The tropical approach to numerical Feynman integration

M. Borinsky, Nikhef Amsterdam

May 17, Radcor & Loopfest 2021

Talk based on arXiv:2008.12310,
to appear in Annales de l'Institut Henri Poincaré D

Motivation

Quantum field theory

Quantum field theory

e.g. $\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 + \lambda\frac{\varphi^4}{4!}$

Quantum field theory

e.g. $\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 + \lambda\frac{\varphi^4}{4!}$

$$\downarrow$$

perturbative expansions

$$\mathcal{O}(\hbar) = \sum_{n\geq 0} A_n \hbar^n = \sum_{\text{graphs } G} \frac{\phi(G)}{|\operatorname{Aut} G|}\hbar^{L_\Gamma}$$

where $\phi(G) = \prod_{\ell} \int \mathrm{d}^D \boldsymbol{k}_\ell \prod_{e\in E} \frac{1}{D_e(\{\boldsymbol{k}\},\{\boldsymbol{p}\},m_e)}$.

A computational question

$$\mathcal{O}(\hbar) = \sum_{n \geq 0} A_n \hbar^n$$

Lower orders $A_0, A_1, A_2, \ldots$ needed to interpret experimental data.

A computational question

$$\mathcal{O}(\hbar) = \sum_{n \geq 0} A_n \hbar^n$$

Lower orders $A_0, A_1, A_2, \ldots$ needed to interpret experimental data.

Practical question

What is the value of $A_0, A_1, A_2, A_3, \ldots$?

How can we calculate them efficiently?

A computational question

$$\mathcal{O}(\hbar) = \sum_{n\geq 0} A_n \hbar^n$$

Lower orders $A_0, A_1, A_2, \ldots$ needed to interpret experimental data.

Practical question

What is the value of $A_0, A_1, A_2, A_3, \ldots$?

How can we calculate them efficiently?

Associated 'meta' question

Is there an algorithm to compute A_n?

What is its runtime?

$$\mathcal{O}(\hbar) = \sum_{n\geq 0} A_n \hbar^n = \sum_{\text{graphs } G} \frac{\phi(G)}{|\,\text{Aut } G|} \hbar^{L_\Gamma}$$

Runtime to compute A_n for n large:

$$\mathcal{O}\Big(\alpha^n \Gamma(n+\beta) \quad \times \quad F(n)\Big)$$

→ number of Feynman graphs with n loops for $n \to \infty$ (α and β depend on theory and observable)

→ time it takes to evaluate a single Feynman integral of order n

$F(n) =$ time to evalute n-loop Feynman integral

'Analytic calculation':

- Unclear: What is an analytic answer for an integral?
- Can ask for an expression within a specific function space
- No function space is known that works for all F. integrals

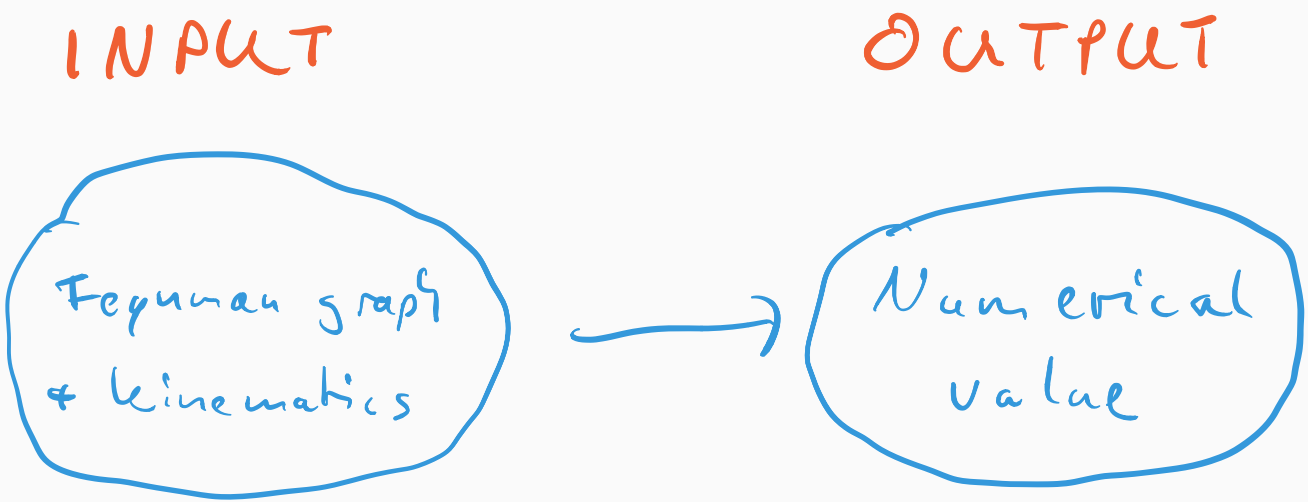
  $\Rightarrow$ complicated to formulate the question

