Renormalization of the flavor-singlet axial-vector current and its anomaly at N^3LO in QCD

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Radcor+LoopFest 2021, May 17

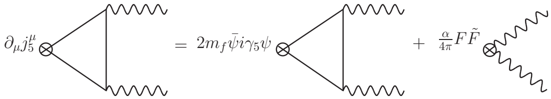
Ref. JHEP05 (2021) 087 [arXiv:2101.09479]
with T. Ahmed and M. Czakon

The Adler-Bell-Jackiw anomaly

The anomalous axial-vector divergence equation [Adler 69; Bell, Jackiw 69]

$$\partial_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi = 2m_f \bar{\psi} i\gamma_5 \psi - \frac{\alpha}{4\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} .$$

Diagrammatically,

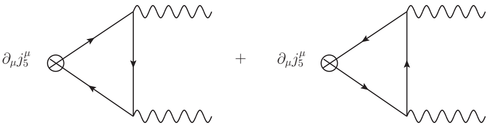


The Adler-Bardeen theorem [Adler, Bardeen 69] : "one-loop" exact

- *gauge/internal anomalies* must cancel !
  - The Standard Model is *anomaly free*
  - Anomaly matching ['t Hooft et al. 80], Spontaneous chiral symmetry breaking ...
- *global/external anomalies* are allowed and important
  - $\pi \to \gamma\gamma$ decay [Steinberger 49; Sutherland, Veltman 67; Adler 69; Bell, Jackiw 69]
  - $U(1)_A/\eta'$ problem [Weinberg 75; 't Hooft 76]
  - Strong CP problem and Axion [Peccei, Quinn 77] ...

Calculating the axial anomaly in DR

The axial anomaly



“vanishes” with *translational invariant loop integrals* and an *anticommuting* γ_5.

In addition,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

is intrinsically a $D = 4$ dimenisional object:

> A fully anticommuting γ_5 is algebraically incompatible with the Dirac algebra in a general $D \neq 4$ dimensions.

Within the Dimensional Regularization, two classes of γ_5 prescriptions:

- A non-anticommuting γ_5 (*constructively* given)
  ['t Hooft,Veltman 72; Breitenlohner,Maison 77; Larin,Vermaseren 91 ...]
- An anticommuting γ_5 (with a careful re-definition of “γ_5-trace”)
  [Bardeen 72, Chanowitz et al. 79; Kreimer 90; Zerf 20 ...]

The γ_5 prescription in use

The HV/BM [72,79] prescription of γ_5 in dimensional regularization:

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\gamma_\mu\gamma_5 \to \frac{1}{2}\left(\gamma_\mu\gamma_5 - \gamma_5\gamma_\mu\right) = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma\,,$$

where the $\epsilon^{\mu\nu\rho\sigma}$ is treated outside the R-operation formally in D dimensions [Larin, Vermaseren 91; Zijlstra, Neerven 92] (conveniently called Larin's prescription).

The γ_5 **no longer anticommutes** with all γ^μ in D dimensions $\Longrightarrow$ "*spurious anomalous terms*" calling for non-trivial UV renormalization [Chanowitz et al. 79; Trueman 79; Kodaira 79; Espriu,Tarrach 82; Collins 84; Larin, Vermaseren 91; Bos 92; Larin 93, ...]

The properly renormalized singlet axial current reads

$$\begin{aligned}[J_5^\mu]_R &= Z_J\,\mu^{4-D}\,\bar{\psi}_B\,\gamma^\mu\gamma_5\,\psi_B \\ &= Z_5^f\,Z_5^{ms}\,\mu^{4-D}\,\bar{\psi}_B\,\frac{-i}{3!}\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_\rho\gamma_\sigma\,\psi_B\end{aligned}$$

Operator mixing under renormalization

The all-order axial-anomaly equation [Adler 69; Adler, Bardeen 69]

$$\left[\partial_\mu J_5^\mu\right]_R = a_s\, n_f\, \mathrm{T}_F\, \left[F\tilde{F}\right]_R$$

in terms of renormalized local composite operators, with $\mathrm{T}_F = 1/2$ and $F\tilde{F} \equiv -\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$ in QCD with n_f massless quarks.

