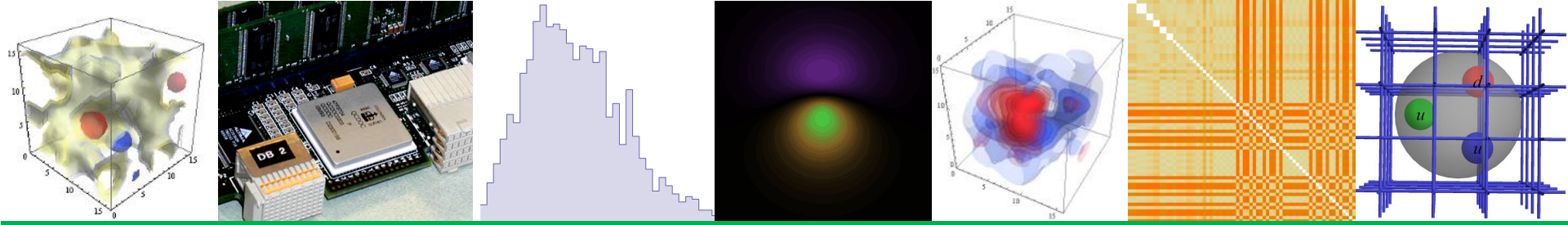


Parton Distributions from Lattice and Impacts on Global QCD Analysis

HUEY-WEN LIN

CTEQ

This work of HL is supported by the NSF under grant PHY 2209424 & 1653405, DOE under DE-SC0024053 and the Research Corporation for Science Advancement through the Cottrell Scholar Award

@LinQCD

Outline

§ Lattice QCD and Parton Distribution Functions

§ Selected *x*-Dependent Parton Distributions

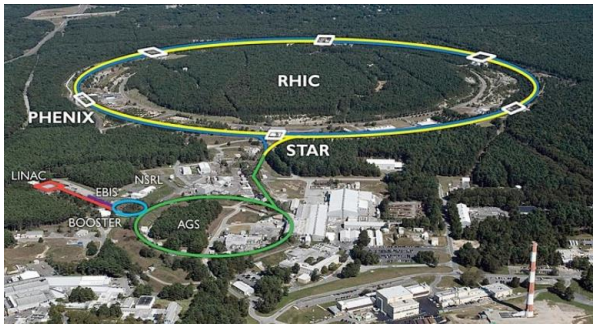
§ Impact of Lattice-QCD PDFs on Global Fits

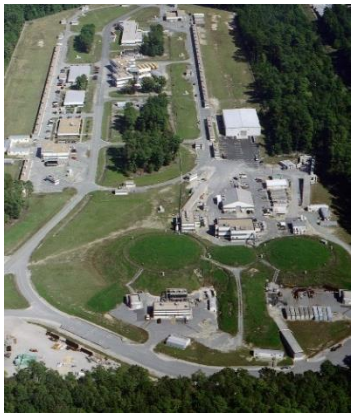
Selected results somewhat biased toward MSULat students and postdocs

Parton Distribution Functions

§ PDFs are universal quark/gluon distributions of nucleon

- Many ongoing/planned experiments
  (BNL, JLab, J-PARC, COMPASS, GSI, EIC, EIcC, LHeC, …)

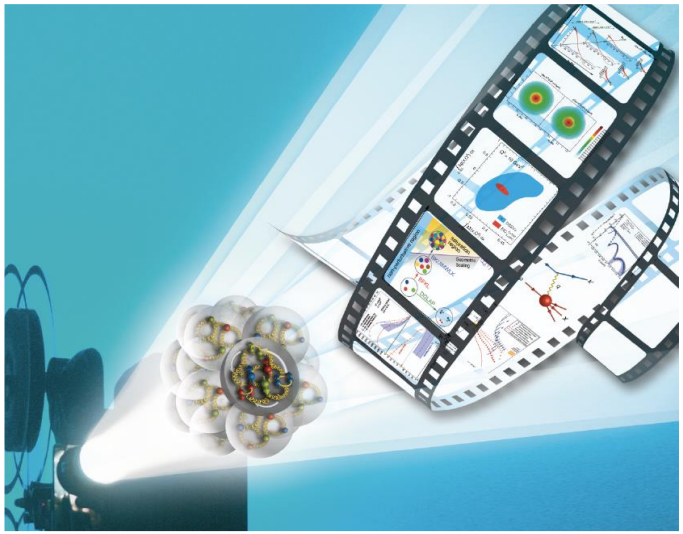




Electron Ion Collider:
The Next QCD Frontier

Imaging of the proton

How are the sea quarks and gluons, and their spins, distributed in space and momentum inside the nucleon?

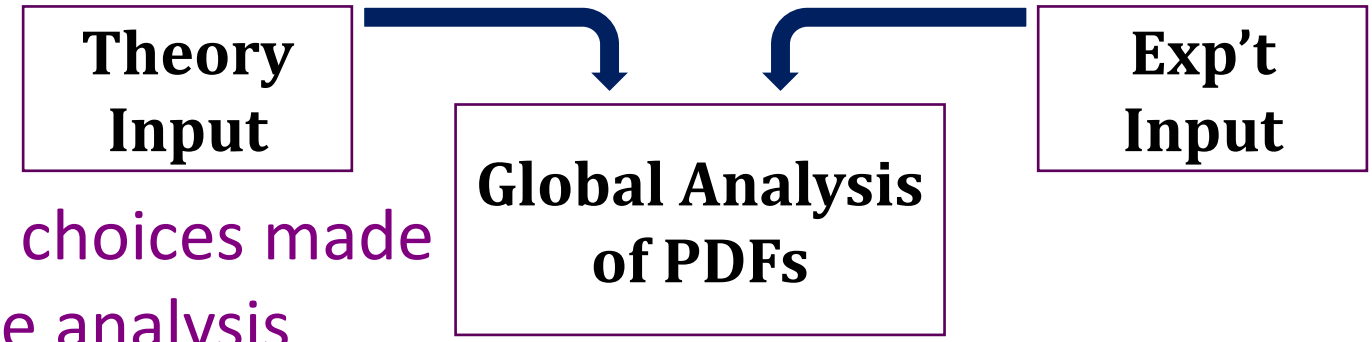


EIC White Paper, 1212.1701; The Present and Future of QCD (2303.02579)

Global Analysis

§ Experiments cover diverse kinematics of parton variables

- Global analysis takes advantage of all data sets



§ Some choices made for the analysis

- Choice of data sets and kinematic cuts
- Strong coupling constant $\alpha_s(M_Z)$
- How to parametrize the distribution

$$xf(x,\mu_0) = a_0 x^{a_1}(1-x)^{a_2} P(x)$$

- Assumptions imposed

  SU(3) flavor symmetry, charge symmetry, strange and sea distributions

$$s = \bar{s} = \kappa(\bar{u} + \bar{d})$$

Lattice QCD 101

§ Lattice QCD is an ideal theoretical tool for investigating the strong-coupling regime of quantum field theories

§ Physical observables are calculated from the path integral

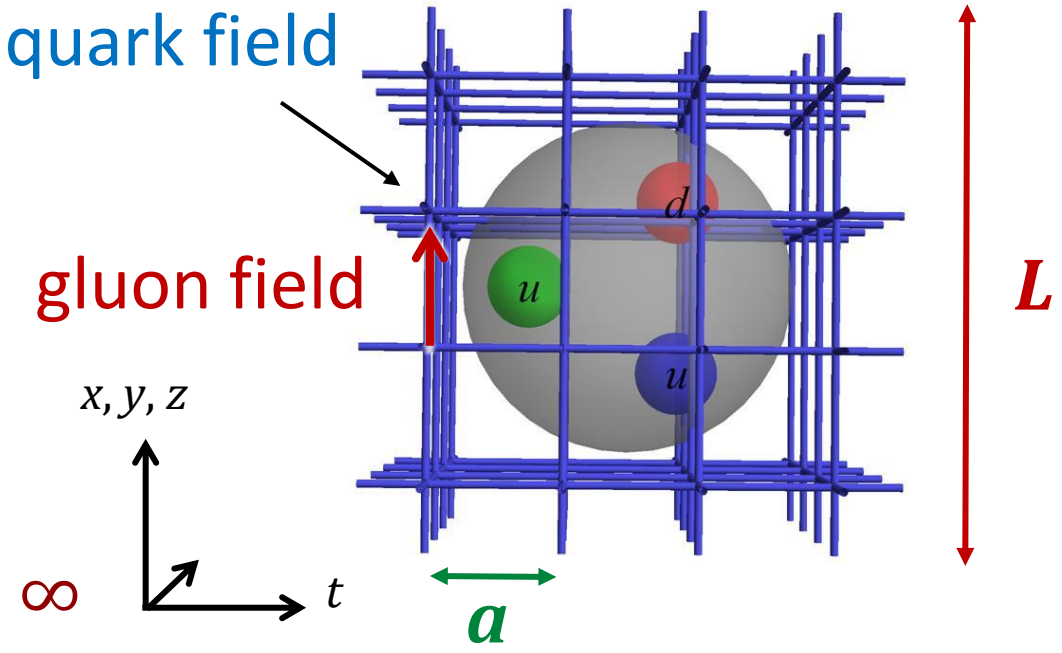
$$\langle 0|O(\bar{\psi},\psi,A)|0\rangle = \frac{1}{Z}\int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{iS(\bar{\psi},\psi,A)} O\,(\bar{\psi},\psi,A)$$

in **Euclidean** space

- Quark mass parameter (described by m_π)
- Impose a UV cutoff
  discretize spacetime
- Impose an infrared cutoff
  finite volume

§ Recover physical limit

$m_\pi \to m_\pi^{\text{phys}}$, $\boldsymbol{a \to 0}$, $\boldsymbol{L \to \infty}$



Lattice QCD 101

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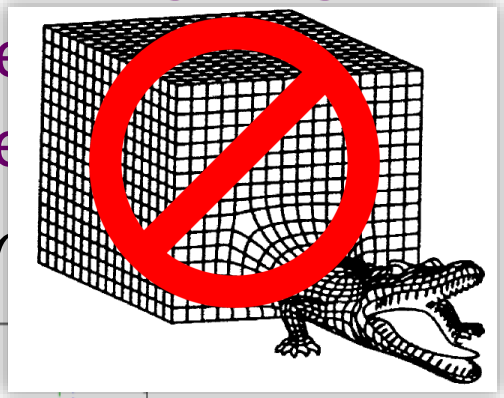
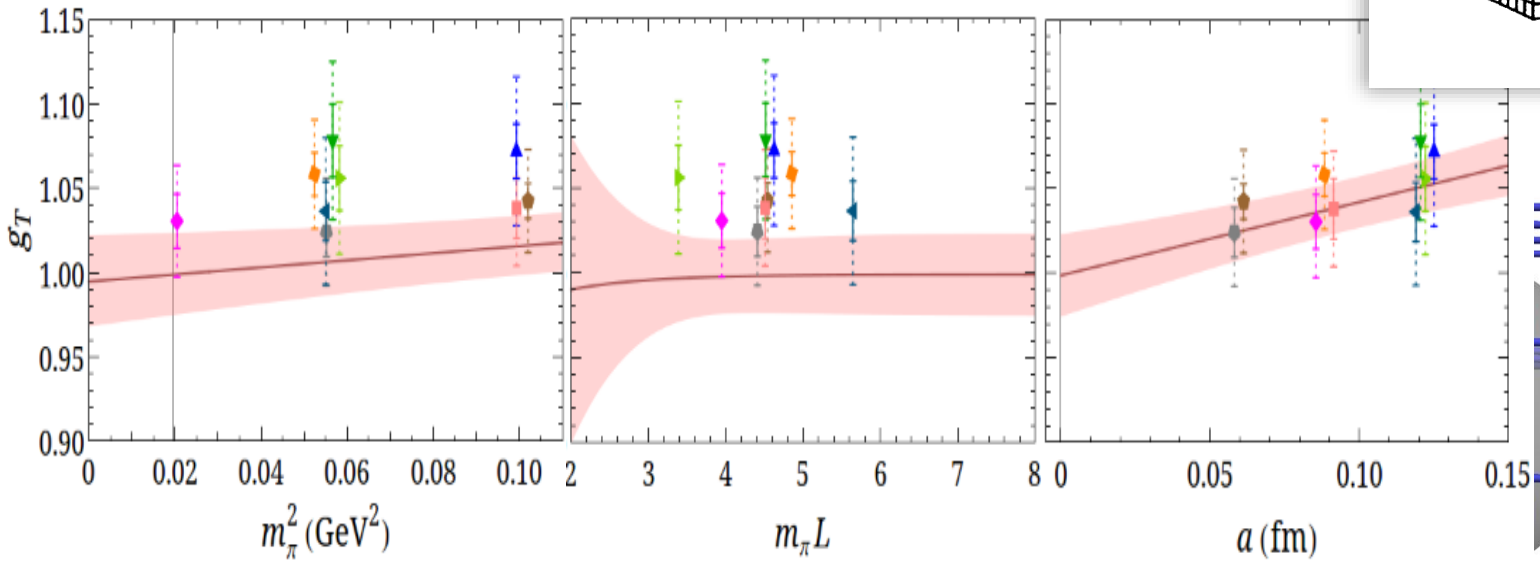
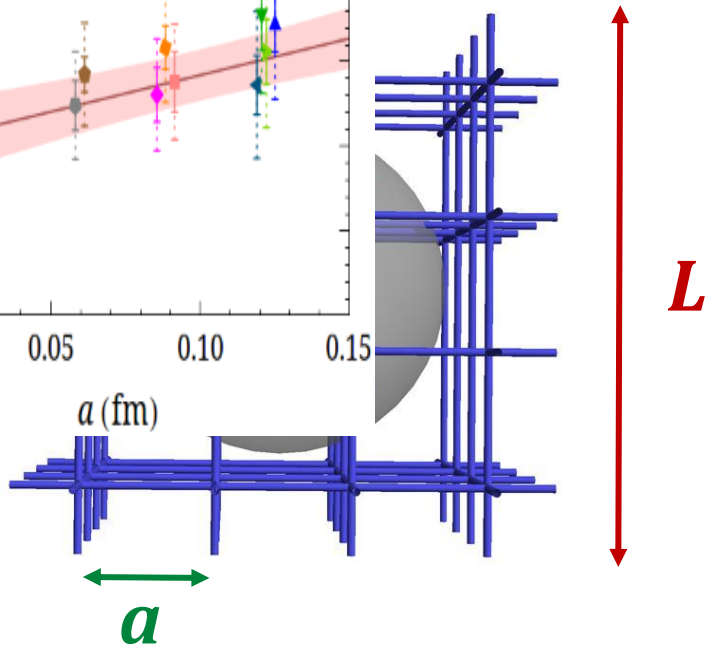
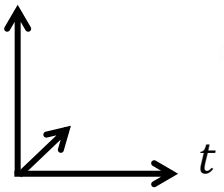