$\Rightarrow$ What counts as an analytic answer?

analytic answer to computation $\Leftrightarrow$ fast algorithm to perform computation numerically

Cut the middle man!

→ direct numerical evaluation



Direct evaluation

$$\mathcal{O}(\hbar) = \sum_{n\geq 0} A_n \hbar^n = \sum_{\text{graphs } G} \frac{\phi(G)}{|\,\text{Aut } G|} \hbar^{L_\Gamma}$$

$$\text{where } \phi(G) = \prod_{\ell} \int \mathrm{d}^D \boldsymbol{k}_\ell \prod_{e\in E} \frac{1}{D_e(\{\boldsymbol{k}\}, \{\boldsymbol{p}\}, m_e)}$$

$$\mathcal{O}(\hbar) = \sum_{n\geq 0} A_n \hbar^n = \sum_{\text{graphs } G} \frac{\phi(G)}{|\operatorname{Aut} G|} \hbar^{L_\Gamma}$$

$$\text{where } \phi(G) = \prod_{\ell} \int \mathrm{d}^D \boldsymbol{k}_\ell \prod_{e\in E} \frac{1}{D_e(\{\boldsymbol{k}\}, \{\boldsymbol{p}\}, m_e)}$$

Via the Schwinger trick we can rewrite the Feynman integral as

$$\phi(G) = \Gamma(\omega_G) \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G(\boldsymbol{x})^{D/2}} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}.$$

with $\omega_G = E - \frac{1}{2} D h_1(G)$.

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G(\boldsymbol{x})^{D/2}} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})}\right)^{\omega_G}$$

where

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G(\boldsymbol{x})^{D/2}} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})}\right)^{\omega_G}$$

where

- Ω is the standard volume form on $\mathbb{P}^{E-1}$:
  $\Omega = \sum_{k=1}^{E}(-1)^k dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_E$.
- $\Psi_G = \sum_T \prod_{e \notin T} x_e$ (sum over spanning trees)
- $\Phi_G = \sum_F \|\boldsymbol{p}(F)\|^2 \prod_{e \notin F} x_e + \Psi_G \sum_e m_e^2 x_e$(sum over 2-forests)
- Ψ_G and Φ_G are homogeneous polynomials in $x_1, \ldots, x_E$.
- We assume that the integral exists.

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

$\Psi_G(\boldsymbol{x})$ and $\Phi_G(\boldsymbol{x})$ exhibit complicated geometric structures.

$\Rightarrow$ These integrals are hard to evaluate

$\Rightarrow$ These integrals are very interesting

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})}\right)^{\omega_G}$$

Obstruction for direct numerical evaluation

Integrand has singularities on the boundary of $\mathbb{P}^{E-1}_{>0}$.

I.e. vanishing locus of Ψ_G and Φ_G meets the boundary of $\mathbb{P}^{E-1}_{>0}$.

