

Type II superstring field theory revisited

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Introduction

- There are mainly three complementary approaches of superstring field theory.
- We have succeeded in completing (at least classically) superstring field theories for all the three approaches **except for the WZW-like type II theory.**

	type I (open)	heterotic (closed)	type II (closed)
WZW-like	○	○	×
A_∞/L_∞ -based	○	○	△
Sen	○	○	○

In this talk,

- We first revisit the L_∞ -algebra-based type II superstring field theory and propose **a new construction method**, which is an extension of **the symmetric construction** given by Erler-Konopka-Sachs for NS-NS sector.
- Then, we construct the WZW-like action through the map defined using the gauge product of the new (symmetric) construction.

Type II superstring field theory with cyclic L_∞ structure [Kunitomo-Sugimoto]

String Field : $\Phi = \Phi_{NS-NS} + \Phi_{R-NS} + \Phi_{NS-R} + \Phi_{R-R} \in \mathcal{H}_{res} \subset \mathcal{H}_{small}$

ghost number = 2

picture numbers = $(-1, -1) + (-1/2, -1) + (-1, -1/2) + (-1/2, -1/2)$

Constraints: (1) $b_0^- \Phi = L_0^- \Phi = 0$, (closed string)

(2) $\mathcal{G}\mathcal{G}^{-1}\Phi = \Phi$, (extra),

where

$$\begin{aligned}\mathcal{G} &= \pi^{(0,0)} + \pi^{(1,0)}X + \pi^{(0,1)}\bar{X} + \pi^{(1,1)}X\bar{X}, \\ \mathcal{G}^{-1} &= \pi^{(0,0)} + \pi^{(1,0)}Y + \pi^{(0,1)}\bar{Y} + \pi^{(1,1)}Y\bar{Y},\end{aligned}$$

with projector onto each component:

$$\begin{aligned}\pi^{(0,0)}\Phi &= \Phi_{NS-NS}, & \pi^{(1,0)}\Phi &= \Phi_{R-NS}, \\ \pi^{(0,1)}\Phi &= \Phi_{NS-R}, & \pi^{(1,1)}\Phi &= \Phi_{R-R}.\end{aligned}$$

and

$$\begin{aligned}
X &= -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0, & Y &= -2\frac{G}{L^+}\delta(\gamma_0), \\
\bar{X} &= -\delta(\bar{\beta}_0)\bar{G}_0 + \delta'(\bar{\beta}_0)\bar{b}_0, & \bar{Y} &= -2\frac{\bar{G}}{L^+}\delta(\bar{\gamma}_0).
\end{aligned}$$

If we expand each component of $\Phi \in \mathcal{H}_{res}$ in the ghost zero-modes, it has the form

$$\begin{aligned}
\Phi_{NS-NS} &= \phi_{NS-NS} - c_0^+ \psi_{NS-NS}, \\
\Phi_{R-NS} &= \phi_{R-NS} - \frac{1}{2}(\gamma_0 + 2c_0^+ G)\psi_{R-NS}, \\
\Phi_{NS-R} &= \phi_{NS-R} - \frac{1}{2}(\bar{\gamma}_0 + 2c_0^+ \bar{G})\psi_{NS-R} \\
\Phi_{R-R} &= \phi_{R-R} - \frac{1}{2}(\gamma_0 \bar{G} - \bar{\gamma}_0 G + 2c_0^+ G \bar{G})\psi_{R-R},
\end{aligned}$$

ϕ s and ψ s correspond to the field and anti-field of the BV formulation in the gauge-fixed bases, respectively.

