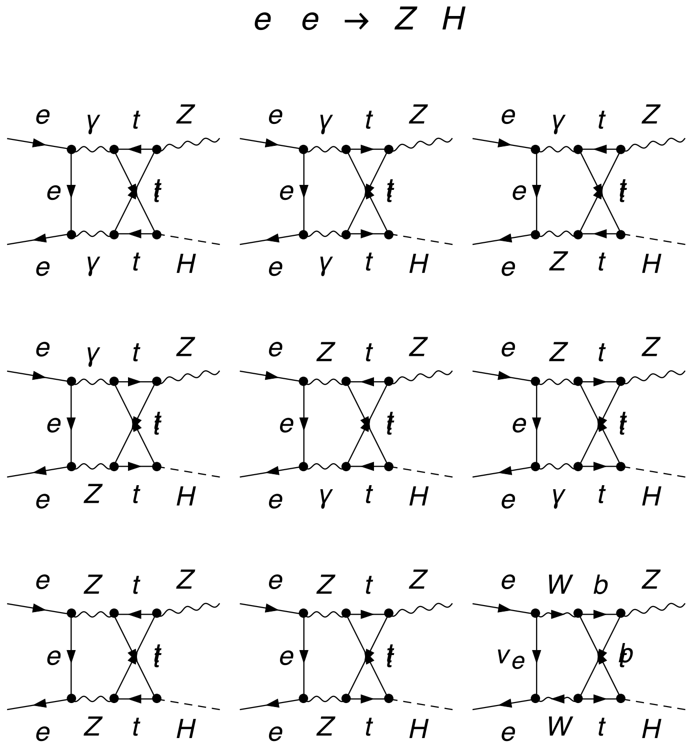
1. Introduction











Planar double-box diagrams

Non-planar double-box diagrams

1. Introduction

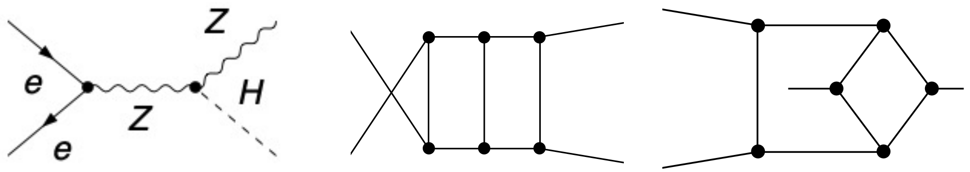
- Analytical calculation: can be done for 1-loop, but difficult in 2-loop: require more knowledge about special functions(harmonic polylogarithmic functions, iterated elliptic integrals)
- Numerical calculation: use Feynman parametrization. Box diagram is equal to integration over 6 Feynman parameters. It requires large computing resources and takes few days because the integrand converges slowly

(F. Yuasa et al; Comput. Phys. Commun. 183, 2136-2144 (2012))

$$I_{planar} = -\int_0^1 d\rho \int_0^1 d\xi \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^1 du_3 \int_0^{1-u3} du_4 \, \frac{\mathcal{C}}{(\mathcal{D} - i\epsilon\mathcal{C})^3} \, \rho^3 \xi^2 (1-\xi)^2,$$

- Our method: simplify the integrand with Feynman parametrization and dispersion relation. The box diagram is reduced to 3-fold integration, which takes few minutes to calculate.

1. Introduction



$$M_0 M_2^* = \iiint dx\,dy\,d\sigma \,(\ldots)$$

- Feynman diagrams → FeynArts (T. Hahn, Comput. Phys. Commun. 140, 418 (2001). [hep-ph/0012260])

- Square amplitude → FeynCalc (dim =4)

(V. Shtabovenko, R. Mertig and F. Orellana, "FeynCalc 9.3: New features and improvements", arXiv:2001.04407.)

- (…) → dispersion relation and Feynman parameterization(Mathematica)

- Numerical calculation → C++, LoopTools, Gauss-Kronrod quadrature In Boost package

(Comput.Phys.Commun.118(1999)153)

(https://www.boost.org/doc/libs/master/libs/math/doc/html/index.html)

2. Evaluation Method – planar diagram

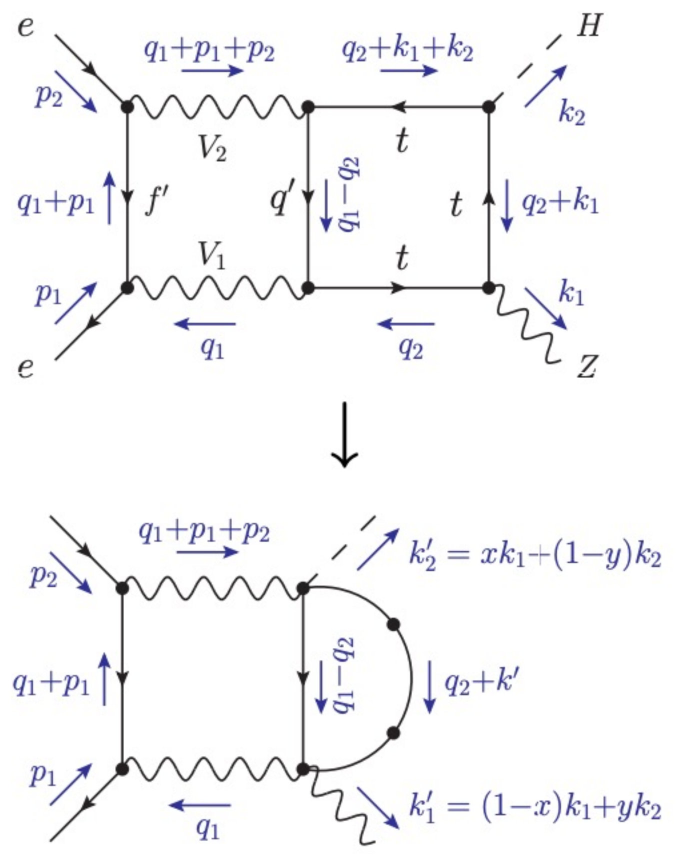
According to Feynman rules, the amplitude for planar diagram can be written as I_{plan}.
Use Feynman parametrization to simplify the denominators only involve q2