$\Rightarrow$ Singularities have to be blown up first

Established solutions

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})}\right)^{\omega_G} \quad \to \quad \int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

Sector decomposition approach

- Algorithms to perform blowups in the general case:
  Binoth, Heinrich ’03; Bogner, Weinzierl ’07; (Hironaka 1964)
- Simple geometric interpretation:
  Kaneko, Ueda ’09

All algorithms are oblivious to the specific structures on the left!

Established solutions

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})}\right)^{\omega_G} \quad \rightarrow \quad \int_{\mathbb{P}_{>0}^{n-1}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

Sector decomposition approach

- Algorithms to perform blowups in the general case:
  Binoth, Heinrich '03; Bogner, Weinzierl '07; (Hironaka 1964)
- Simple geometric interpretation:
  Kaneko, Ueda '09

All algorithms are oblivious to the specific structures on the left!

previous

Established solutions

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G} \quad \rightarrow \quad \int_{\mathbb{P}_{>0}^{n-1}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

Numerical evaluation using sector decomposition for blowups:

- Runtime to evaluate the integral up to δ-accuracy

$$\approx \mathcal{O}(V^2 \cdot \delta^{-2})$$

  where V is the number of monomials in $\frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}}$.
- For Feynman integrals V grows $\approx$ exponentially with n.

Results MB 2020:

1. Numerical integration is an exercise in tropical geometry.
2. The general (oblivious) approach can be accelerated:

$$\mathcal{O}(V^2 \cdot \delta^{-2}) \quad \rightarrow \quad \mathcal{O}(V^2 + \delta^{-2})$$

$\Rightarrow$ achievable accuracy 'decouples' from integral complexity.

3. Euclidean Feynman integration can be accelerated extremely:

$$\mathcal{O}(V^2 \cdot \delta^{-2}) \approx \mathcal{O}(2^{cn} \cdot \delta^{-2}) \quad \rightarrow \quad \mathcal{O}(n 2^n + \delta^{-2})$$

with $c \gg 1$ where n is the number of edges of the graph.

Theorem MB 2020

There is a ‘fast’ algorithm to approximate the Feynman integral.

Theorem MB 2020

There is a ‘fast’ algorithm to approximate the Feynman integral.

- 3 loops is already a tough challenge for existing programs.

Theorem MB 2020

There is a ‘fast’ algorithm to approximate the Feynman integral.

- 3 loops is already a tough challenge for existing programs.
- New: ≥ 17 loops possible (with basic implementation).

Theorem MB 2020

There is a ‘fast’ algorithm to approximate the Feynman integral.

- 3 loops is already a tough challenge for existing programs.
- New: ≥ 17 loops possible (with basic implementation).
- Caveat: Only Euclidean - no Minkowski regime (so far).