Symplectic forms:

$$\text{In } \mathcal{H}_{small} \quad \omega_s(\Phi_1, \Phi_2) = \langle \Phi_1 | c_0^- | \Phi_2 \rangle,$$

$$\begin{aligned} \text{In } \mathcal{H}_{res} \quad \Omega(\Phi_1, \Phi_2) &= \omega_s(\Phi_1, \mathcal{G}^{-1}\Phi_2) \\ &= \omega_s(\Phi_{1NS-NS}, \Phi_{2NS-NS}) + \omega_s(\Phi_{1R-NS}, Y\Phi_{2R-NS}) \\ &\quad + \omega_s(\Phi_{1NS-R}, \bar{Y}\Phi_{2NS-R}) + \omega_s(\Phi_{1R-R}, Y\bar{Y}\Phi_{2R-R}) \\ &= \sum_i (\langle \phi_i | \psi_i \rangle + \langle \psi_i | \phi_i \rangle). \end{aligned}$$

A symplectic form ω_l in \mathcal{H}_{large} is similarly defined to ω_s , and we have

$$\omega_l(\Phi_1, \Phi_2) = \omega_s(\xi_0 \bar{\xi}_0 \Phi_1, \Phi_2) \quad \text{for } \Phi_1, \Phi_2 \in \mathcal{H}_{res}.$$

$(n + 1)$ -string interaction is represented by

$$\text{multi-linear map: } L_n : (\mathcal{H}_{res})^{\wedge n} \longrightarrow \mathcal{H}_{res}, \quad (n \geq 2)$$

where $(\mathcal{H}_{res})^{\wedge n}$ is the space of the symmetrized tensor product:

$$(\mathcal{H}_{res})^{\wedge n} \ni \Phi_1 \wedge \cdots \wedge \Phi_n = \sum_{\sigma} (-1)^{\epsilon(\sigma)} \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(n)},$$

and we identify $L_1\Phi = Q\Phi$.

If $\{L_n\}$ satisfy

$$\sum_{\sigma} \sum_{m=1}^n \frac{(-1)^{\epsilon(\sigma)}}{m!(n-m)!} L_{n-m+1}(L_m(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}) = 0,$$

$$\Omega(\Phi_1, L_n(\Phi_2, \cdots, \Phi_{n+1})) = -(-1)^{|\Phi_1|} \Omega(L_n(\Phi_1, \cdots, \Phi_n), \Phi_{n+1}),$$

it is called a **cyclic L_{∞} algebra**, $(\mathcal{H}_{res}, \Omega, \{L_n\})$.

If we have $(\mathcal{H}^{res}, \Omega, \{L_n\})$ (with appropriate ghost and picture numbers), the action is given by

$$I = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \Omega(\Phi, L_{n+1}(\underbrace{\Phi, \dots, \Phi}_{n+1})),$$

which is invariant under the gauge transformation

$$\delta\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} L_{n+1}(\underbrace{\Phi, \dots, \Phi}_n, \Lambda).$$

- $\{L_n\}$ with appropriate ghost numbers, but no picture number, can be constructed similarly to the bosonic closed string field theory.
- We must find how to construct $\{L_n\}$ with appropriate picture numbers from those without picture numbers, (*bosonic* $\{L_n\}$).

How to construct $(\mathcal{H}_{res}, \Omega, \{L_n\})$

Coalgebra representation:

Symmetrized tensor algebra: $\mathcal{SH}_{res} = (\mathcal{H}_{res})^{\wedge 0} \oplus (\mathcal{H}_{res})^{\wedge 1} \oplus (\mathcal{H}_{res})^{\wedge 2} \oplus \dots$

Coderivation: $\mathbf{L}_n : \mathcal{SH}_{res} \longrightarrow \mathcal{SH}_{res}$

defined by

$$\mathbf{L}_n \Phi_1 \wedge \dots \wedge \Phi_m = 0, \quad \text{for } m < n,$$

$$\mathbf{L}_n \Phi_1 \wedge \dots \wedge \Phi_m = L_n(\Phi_1 \wedge \dots \wedge \Phi_m), \quad \text{for } m = n,$$

$$\mathbf{L}_n \Phi_1 \wedge \dots \wedge \Phi_m = (L_n \wedge \mathbb{I}_{m-n})\Phi_1 \wedge \dots \wedge \Phi_m, \quad \text{for } m < n.$$

The L_∞ algebra is represented by a (degree odd) coderivation $\mathbf{L} = \sum_{n=0}^{\infty} \mathbf{L}_{n+1}$ satisfying

$$[\mathbf{L}, \mathbf{L}] = 0. \quad \left([\cdot, \cdot] : \text{graded commutator} \right)$$