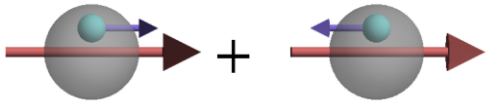
§ Recover physical limit

$$m_\pi \to m_\pi^{\text{phys}},\ a \to 0,\ L \to \infty$$



PDFs on the Lattice

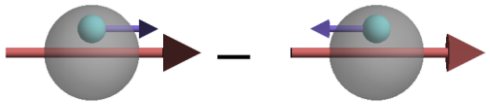
§ Traditional lattice calculations rely on operator product expansion, only provide moments



spin-averaged/unpolarized

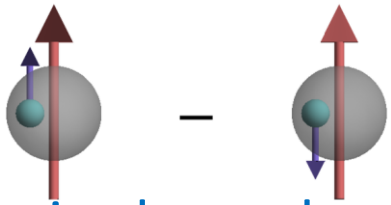
$$\langle x^{n-1}\rangle_q = \int_{-1}^{1} dx\, x^{n-1} q(x)$$

most well known



spin-dependent longitudinally polarized

$$\langle x^{n-1}\rangle_{\Delta q} = \int_{-1}^{1} dx\, x^{n-1} \Delta q(x)$$



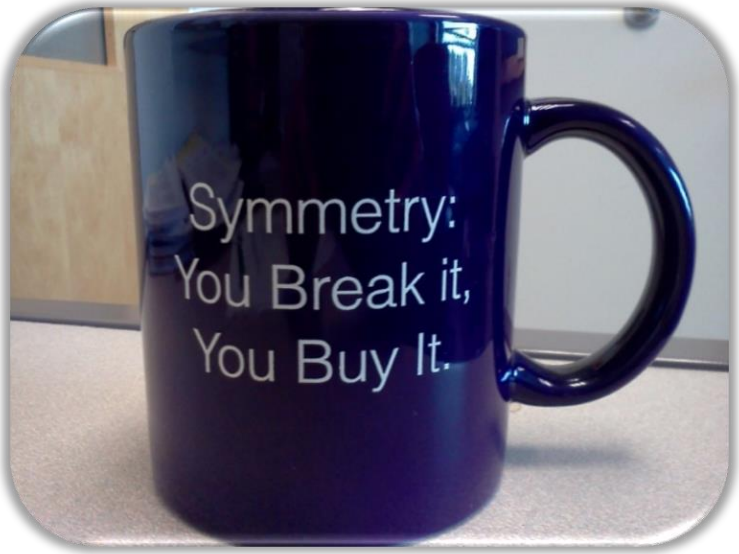
spin-dependent transversely polarized

$$\langle x^{n-1}\rangle_{\delta q} = \int_{-1}^{1} dx\, x^{n-1} \delta q(x)$$

very poorly known

§ True distribution can only be recovered with all moments

Lattice Structure Limitation



§ Lattice calculations rely on operator product expansion, only provide moments

$$\langle x^{n-1} \rangle_q = \int_{-1}^{1} dx\, x^{n-1} q(x)$$

§ Limited to the lowest few moments

- For higher moments, all ops mix with lower-dimension ops
- Novel proposals to overcome this problem

W. Detmold and C. Lin, Phys. Rev. D73 (2006) 014501

Z. Davoudi and M. J. Savage, Phys. Rev. D86 (2012) 054505

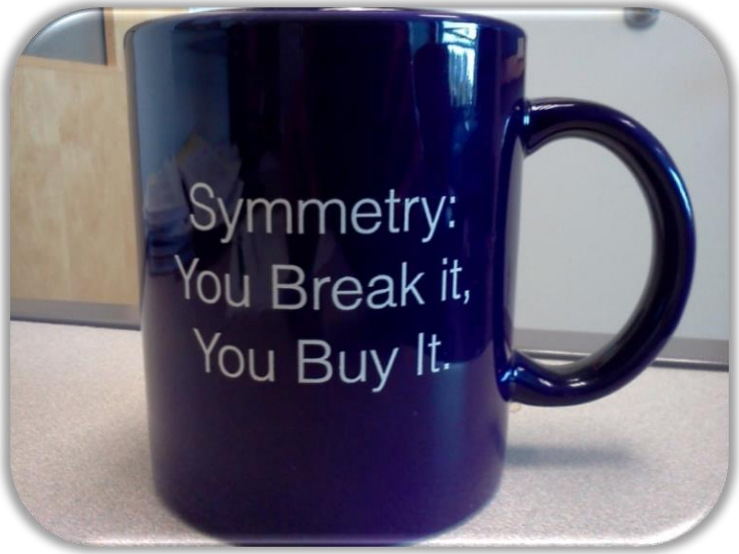
A. Shindler, arXiv:2311.18704

THE HOPE COLLABORATION

David Lin (TMD CS-kernel) @ Mon. 2:30 PM

Dimitra Pefkou (PDF higher moments) @ Mon. 3:30 PM

Lattice Structure Limitation



§ Lattice calculations rely on operator product expansion, only provide moments

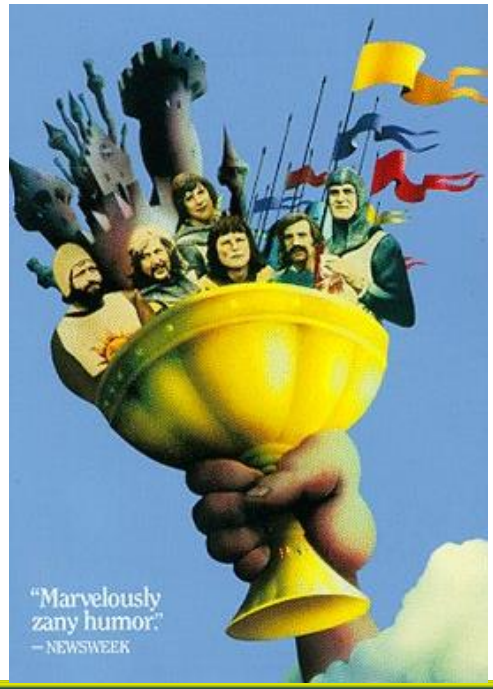
$$\langle x^{n-1} \rangle_q = \int_{-1}^{1} dx\, x^{n-1} q(x)$$

§ Longstanding obstacle!

- Holy grail of structure calculations

§ Applies to many structure quantities:

- Parton distribution functions (PDF)
- Generalized parton distributions (GPD)
- Transverse-momentum distributions (TMD)
- ⋮



A NEW HOPE

It is a period of war and economic uncertainty.

Turmoil has engulfed the galactic republics.

Basic truths at foundation of the human civilization are disputed by the dark forces of the evil empire.

A small group of QCD Knights from United Federation of Physicists has gathered in a remote location on the third planet of a star called Sol on the inner edge of the Orion–Cygnus arm of the galaxy.

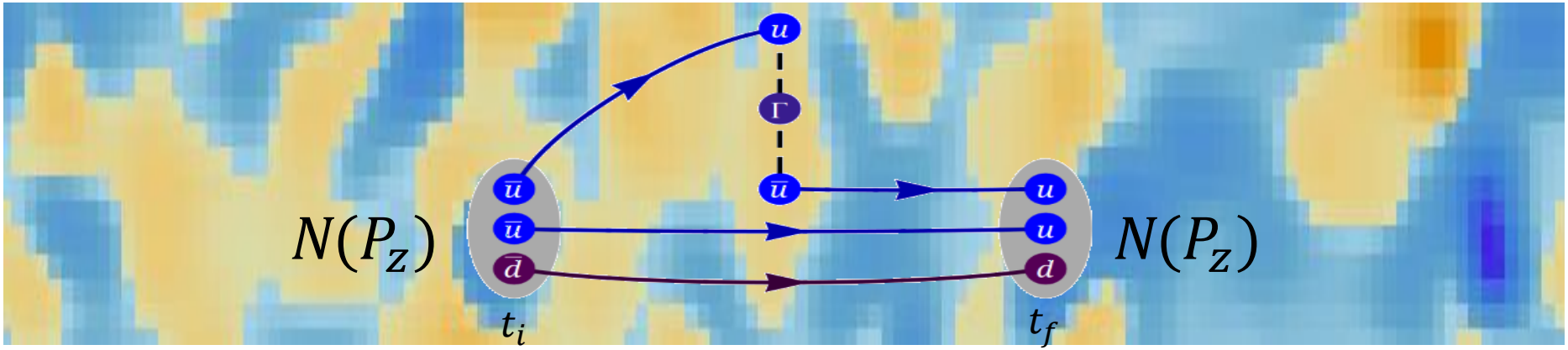
The QCD Knights are the only ones who can tame the power of the Strong Force, responsible for holding atomic nuclei together, for giving mass and shape to matter in the Universe.

They carry secret plans to build the most powerful

Lattice Parton Method

§ Large-momentum effective theory (LaMET)/quasi-PDF

(X. Ji, 2013; See 2004.03543 for review)



§ Compute quasi-distribution via

$$\tilde{q}(x,\mu,P_z) = \int \frac{dz}{4\pi} e^{-izk_z} \left\langle P \left| \bar{\psi}(z)\Gamma \exp\left(-ig\int_0^z dz' A_z(z')\right)\psi(0) \right| P \right\rangle$$

§ Recover true distribution (take $P_z \to \infty$ limit)

$$\tilde{q}(x,\mu,P_z) = \int_{-\infty}^{\infty} \frac{dy}{|y|} C\left(\frac{x}{y},\frac{\mu}{P_z}\right) \boldsymbol{q(y,\mu)} + \mathcal{O}\left(\frac{M_N^2}{\boldsymbol{P_z^2}}, \frac{\Lambda_{\rm QCD}^2}{(x\boldsymbol{P_z})^2}, \frac{\Lambda_{\rm QCD}^2}{((1-x)\boldsymbol{P_z})^2}\right)$$

X. Xiong e.a., 1310.7471; J.-W. Chen e.a., 1603.06664

Lattice Parton Method

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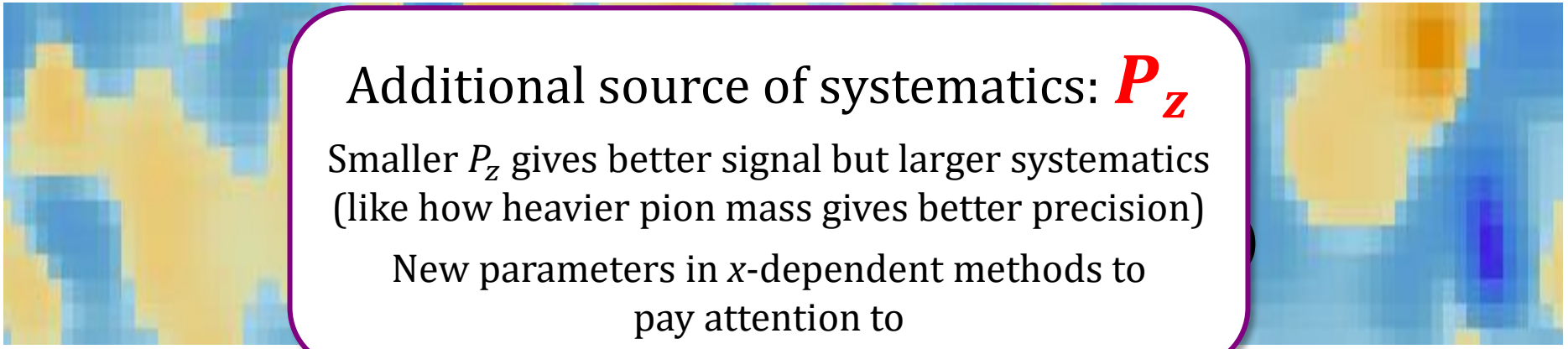
(X. Ji, 2013; See 2004.03543 for review)



Additional source of systematics: $\boldsymbol{P_z}$

Smaller P_z gives better signal but larger systematics (like how heavier pion mass gives better precision)

New parameters in x-dependent methods to pay attention to

§ Compute quasi-distribution via

$$\tilde{q}(x,\mu,P_z) = \int \frac{dz}{4\pi} e^{-izk_z} \left\langle P \left| \bar{\psi}(z)\Gamma \exp\left(-ig\int_0^z dz' A_z(z')\right)\psi(0) \right| P \right\rangle$$

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X. Xiong e.a., 1310.7471; J.-W. Chen e.a., 1603.06664

Direct x-Dependent Structure

§ Longstanding obstacle to lattice calculations!

