Operator mixing in massless QCD-like theories and Poincare'-Dulac theorem

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Outline of the talk

- Motivations
- *Brief description of operator mixing*
- *Example of operator mixing*
- *Poincare'-Dulac theorem*
- *Mixing analysis based on Poincare'-Dulac theorem*
- *Conclusions*

Motivations

- To establish conditions under which operator mixing reduces to the multiplicatively renormalizable case
- To revisit the problem of operator mixing with a more systematic approach [Buras '80, Sonoda '91]
- Operator mixing occurs in a number of applications
- Applications within the framework of establishing ultraviolet constraints on a candidate solution to QCD in the large-N limit [Bochicchio '17] [MB,Bochicchio,Papinutto,Scardino '21]

Operator mixing

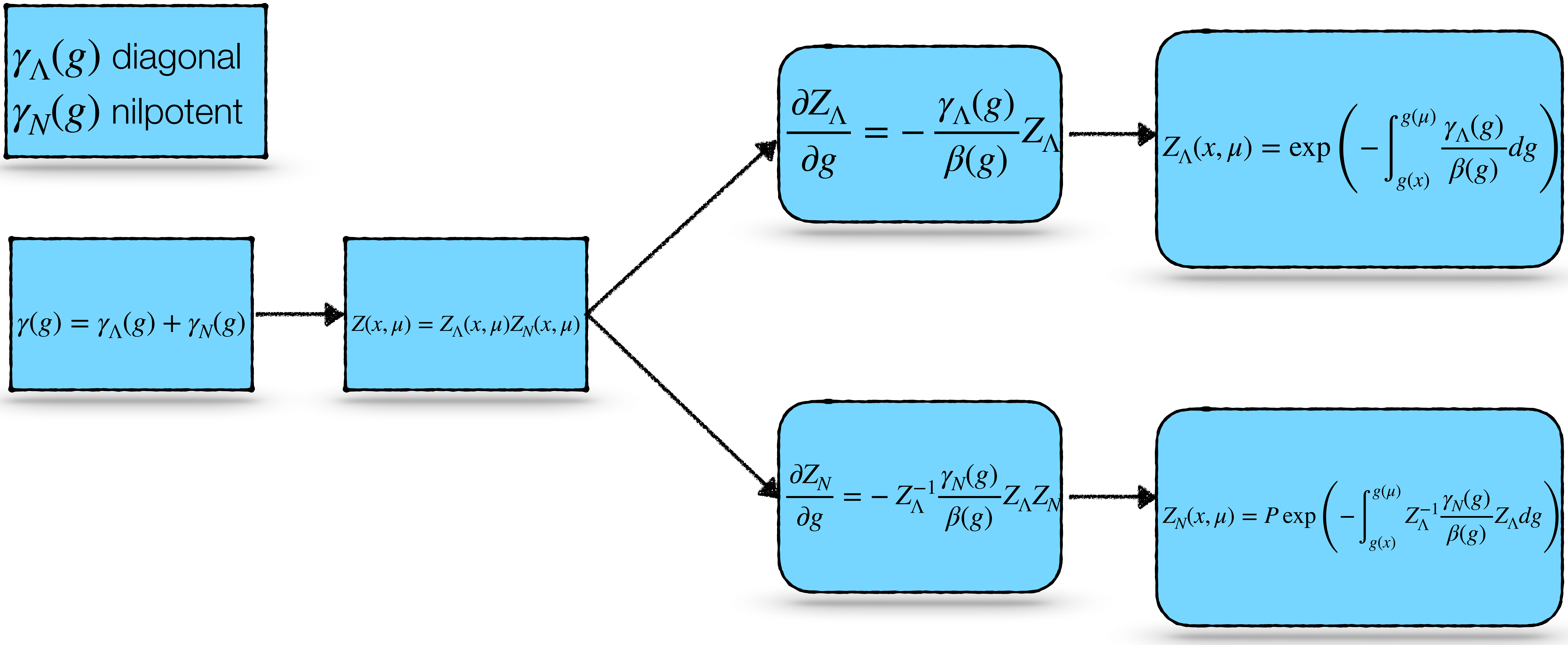
A set of local operators $\mathcal{O}_i^B(x)$ mix under renormalization if:

$$\mathcal{O}_i^R(x) = Z_{ik}(\Lambda, \mu)\mathcal{O}_k^B(x)$$

The mixing matrix $Z(\Lambda, \mu)$ satysfies the ODE:

$$-\frac{\partial Z}{\partial \log \mu} Z^{-1} = \gamma(g) = \gamma_0 g^2 + \gamma_1 g^4 + \cdots$$