How to construct L :

L with appropriate picture numbers:

$$L = \sum_{p,r=0}^{\infty} \sum_{\bar{p},\bar{r}=0}^{\infty} L_{p+r+1,\bar{p}+\bar{r}+1}^{(p,\bar{p})} |_{(2r,2\bar{r})},$$

$$\left(L_{p+r+1,\bar{p}+\bar{r}+1}^{(p,\bar{p})} |_{(2r,2\bar{r})} \equiv \delta_{p+r,\bar{p}+\bar{r}} L_{p+r+1}^{(p,\bar{p})} |_{(2r,2\bar{r})} \right),$$

(p, \bar{p}) : picture numbers

$(2r, 2\bar{r})$: Ramond numbers = # of Ramond inputs – # of Ramond outputs

However, it is not easy to construct cyclic L directly because the Ramond number is not invariant under the cyclic permutation.

So, instead, we consider degree odd coderivation

$$\mathbf{B} = \sum_{p,r=0}^{\infty} \sum_{\bar{p},\bar{r}=0}^{\infty} \mathbf{B}_{p+r+1,\bar{p}+\bar{r}+1}^{(p,\bar{p})} |^{(2r,2\bar{r})},$$

$$\left(\mathbf{B}_{p+r+1,\bar{p}+\bar{r}+1}^{(p,\bar{p})} |^{(2r,2\bar{r})} \equiv \delta_{p+r,\bar{p}+\bar{r}} \mathbf{B}_{p+r+1}^{(p,\bar{p})} |^{(2r,2\bar{r})}, \quad \mathbf{B}_1 \equiv 0 \right).$$

$(2r, 2\bar{r})$ (superscript):

cyclic Ramond numbers = $\#$ of Ramond inputs + $\#$ of Ramond output,

which is **invariant unber cyclic permutation**.

And suppose that $(\mathcal{H}_l, \mathcal{D})$ with

$$\pi_1 \mathcal{D} = \pi_1(\mathbf{Q} - \boldsymbol{\eta} - \bar{\boldsymbol{\eta}}) + (\pi_1^{(0,0)} + \pi_1^{(1,0)} + \pi_1^{(0,1)})\mathbf{B} + \frac{1}{2}(X + \bar{X})\pi_1^{(1,1)}\mathbf{B},$$

is an L_∞ algebra: $[\mathcal{D}, \mathcal{D}] = 0$.

Or equivalently $(\mathcal{H}_l, \mathcal{D})$, $(\mathcal{H}_l, \mathcal{C})$, and $(\mathcal{H}_l, \bar{\mathcal{C}})$ with

$$\pi_1 \mathcal{D} = \pi_1 \mathbf{Q} + \pi_1^{(0,0)} \mathbf{B},$$

$$\pi_1 \mathcal{C} = \pi_1 \boldsymbol{\eta} + \pi_1^{(1,0)} \mathbf{B} - \frac{1}{2} \bar{X} \pi_1^{(1,1)} \mathbf{B}, \quad \pi_1 \bar{\mathcal{C}} = \pi_1 \bar{\boldsymbol{\eta}} + \pi_1^{(0,1)} \mathbf{B} - \frac{1}{2} X \pi_1^{(1,1)} \bar{\mathbf{B}}$$

are three (mutually commutative) L_∞ algebras:

$$[\mathcal{D}, \mathcal{D}] = [\mathcal{C}, \mathcal{C}] = [\bar{\mathcal{C}}, \bar{\mathcal{C}}] = [\mathcal{D}, \mathcal{C}] = [\mathcal{D}, \bar{\mathcal{C}}] = [\mathcal{C}, \bar{\mathcal{C}}] = 0.$$