$$\text{Feynman parametrizaiton:} \frac{1}{abc} = \int_0^1 dx \int_0^{1-x} dy \frac{1}{(ax+by+c(1-x-y))^3}$$

$$I_{plan} = \int d_{q_1}^D d_{q_2}^D \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2}$$

$$\underbrace{\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}}_{\int_0^1 dx \int_0^{1-x} dy \frac{1}{((q_2+k')^2 - m'^2)^3} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \frac{1}{(q_2+k')^2 - m'^2}}$$

$$= \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int d_{q_1}^D \frac{B_0((q_1+k')^2, m_{q'}^2, m'^2)}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$



Loop momentum q1 appears in B0 functions, so cannot integrate over q1.
→ use dispersion relation to put q1 outside B0 function

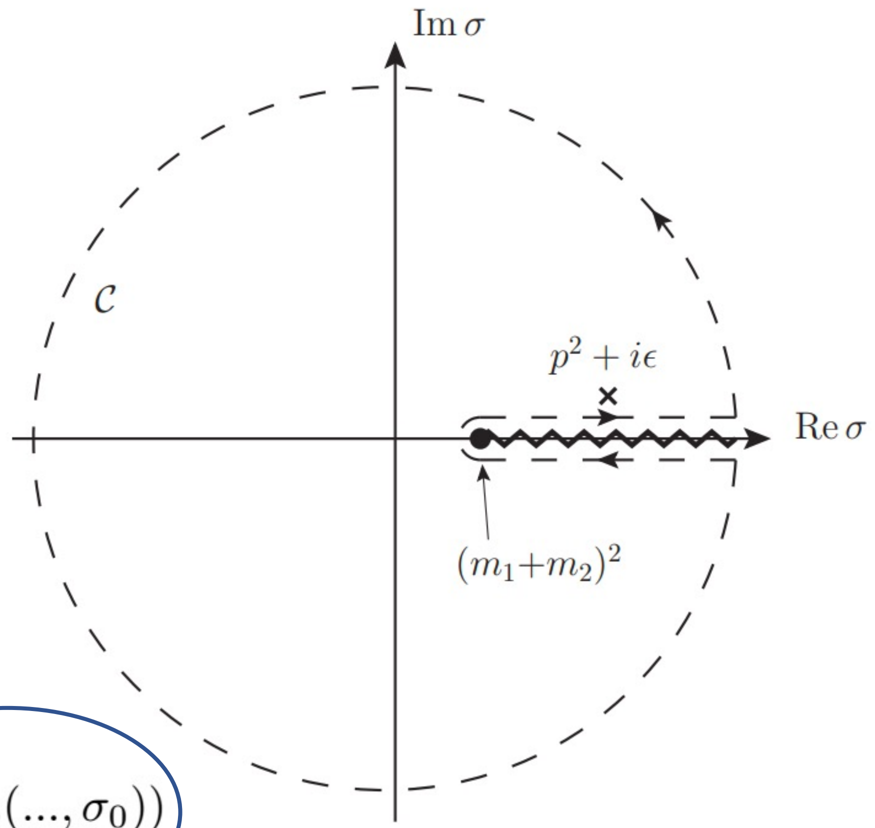
2. Evaluation Method – planar diagram

dispersion relation:

$$B_0(p^2,m_1^2,m_2^2) = \frac{1}{2\pi i}\oint_C d_\sigma \frac{B_0(\sigma,m_1^2,m_2^2)}{\sigma - p^2 - i\varepsilon}$$

$$= \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{\Delta B_0(\sigma,m_1^2,m_2^2)}{\sigma - p^2 - i\epsilon}$$

$$= \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{1}{\pi}\frac{ImB_0(\sigma,m_1^2,m_2^2)}{\sigma - p^2 - i\epsilon}$$