Operator mixing



Geometrical Interpretation

- $Z(x,\mu)$ satisfies the ODE:

$$\mathscr{D}\, Z(x,\mu) = 0, \quad \mathscr{D} = \frac{\partial}{\partial g} - A(g)$$

- $\mathscr{D}$ is a covariant derivative and $A(g)$ a meromorphic connection with Fuchsian singularity at $g = 0$

$$A(g) = -\frac{\gamma(g)}{\beta(g)} = \frac{1}{g}\left(A_0 + \sum_{k=1}^{\infty} A_{2k} g^{2k}\right)$$

- $Z(x,\mu)$ can be seen as a Wilson line:

$$Z(x,\mu) = P\exp\left(\int_{g(x)}^{g(\mu)} A(g)dg\right) = P\exp\left(\int_{g(x)}^{g(\mu)} -\frac{\gamma(g)}{\beta(g)}dg\right)$$

- $Z(x,\mu)$ transforms as:

$$Z'(x,\mu) = S(g(\mu))Z(x,\mu)S^{-1}(g(x))$$

- $S(g)$ holomorphic gauge transformation

Example: 2×2 systems with elementary methods

- We consider the system of 2 operators that mix under renormalization:

$$\frac{\partial Z}{\partial g} = \left(A_0 g^{-1} + N_{2k} g^{2k-1}\right) Z, \quad A_0 = \Lambda + N_0$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad N_0 = \begin{pmatrix} 0 & \nu_{12} \\ 0 & 0 \end{pmatrix} \quad N_{2k} = \begin{pmatrix} 0 & \nu_{12} \\ 0 & 0 \end{pmatrix}$$

Nonresonant diagonalizable

- We refer to nonresonant diagonalizable mixing as:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad N_0 = 0$$

$\lambda_1 - \lambda_2 \neq 2k$, with k a positive integer

- Z is gauge equivalent to the diagonal matrix:

$$Z_\Lambda(x,\mu) = \begin{pmatrix} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} & 0 \\ 0 & \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_2} \end{pmatrix}$$

$$Z(x,\mu) = \begin{pmatrix} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} & \frac{\nu_{12} g^{2k}(x)}{\lambda_1 - \lambda_2 - 2k}\left(\left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} - \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_2+2k}\right) \\ 0 & \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_2} \end{pmatrix}$$

$$Z(x,\mu) = S(g(\mu)) Z_\Lambda(x,\mu) S^{-1}(g(x))$$

$S(g)$ is the holomorphic gauge transformation

$$S(g) = \begin{pmatrix} 1 & \frac{\nu_{12} g^{2k}}{2k - \lambda_1 + \lambda_2} \\ 0 & 1 \end{pmatrix}$$

Resonant diagonalizable

- We refer to resonant diagonalizable mixing as:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad N_0 = 0$$

$\lambda_1 - \lambda_2 = 2k$, with k positive integer

- $Z(x, \mu)$ is not diagonalizable by a holomorphic gauge transformation

$$Z(x,\mu) = \begin{pmatrix} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} & \nu_{12} g^{2k}(x) \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} \log \frac{g(\mu)}{g(x)} \\ 0 & \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_2} \end{pmatrix}$$

Nondiagonalizable

- We refer to nondiagonalizable mixing as:

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad N_0 = \begin{pmatrix} 0 & \nu_{12} \\ 0 & 0 \end{pmatrix} \quad N_{2k} = 0$$

- $Z(x, \mu)$ is not diagonalizable by a holomorphic gauge transformation.

$$Z(x,\mu) = \begin{pmatrix} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda} & \nu_{12}\left(\frac{g(\mu)}{g(x)}\right)^{\lambda} \log\frac{g(\mu)}{g(x)} \\ 0 & \left(\frac{g(\mu)}{g(x)}\right)^{\lambda} \end{pmatrix}$$