Interesting example

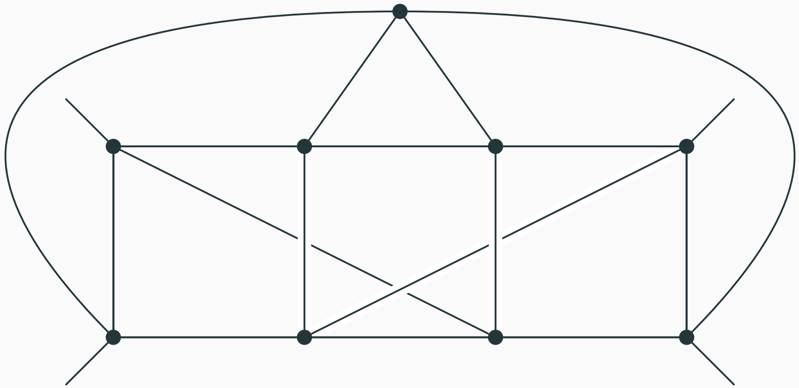


Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

Interesting example

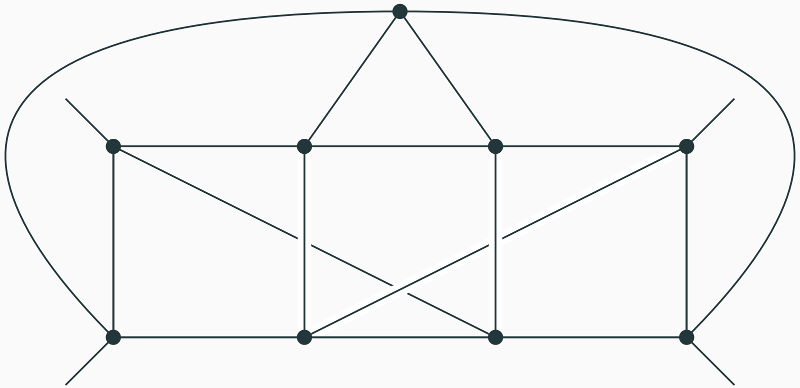


Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

$$\Gamma(\varepsilon) \int_{\mathbb{P}^{E-1}_{>0}} \frac{1}{\Psi_G(\boldsymbol{x})^{2-\varepsilon}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\varepsilon} \Omega \approx \frac{1}{\epsilon}\, 422.9610 \cdot (1 \pm 10^{-6}) + \dots$$

Interesting example

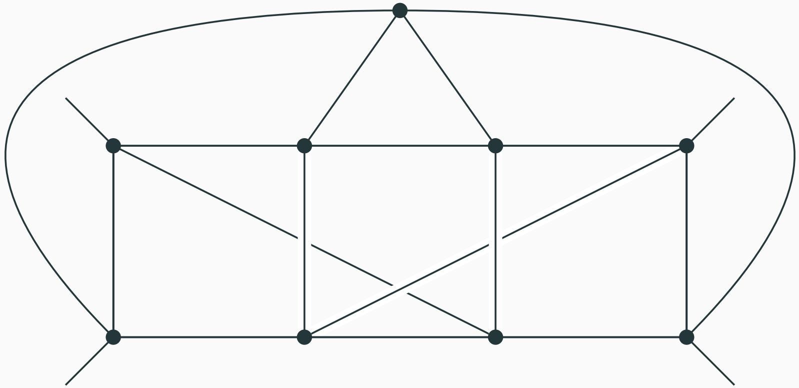


Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

$$\Gamma(\varepsilon)\int_{\mathbb{P}^{E-1}_{>0}} \frac{1}{\Psi_G(\boldsymbol{x})^{2-\varepsilon}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\varepsilon} \Omega \approx \frac{1}{\epsilon}\, 422.9610 \cdot (1 \pm 10^{-6}) + \ldots$$

- $\sim$ 10 CPU secs to compute up to 10^{-3}-accuracy at 8 loops.

Interesting example

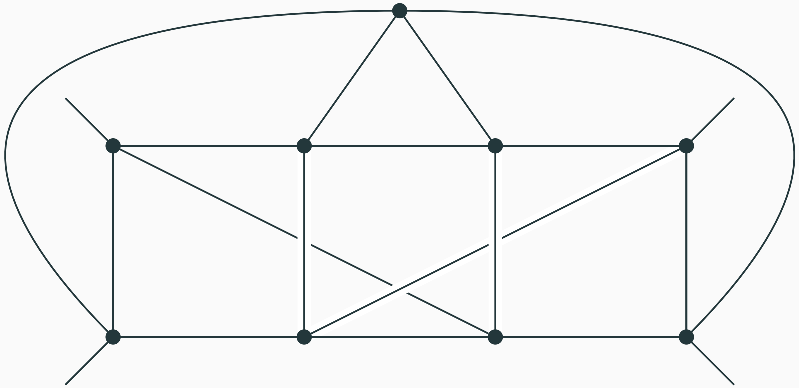


Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

$$\Gamma(\varepsilon)\int_{\mathbb{P}^{E-1}_{>0}} \frac{1}{\Psi_G(\boldsymbol{x})^{2-\varepsilon}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\varepsilon} \Omega \approx \frac{1}{\epsilon}\, 422.9610 \cdot (1 \pm 10^{-6}) + \ldots$$

- $\sim$ 10 CPU secs to compute up to 10^{-3}-accuracy at 8 loops.
- $\sim$ 30 CPU days to compute up to 10^{-6}-accuracy at 8 loops.