If we have $(\mathcal{H}_l, \mathbf{D})$, $(\mathcal{H}_l, \mathbf{C})$, and $(\mathcal{H}_l, \bar{\mathbf{C}})$, then cohomomorphism (L_∞ -morphism)

$$\pi_1 \hat{\mathbf{F}}^{-1} = \pi_1 \mathbb{I} - \left(\Xi \pi^{(1,0)} + \bar{\Xi} \pi^{(0,1)} + \frac{1}{2} (\Xi \bar{X} + X \bar{\Xi}) \pi^{(1,1)} \right) \pi_1 \mathbf{B} \quad (1)$$

maps them as

$$\pi_1 \hat{\mathbf{F}}^{-1} \mathbf{D} \hat{\mathbf{F}} = \pi_1 \mathbf{Q} + \mathcal{G} \pi_1 \mathbf{B} \hat{\mathbf{F}} \equiv \pi_1 \mathbf{L}, \quad (2a)$$

$$\pi_1 \hat{\mathbf{F}}^{-1} \mathbf{C} \hat{\mathbf{F}} = \pi_1 \boldsymbol{\eta}, \quad \pi_1 \hat{\mathbf{F}}^{-1} \bar{\mathbf{C}} \hat{\mathbf{F}} = \pi_1 \bar{\boldsymbol{\eta}}, \quad (2b)$$

satisfying $[\mathbf{L}, \mathbf{L}] = [\boldsymbol{\eta}, \mathbf{L}] = [\bar{\boldsymbol{\eta}}, \mathbf{L}] = 0$ and

$$\Omega(\Phi_1, L_n(\Phi_2, \dots, \Phi_{n+1})) = \omega_s(\Phi_1, b_n(\Phi_2, \dots, \Phi_{n+2})), \quad (\mathbf{b} = \mathbf{B} \hat{\mathbf{F}}).$$

We can show that if \mathbf{B} is cyclic wrt ω_l , then \mathbf{b} is cyclic wrt ω_s , and thus, \mathbf{L} is cyclic wrt Ω . $(\mathcal{H}_{res}, \Omega, \mathbf{L})$ is the desired cyclic L_∞ algebra.

Explicit construction of $(\mathcal{H}_l, \mathcal{D})$

$[\mathcal{D}, \mathcal{D}] = 0$ is equivalent to

$$\begin{aligned}
 [\mathcal{Q}, \mathcal{B}] + \frac{1}{2}[\mathcal{B}, \mathcal{B}]^{11} &= 0, \\
 [\eta, \mathcal{B}] - \frac{1}{2}[\mathcal{B}, \mathcal{B}]^{21} - \frac{1}{4}[\mathcal{B}, \mathcal{B}]_{\bar{X}}^{22} &= 0, \\
 [\bar{\eta}, \mathcal{B}] - \frac{1}{2}[\mathcal{B}, \mathcal{B}]^{12} - \frac{1}{4}[\mathcal{B}, \mathcal{B}]_X^{22} &= 0,
 \end{aligned} \tag{3}$$

where

$$\begin{aligned}
 [\mathcal{A}, \mathcal{B}]^{11} &= \sum_n \left(\mathcal{A}_n \left(\pi_1^{(0,0)} \mathcal{B} \wedge \mathbb{I}_{n-1} \right) - (-1)^{AB} \mathcal{B}_n \left(\pi_1^{(0,0)} \mathcal{A} \wedge \mathbb{I}_{n-1} \right) \right), \\
 [\mathcal{A}, \mathcal{B}]^{21} &= \sum_n \left(\mathcal{A}_n \left(\pi_1^{(1,0)} \mathcal{B} \wedge \mathbb{I}_{n-1} \right) - (-1)^{AB} \mathcal{B}_n \left(\pi_1^{(1,0)} \mathcal{A} \wedge \mathbb{I}_{n-1} \right) \right), \\
 [\mathcal{A}, \mathcal{B}]_X^{22} &= \sum_n \left(\mathcal{A}_n \left(X \pi_1^{(1,1)} \mathcal{B} \wedge \mathbb{I}_{n-1} \right) - (-1)^{AB} \mathcal{B}_n \left(X \pi_1^{(1,1)} \mathcal{A} \wedge \mathbb{I}_{n-1} \right) \right),
 \end{aligned}$$

and so on.