Poincare'-Dulac Theorem

The most general ODE system with Fuchsian singularity at $g = 0$, with meromorphic connection $A(g)$

$$A(g) = \frac{1}{g}\left(A_0 + \sum_{n=1}^{\infty} A_n g^n\right)$$

can be set, by a holomorphic gauge transformation, in the Poincare'-Dulac-Levelt normal form:

$$A'(g) = \frac{1}{g}\left(\Lambda + N_0 + \sum_{k=1} N_k g^k\right)$$

Poincare'-Dulac Theorem

- $\Lambda + N_0$ is the Jordan normal form of A_0;
- $\Lambda = diag(\lambda_1, \lambda_2, \cdots)$, with $\lambda_1 \geq \lambda_2 \geq \cdots$;
- $N_0,\ N_k$ are nilpotent upper triangular;
- For $k = 1,2,\cdots\ g^{\Lambda} N_k g^{-\Lambda} = g^k N_k$, i.e. $(N_k)_{ij}$ may be non vanishing only if the resonant condition $\lambda_i - \lambda_j = k$ holds, with $i \leq j$ and k some positive integer.

Fundamental solution

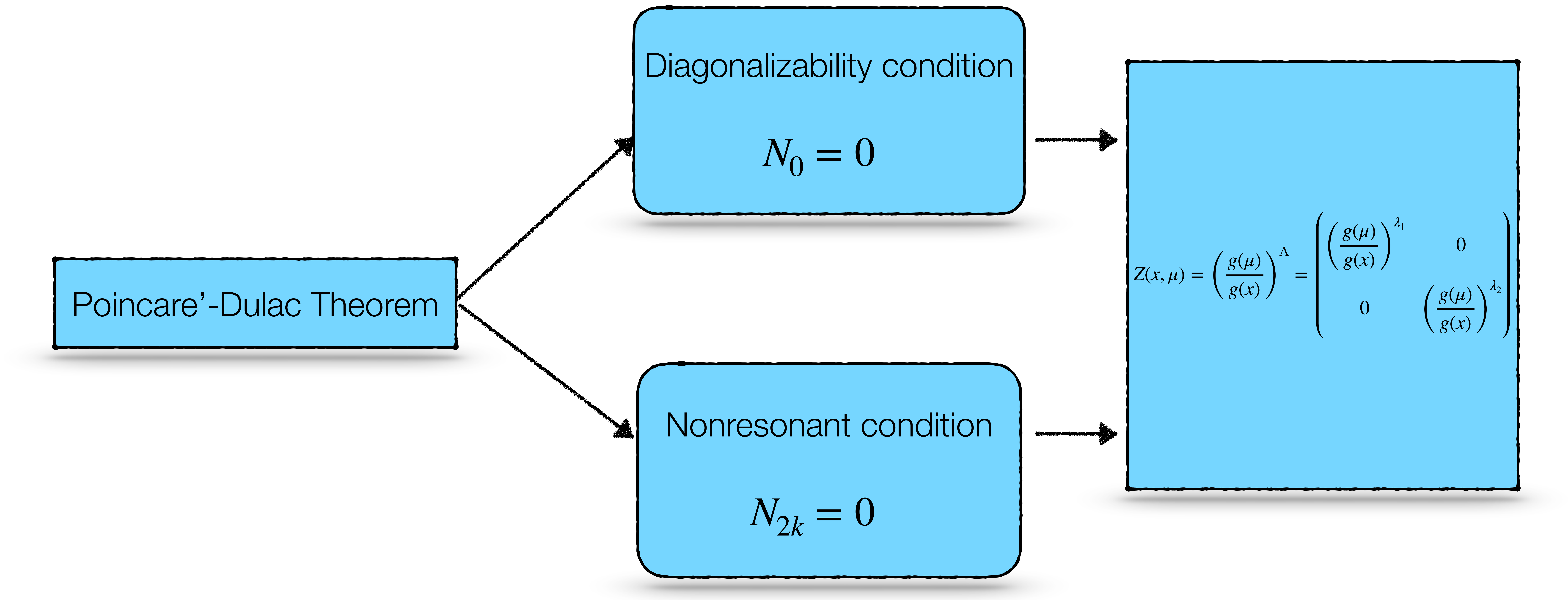
- A fundamental solution to a linear system in the Poincare'-Dulac-Levelt form is:

$$Z = g^{\Lambda} g^{N} \text{ with } N = N_0 + \sum_{k=1} N_k$$

- The solution that reduces to the identity for $g(x) = g(\mu)$ is

$$Z(x,\mu) = \left(\frac{g(\mu)}{g(x)}\right)^{\Lambda} e^{\sum_{k=0} g^{2k}(x) N_{2k} \log \frac{g(\mu)}{g(x)}}$$