- The renormalization of the operators involved: [Adler 69; Espriu,Tarrach 82; Breitenlohner,Maison,Stelle 84; Bos 92; Larin 93 ...]

$$\begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_R \\ \left[F\tilde{F}\right]_R \end{pmatrix} = \mu^{4-D} \begin{pmatrix} Z_J & 0 \\ Z_{FJ} & Z_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_B \\ \left[F\tilde{F}\right]_B \end{pmatrix}$$

- with the matrix of *anomalous dimensions*:

$$\frac{\mathrm{d}}{\mathrm{d}\,\ln\mu^2} \begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_R \\ \left[F\tilde{F}\right]_R \end{pmatrix} = \begin{pmatrix} \gamma_J & 0 \\ \gamma_{FJ} & \gamma_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_R \\ \left[F\tilde{F}\right]_R \end{pmatrix}$$

Form factor decomposition of the AVV amplitude

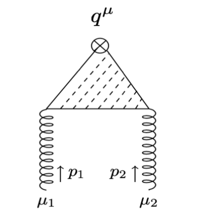
Determine UV Z_5s via computing the 2-gluon matrix elements of $\left[\partial_\mu J_5^\mu\right]_R = a_s\, n_f\, \mathrm{T}_F\, \left[F\tilde{F}\right]_R$

The 1PI AVV amplitude without external polarization vectors:

$$\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y}\, \langle 0|\hat{\mathrm{T}}\left[J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|0\rangle|_{\mathrm{amp}}$$

Form factor decomposition:

$$\begin{aligned}\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) &= F_1\,\epsilon^{\mu\,\mu_1\,\mu_2\,(p_2-p_1)} \\ &+ F_2\left(p_1^{\mu_1}\epsilon^{\mu\,\mu_2\,p_1\,p_2} - p_2^{\mu_2}\epsilon^{\mu\,\mu_1\,p_1\,p_2}\right) \\ &+ F_3\left(p_1^{\mu_2}\epsilon^{\mu\,\mu_1\,p_1\,p_2} - p_2^{\mu_1}\epsilon^{\mu\,\mu_2\,p_1\,p_2}\right)\end{aligned}$$



taking into account the odd parity and Bose symmetry w.r.t gluons ($p_1 \leftrightarrow p_2\,,\mu_1 \leftrightarrow \mu_2$).

The zero momentum insertion limit

The *axial anomaly* reads

$$q_\mu \Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1, p_2) = 2F_1\, \epsilon^{\mu_1\,\mu_2\,p_1\,p_2}$$

with $q = p_1 + p_2$ and on-shell kinematics $p_1^2 = p_2^2 = 0$.

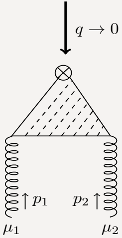
In addition to the apparent vanishing of the anomaly at $q^\mu = 0$, there is the *low-energy PCAC* theorem $F_1(q^2)|_{q^2=0} = 0$ [Sutherland, Veltman 67] (gauge invariance and analyticity).

Despite $q_\mu \Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1, -p_1) = 0$, the form factor F_1 is not zero if the p_1 is set **off-shell**.

$$\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1, -p_1) = -2F_1\, \epsilon^{\mu\,\mu_1\,\mu_2\,p_1},$$

$$\mathcal{P}_{\mu\mu_1\mu_2} = -\frac{1}{6\, p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2\nu}\, p_1^\nu,$$

$$\mathcal{M}_{lhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1, -p_1) \propto -2F1$$



$\frac{1}{(6-11D+6D^2-D^3)\, p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2 p_1} \longrightarrow \frac{-1}{6\, p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2 p_1}$ albeit with indices in D [LC 19; Ahmed et al. 19; Peraro,Tancredi 20]

The matrix element at $q = 0$

The reduced anomaly matrix element [Bos 92; Larin 93]

$$\mathrm{M} = R_{\overline{\mathrm{MS}}}\, \epsilon_{\mu_1\mu_2\rho\sigma}\, \frac{p_1^\rho}{p_1^2}\, \frac{\partial}{\partial q_\sigma} \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y}\, \langle 0|\hat{\mathrm{T}}\left[\frac{\partial}{\partial y^\mu} J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|0\rangle|_{q\to 0}\,,$$

$$= R_{\overline{\mathrm{MS}}}\, \epsilon_{\mu_1\mu_2\rho\sigma}\, \frac{p_1^\rho}{p_1^2}\, i\Big(\Gamma_{lhs}^{\rho\mu_1\mu_2}(p_1, q-p_1) + \cancel{q_\mu \frac{\partial}{\partial q_\sigma}\Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, q-p_1)}\Big)|_{q\to 0}\,,$$

where $R_{\overline{\mathrm{MS}}}$ denotes the $\overline{\mathrm{MS}}$ R-operation.