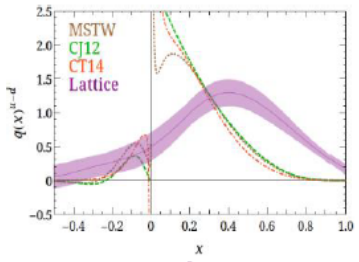
- **Quasi-PDF**/large-momentum effective theory (LaMET) (X. Ji, 2013; See 2004.03543 for review)
- **Pseudo-PDF** method: differs in FT (A. Radyushkin, 2017)
- Lattice cross-section method (**LCS**) (Y Ma and J. Qiu, 2014, 2017)
- Compton amplitude method (A.J. Chambers et al., 1703.01153)
- Hadronic tensor currents (Liu et al., hep-ph/9806491, … 1603.07352)
- Euclidean correlation functions (RQCD, 1709.04325)

Lattice Parton Calculations

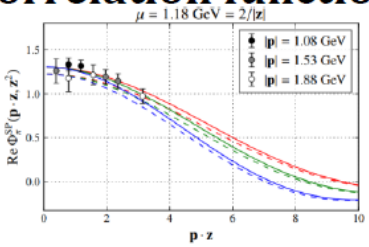
§ Rapid developments!

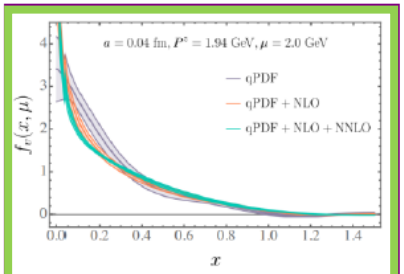
HL, Few Body Syst. 64 (2023) 3, 58

LaMET/quasi-PDF lattice calculation



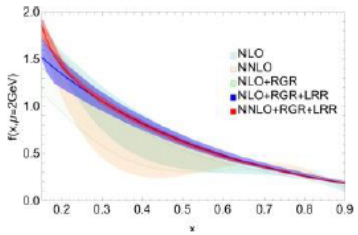
Euclidean correlation functions





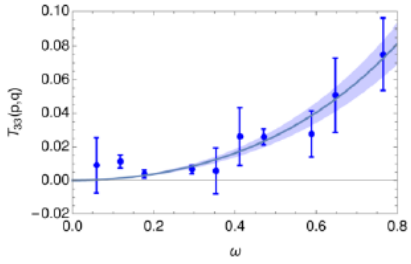
1st NNLO PDF

1stPDF w/ LRR+RGR

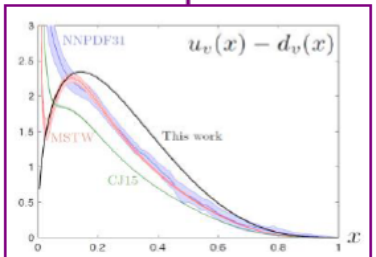


2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023 2024

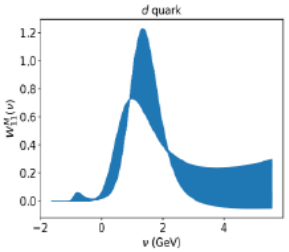
Compton amplitude



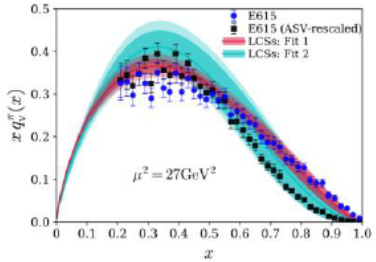
pseudo-PDFs



Hadronic tensor



LCS



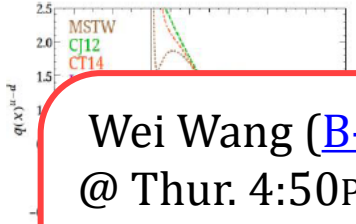
CSSM/QCDSF

Lattice Parton Calculations

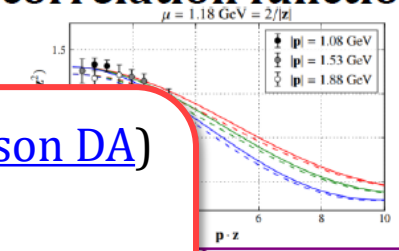
§ Rapid developments!

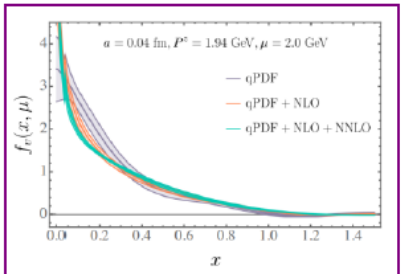
HL, Few Body Syst. 64 (2023) 3, 58

LaMET/quasi-PDF lattice calculation



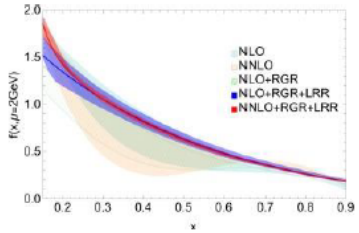
Euclidean correlation functions





1st NNLO PDF

1stPDF w/ LRR+RGR

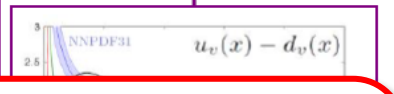


Wei Wang (B-meson DA)
@ Thur. 4:50PM

Jun Zeng (octet and decuplet baryons DAs)
@ Thur. 5:10PM

2017 2018 2019 2020 2021 2022 2023 2024

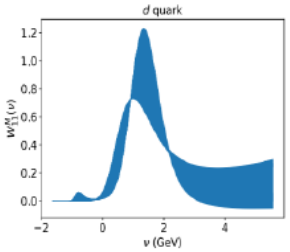
Compton amplitude



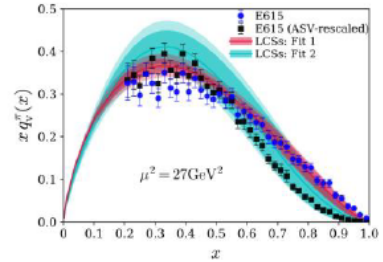
Matthew Rumley (transition GPDs)
@ Wed. 5:50PM

K. Utku Can (parity-odd structure)
@ Thur. 3:00PM

Hadronic tensor



LCS

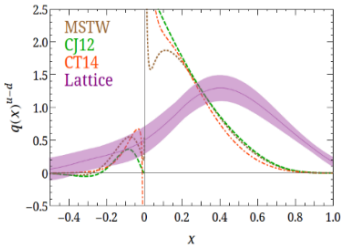


CSSM/QCDSF

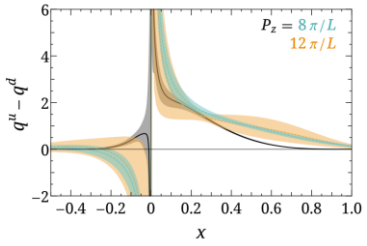
Lattice Parton Calculations

§ Physics-quantity milestones

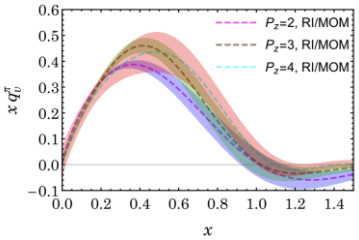
First unpol. lattice PDF



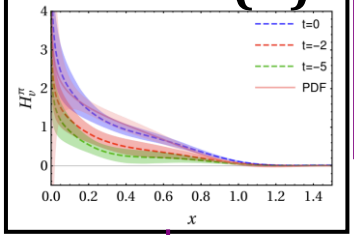
First PDFs at $M_\pi^{\rm phys}$



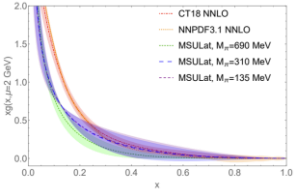
Pion v-PDF



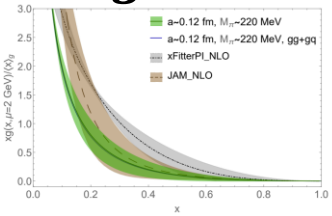
1st GPD (π)



N g-PDF

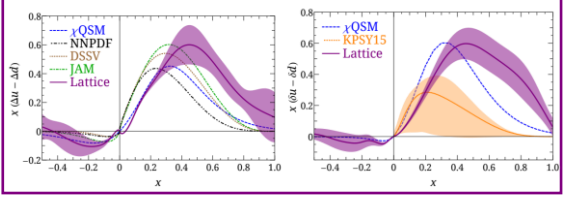


π g-PDF

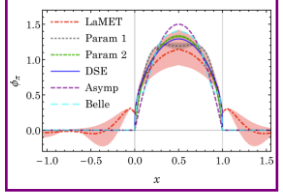


2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023

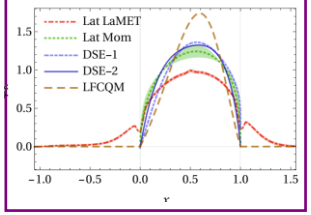
Pol. PDFs and mass corrections



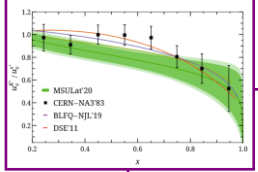
Pion DA



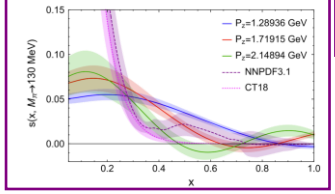
Kaon DA



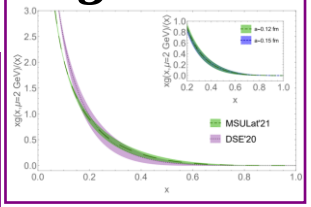
K PDF



s,c PDF



Kaon g-PDF

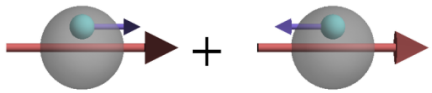


HL, Few Body Syst. 64 (2023) 3, 58

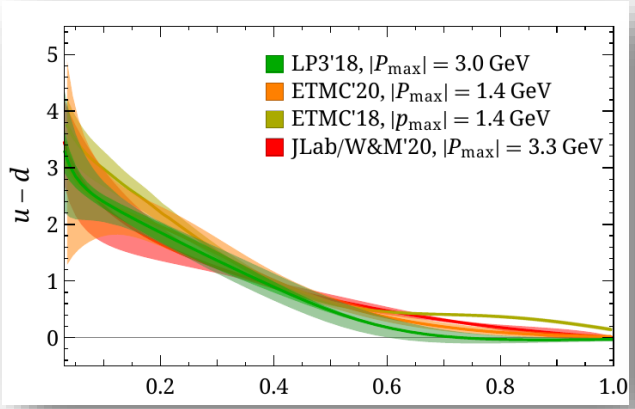
Lattice Example Results

§ Summary of PDF results at physical pion mass

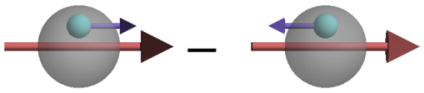
unpolarized



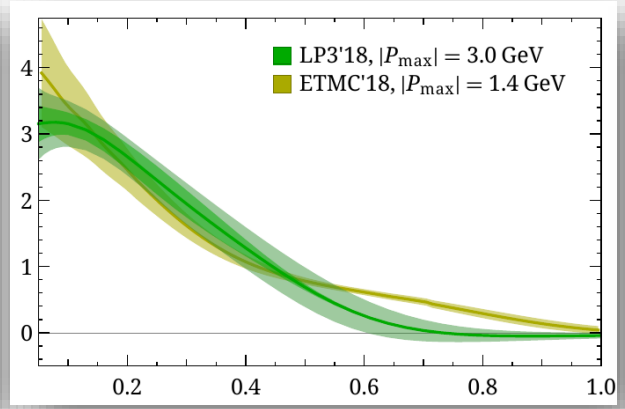
$u(x) - d(x)$



longitudinally polarized

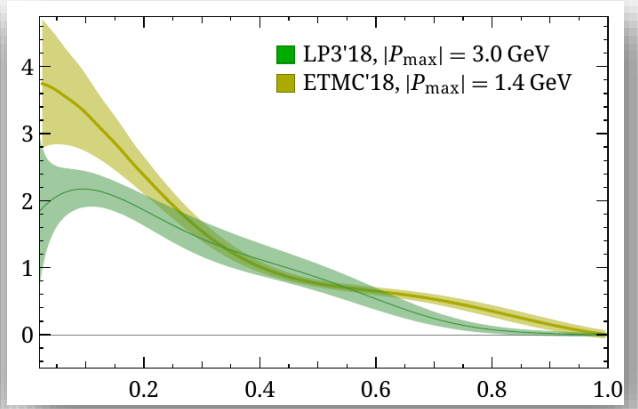


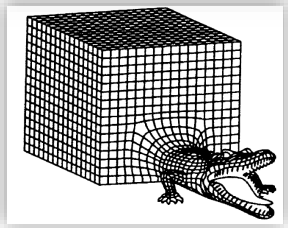
$\Delta u(x) - \Delta d(x)$



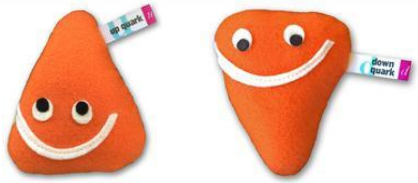
transversely polarized

$\delta u(x) - \delta d(x)$





Finite volume,
Discretization,
…

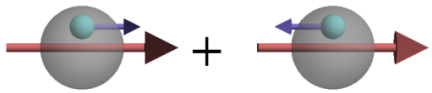


2006.08636 (PDFLattice2019)