Nonresonant diagonalizable



Resonant diagonalizable

Poincare'-Dulac Theorem

Diagonalizability condition

$$N_0 = 0$$

resonant condition

$$N_{2k} = \begin{pmatrix} 0 & \nu_{12} \\ 0 & 0 \end{pmatrix}$$

$$Z(x,\mu) = \left(\frac{g(\mu)}{g(x)}\right)^{\Lambda} e^{g^{2k}(x) N_{2k} \log \frac{g(\mu)}{g(x)}}$$

$$= \begin{pmatrix} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} & \nu_{12} g^{2k}(x) \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_1} \log \frac{g(\mu)}{g(x)} \\ 0 & \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_2} \end{pmatrix}$$

Nondiagonalizable

Poincare'-Dulac Theorem

Nondiagonalizability condition

$$N = N_0 = \begin{pmatrix} 0 & \nu_{12} \\ 0 & 0 \end{pmatrix}$$

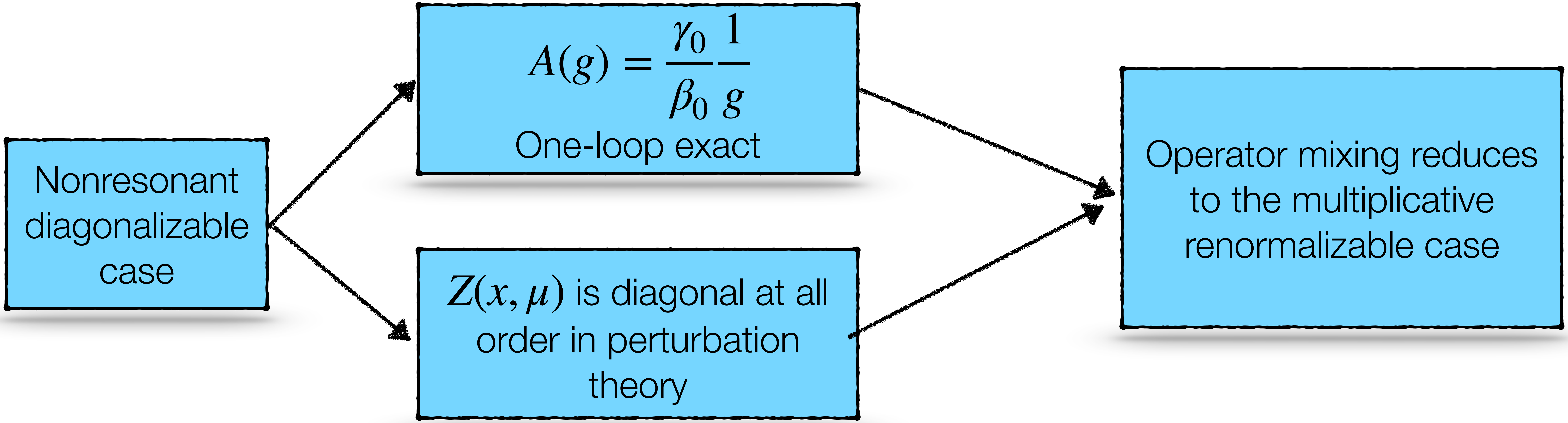
$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$Z(x,\mu) = \left(\frac{g(\mu)}{g(x)}\right)^{\Lambda} e^{N_0 \log \frac{g(\mu)}{g(x)}}$$