We define generating functions by

$$B(s, \bar{s}, t) = \sum_{p,m,r=0}^{\infty} \sum_{\bar{p},\bar{m},\bar{r}=0}^{\infty} t^{p+\bar{p}} s^m \bar{s}^{\bar{m}} B_{p+m+r+1, \bar{p}+\bar{m}+\bar{r}+1}^{(p,\bar{p})} \Big|_{(2r,2\bar{r})}, \quad (4a)$$

$$\lambda(s, \bar{s}, t) = \sum_{p,m,r=0}^{\infty} \sum_{\bar{p},\bar{m},\bar{r}=0}^{\infty} t^{p+\bar{p}} s^m \bar{s}^{\bar{m}} \lambda_{p+m+r+2, \bar{p}+\bar{m}+\bar{r}+1}^{(p+1,\bar{p})} \Big|_{(2r,2\bar{r})}, \quad (4b)$$

$$\bar{\lambda}(s, \bar{s}, t) = \sum_{p,m,r=0}^{\infty} \sum_{\bar{p},\bar{m},\bar{r}=0}^{\infty} t^{p+\bar{p}} s^m \bar{s}^{\bar{m}} \bar{\lambda}_{p+m+r+1, \bar{p}+\bar{m}+\bar{r}+2}^{(p,\bar{p}+1)} \Big|_{(2r,2\bar{r})}. \quad (4c)$$

For later use, we denote the gauge products restricted on NS-NS sector as

$$\lambda^{NS-NS}(s, \bar{s}, t) = \sum_{p,m=0}^{\infty} \sum_{\bar{p},\bar{m}=0}^{\infty} t^{p+\bar{p}} s^m \bar{s}^{\bar{m}} \lambda_{p+m+2, \bar{p}+\bar{m}+1}^{(p+1,\bar{p})} \Big|_{(0,0)},$$

$$\bar{\lambda}^{NS-NS}(s, \bar{s}, t) = \sum_{p,m=0}^{\infty} \sum_{\bar{p},\bar{m}=0}^{\infty} t^{p+\bar{p}} s^m \bar{s}^{\bar{m}} \bar{\lambda}_{p+m+1, \bar{p}+\bar{m}+2}^{(p,\bar{p}+1)} \Big|_{(0,0)}.$$

If generating functions (4) satisfy the differential equations

$$\begin{aligned}
\partial_t \mathbf{B}(s, \bar{s}, t) &= [\mathbf{Q}, (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)] + [\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]^{11} \\
&\quad + s \left([\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]^{21} + t [\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]_{\bar{X}}^{22} \right) \\
&\quad + \bar{s} \left([\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]^{12} + t [\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]_X^{22} \right) \\
&\quad + s\bar{s} [\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]^{22} \\
&\quad + \frac{s}{2} [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]_{\Xi}^{22} + \frac{\bar{s}}{2} [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]_{\Xi}^{22}, \quad (5a)
\end{aligned}$$

$$\begin{aligned}
\partial_s \mathbf{B}(s, \bar{s}, t) &= [\boldsymbol{\eta}, \boldsymbol{\lambda}(s, \bar{s}, t)] - t \left([\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]^{21} \right. \\
&\quad \left. + \frac{t}{2} [\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]_{\bar{X}}^{22} \right), \quad (5b)
\end{aligned}$$

$$\begin{aligned}
\partial_{\bar{s}} \mathbf{B}(s, \bar{s}, t) &= [\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\lambda}}(s, \bar{s}, t)] - t \left([\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]^{12} \right. \\
&\quad \left. + \frac{t}{2} [\mathbf{B}(s, \bar{s}, t), (\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}})(s, \bar{s}, t)]_X^{22} \right), \quad (5c)
\end{aligned}$$

the relations

$$\begin{aligned}
& [\mathbf{Q}, \mathbf{B}(s, \bar{s}, t)] + \frac{1}{2} [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]^{11} \\
& + \frac{s}{2} \left([\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]^{21} + t [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]_{\bar{X}}^{22} \right) \\
& + \frac{\bar{s}}{2} \left([\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]^{12} + t [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]_X^{22} \right) \\
& + \frac{s\bar{s}}{2} [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]^{22} = 0, \tag{6a}
\end{aligned}$$