$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s,m'^2,m_{q'}^2) D_0(\ldots,\sigma)$$

$$= \int_0^1 dx \int_0^{1-x} dy \int_{(m'+m_{q'})^2}^{\infty} d\sigma \partial^2_{m'^2} \Delta B_0(s,m'^2,m_{q'}^2)\left(D_0(\ldots,\sigma) - \frac{\sigma_0}{\sigma} D_0(\ldots,\sigma_0)\right)$$

$$+ \int_0^1 dx \int_0^{1-x} dy \sigma_0 D_0(\ldots,\sigma_0) \partial^2_{m'^2} B_0(0,m'^2,m_{q'}^2)$$

cancel the divergence at lower bound

2. Evaluation Method – non-planar diagram

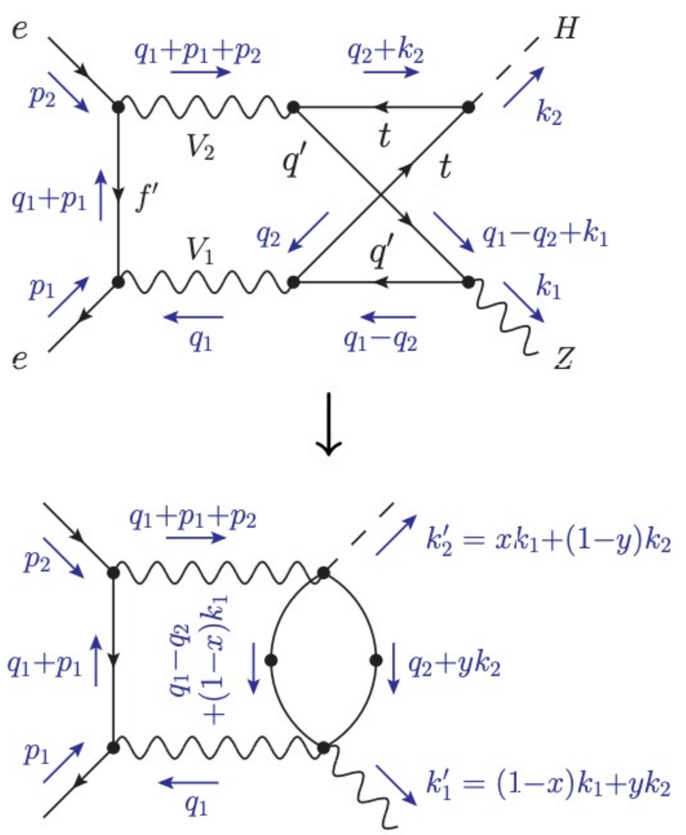
Similarly, use Feynman parametrization to simplify the denominators including q2 and integrating loop momentum q2 gives B0 function

$$I_{NP} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$

$$\frac{1}{\underbrace{((q_1-q_2)^2 - m_{q'}^2)((q_1-q_2+k_1)^2 - m_{q'}^2)}_{\int_0^1 dx \partial_{m'^2_1} \frac{1}{(q_1-q_2+(1-x)k_1)^2 - m'^2_1}} \underbrace{(q_2^2 - m_t^2)((q_2+k_2)^2 - m_t^2)}_{\int_0^1 dy \partial_{m'^2_2} \frac{1}{(q_2+yk_2)^2 - m'^2_2}}}$$

$$= \int_0^1 dx \int_0^1 dy \partial_{m'^2_1} \partial_{m'^2_2} \int d^D_{q_1} B_0((q1+(1-x)k_1+yk_2)^2, m'^2_1, m'^2_2)$$

$$\frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$



2. Evaluation Method – non-planar diagram

- For non-planar double box including $\gamma\gamma, \gamma Z, ZZ$, use the same dispersion relation as planar diagram
- For non-planar double-box including WW:

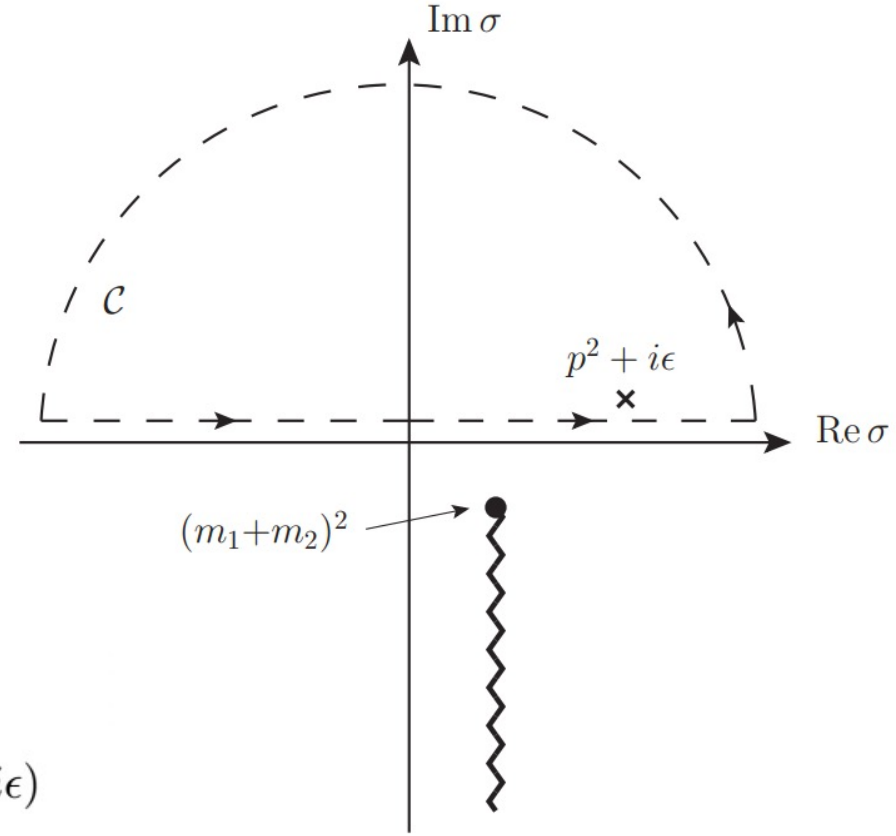
$$m_1'^2 = m_b^2 - x(1-x)m_Z^2 < 0$$

Branch cut changes , we use a new dispersion relation

$$B_0(p^2, m'^2_1, m'^2_2) = \frac{1}{2\pi i}\oint_C d\sigma \frac{B_0(\sigma, m'^2_1, m'^2_2)}{\sigma - p^2 - i\varepsilon}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\sigma \frac{B_0(\sigma, m'^2_1, m'^2_2)}{\sigma - p^2 - i\epsilon}$$

$$I_{NP-WW} = \frac{-1}{2\pi i}\int_0^1 dx \int_0^1 dy \int_{-\infty}^{\infty} d\sigma \partial_{m'^2_1}\partial_{m'^2_2} B_0(\sigma, m'^2_1, m'^2_2) D0(\dots, \sigma - i\epsilon)$$



2. Evaluation Method

$$I = \underbrace{\int dx \int dy \int d\sigma}_{\text{Gauss-kronrod quadrature(Boost)}} \quad \underbrace{B_0(\sigma, m_1^2, m_2^2)\text{or}\Delta B_0(\sigma, m_1^2, m_2^2)}_{\text{analytical expression is known}} \times \underbrace{(c_A A_0 + c_B B_0 + c_C C_0 ...)}_{\text{LoopTools package}}$$

- Programming using C++
- Running time: few minutes to half an hour(non-planar WW)
- Advantages: low requirement of computer,

  short running time
- Precision: 4-digit, the precision is confined by the Looptools(double-precision)
- Stability: integrand is smooth

3. Result: instability

- Upper and lower bound of the integrand, $\delta \sim 10^{-3}, \Lambda \sim 10^{8}$
  $\int_{\sigma_0}^{\infty} f(\sigma) = \int_{\sigma_0(1+\delta)}^{\Lambda} f(\sigma) + 2\sigma_0 \delta f(\sigma_0 \delta) + \Lambda f(\Lambda)$
- For non-planar diagram, the Gram determinants(tensor decomposition arXiv:0812.2134[hep-ph])for some Passarino-Veltman tensor functions vanish when x is equal to y, and Looptools is not able to give a number.
  a) separate the integration region of x: (0,0.5) (0.5,1)
  b) separate the integration region of y: (0,x- δ),(x- δ,x+δ),(x+ δ,1) $\delta = 10^{-2,-3,\ldots}$
- For non-planar diagram with W bosons, $\sigma - i\epsilon, \epsilon \sim 10^{-9}|\sigma|$ or $\epsilon \sim 10^{-5}$

3. Result

Parameter	Value
M_Z	91.1876 GeV
M_W	80.379 GeV
M_H	125.1 GeV
m_t	172.76 GeV
α	$1/137$
E_{CM}	240 GeV
m_γ	10^{-6} GeV
θ	$\pi/2$