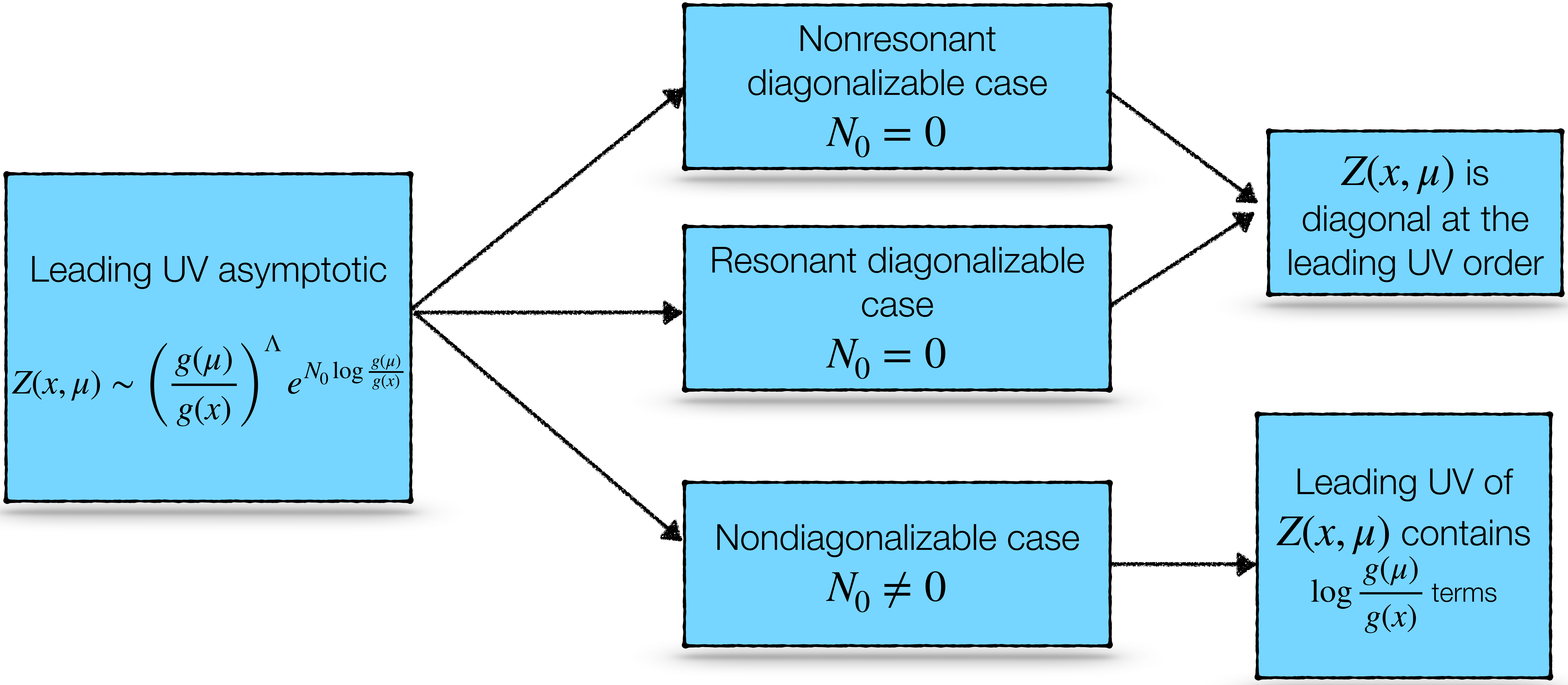
$$= \begin{pmatrix} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda} & \nu_{12} \left(\frac{g(\mu)}{g(x)}\right)^{\lambda} \log \frac{g(\mu)}{g(x)} \\ 0 & \left(\frac{g(\mu)}{g(x)}\right)^{\lambda} \end{pmatrix}$$

Mixing vs Multiplicative renormalizability

Poincare'-Dulac theorem allows us to establish a sufficient condition such that $-\frac{\gamma(g)}{\beta(g)}$ is diagonalizable at all orders in perturbation theory.



Ultraviolet asymptotic behaviour



Unitarity constraint for massless QCD-like theories

- Massless QCD-like theories are conformal invariant up to order g^2 in perturbation theory
- Nondiagonalizable mixing can happen also in conformal field theories (CFTs) if Δ, the conformal dimension matrix, is nondiagonalizable [Gurarie '93] [Hogervorst, Paulos, Vichi '17]
- CFTs with nondiagonalizable Δ are non unitary theories [Gurarie '93] [Hogervorst, Paulos, Vichi '17]
- Unitarity

  $\Rightarrow$

  γ_0 is always diagonalizable for a system of Hermitian gauge-invariant operators in a massless QCD-like theory

Conclusions

- The Poincare'-Dulac theorem allows us to classify operator mixing in a systematic way
- It is possible to establish from a one-loop computation whether it exists an operator basis where the mixing matrix is diagonal to all orders in perturbation theory
- Unitarity rules out the nondiagonalizable mixing case for Hermitian gauge-invariant operators in massless QCD-like theories

Thank you for your attention!