Lattice Example Results

§ Summary of PDF results at physical pion mass

unpolarized



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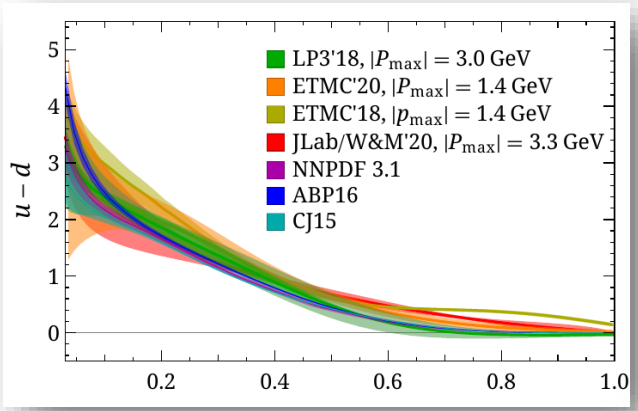
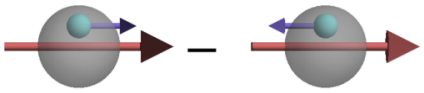
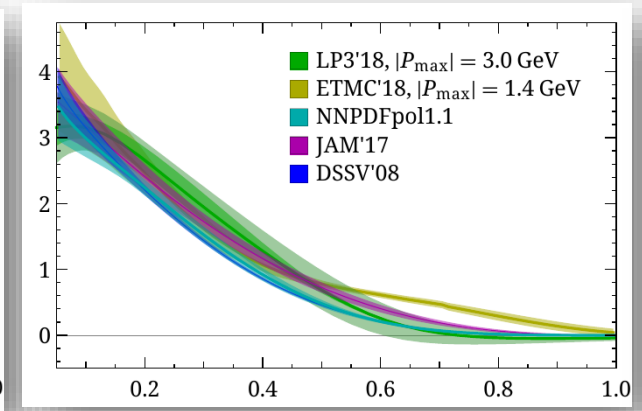


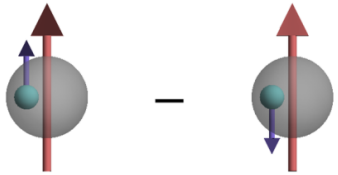
longitudinally polarized



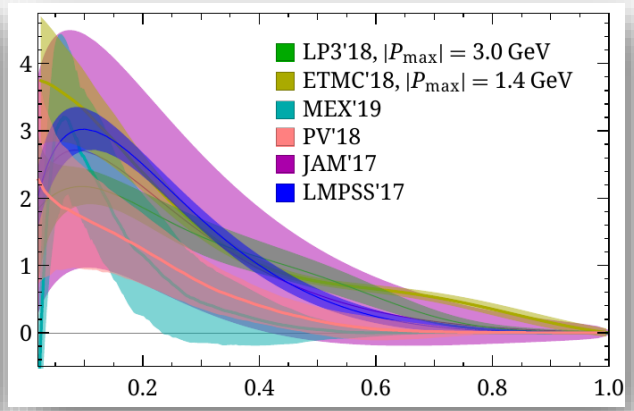
$\Delta u(x) - \Delta d(x)$

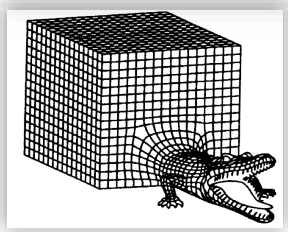


transversely polarized



$\delta u(x) - \delta d(x)$





Finite volume,
Discretization,
…

2006.08636 (PDFLattice2019)

Lattice Example Results

§ Summary of PDF results at physical pion mass

unpolarized

$u(x) - d(x)$

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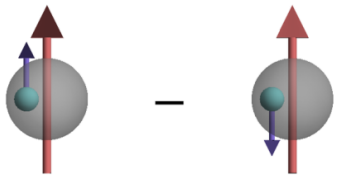
$\Delta u(x) - \Delta d(x)$

transversely polarized



$\delta u(x) - \delta d(x)$

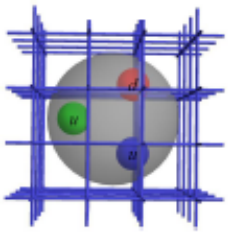
§ Complementary lattice inputs for best PDFs

Theory Input

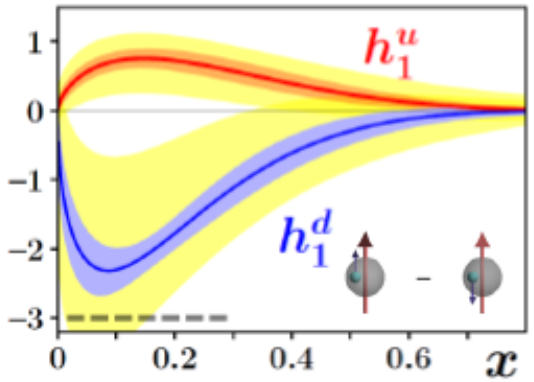
Exp't Input

Global Analysis of PDFs

+



Work has been made in this direction



Phys. Rev. Lett. 120, 152502 (2018)

Isovector PDFs Update

§ Nucleon isovector PDF calculated directly at **physical pion mass**

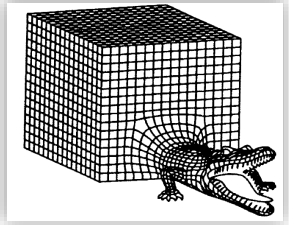
- NNLO matching & treat leading-renormalon effects
  - Leading-renormalon resummation (LRR)
  - Renormalization-group resummation (RGR)

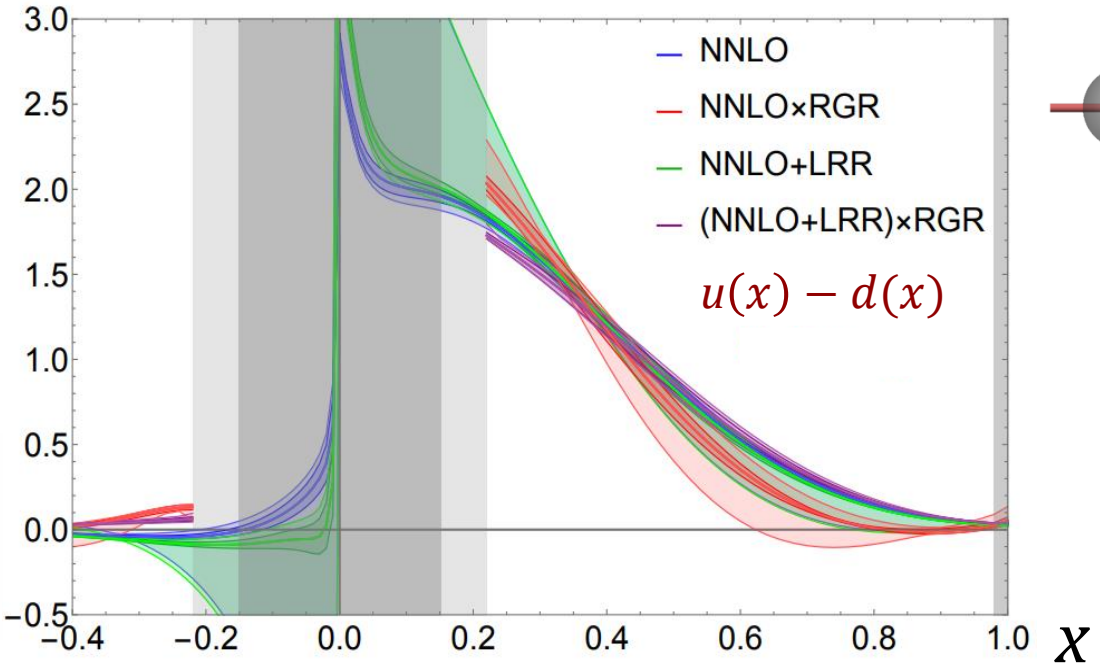
R. Zhang, et. al.
PLB 844, 138081 (2023)

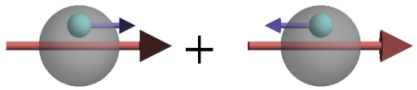
- N_f= 2+1+1 clover/HISQ, $a \approx 0.09$ fm, $P_z \approx 2$ GeV

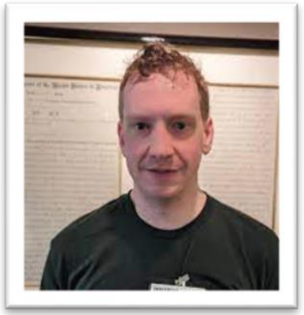
J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Wanted PDFs, GPDs, etc…









P: Jack Holligan

Isovector PDFs Update

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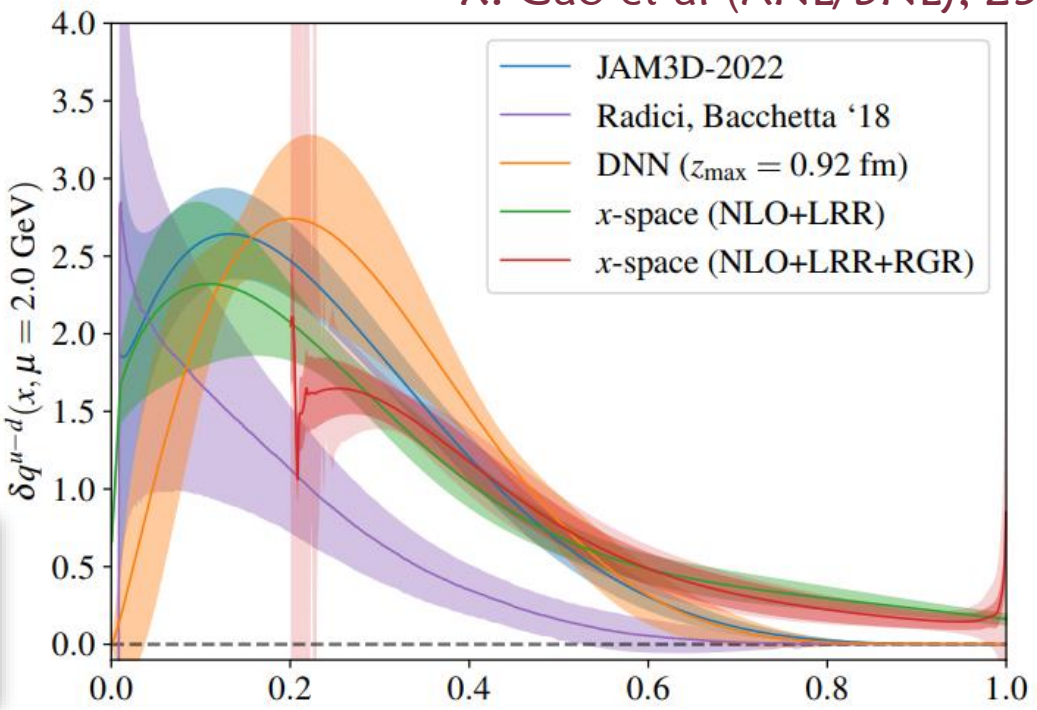
  - Renormalization-group resummation (RGR)

R. Zhang, et. al.
PLB 844, 138081 (2023)

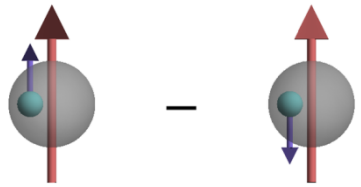
- N_f= 2+1 clover/HISQ, $a \approx 0.076$ fm, $P_z \approx 1.5$ GeV

X. Gao et al (ANL/BNL), 2310.19047 [hep-lat]

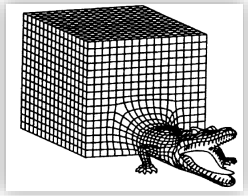
Wanted PDFs, GPDs, etc…



Transversity



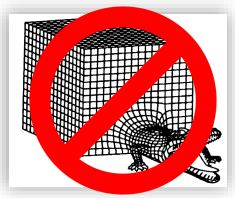
$\delta u(x) - \delta d(x)$

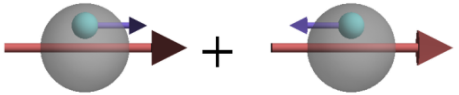


Continuum PDF

§ Nucleon PDFs using quasi-PDFs in the continuum limit

- Lattice details: clover/2+1+1 HISQ (MSULat)
  - $a \approx \{0.06, 0.09, 0.12\}$ fm
  - $M_\pi \in \{\mathbf{135}, 220, 310\}$-MeV pion
  - $M_\pi L \in \{3.3, 5.5\}$
  - $P_z \approx 2$ GeV

