Reduction of Discretisation Artifacts in $S_1(Q^2)$ Calculation

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Motivation

- Within the last decade, determinations of both the proton and deuteron charge radii exhibited significant discrepancies.
- Similarly, tensions arose over the electromagnetic contribution to the proton-neutron mass difference, with earlier calculations suggesting $m_{\mathrm{QED}} = 0.76(30)\,\mathrm{MeV}$ but more recent evaluations proposing larger values.

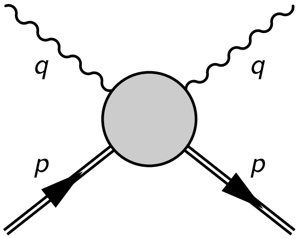
Forward Compton Amplitude

- We define the forward VVCS amplitude as

$$T_{\mu\nu} = i \int d^4z \; e^{iq\cdot z} \langle p | \mathcal{T}\{ j_\mu(z) j_\nu(0) \} | p \rangle,$$

where j_μ denote an electromagnetic current.

- Amplitude describes process of photon-proton scattering $\gamma^*(q)p(p) \to \gamma^*(q)p(p)$.



Amplitude Decomposition

- $T_{\mu\nu}$ can be decomposed into two structure functions, T_1 and T_2:

$$T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(\omega, Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{T_2(\omega, Q^2)}{m_p^2},$$

where $Q^2 = -q^2$ and $\omega = 2p\cdot q/Q^2$.

Subtraction Function

- In turn, T_1 and T_2 satisfy the dispersion relations:

$$T_1(\omega, Q^2) = T_1(0, Q^2) + \frac{2\omega^2}{\pi} \int_1^\infty d\omega' \frac{\operatorname{Im} T_1(\omega', Q^2)}{\omega'(\omega'^2 - \omega^2 - i\epsilon)}$$

$$T_2(\omega, Q^2) = \frac{2\omega}{\pi} \int_1^\infty d\omega' \frac{\operatorname{Im} T_2(\omega', Q^2)}{\omega'^2 - \omega^2 - i\epsilon}.$$

- $T_1(0, Q^2)$ is called the **subtraction function**: $S_1(Q^2)$.

Relevance of $S_1(Q^2)$

- $T_{\mu\nu}$ is an input for both the μH Lamb shift calculation of the proton radius, and for the calculation of the proton-neutron mass difference.
- Unlike $\mathrm{Im}\, T_1(\omega, Q^2)$ and $\mathrm{Im}\, T_2(\omega, Q^2)$, $S_1(Q^2)$ cannot be accessed from inclusive processes.
- $S_1(Q^2)$ thus contributes significant theoretical uncertainty.

Models of $S_1(Q^2)$

- For $Q^2 << m_P^2$, $S_1(Q^2)$ can be calculated in a variety of formalisms but these calculations have sizable errors and are not always consistent.

- For $Q^2 >> \Lambda_{QCD}^2$, $S_1(Q^2)$ can be evaluated model-independently using the Operator Product Expansion (OPE).

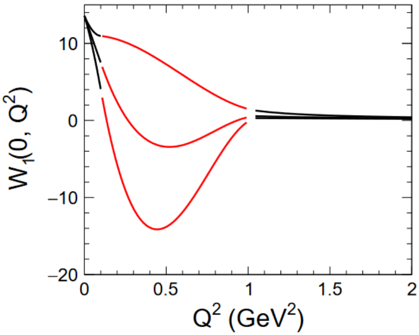


Figure 1: Hill & Paz, 2017

Lattice Determination

- This uncertainty motivated a first-principles determination of $S_1(Q^2)$ using lattice QCD and the Feynman-Hellmann method.

- Employed simplest kinematics to extract $S_1(Q^2)$: $\mu = \nu = 3$ and $p_3 = q_3 = 0$.

Prior $S_1(Q^2)$ Result

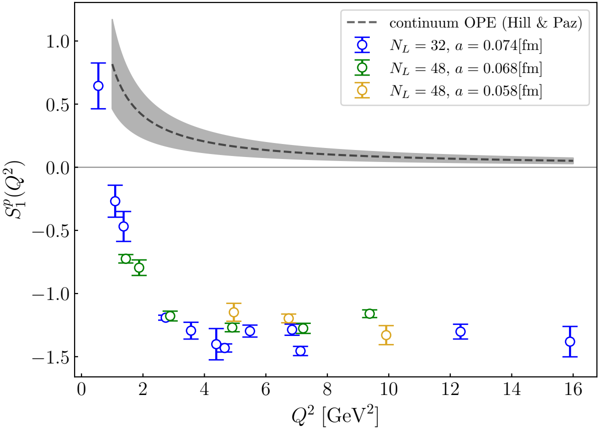


Figure 2: Hannaford-Gunn, et al. (preliminary)