Evaluating $\mathcal{M}_{lhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, -p_1)$ at $q = 0$ with off-shell gluon momenta $p_1^2 \neq 0$:

- possible IR divergences nullified owing to the *IR-rearrangement* [Vladimirov 79]
- 4-loop massless *propagator-type* master integrals available [Smirnov, Tentyukov 10, Baikov, Chetyrkin 10; Lee, Smirnov, Smirnov 11]
- gauge-dependent $\mathcal{M} \Longrightarrow$ UV renormalization of *gauge parameter* ξ !

Treatment of the operator $F\tilde{F}$

The axial-anomaly (topological-charge density) operator $F\tilde{F}$ with the *Chern-Simons current* K^μ

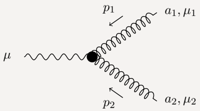
$$
\begin{aligned}
F\tilde{F} &= \partial_\mu K^\mu \\
&= \partial_\mu \left(-4\, \epsilon^{\mu\nu\rho\sigma} \left(A^a_\nu \partial_\rho A^a_\sigma + g_s \frac{1}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma \right) \right)
\end{aligned}
$$

by the virtue of total antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$ [Bardeen 74].
Unlike J^μ_5, the current K^μ is not gauge-invariant.

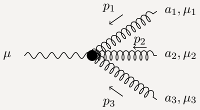
$$
\Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1,p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y} \langle 0|\hat{\mathrm{T}}\left[K^\mu(y)\, A^{\mu_1}_a(x)\, A^{\mu_2}_a(0)\right]|0\rangle|_{\mathrm{amp}}
$$

$$
\mathcal{M}_{rhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1,-p_1)\,.
$$

The *Feynman Rules* in use:



$= 4\, \delta_{a_1a_2}\, \varepsilon_{\mu_1\mu_2\mu(p_2-p_1)}$



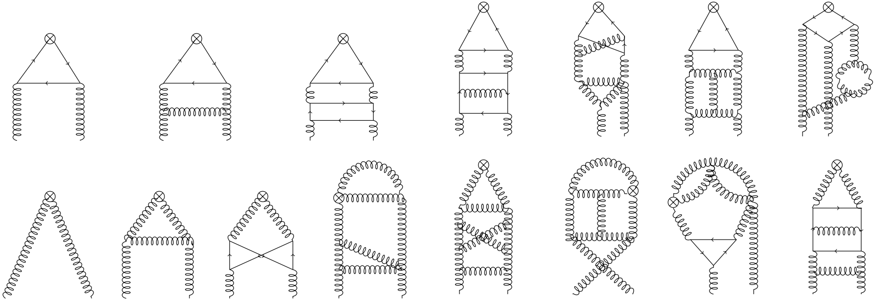
$= 8\, i\, g_s\, f^{a_1a_2a_3}\, \varepsilon_{\mu_1\mu_2\mu_3\mu}$

Feynman diagrams

The Work Flow:

- Generating Feynman diagrams
- Applying Feynman Rules, Dirac/Lorentz algebra, Color algebra
- IBP reduction of loop integrals
- Inserting Master integrals

Generator / Loop order	DiaGen		Qgraf	
	l.h.s.	r.h.s.	l.h.s.	r.h.s.
1	2	3	2	4
2	20	57	21	64
3	429	1361	447	1488
4	11302	37730	11714	40564



IBP reduction and master integrals

Loop integrals in diagrams, reduced by IBP [Tkachov 81; Chetyrkin,Tkachov 81];

Analytic results of p-master integrals, up to 4 loop [Baikov, Chetyrkin 10; Lee, Smirnov,Smirnov 12].