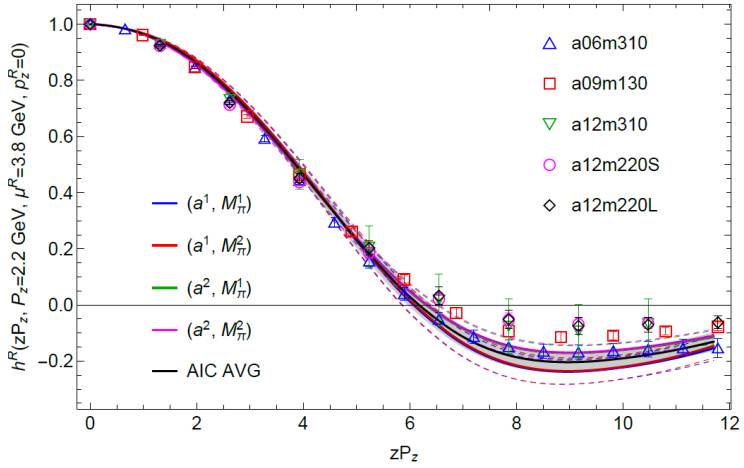


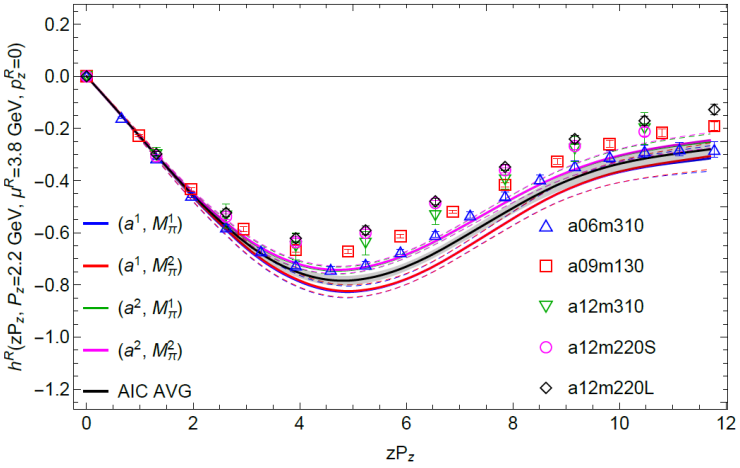


2011.14971, HL et al. (MSULat)

- Naïve extrapolation to physical-continuum limit

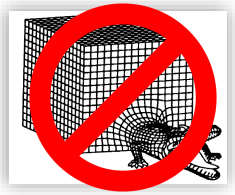
Quantities that can be calculated on the lattice





Continuum PDF

§ Nucleon PDFs using quasi-PDFs in the continuum limit



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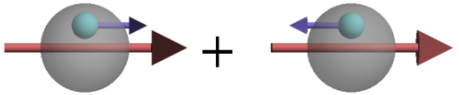
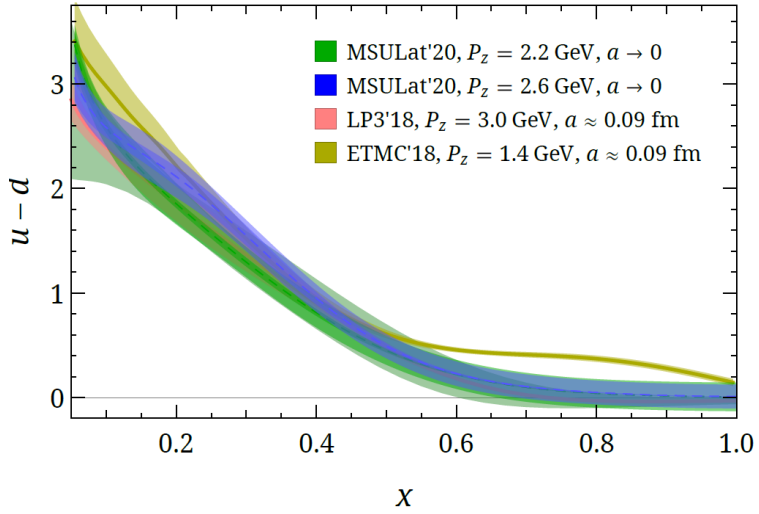
2011.14971, HL et al. (MSULat)

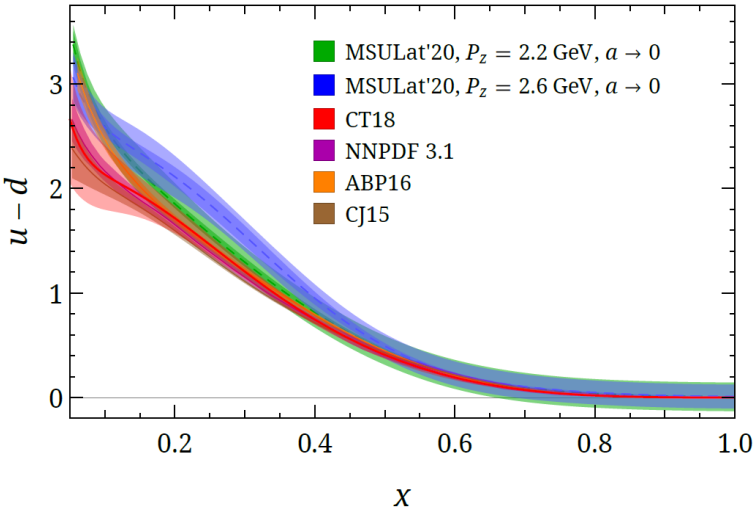


$u(x) - d(x)$

- Naïve extrapolation to physical-continuum limit

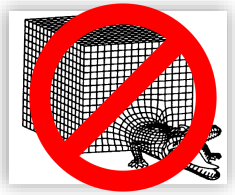
Wanted PDFs, GPDs, etc…





Continuum PDF

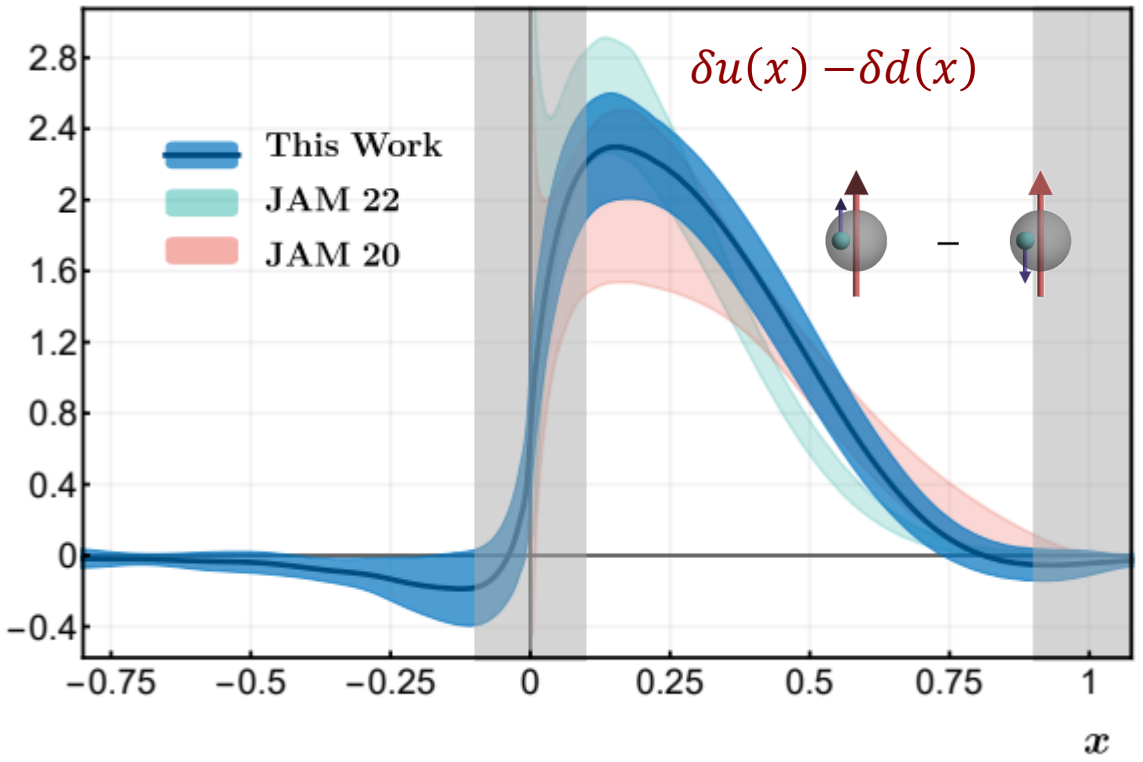
§ Nucleon PDFs using quasi-PDFs in the continuum limit



- Lattice details: clover/2+1 clover (LPC)
  - a ≈ {0.49, 0.64, 0.85, 0.98} fm,
  - $M_\pi \in [222, 354]$-MeV pion,
  - $M_\pi L \in [3.9, 8.1]$
  - $P_z \in [1.8, 2.8]$

F. Yao et al (LPC), 2208.08008

Wanted
PDFs, GPDs,
etc…



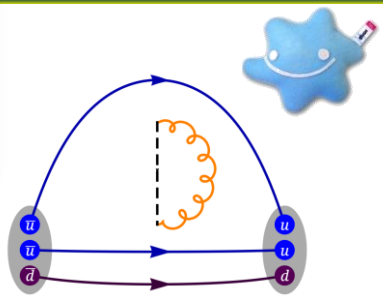
Gluon PDF in Nucleon

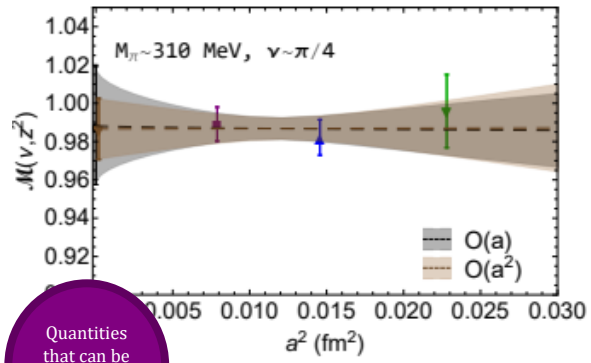
§ Continuum Gluon PDF w/ pseudo-PDF

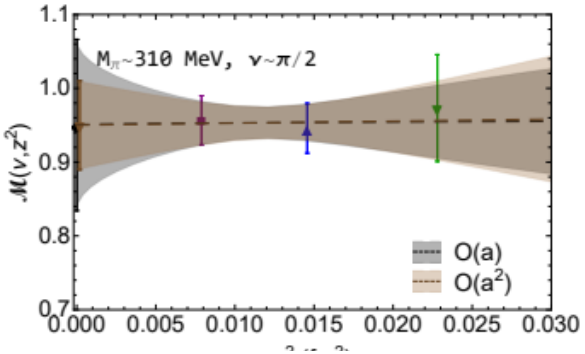
- 2+1+1 HISQ {0.09, 0.12, 0.15} fm

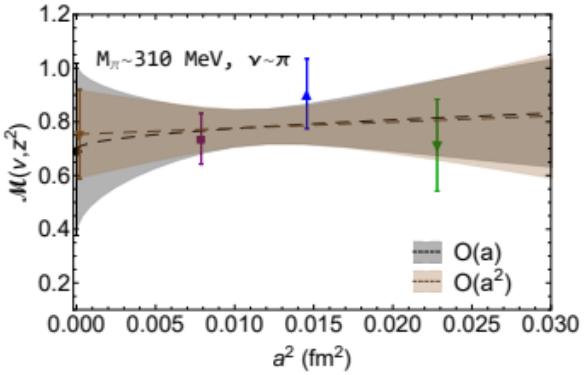
  [220, 310, 700]-MeV pion, 10^5–10^6 statistics