Interesting example

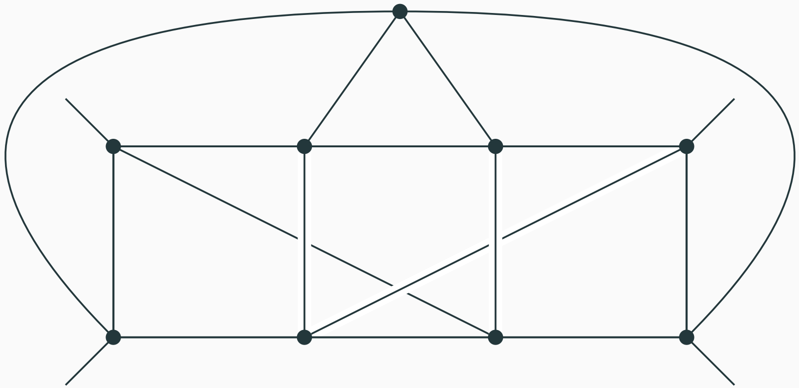


Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

$$\Gamma(\varepsilon) \int_{\mathbb{P}_{>0}^{E-1}} \frac{1}{\Psi_G(\boldsymbol{x})^{2-\varepsilon}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\varepsilon} \Omega \approx \frac{1}{\epsilon}\, 422.9610 \cdot (1 \pm 10^{-6}) + \ldots$$

- $\sim$ 10 CPU secs to compute up to 10^{-3}-accuracy at 8 loops.
- $\sim$ 30 CPU days to compute up to 10^{-6}-accuracy at 8 loops.
- Higher orders in ϵ can also be computed.

The Tropical Approach

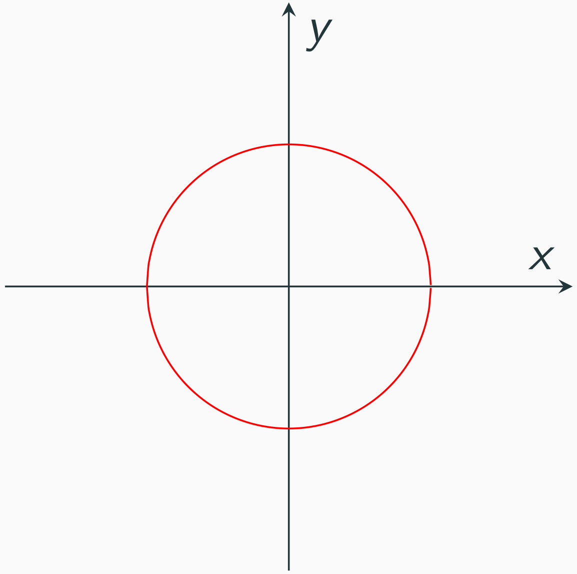
Tropical geometry

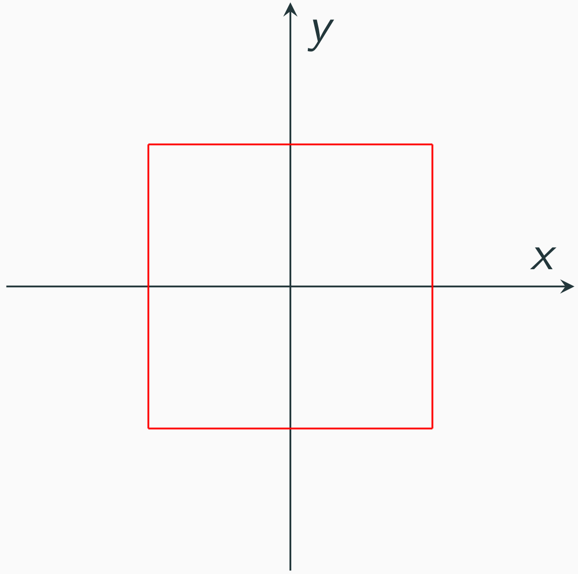
Philosophy

Deform geometry to sacrifice smoothness for simplicity.