- DiaGen/IdSolver [Czakon] + Forcer [Ruijl,Ueda,Vermaseren]
  - **Amplitude projection**: about $3+6$ days @ 24 cores (Intel® Xeon® Silver 4116)
  - **Forcer (pre-solved IBP)**: about $12+24$ hours @ 8 cores (Intel® Xeon® E3-1275 V2)
- QGRAF [Nogueira] + FORM [Vermaseren] + Reduze 2 [Manteuffel, Studerus] + FIRE [Smirnov] combined with LiteRed [Lee]
  - **IBP (by Laporta)**: about one month @ 32 cores (Intel® Xeon® Silver 4216)
  - a few hundred GB RAM

At 4-loop: $\sim 10^5$ loop integrals in Feynman amplitudes reduced to **28** masters.

The analytical results were found to be identical between the two set-ups.

UV renormalization

$$\overline{\text{MS}}: \quad \hat{a}_s\, S_\epsilon = Z_{a_s}(\mu^2)\, a_s(\mu^2)\, \mu^{2\epsilon}\,, \quad a_s \equiv \frac{\alpha_s}{4\pi} = \frac{g_s^2}{16\pi^2}\,, \quad S_\epsilon = (4\pi)^\epsilon\, e^{-\epsilon\gamma_E}$$

By the multiplicative renormalizability of the QCD Lagrangian and of J_5^μ and K^μ:

$$\begin{aligned}
\mathcal{M}_{lhs} &= Z_J\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(\hat{a}_s\,,\, \hat{\xi}) \\
&= Z_5^f\, Z_5^{ms}\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\, \xi) \equiv Z_5^f\, \bar{\mathcal{M}}_{lhs}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{rhs} &= Z_{F\tilde{F}}\, Z_3\, \hat{\mathcal{M}}_{rhs}\,(\hat{a}_s\,,\, \hat{\xi}) + Z_{FJ}\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(\hat{a}_s\,,\, \hat{\xi}) \\
&= Z_{F\tilde{F}}\, Z_3\, \hat{\mathcal{M}}_{rhs}\,(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\, \xi) + Z_{FJ}\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\, \xi)
\end{aligned}$$

where the QCD gauge-fixing parameter ξ is defined via $\frac{i}{k^2}\left(-g^{\mu\nu} + \xi\, \frac{k^\mu k^\mu}{k^2}\right)$.

- The renormalization $1 - \hat{\xi} = Z_3(1 - \xi)$ is crucial here.
- The overall wavefunction Z_3 is not necessary (for determining Z_5).

Results on $Z_J \equiv Z_5^{ms} Z_5^f$

The $\overline{\rm MS}$ renormalization constant Z_5^{ms} can be extracted from the ϵ poles of the 4-loop expression of $\hat{\mathcal{M}}_{lhs}$:

$$Z_5^{ms} = 1 + a_s^2 \Big\{ C_A C_F \Big(\frac{22}{3\epsilon}\Big) + C_F n_f \Big(\frac{5}{3\epsilon}\Big) \Big\} + a_s^3 \Big\{ C_A^2 C_F \Big(\frac{3578}{81\epsilon} - \frac{484}{27\epsilon^2}\Big) + C_A C_F n_f \Big(\frac{149}{81\epsilon} - \frac{22}{27\epsilon^2}\Big) + C_A C_F^2 \Big(-\frac{308}{9\epsilon}\Big) + C_F^2 n_f \Big(-\frac{22}{9\epsilon}\Big) + C_F n_f^2 \Big(\frac{20}{27\epsilon^2} + \frac{26}{81\epsilon}\Big) \Big\} .$$

agreement with [Larin,Vermaseren, 91] was found.

By perturbatively expanding the ratio of the ***finite*** $\bar{\mathcal{M}}_{lhs}$ and $a_s n_f \mathrm{T}_F \mathcal{M}_{rhs}$ to $\mathcal{O}(a_s^3)$,

$$Z_5^f = 1 + a_s \Big\{ -4 C_F \Big\} + a_s^2 \Big\{ C_A C_F \Big(-\frac{107}{9}\Big) + C_F^2 \Big(22\Big) + C_F n_f \Big(\frac{31}{18}\Big) \Big\} + a_s^3 \Big\{ C_A^2 C_F \Big(56\zeta_3 - \frac{2147}{27}\Big) + C_A C_F^2 \Big(\frac{5834}{27} - 160\zeta_3\Big) + C_A C_F n_f \Big(\frac{110}{3}\zeta_3 - \frac{133}{81}\Big) + C_F^3 \Big(96\zeta_3 - \frac{370}{3}\Big) + C_F^2 n_f \Big(\frac{497}{54} - \frac{104}{3}\zeta_3\Big) + C_F n_f^2 \Big(\frac{316}{81}\Big) \Big\} .$$