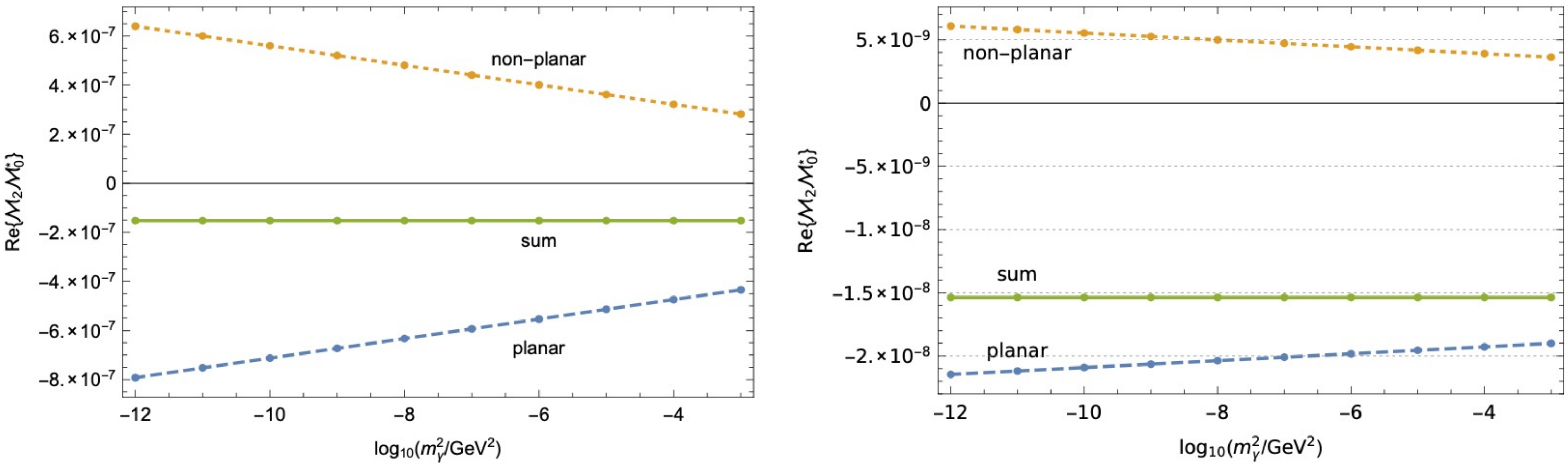
few minutes

V_1V_2 diagr. class	$\mathrm{Re}\{\mathcal{M}_2\mathcal{M}_0^*\}$
$\gamma\gamma$	$-1.524(1)\times 10^{-7}$
γZ	$-1.537(1)\times 10^{-8}$
ZZ planar	$-4.402(4)\times 10^{-8}$
ZZ non-planar	$1.724(2)\times 10^{-8}$
WW planar	$-1.1392(8)\times 10^{-6}$
WW non-planar	$-5.577(5)\times 10^{-7}$

~ 30 minutes

regulate the IR divergence,
no UV divergence

3. Result



Dependence of the $\gamma\gamma$(left) and γZ(right) two-loop boxes on the photon mass m_γ

3. Result

Parameter	Value
M_Z	91.1876 GeV
M_W	80.379 GeV
M_H	125.1 GeV
m_t	172.76 GeV
α	$1/137$
E_{CM}	240 GeV
m_γ	10^{-6} GeV
θ	$\pi/2$

regulate the IR divergence,
no UV divergence

few minutes

V_1V_2 diagr. class	$\mathrm{Re}\{\mathcal{M}_2\mathcal{M}_0^*\}$
$\gamma\gamma$	$-1.524(1)\times 10^{-7}$
γZ	$-1.537(1)\times 10^{-8}$
ZZ planar	$-4.402(4)\times 10^{-8}$
ZZ non-planar	$1.724(2)\times 10^{-8}$
WW planar	$-1.1392(8)\times 10^{-6}$
WW non-planar	$-5.577(5)\times 10^{-7}$

~ 30 minutes

3. Summary

- Double-box diagrams can be efficiently evaluated by Feynman parametrization and dispersion relation (For non-planar diagrams with 2 W bosons, dispersion relation is different from other diagrams)
- Takes few minutes for numerical calculation. For non-planar diagrams with 2 W bosons, takes half an hour.
- IR divergence is controlled by giving photon a small mass without loss of numerical precision.
- The evaluation method can also be applied for the calculation of electroweak corrections to other $2 \rightarrow 2$ process, such as $e^+e^- \rightarrow W^+W^-$

Thank you!

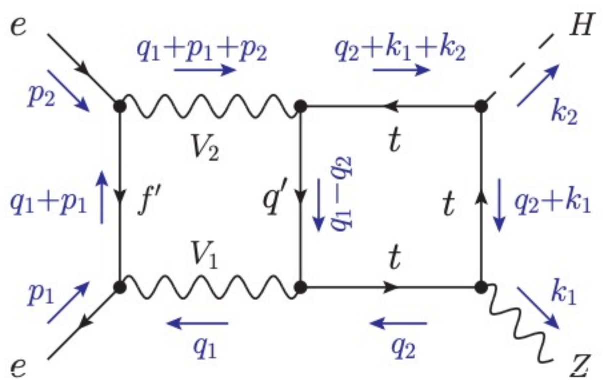
2. Evaluation Method – planar diagram

According to Feynman rules, the amplitude for planar diagram can be written as I_{plan}.
Use Feynman parametrization to simplify the denominators only involve q2

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2}$$

$$\underbrace{\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}}$$

$$\int_0^1 dx \int_0^{1-x} dy \frac{1}{((q_2+k')^2 - m'^2)^3} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \frac{1}{(q_2+k')^2 - m'^2}$$



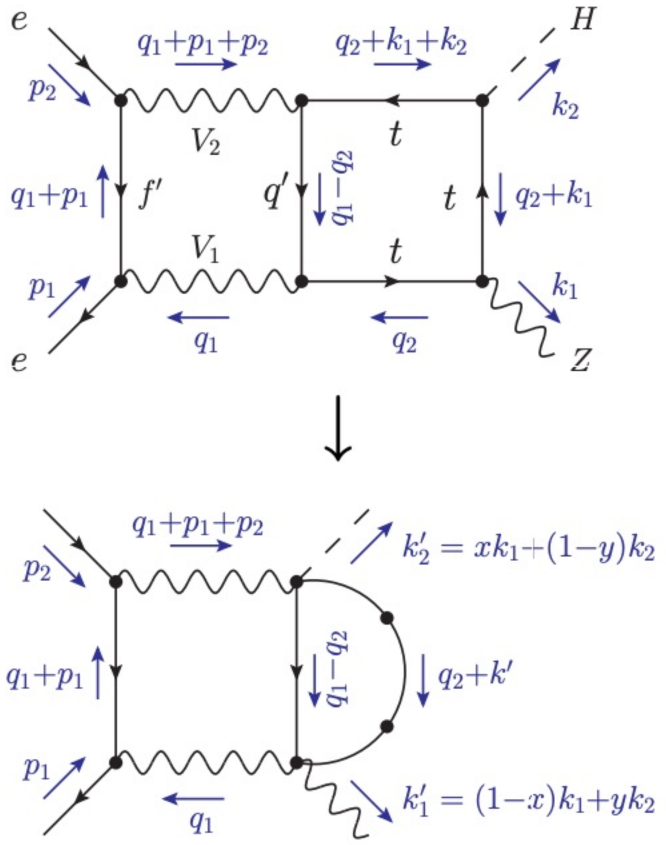
$$\text{Feynman parametrizaiton:} \frac{1}{abc} = \int_0^1 dx \int_0^{1-x} dy \frac{1}{(ax+by+c(1-x-y))^3}$$