Various applications in algebraic geometry.

$$1 = x^2 + y^2 \quad \to \quad 1 = (x^2 + y^2)^{\mathrm{tr}} = \max\{x^2, y^2\}$$





Application to Feynman graph polynomials

$$\Psi_G = \sum_T \prod_{e \notin T} x_e \qquad \Rightarrow \qquad \Psi_G^{\mathrm{tr}} = \max_T \prod_{e \notin T} x_e$$

Application to Feynman graph polynomials

$$\Psi_G = \sum_T \prod_{e \notin T} x_e \qquad \Rightarrow \qquad \Psi_G^{\mathrm{tr}} = \max_T \prod_{e \notin T} x_e$$

$$\Phi_G = \sum_F \|p(F)\|^2 \prod_{e \notin F} x_e \qquad \Rightarrow \qquad \Phi_G^{\mathrm{tr}} = \max_{\substack{F \\ \text{s.t. } \|p(F)\|^2 \neq 0}} \prod_{e \notin F} x_e$$

Feynman integral: $$\phi(G) = \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{(\Psi_G)^{D/2}} \left(\frac{\Psi_G}{\Phi_G} \right)^{\omega_G}$$

Feynman integral: $$\phi(G) = \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{(\Psi_G)^{D/2}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\omega_G}$$

$\Rightarrow$ Tropicalized version: $$\phi^{\mathrm{tr}}(G) = \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{(\Psi^{\mathrm{tr}}_G)^{D/2}} \left(\frac{\Psi^{\mathrm{tr}}_G}{\Phi^{\mathrm{tr}}_G}\right)^{\omega_G}$$

Feynman integral: $$\phi(G) = \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{(\Psi_G)^{D/2}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\omega_G}$$

$\Rightarrow$ Tropicalized version: $$\phi^{\mathrm{tr}}(G) = \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{(\Psi^{\mathrm{tr}}_G)^{D/2}} \left(\frac{\Psi^{\mathrm{tr}}_G}{\Phi^{\mathrm{tr}}_G}\right)^{\omega_G}$$

QFT tropicalization

Replace all instances of Ψ and Φ with their tropicalized versions.

Tropical approach

Computations are easy for the tropicalized QFTs:

- Tropicalized Feynman integrals are easily calculated exactly.

Tropical approach

Computations are easy for the tropicalized QFTs:

- Tropicalized Feynman integrals are easily calculated exactly.
- They correspond to volumes of certain polytopes.

Tropical approach

Computations are easy for the tropicalized QFTs:

- Tropicalized Feynman integrals are easily calculated exactly.
- They correspond to volumes of certain polytopes.
- All observables are rational numbers/functions.

  Panzer 2019; MB 2020

Tropical approach

Computations are easy for the tropicalized QFTs:

- Tropicalized Feynman integrals are easily calculated exactly.
- They correspond to volumes of certain polytopes.
- All observables are rational numbers/functions.

  Panzer 2019; MB 2020

→ Hepp-Bound

Tropical approach

Computations are easy for the tropicalized QFTs:

- Tropicalized Feynman integrals are easily calculated exactly.
- They correspond to volumes of certain polytopes.
- All observables are rational numbers/functions.
  Panzer 2019; MB 2020
- When the tropical version is known exactly, numerical integration of the original integrals is just an extra step.
  MB 2020

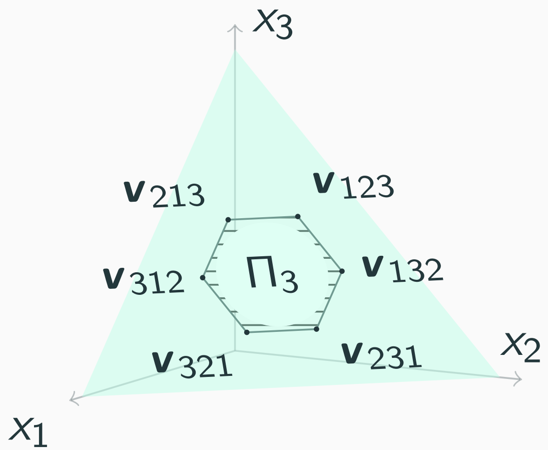
Tropical approach

Computations are easy for the tropicalized QFTs:

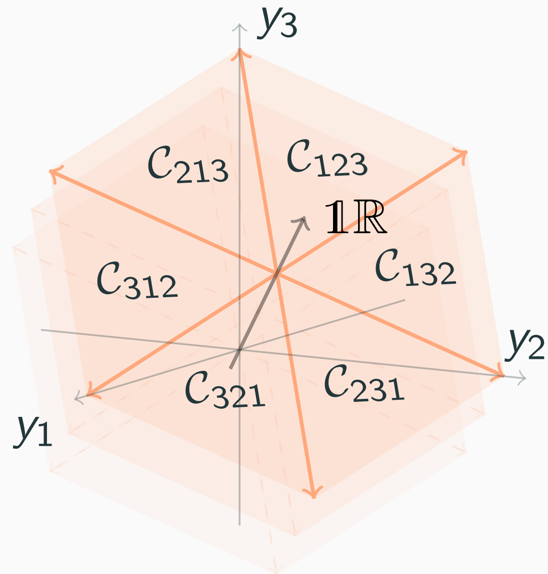
- Tropicalized Feynman integrals are easily calculated exactly.
- They correspond to volumes of certain polytopes.
- All observables are rational numbers/functions.
  Panzer 2019; MB 2020
- When the tropical version is known exactly, numerical integration of the original integrals is just an extra step.
  MB 2020

$\Rightarrow$ Better understanding of tropical geometry leads to faster numerical integration.

Relevant polytopes: Generalized permutahedra



(a) The permutahedron $\Pi_3 \subset \mathbb{R}^3$.



(b) Dual of Π_3: The corresponding braid arrangement fan.

Gen. permutahedra are well understood thanks to **Postnikov 2008** and **Aguiar, Ardila 2017**.

Current limitations

Problem: Non-Euclidean kinematic regions are not as fast, because

- The generalized permutahedron structure breaks down at singular momentum configurations (IR singularities).

Current limitations

Problem: Non-Euclidean kinematic regions are not as fast, because

- The generalized permutahedron structure breaks down at singular momentum configurations (IR singularities).
- Φ_G can vanish in the integration domain ($\Rightarrow$ analytic continuation is necessary).

Conclusions

Tropical approach to Feynman integration

- Fast numerical evaluation of Euclidean Feynman integrals

Tropical approach to Feynman integration

- Fast numerical evaluation of Euclidean Feynman integrals
- Loop order ≈ 15 or ≈ 30 edges are easily possible.
- Applications:
  - Calculations in the Euclidean regime.
  - Renormalization group calculations.
  - (Massive) form factor calculations.
  - ...

Tropical approach to Feynman integration

- Fast numerical evaluation of Euclidean Feynman integrals
- Loop order ≈ 15 or ≈ 30 edges are easily possible.
- Applications:
  - Calculations in the Euclidean regime.
  - Renormalization group calculations.
  - (Massive) form factor calculations.
  - ...
- Much left to explore:
  - Tropical amplitudes
  - What is the role of the generalized permutahedra?
  - IR singularities/Minkowski space
  - ...

Tropical approach to Feynman integration

- Fast numerical evaluation of Euclidean Feynman integrals
- Loop order ≈ 15 or ≈ 30 edges are easily possible.
- Applications:
    - Calculations in the Euclidean regime.
    - Renormalization group calculations.
    - (Massive) form factor calculations.
    - ...
- Much left to explore:
    - Tropical amplitudes
    - What is the role of the generalized permutahedra?
    - IR singularities/Minkowski space
    - ...

CODE AVAILABLE AT

https://github.com/michibo/tropical-feynman-quadrature