The first application of the new result:

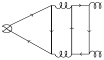
the 3-loop singlet contribution to the massless axial quark form factor [Gehrmann, Primo 21]

The anomalous dimension of the axial current

The difference is proportional to $n_f\, C_F$:

$$Z_5^{ms} - Z_{5,\mathrm{NS}}^{ms} = C_F\, n_f \Big(\frac{3}{\epsilon} a_s^2 + \frac{1}{9\epsilon^2} \big((-66 + 109\epsilon) C_A - 54\epsilon C_F + (12 + 2\epsilon) n_f \big) a_s^3 \Big) + \mathcal{O}(a_s^4)\,,$$

$$Z_5^{f} - Z_{5,\mathrm{NS}}^{f} = C_F\, n_f \Big(\frac{3}{2} a_s^2 + \frac{1}{54} \big((-326 + 1404\zeta_3) C_A + (621 - 1296\zeta_3) C_F + 176 n_f \big) a_s^3 \Big) + \mathcal{O}(a_s^4)\,,$$



The result for Z_5^f in QED: $a_s \to \frac{\alpha}{4\pi}, \frac{n_f}{2} \to n_f, C_A \to 0, C_F \to 1.$

The *anomalous dimension* γ_J of the $\left[J_5^{\mu}\right]_R$:

$$\begin{aligned}
\gamma_J &= \epsilon + \mu^2 \frac{\mathrm{d}}{\mathrm{d}\,\mu^2} \ln \left(Z_5^f\, Z_5^{ms} \right) \\
&= \epsilon + a_s \Big\{ 4 C_F \epsilon \Big\} + a_s^2 \Big\{ C_A C_F \Big(\frac{214}{9}\epsilon \Big) - C_F^2 \Big(28\epsilon \Big) + C_F n_f \Big(-\frac{31}{9}\epsilon - 6 \Big) \Big\} \\
&+ a_s^3 \Big\{ C_A^2 C_F \Big(\frac{2147}{9}\epsilon - 168\epsilon\zeta_3 \Big) + C_A C_F^2 \Big(480\epsilon\zeta_3 - \frac{4550}{9}\epsilon \Big) \\
&+ C_A C_F n_f \Big(-110\epsilon\zeta_3 + \frac{133}{27}\epsilon - \frac{142}{3} \Big) + C_F^3 (170\epsilon - 288\epsilon\zeta_3) \\
&+ C_F^2 n_f \Big(104\epsilon\zeta_3 - \frac{869}{18}\epsilon + 18 \Big) + C_F n_f^2 \Big(\frac{4}{3} - \frac{316}{27}\epsilon \Big) \Big\}\,.
\end{aligned}$$

- The 4-dimensional limit ($\epsilon = 0$) is in agreement with [Larin 93].
- The ϵ-dependent parts to $\mathcal{O}(a_s^2)$ agree with [Ahmed, Gehrmann, Mathews, Rana, Ravindran 15].

The (non-Abelian) Adler-Bardeen theorem

The equality verified to 4-loop order in QCD for the first time:

$$Z_{F\tilde{F}} = Z_{a_s}$$

where non-quadratic Casimirs start to appear.

The axial-anomaly equation in QCD in terms of the bare fields:

$$\mu^{2\epsilon} \left(Z_J - n_f \, \mathrm{T}_F \, a_s \, Z_{FJ} \right) \left[\partial_\mu J_5^\mu \right]_B = \hat{a}_s \, n_f \, \mathrm{T}_F \left[F\tilde{F} \right]_B$$

- In an Abelian theory in Pauli-Villar regularization (with an anticommuting γ_5), the coefficient is 1 to all orders [Adler 69; Adler, Bardeen 69]
- The coefficient is not 1 with a non-anticommuting γ_5 in DR in QCD, but the LHS current remains ***RG-invariant*** (albeit in D=4 limit):

$$\gamma_{F\tilde{F}} = -\mu^2 \frac{\mathrm{d} \ln a_s}{\mathrm{d} \mu^2} = -\beta + \epsilon \,, \;\; \gamma_J = n_f \, \mathrm{T}_F \, a_s \, \gamma_{FJ} \,.$$

- An all-order argument of the non-Abelian extension was sketched [Breitenlohner, Maison, Stelle 84]; Finally, a proof is completed only recently (!) [Lúscher, Weisz 21]
- However, Z_J is not predicted and still needs to be computed order by order ...