2. Evaluation Method – planar diagram

Integrating over loop momentum q2 gets B0 function:

$$I_{plan} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)} \frac{1}{((q_1-q_2)^2 - m_{q'}^2)} \frac{1}{(q_2+k')^2 - m'^2}$$

$$= \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int d^D_{q_1} \frac{B_0((q_1+k')^2, m_{q'}^2, m'^2)}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$



Loop momentum q1 appears in B0 functions, so cannot integrate over q1.
→ use dispersion relation to put q1 outside B0 function

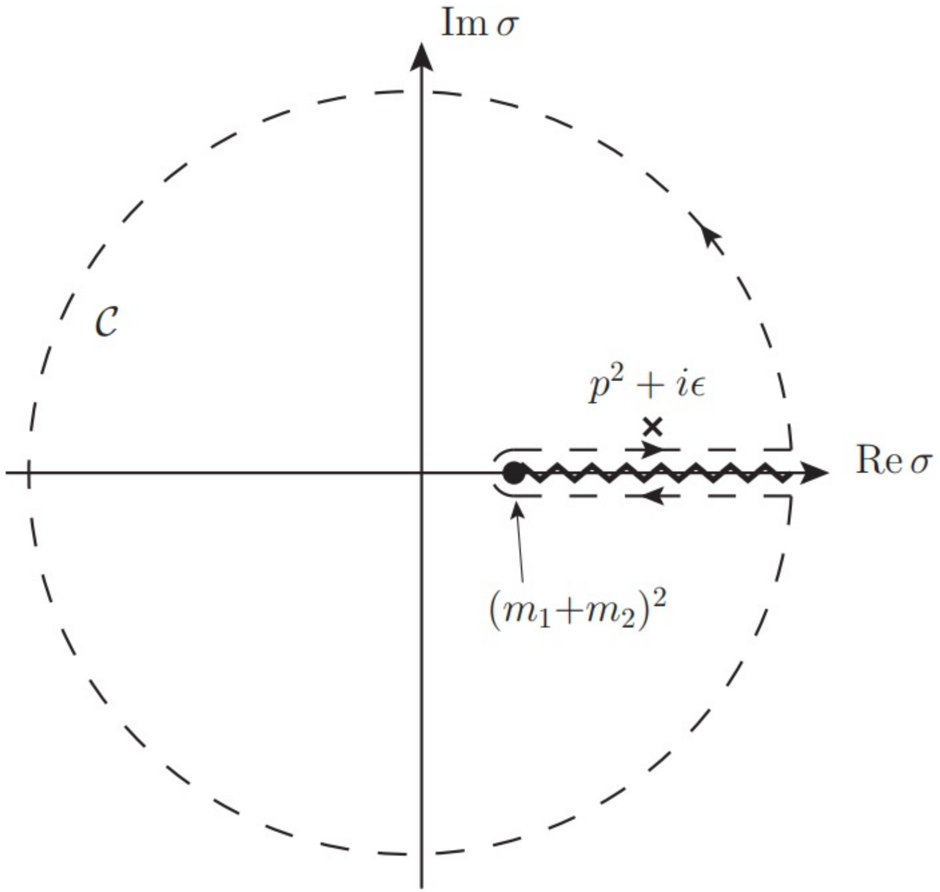
2. Evaluation Method – planar diagram

dispersion relation:

$$B_0(p^2, m_1^2, m_2^2) = \frac{1}{2\pi i}\oint_C d_\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\varepsilon}$$

$$= \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{\Delta B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$

$$= \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{1}{\pi} \frac{Im B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$



2. Evaluation Method – planar diagram

Integrating over q1 gets the D0 function.
Use Leibiniz's rule to put the derivative inside the integral: ΔB_0 is divergent at the lower bound, it can be fixed by subtracting one term to make the integrand become 0 at the lower bound.

$$I_{plan} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \int d^D_{q_1} \Delta B_0(s, m'^2, m^2_{q'})$$

$$\frac{1}{(q_1^2 - m^2_{V_1})((q_1+p_1)^2 - m^2_{f'})((q_1+p_1+p_2)^2 - m^2_{V_2})(s-(q_1+k')^2)}$$

$$= -\int_0^1 dx \int_0^{1-x} \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'}) D_0(p_1^2, p_2^2, {k'}_2^2, {k'}_1^2, s, t, m^2_{V_1}, m^2_{f'}, m^2_{V_2}, \sigma)$$