Z. Fan et al (MSULat), 2210.09985

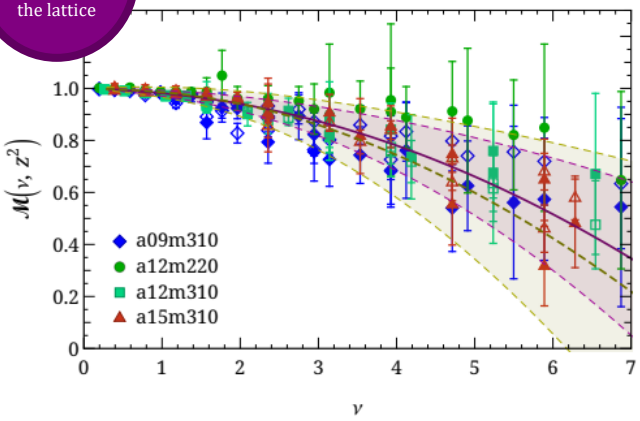


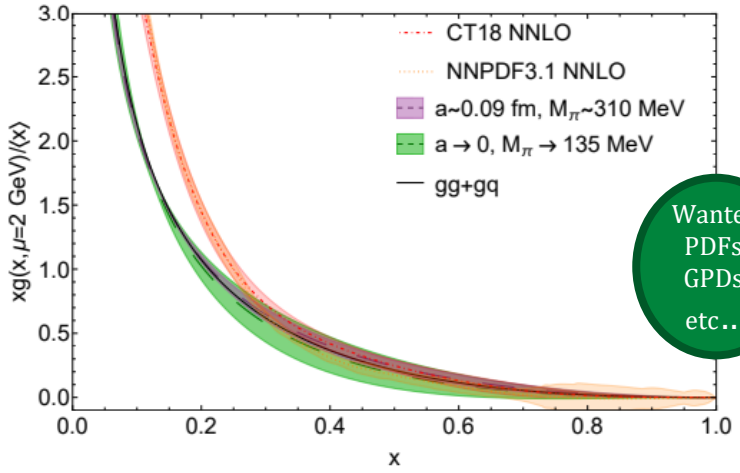






Quantities that can be calculated on the lattice



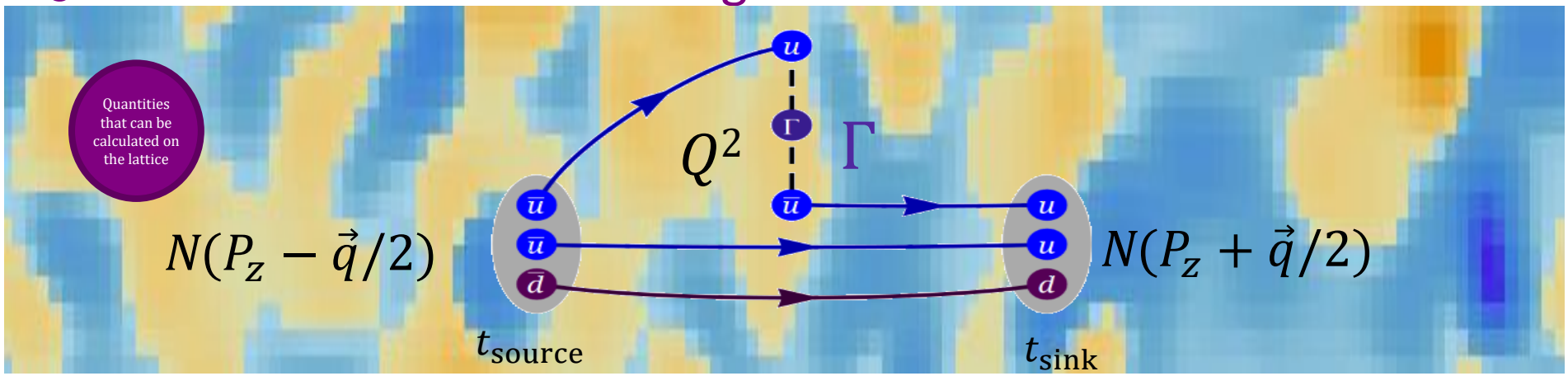


Wanted PDFs, GPDs, etc…

G: Bill Good

Generalized Parton Distributions

- Nucleon/pion GPDs using quasi-PDFs at **physical pion mass** calculated at Breit Frame
- One calculates the following matrix elements on the lattice



Nucleon unpolarized GPDs: H and E

$$\tilde{F}\left(x,\tilde{\xi},t,\bar{P}_z\right) = \frac{\bar{P}_z}{\bar{P}_0}\int\frac{dz}{4\pi}e^{ixz\bar{P}z}\langle P'|\tilde{O}_{\gamma_0}(z)|P\rangle = \frac{\bar{u}(P')}{2\bar{P}^0}\left(\boldsymbol{H}\left(x,\tilde{\xi},t,\bar{P}_z\right)\gamma^0 + \boldsymbol{E}\left(x,\tilde{\xi},t,\bar{P}_z\right)\frac{i\sigma^{0\mu}\Delta_\mu}{2M}\right)u(P'')$$

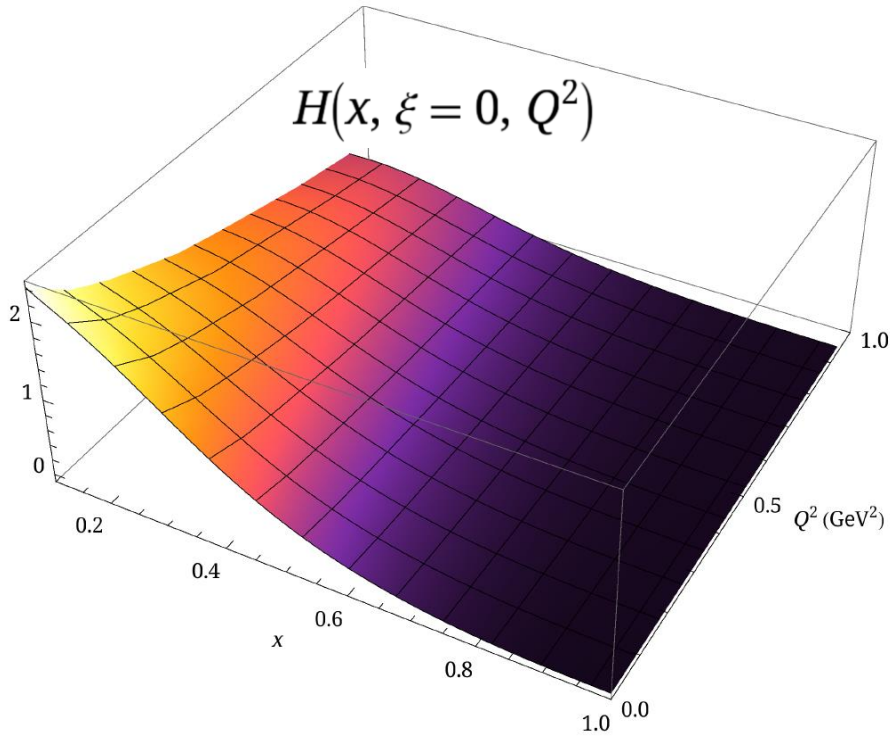
$$p^\mu = \frac{p''^\mu + p'^\mu}{2}, \qquad \Delta^\mu = p''^\mu - p'^\mu, \qquad t = \Delta^2, \qquad \xi = \frac{p''^+ - p'^+}{p''^+ + p'^+}$$

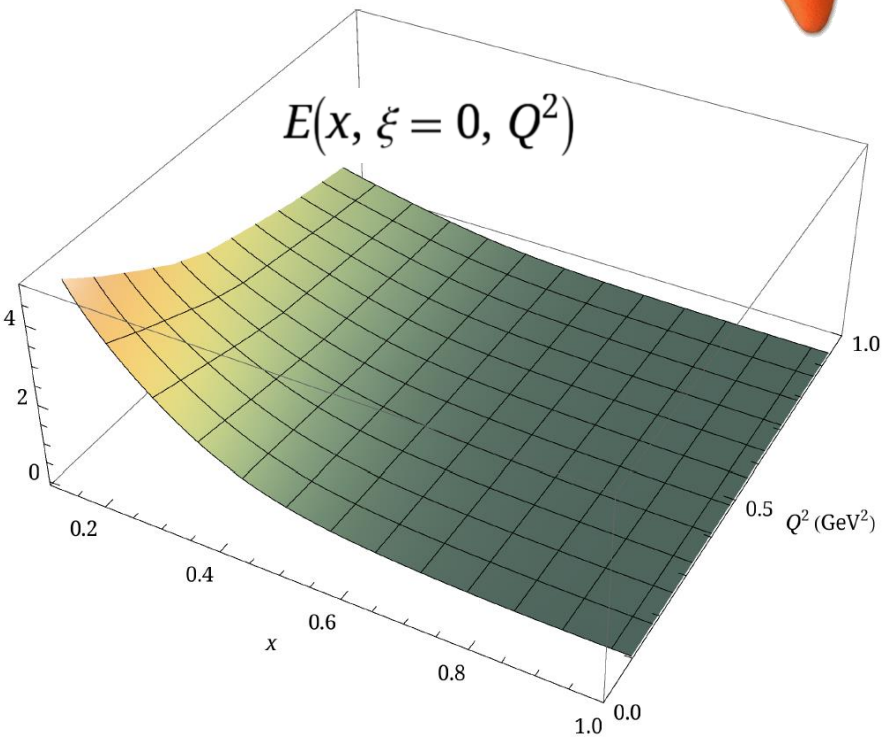
HL, Phys. Rev. Lett. 127 (2021) 18, 182001

Isovector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at **physical pion mass**

- MSULat: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results





HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Isovector Nucleon GPDs

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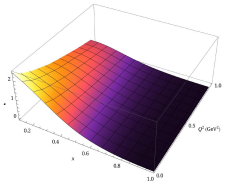
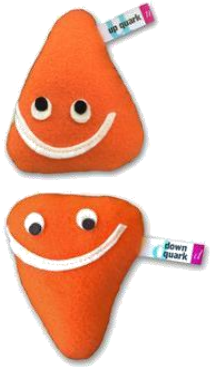
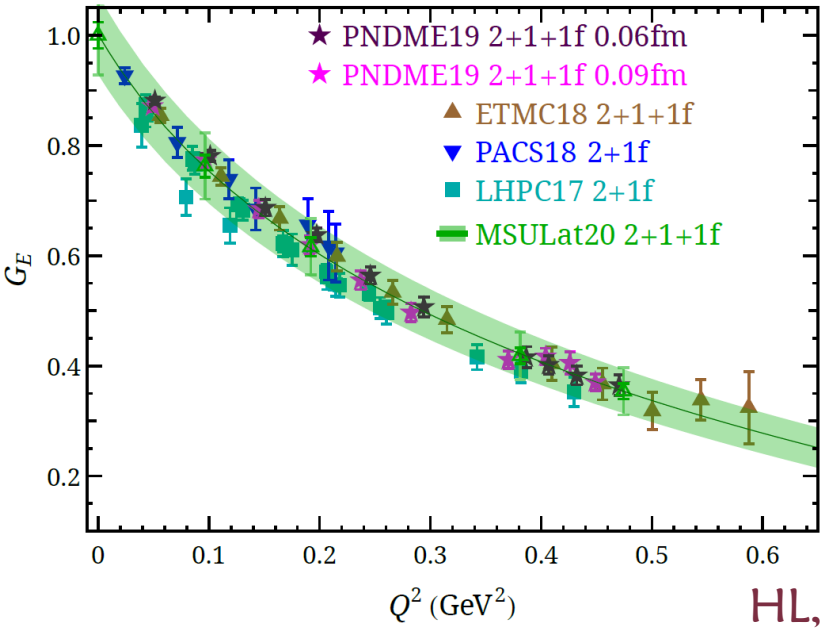
- Lattice details: clover/2+1+1 HISQ (MSULat)
  0.09 fm, **135-MeV** pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results

$$\int_{-1}^{+1} dx\, x^{n-1} \; = \sum_{i=0,\text{even}}^{n-1} (-2\xi)^i A_{ni}^q(t) + (-2\xi)^n C_{n0}^q(t)\Big|_{n\ \text{even}}$$





$n = 1$



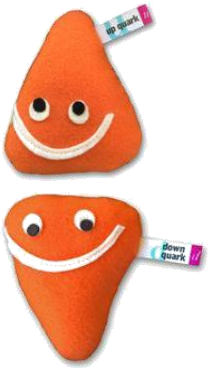
HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Isovector Nucleon GPDs

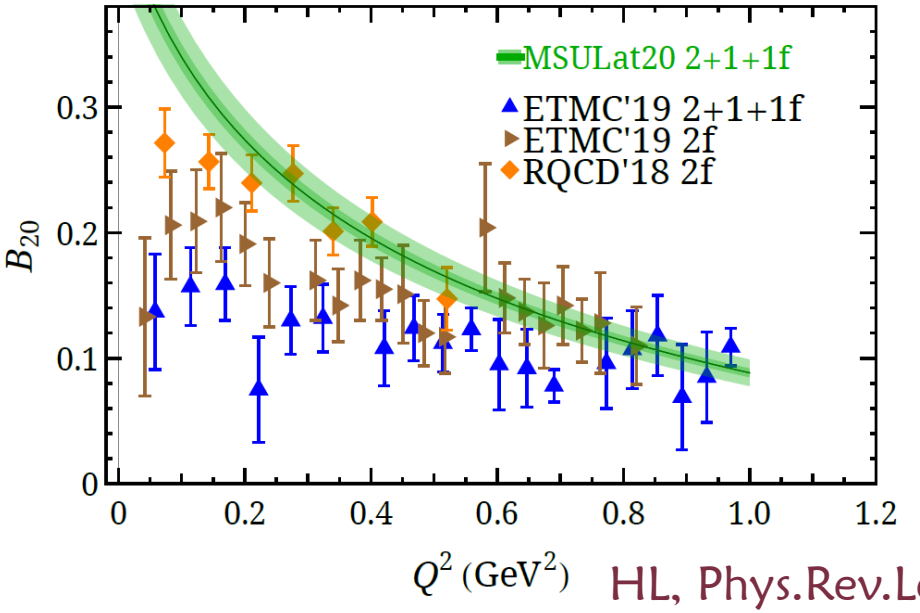
§ Nucleon GPD using quasi-PDFs at physical pion mass