Summary and Outlook

- We have described a set-up for computing the renormalization constants of axial-vector currents in QCD with a non-anticommuting γ_5 in dimensional regularization.
- We have extended the result of Z_J, and in particular, of the finite non-$\overline{\mathrm{MS}}$ factor Z_5^f, of the flavor-singlet axial-vector current to $\mathcal{O}(\alpha_s^3)$.
- Furthermore, we have verified explicitly up to 4-loop order $Z_{F\tilde{F}} = Z_{\alpha_s}$ in the $\overline{\mathrm{MS}}$ scheme, from which follows $\gamma_J = a_s\, n_f\, \mathrm{T}_F\, \gamma_{FJ}$ valid to $\mathcal{O}(\alpha_s^4)$.
- A proof of $Z_{F\tilde{F}} = Z_{\alpha_s}$ in dimensionally regularized QCD to all orders is recently completed [Lúscher, Weisz 21].
- Our result has found its first practical application in the computation of the 3-loop singlet contribution to the massless axial quark form factor [Gehrmann, Primo 21].
- It could be used also in places such as singlet contributions to the polarized structure functions in charged-current deep-inelastic scattering...

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THANK YOU

Backup Slides

The r.h.s. of eq. (19) of [Phys.Lett.B303 113] reads

$$M = R_{\overline{\rm MS}}\,\epsilon_{\mu_1\mu_2\rho\sigma}\,\frac{p_1^\rho}{p_1^2}\,\frac{\partial}{\partial q_\sigma}\,\int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y}\,\langle 0|\hat{T}\left[\frac{\partial}{\partial y^\mu} J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|0\rangle|_{\rm amp}\,,$$

where $R_{\overline{\rm MS}}$ denotes the R-operation in the $\overline{\rm MS}$ scheme.
Since the equal-time commutator between $J_5^0(y) = \sum_\psi \bar{\psi}(y)\gamma^0\gamma_5\psi(y)$ and $A_a^\mu(x)$ vanishes, the derivative w.r.t y can be pulled in front of the integral

$$\begin{aligned} M &= R_{\overline{\rm MS}}\,\epsilon_{\mu_1\mu_2\rho\sigma}\,\frac{p_1^\rho}{p_1^2}\,\frac{\partial}{\partial q_\sigma}\,\int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y}\,\frac{\partial}{\partial y^\mu}\langle 0|\hat{T}\left[J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|0\rangle|_{\rm amp} \\ &= R_{\overline{\rm MS}}\,\epsilon_{\mu_1\mu_2\rho\sigma}\,\frac{p_1^\rho}{p_1^2}\,\frac{\partial}{\partial q_\sigma}\,\int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y} \\ &\quad \frac{\partial}{\partial y^\mu}\left(\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\, e^{ik_1\cdot x + iQ\cdot y}\,\Gamma_{lhs}^{\mu\mu_1\mu_2}(k_1, -Q-k_1)\right) \\ &= R_{\overline{\rm MS}}\,\epsilon_{\mu_1\mu_2\rho\sigma}\,\frac{p_1^\rho}{p_1^2}\,\frac{\partial}{\partial q_\sigma}\left(iq_\mu\Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, q-p_1)\right) \\ &= R_{\overline{\rm MS}}\,\epsilon_{\mu_1\mu_2\rho\sigma}\,\frac{p_1^\rho}{p_1^2}\, i\left(\Gamma_{lhs}^{\rho\mu_1\mu_2}(p_1, q-p_1) + q_\mu\frac{\partial}{\partial q_\sigma}\Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, q-p_1)\right)\,, \end{aligned}$$

where in the second line the Fourier transformation of the correlation function in coordinate space in terms of the momentum-space matrix element $\Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, p_2)$ has been inserted.
The second term should not contribute in the limit $q \to 0$ assuming $\Gamma_{lhs}^{\rho\mu_1\mu_2}(p_1, q-p_1)$ has no power divergence $q \to 0$