Leibiniz's rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt + f(x, b(x))\frac{db(x)}{dx} - f(x, a(x))\frac{da(x)}{dx}$$

2. Evaluation Method – planar diagram

$$\partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m_{q'}^2)(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0) + \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$$= \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m_{q'}^2)(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$\rightarrow 0$ at the lower bound, so derivative van be put inside the integral

$$+ \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m_{q'}^2)\frac{\sigma_0}{\sigma}D_0(...,\sigma_0)$$

$\rightarrow$ integrate over σ gives $B_0(0, m'^2, m_{q'}^2)$ (dispersion relation)

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m_{q'}^2) D_0(p_1^2, p_2^2, k'^2_2, k'^2_1, s, t, m_{V_1}^2, m_{f'}^2, m_{V_2}^2, \sigma)$$

$$= \int_0^1 dx \int_0^{1-x} dy \int_{(m'+m_{q'})^2}^{\infty} d\sigma \partial^2_{m'^2}\Delta B_0(s, m'^2, m_{q'}^2)(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$$+ \int_0^1 dx \int_0^{1-x} dy \sigma_0 D_0(...,\sigma_0)\partial^2_{m'^2}B_0(0, m'^2, m_{q'}^2)$$

2. Evaluation Method – planar diagram

If the numerator is not equal to 1, integrating over q2 gets B1,B00,B11 functions. They have same dispersion relation as B0.
For example, num = $p_1 \cdot q_2$

$$\text{num} = p_1 \cdot q_2, (p_1 \cdot q_2)(k_1 \cdot q_2)... \neq 1$$

$$\int d^D q_2 \frac{p_1 \cdot q_2}{(q_2^2 - m_{q'}^2)(q_2 + q_1 + k')^2 - m'^2} = p_{1\mu} \int d^D q_2 \frac{q_2^\mu}{(q_2^2 - m_{q'}^2)(q_2 + q_1 + k')^2 - m'^2}$$

$$= p_{1\mu} B_\mu((q_1 + k')^2, m'^2, m_{q'}^2)$$

$$= \underbrace{p_{1\mu}(p_1 + k')_\mu}_{p_1 \cdot (p_1 + k')} B_1((q_1 + k')^2, m'^2, m_{q'}^2) \Rightarrow \frac{p_1 \cdot (p_1 + k')}{\sigma - (q_1 + k')^2} \int d\sigma \Delta B_1$$

similarly,

$$\int d^D q_2 \frac{q_2^\mu q_2^\nu}{(q_2^2 - m_{q'}^2)(q_2 + q_1 + k')^2 - m'^2} \quad \Rightarrow B_{00}, B_{11} \Rightarrow \int d\sigma \Delta B_{00}, \int d\sigma \Delta B_{11}$$

$$\int d^D q_2 \frac{q_2^\mu q_2^\nu, q_2^\rho}{(q_2^2 - m_{q'}^2)(q_2 + q_1 + k')^2 - m'^2} \quad \Rightarrow B_{001}, B_{111} \Rightarrow \int d\sigma \Delta B_{001}, \int d\sigma \Delta B_{111}$$

2. Evaluation Method – non-planar diagram

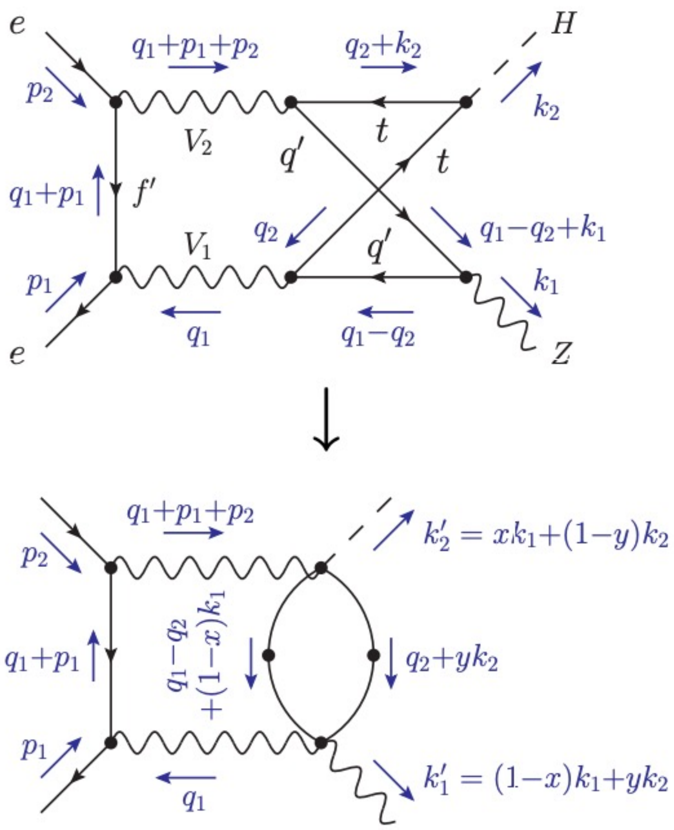
Similarly, use Feynman parametrization to simplify the denominators including q2 and integrating loop momentum q2 gives B0 function