Cause of Discrepancy

- Result suffers from either discretisation artifacts or finite volume effects (ΔS_1).
  - EFT calculation has shown $T_{\mu\nu}$ varies minimally with volume[1].
  - Naturally turn to lattice artifacts as the cause of the discrepancy.

$$S_1^{\text{phy}}(Q^2) = S_1^{\text{latt}}(Q^2) - \Delta S_1.$$

- This motivates a Lattice Operator Product Expansion of the Compton amplitude[2].

[1] Lozano, Agadjanov, Gegelia, et al., 2021
[2] Extending work done by QCDSF collaboration - see Göckeler, Horsley, Perlt, et al., 2006

Operator Product Expansion

- An OPE expresses two operators $\mathcal{O}_1(x)\mathcal{O}_1(0)$, in the limit $x \to 0$, as a linear combination of local operators:

$$\lim_{q\to\infty} \int d^4x\, e^{iq\cdot x} \mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_n C_n(q)\mathcal{O}_n(0)$$

- All q-dependence is isolated in the 'Wilson coefficients', $C_n(q)$.

Method

- By Wick-rotating and discretising, we convert $T_{\mu\nu} \rightarrow T^L_{\mu\nu}$. At tree level, we need only consider two contributions:

$$T^L_{\mu\nu} = \bar{u}(p)\gamma_\mu \tilde{S}_W(p+q)\gamma_\nu u(p) + \bar{u}(p)\gamma_\nu \tilde{S}_W(p-q)\gamma_\mu u(p),$$

where $\tilde{S}_W$ is the Wilson fermion propagator

$$\tilde{S}_W(k) = a\frac{M(k,m_0) - i\sum_\mu \gamma_\mu \sin(ak_\mu)}{M(k,m_0)^2 + \sum_\mu \sin^2(ak_\mu)}$$

with $M(k,m_0) = m_0 + \frac{r}{a}\sum_\rho \left[1 - \cos(ak_\rho)\right]$.

Method

- Since $\sin(ap_\mu) \ll \sin(aq_\mu)$, Taylor expand in $\sin(ap_\mu)$.
- Simplify Dirac gamma matrices using Euclidean relations.
- Associate $\sin(ap_\mu) \leftrightarrow -ia\overleftrightarrow{D}_\mu$.

OPE Result

$$
\begin{aligned}
T^L_{\mu\nu} = {} & \frac{2a^2}{\Omega(q,m_0)}\left[\left[M(q,m_0)-m_0\right]\delta_{\mu\nu}\langle p|\bar{\psi}\psi|p\rangle - \frac{i}{a}\sum_{\tau,\sigma}\epsilon_{\sigma\mu\tau\nu}\sin(aq_\tau)\langle p|\bar{\psi}\gamma_\sigma\gamma_5\psi|p\rangle\right] \\
& - \frac{2a^2}{\Omega(q,m_0)}\left[r\sum_\tau \sin(aq_\tau)\langle p|\bar{\psi}\sigma_{\mu\nu}\overleftrightarrow{D}_\tau\psi|p\rangle + \cos(aq_\mu)\langle p|\bar{\psi}\gamma_\nu\overleftrightarrow{D}_\mu\psi|p\rangle + \cos(aq_\nu)\langle p|\bar{\psi}\gamma_\mu\overleftrightarrow{D}_\nu\psi|p\rangle\right. \\
& \qquad \left. + 2\delta_{\mu\nu}\sum_\tau \sin^2(aq_\tau/2)\langle p|\bar{\psi}\gamma_\tau\overleftrightarrow{D}_\tau\psi|p\rangle\right] \\
& + \frac{4a^4}{\Omega^2(q,m_0)}\left[M(q,m_0)\sum_\tau\frac{1}{a}\sin(aq_\tau)\left[\cos(aq_\tau)+raM(q,m_0)\right]\langle p|\bar{\psi}\sigma_{\mu\nu}\overleftrightarrow{D}_\tau\psi|p\rangle\right] \\
& + \frac{4a^4}{\Omega^2(q,m_0)}\left[\sum_{\alpha,\tau}\frac{1}{a}\sin(aq_\alpha)\frac{1}{a}\sin(aq_\tau)\left[\cos(aq_\tau)+raM(q,m_0)\right]\left[\delta_{\mu\alpha}\langle p|\bar{\psi}\gamma_\nu\overleftrightarrow{D}_\tau\psi|p\rangle\right.\right. \\
& \qquad \left.\left. + \delta_{\nu\alpha}\langle p|\bar{\psi}\gamma_\mu\overleftrightarrow{D}_\tau\psi|p\rangle - \delta_{\mu\nu}\langle p|\bar{\psi}\gamma_\alpha\overleftrightarrow{D}_\tau\psi|p\rangle\right]\right] + \mathcal{O}(\overleftrightarrow{D}^2).
\end{aligned}
$$