$$\begin{aligned}
& [\boldsymbol{\eta}, \mathbf{B}(s, \bar{s}, t)] \\
& - \frac{t}{2} \left([\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]^{21} + \frac{t}{2} [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]_{\bar{X}}^{22} \right) = 0, \tag{6b}
\end{aligned}$$

$$\begin{aligned}
& [\bar{\boldsymbol{\eta}}, \mathbf{B}(s, \bar{s}, t)] \\
& - \frac{t}{2} \left([\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]^{12} + \frac{t}{2} [\mathbf{B}(s, \bar{s}, t), \mathbf{B}(s, \bar{s}, t)]_X^{22} \right) = 0 \tag{6c}
\end{aligned}$$

hold, which reduce to relations (3) at $(s, \bar{s}, t) = (0, 0, 1)$.

Note: In NS-NS sector, the differential equations (5) and the relations (6) reduce to those of symmetric construction given by EKS.

The differential equations (5) can recursively be solved under **intial condition**

$$\mathbf{B}(s, \bar{s}, 0) = \sum_{m, \bar{m}, r, \bar{r}} s^m \bar{s}^{\bar{m}} \mathbf{L}_{m+r+1, \bar{m}+\bar{r}+1}^B |^{(2r, 2\bar{r})},$$

by inserting ξ_0, X_0 etc. into *bosonic product* \mathbf{L}_{n+1}^B so as to respect cyclicity.

Using $\mathbf{B} = \mathbf{B}(0, 0, 1)$, we obtain $(\mathcal{H}_l, \mathcal{D})$, and then, $(\mathcal{H}^{res}, \Omega, \mathbf{L})$ by cohomomorphism (1) as

$$\pi_1 \mathbf{L} = \pi_1 \mathbf{Q} + \mathcal{G} \pi_1 \mathbf{B} \hat{\mathbf{F}},$$

$$\pi_1 \hat{\mathbf{F}} = \pi_1 \mathbb{I} + \left(\Xi \pi^{(1,0)} + \bar{\Xi} \pi^{(0,1)} + \frac{1}{2} (\Xi \bar{X} + X \bar{\Xi}) \pi^{(1,1)} \right) \pi_1 \mathbf{B} \hat{\mathbf{F}}.$$

Map to WZW-like action

L_∞ triplet characterize the type II superstring field theory (Erler)(Matsunaga)

Basic triplet $(\eta, \bar{\eta}; L)$:

$$\eta\Phi = 0, \quad \bar{\eta}\Phi = 0, \quad (\text{constraints}), \quad (7a)$$

$$L(e^{\wedge\Phi}) = 0, \quad (\text{eom}). \quad (7b)$$

Constraints (7a) imply $\Phi \in \mathcal{H}_{small}$.

Cohomomorphism (L_∞ morphism)

$$\hat{g} = \vec{\mathcal{P}} \exp \left(\int_0^1 dt (\lambda + \bar{\lambda})^{NS-NS} (0, 0, t), \right)$$

maps $(\eta, \bar{\eta}; L)$ to dual triplet $(L^\eta, L^{\bar{\eta}}; \tilde{L})$:

$$L^\eta = \hat{g}\eta\hat{g}^{-1}, \quad L^{\bar{\eta}} = \hat{g}\bar{\eta}\hat{g}^{-1}, \quad \tilde{L} = \hat{g}L\hat{g}^{-1}.$$

Note that since \hat{g} is non-trivial **only in NS-NS sector**, it maps Φ as

$$\hat{g}(e^{\wedge\Phi}) = e^{\wedge(G+\Phi_{R-NS}+\Phi_{NS-R}+\Phi_{R-R})}, \quad G = \pi_1\hat{g}(e^{\wedge\Phi_{NS-NS}}).$$