- Lattice details: clover/2+1+1 HISQ (MSULat)
  0.09 fm, **135-MeV** pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results

$$\int_{-1}^{+1} dx\, x^{n-1} \; = \sum_{i=0,\text{even}}^{n-1} (-2\xi)^i B^q_{ni}(t) - (-2\xi)^n C^q_{n0}(t)\Big|_{n\ \text{even}}$$



$n = 2$

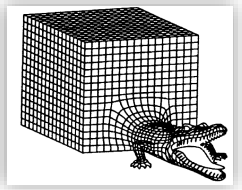


HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Nucleon Tomography

§ Nucleon GPD using quasi-PDFs at physical pion mass

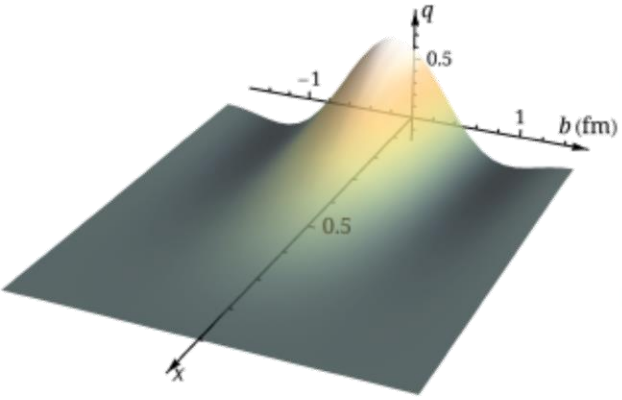
- Lattice details: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results

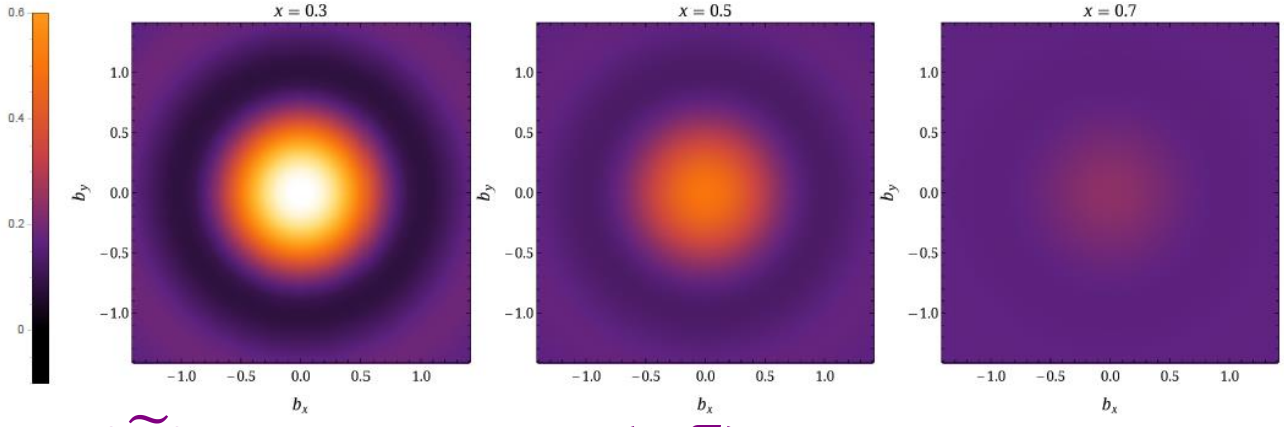


finite-volume, discretization,

$$q(x,b) = \int \frac{d\vec{q}}{(2\pi)^2} H(x, \xi = 0, t = -\vec{q}^2) e^{i\vec{q}\cdot\vec{b}}$$

HL, Phys. Rev. Lett. 127 (2021) 18, 182001





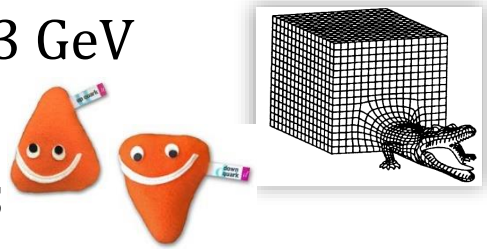
§ Nucleon helicity GPD ($\widetilde{H}$) and pion GPD (H^π) using quasi-PDFs at **physical pion mass**

HL (MSULat), Phys.Lett.B 824 (2022) 136821;
Phys. Lett. B 846 (2023) 138181

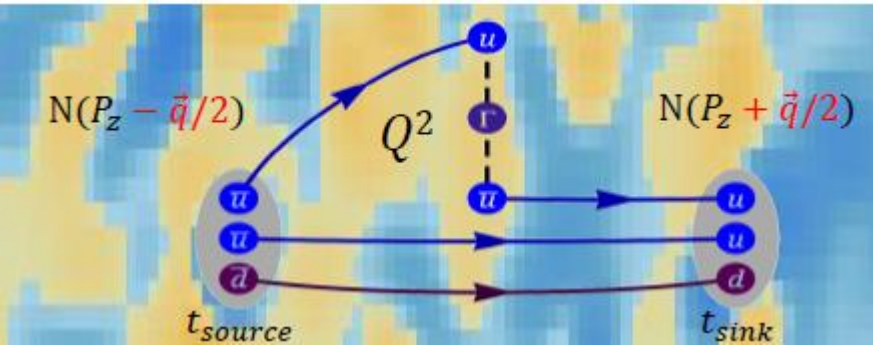
Asymmetric-Frame GPD

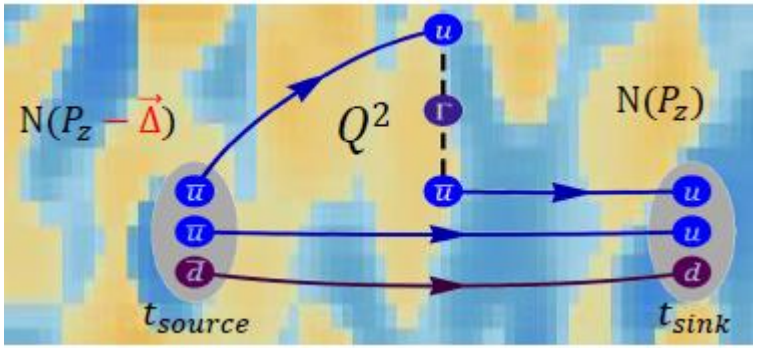
§ New calculations by ANL/BNL/ETMC using asymmetric frame

- 2+1+1 twisted-Wilson, 0.09 fm, $P_z \approx 1.3$ GeV
  260-MeV pion, one source-sink used
- $\xi = 0$ isovector nucleon GPD results



finite-volume, discretization, *heavy quark, excited-state, …*





$$\tilde{F}(x,\tilde{\xi},t,\bar{P}_z) = \frac{\bar{u}(P')}{2\bar{P}^0}\left(\boldsymbol{H}(x,\tilde{\xi},t,\bar{P}_z)\gamma^0 + \boldsymbol{E}(x,\tilde{\xi},t,\bar{P}_z)\frac{i\sigma^{0\mu}\Delta_\mu}{2M}\right)u(P'')$$

$$F^\mu(z,P,\Delta) = \bar{u}(p_f,\lambda')\left[\frac{P^\mu}{m}\boldsymbol{A_1} + mz^\mu\boldsymbol{A_2} + \frac{\Delta^\mu}{m}\boldsymbol{A_3} + im\sigma^{\mu z}\boldsymbol{A_4} + \frac{i\sigma^{\mu\Delta}}{m}\boldsymbol{A_5} + \frac{P^\mu i\sigma^{z\Delta}}{m}\boldsymbol{A_6} + mz^\mu i\sigma^{z\Delta}\boldsymbol{A_7} + \frac{\Delta^\mu i\sigma^{z\Delta}}{m}\boldsymbol{A_8}\right]u(p_i,\lambda)$$

$$H(z\cdot P^{s/a}, z\cdot\Delta^{s/a}, (\Delta^{s/a})^2) = A_1 + \frac{\Delta^{s/a}\cdot z}{P^{s/a}\cdot z}A_3$$

ANL/BNL/ETMC, 2209.05373 ,2310.13114

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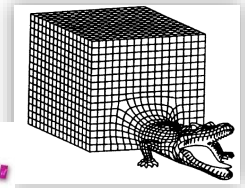
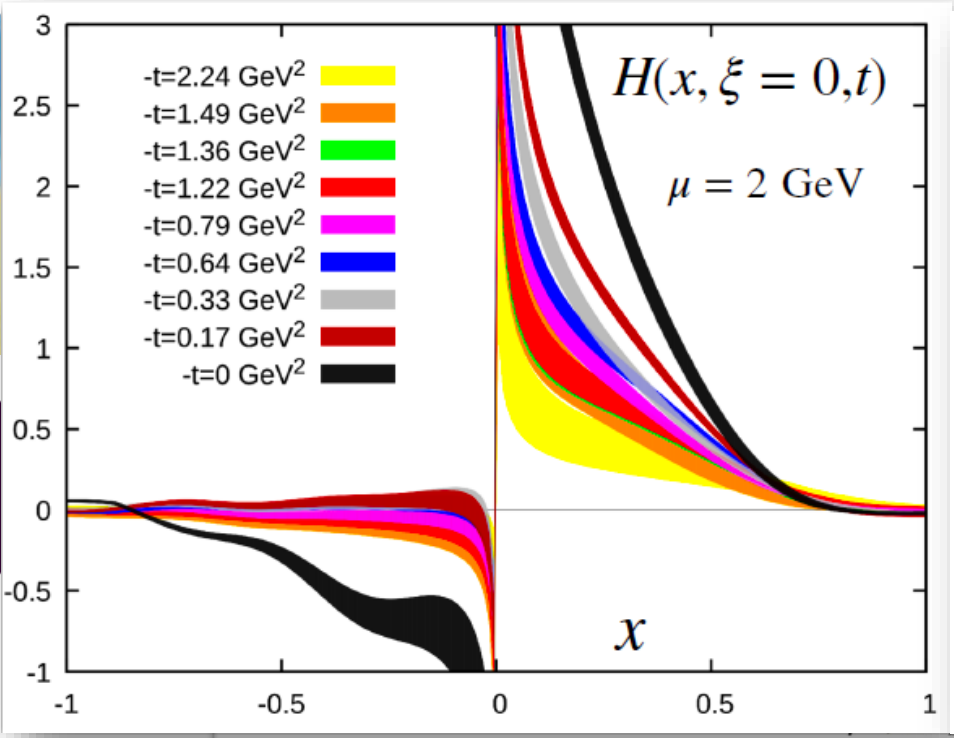
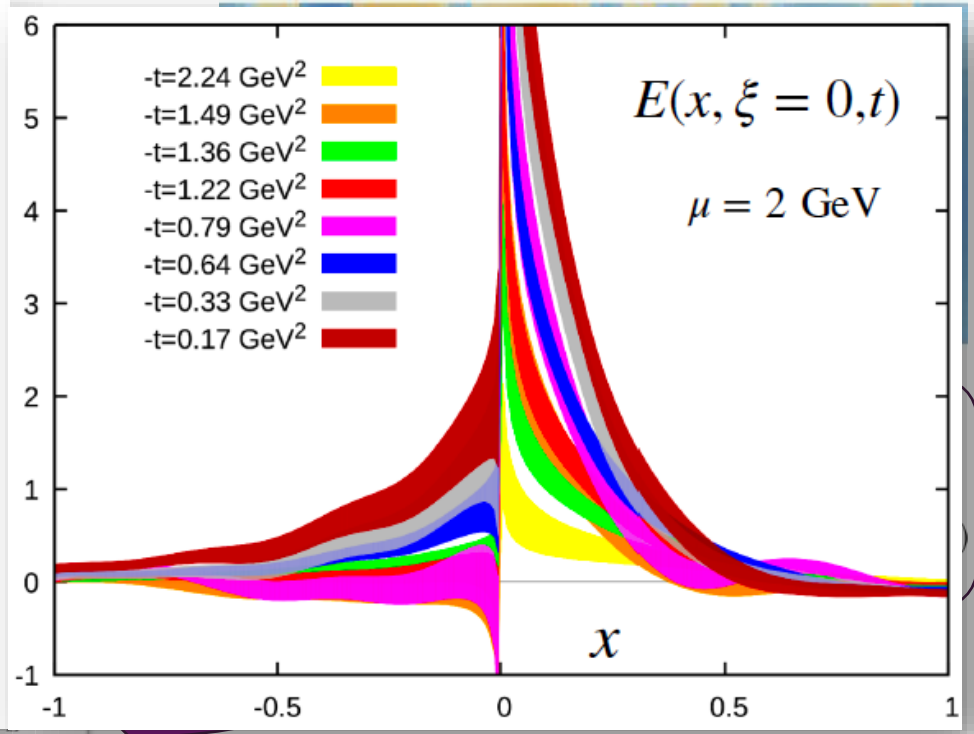


finite-volume, discretization, *heavy quark, excited-state, …*





See Shohini Bhattacharya' talks@ Lattice 2024

ANL/BNL/ETMC, 2209.05373 ,2310.13114

GPD Systematic Update

§ Nucleon isovector GPDs calculated directly at physical pion mass

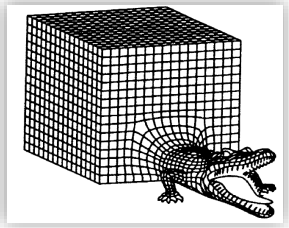
- NNLO matching & treat leading-renormalon effects
  - Leading-renormalon resummation (LRR)
  - Renormalization-group resummation (RGR)