OPE Result

- Taking $\mu = \nu = 3$ abd $q_3 = p_3 = 0$:

$$\boxed{T_{33}^{L\,(0)} = \frac{2ar\sum_\rho\left[\cos(aq_\rho) - 1\right]}{\sum_\rho \sin^2(aq_\rho) + a^2M^2(q, m_0)}\langle p|\bar{\psi}\psi|p\rangle}$$

- In limit $a \to 0$, $T_{33}^{L\,(0)} \to 0$.

Structureless Free Fermion Correction

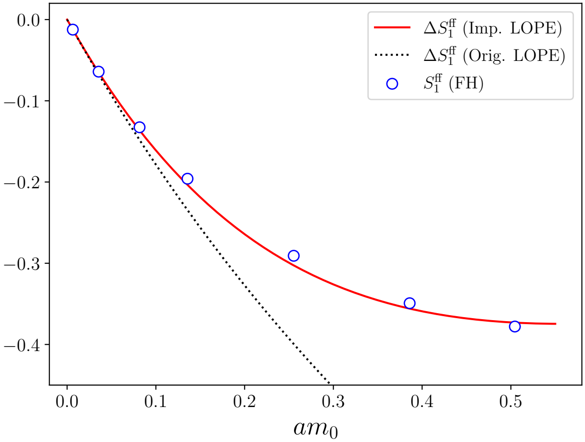
- The correction for a free fermion is given by:

$$\Delta S_1 = \frac{r \sum_\rho \left[\cos(aq_\rho) - 1\right]}{\sum_\rho \sin^2(aq_\rho) + a^2 M^2(q, m_0)} 4am_0 g_S^{\text{bare}}.$$

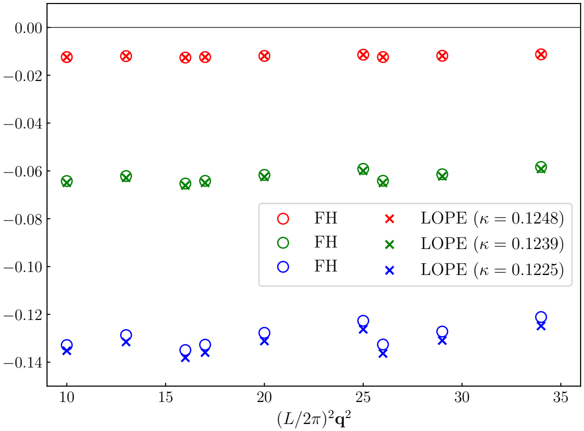
- For a structureless fermion, $S_1^{\text{phy}}(Q^2) = 0$, thus

$$S_1^{\text{latt}}(Q^2) = \Delta S_1.$$

Structureless Free Fermion Result



Structureless Free Fermion Result



Proton Correction

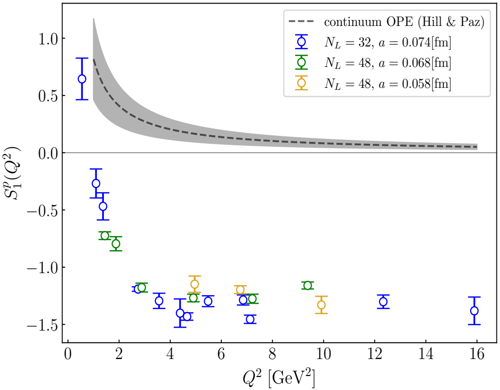
- The correction for the proton is given by:

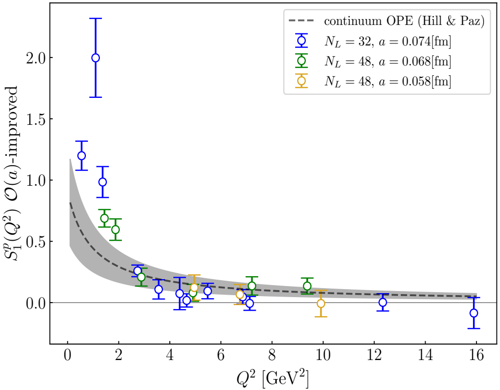
$$\Delta S_1 = \frac{r\sum_\rho \left[\cos(aq_\rho) - 1\right]}{\sum_\rho \sin^2(aq_\rho) + a^2 M^2(q, m_0)} 4am_p Z_V^2 g_S^{\text{bare}},$$

where the bare scalar charge is determined from Feynman-Hellmann[3].

[3]Rose, Batelaan, Horsley, et al., 2023

Final Proton Result





Lattice Specifications

N_f	$L^3 \times T$	β	κ	a[fm]	m_π[MeV]	Z_V
$2+1$	$32^3 \times 64$	5.50	0.120900	0.074	470	0.86
$2+1$	$48^3 \times 96$	5.65	0.122005	0.068	410	0.87
$2+1$	$48^3 \times 96$	5.80	0.122810	0.058	430	0.88

Lattice Specifications

$\kappa = 0.120900$, $\beta = 5.50$

q $[2\pi/L]$	Q^2 [GeV^2]
$(1,1,0)$	0.55
$(2,0,0)$	1.10
$(2,1,0)$	1.37
$(3,1,0)$	2.74
$(3,2,0)$	3.57
$(4,0,0)$	4.39
$(4,1,0)$	4.66
$(4,2,0)$	5.48
$(4,3,0)$	6.86
$(5,1,0)$	7.13
$(6,3,0)$	12.3
$(7,3,0)$	15.9

$\kappa = 0.122005$, $\beta = 5.65$

q $[2\pi/L]$	Q^2 [GeV^2]
$(3,1,0)$	1.43
$(3,2,0)$	1.86
$(4,2,0)$	2.86
$(4,3,0)$	4.86
$(7,1,0)$	7.14
$(7,4,0)$	9.29

$\kappa = 0.122810$, $\beta = 5.80$

q $[2\pi/L]$	Q^2 [GeV^2]
$(4,3,0)$	4.81
$(5,3,0)$	6.54
$(7,1,0)$	9.62

Feynman-Hellmann Specifications

- In the Feynman-Hellmann method, we perturb the action S by an operator $\mathcal{O}$ with some unphysical coupling λ: $S \to S + \lambda\mathcal{O}$. In our context:

$$S(\lambda) = S + \lambda \int d^3z \left(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}}\right) j_\mu(z),$$

where j_μ is the local Minkowski vector current: $j_\mu = Z_V \bar{\psi}\gamma_\mu\psi$.

- The FH theorem gives the following relationship:

$$\left.\frac{\partial^2 E_\lambda}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{T_{33}(\mathbf{p},\mathbf{q})}{E_N(\mathbf{p})}$$

Covariant Derivative & $\sin(ap_\mu)$

- The continuum Euclidean Dirac equation is given by $(\partial\!\!\!/ + m)\psi = 0$.
- Taking positive plane-wave solution $\psi = u(\mathbf{p})e^{ip\cdot z}$, this becomes $(ip\!\!\!/ + m)u(\mathbf{p}) = 0$.
- Thus associate $p_\mu \leftrightarrow -i\partial_\mu$.
- Adjusting for lattice and gauge fields, we have:

$$\frac{1}{a}\sin(ap_\mu) \leftrightarrow -i\overleftrightarrow{D}_\mu$$

Forward Compton Amplitude of Structureless Fermion

- From standard QED, the VVCS amplitude of a structureless fermion is given by

$$T_{\mu\nu} = \frac{2\omega^2}{1-\omega^2}\left[\frac{p_\mu q_\nu + p_\nu q_\mu}{p\cdot q} + \frac{Q^2}{(p\cdot q)^2} p_\mu p_\nu - g_{\mu\nu}\right] + \frac{2\omega}{1-\omega^2}\frac{i}{p\cdot q}\epsilon_{\mu\nu\tau\sigma} q^\tau s^\sigma .$$

- Taking $\mu = \nu = 3$, $p_3 = q_3 = 0$ and $\omega = 0$:

$$S_1(Q^2) = T_1(0, Q^2) = T_{33} = \left.\frac{2\omega^2}{1-\omega^2}\right|_{\omega=0} = 0$$