$$I_{NP} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$

$$\frac{1}{\underbrace{((q_1-q_2)^2 - m_{q'}^2)((q_1-q_2+k_1)^2 - m_{q'}^2)}_{\int_0^1 dx \partial_{m'^2_1} \frac{1}{(q_1-q_2+(1-x)k_1)^2 - m'^2_1}} \underbrace{(q_2^2 - m_t^2)((q_2+k_2)^2 - m_t^2)}_{\int_0^1 dy \partial_{m'^2_2} \frac{1}{(q_2+yk_2)^2 - m'^2_2}}}$$

$$= \int_0^1 dx \int_0^1 dy \partial_{m'^2_1} \partial_{m'^2_2} \int d^D_{q_1} B_0((q1 + (1-x)k_1 + yk_2)^2, m'^2_1, m'^2_2)$$

$$\frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$



2. Evaluation Method – non-planar diagram

For non-planar double box including $\gamma\gamma, \gamma Z, ZZ$, use the same dispersion relation. Put the derivative inside the integral by subtracting $\frac{\sigma_0}{\sigma} D_0$ and add it back

$$I_{NP(\gamma\gamma,\gamma Z,ZZ)} = \int_0^1 dx \int_0^1 dy \int_{(m_1'+m_2')^2}^{\infty} d\sigma \partial_{m_1'^2} \partial_{m_2'^2} \Delta B_0(\sigma, m_1'^2, m_2'^2)(D_0(..., \sigma) - \frac{\sigma_0}{\sigma} D_0(..., \sigma_0))$$
$$+ \int_0^1 dx \int_0^1 dy \sigma_0 D_0(..., \sigma_0) \partial_{m_1'^2} \partial_{m_2'^2} B_0(0, m_1'^2, m_2'^2)$$

2. Evaluation Method – non-planar diagram

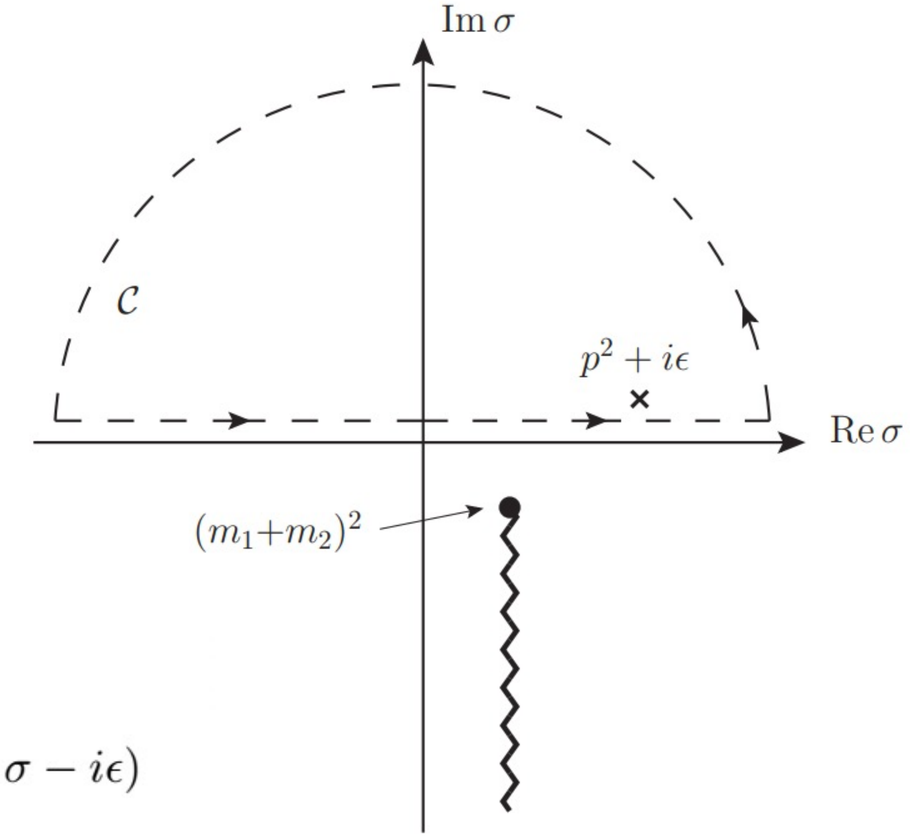
For non-planar double-box including WW:

$m_1'^2 = m_b^2 - x(1-x)m_Z^2 < 0$

Branch cut changes , we use a new dispersion relation.

$$B_0(p^2, {m'}_1^2, {m'}_2^2) = \frac{1}{2\pi i}\oint_C d\sigma \frac{B_0(\sigma, {m'}_1^2, {m'}_2^2)}{\sigma - p^2 - i\varepsilon}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\sigma \frac{B_0(\sigma, {m'}_1^2, {m'}_2^2)}{\sigma - p^2 - i\epsilon}$$



$$I_{NP-WW} = \frac{-1}{2\pi i}\int_0^1 dx \int_0^1 dy \int_{-\infty}^{\infty} d\sigma \partial_{{m'}_1^2}\partial_{{m'}_2^2} B_0(\sigma, {m'}_1^2, {m'}_2^2) D0(..., \sigma - i\epsilon)$$