For dual triplet $(\mathbf{L}^\eta, \mathbf{L}^{\bar{\eta}}; \tilde{\mathbf{L}})$,

$$\mathbf{L}^\eta(e^{\wedge G}) + \eta\Phi_{R-NS} + \eta\Phi_{NS-R} + \eta\Phi_{R-R} = 0, \quad (\text{constraints}), \quad (8a)$$

$$\mathbf{L}^{\bar{\eta}}(e^{\wedge G}) + \bar{\eta}\Phi_{R-NS} + \bar{\eta}\Phi_{NS-R} + \bar{\eta}\Phi_{R-R} = 0, \quad (\text{constraints}), \quad (8b)$$

$$\tilde{\mathbf{L}}(e^{\wedge(G+\Phi_{R-NS}+\Phi_{NS-R}+\Phi_{R-R})}) = 0 \quad (\text{eom}). \quad (8c)$$

Constrants (8a) and (8b) can be solved by setting

$$G = G_{\eta\bar{\eta}}(V), \quad \Phi_{R-NS} = \Psi, \quad \Phi_{NS-R} = \bar{\Psi}, \quad \Phi_{R-R} = \Sigma,$$

where $(V, \Psi, \bar{\Psi}, \Sigma) \equiv \mathcal{V}$ are the string fields in WZW-like fomulation, and $G_{\eta\bar{\eta}}(V)$ is the **(dual) pure-gauge field (Matsunaga)**

$$G_{\eta\bar{\eta}}(V) = \eta\bar{\eta}V + \frac{1}{2}(L_2^\eta(\eta\bar{\eta}V, \bar{\eta}V) + \eta L_2^{\bar{\eta}}(\eta\bar{\eta}V, V)) + \dots .$$

The WZW-like action is given by

$$I_{WZW} = \int_0^1 dt \omega_l \left(\mathcal{B}_t(\mathcal{V}(t)), \mathcal{G}^{-1} \pi_1 \tilde{\mathbf{L}}(e^{\wedge G(\mathcal{V}(t))}) \right), \quad (9)$$

where

$$\begin{aligned} G(\mathcal{V}(t)) &= G_{\eta\bar{\eta}}(V(t)) + \Psi(t) + \bar{\Psi}(t) + \Sigma(t), \\ \mathcal{B}_t(\mathcal{V}(t)) &= B_t(V(t)) + \xi_0 \bar{\xi}_0 \partial_t \Psi(t) + \xi_0 \bar{\xi}_0 \partial_t \bar{\Psi}(t) + \xi_0 \bar{\xi}_0 \partial_t \Sigma(t). \end{aligned}$$

The action (9) is invariant under the gauge transformation

$$\begin{aligned} \mathcal{B}_\delta(\mathcal{V}) &= \pi_1^{(0,0)} \tilde{\mathbf{L}}(e^{\wedge G(\mathcal{V})} \wedge \hat{\Lambda}) + \pi_1^{(0,0)} \mathbf{L}^\eta(e^{\wedge G(\mathcal{V})} \wedge \Omega) + \pi_1^{(0,0)} \mathbf{L}^{\bar{\eta}}(e^{\wedge G(\mathcal{V})} \wedge \bar{\Omega}) \\ &\quad + \xi_0 \bar{\xi}_0 (\pi_1^{(1,0)} + \pi_1^{(0,1)} + \pi_1^{(1,1)}) \tilde{\mathbf{L}}(e^{\wedge G(\mathcal{V})} \wedge \eta\bar{\eta}\hat{\Lambda}), \end{aligned}$$

where $\hat{\Lambda} = \Lambda + \xi_0 \bar{\xi}_0 (\lambda + \bar{\lambda} + \rho)$.

Summary and Outlook

Summary:

- ◇ We revisited type II superstring field theory and propose a new construction method extending the symmetric construction for NS-NS sector.
- ◇ Using gauge product of new construction, we give a map between string fields of L_∞ -based and WZW-like formulations and construct WZW-like action.
- ◇ We filled a gap and completed the table of three complementary formulations for the superstring field theory.

Outlook:

♠ Concrete analysis using superstring field theory

non-perturbative: classical solutions, dualities, AdS/CFT, D-instanton, ...
perturbative: off-shell amplitudes, infrared divergence, ...

♠ Quantization

gauge fixing, loop homotopy algebra, quantum BV master action, ...

We have still many remaining issues to study!