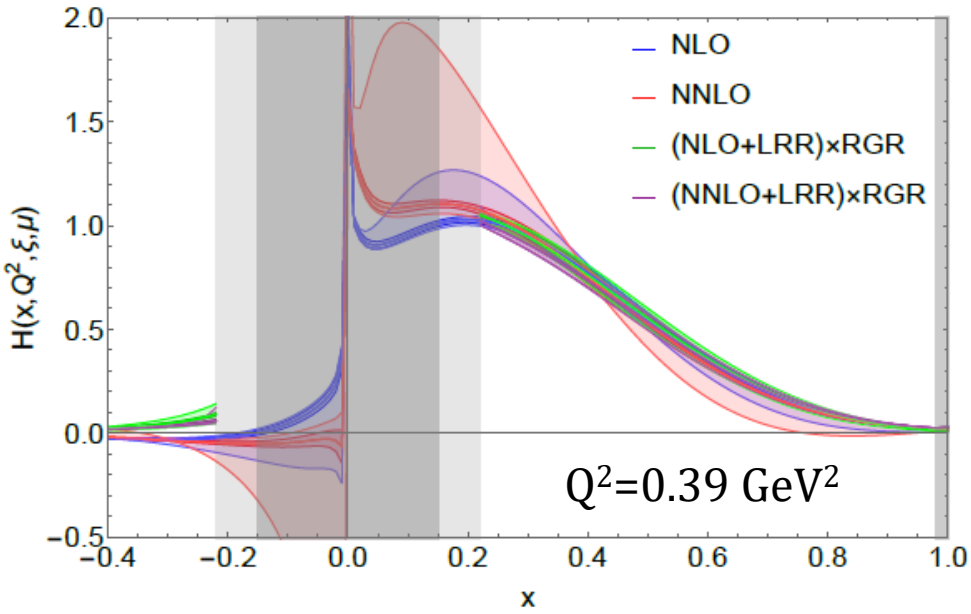
R. Zhang, et. al.
PLB 844, 138081 (2023)

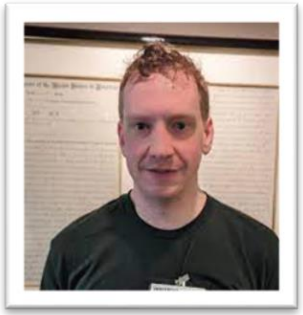
- N_f= 2+1+1 clover/HISQ, $a \approx 0.09$ fm, 135-MeV pion, $P_z \approx 2$ GeV

J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Wanted PDFs, GPDs, etc…







P: Jack Holligan

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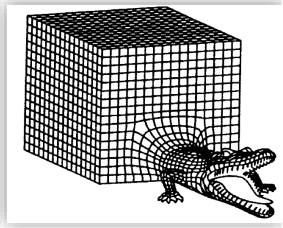

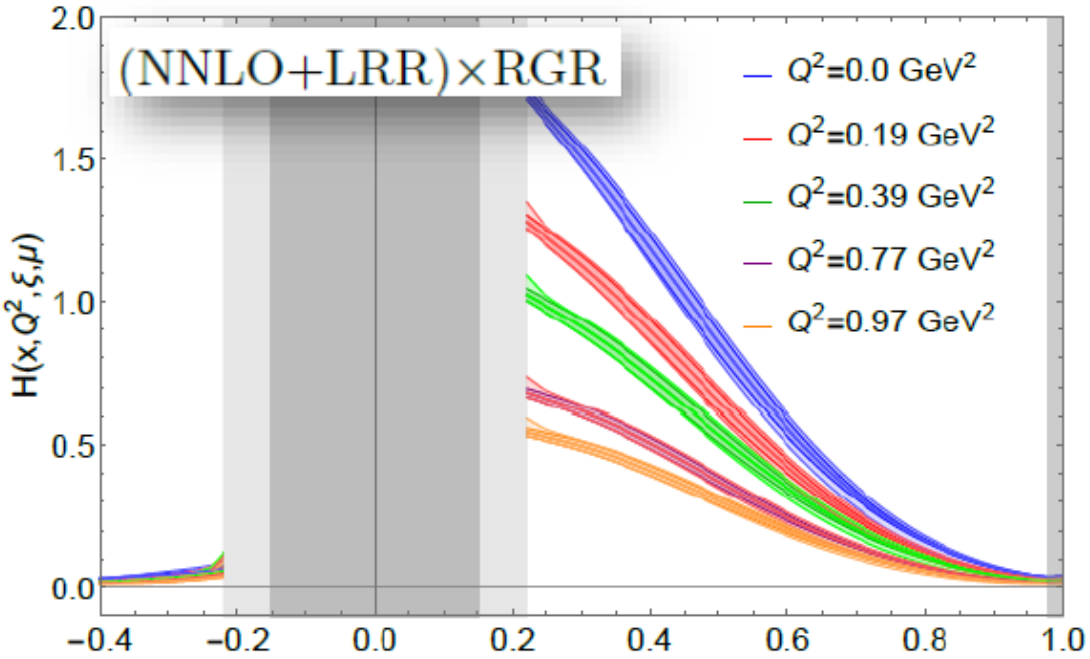

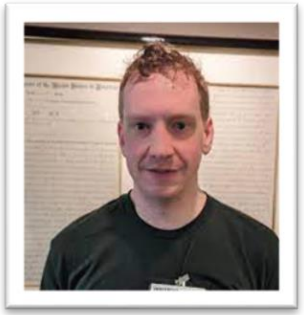

P: Jack Holligan

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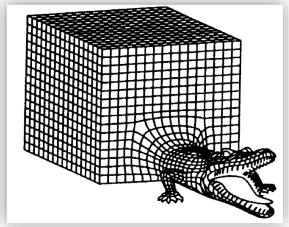
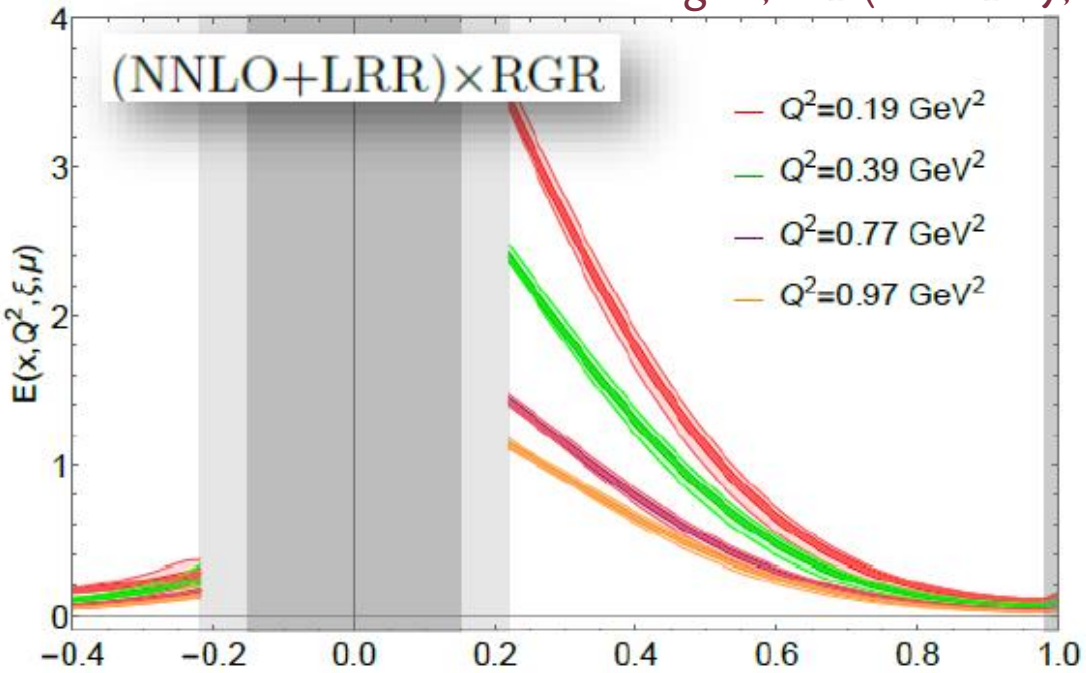
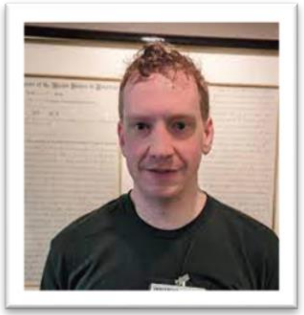




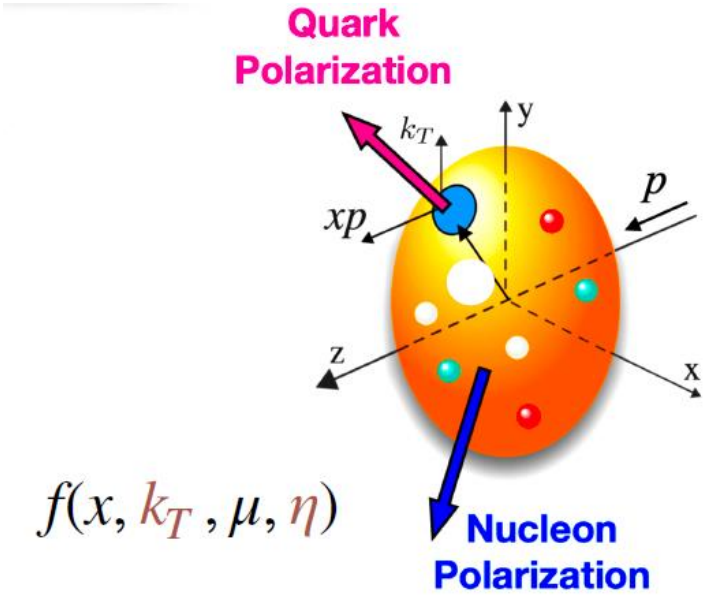


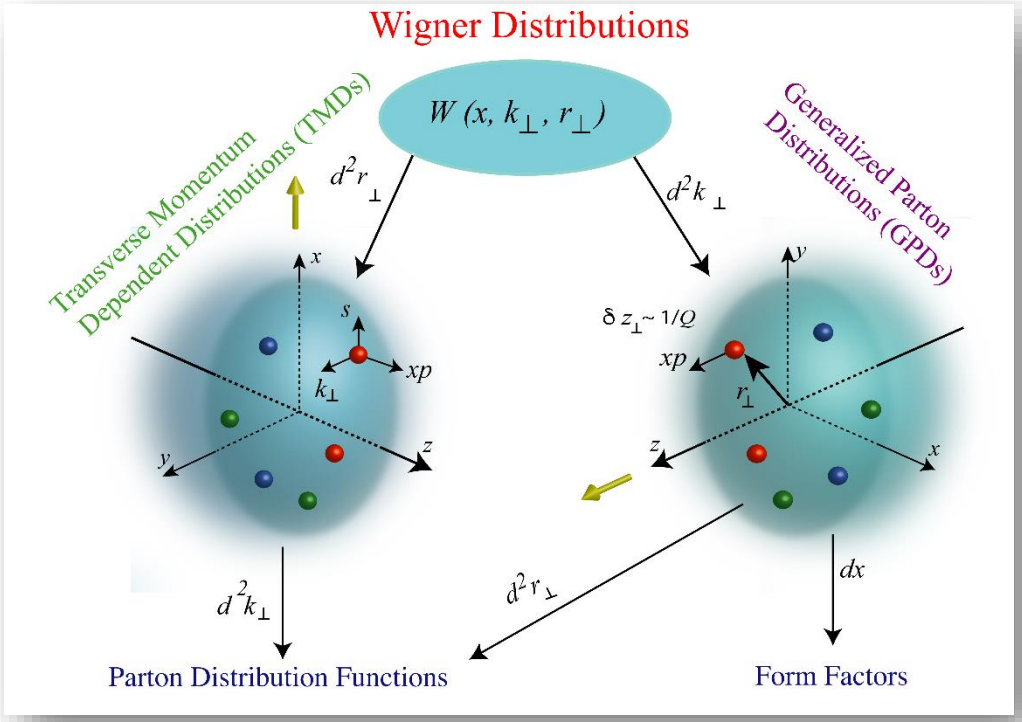
P: Jack Holligan

Transverse Momentum Distributions

§ TMDs inform on confined motion of partons in hadron

- Complementary to structure provided by PDFs and GPDs





Picture from INT Program INT–17–3

§ Mini-review by David Lin @ Mon. 2:30PM

Transverse Momentum Distributions

§ TMDs inform on confined motion of partons in hadron

- Complementary to structure provided by PDFs and GPDs

Relating quasi-TMDPDF to TMDPDF

M.A. Ebert, S.T. Schindler, I.W. Stewart, Y. Zhao, JHEP 04 (2022) 178

$$\tilde{f}^{\mathrm{TMD}}(x,\vec{b}_T,\mu,P^z) = \underbrace{C^{\mathrm{TMD}}(\mu,xP^z)}_{\text{pertub. theo.}}\, g_S(b_T,\mu)\exp\left[\frac{1}{2}K(b_T,\mu)\log\frac{(2xP^z)^2}{\zeta}\right] \times f^{\mathrm{TMD}}(x,\vec{b}_T,\mu,\zeta) + \mathcal{O}\left(\frac{q_T^2}{P_z^2},\frac{\Lambda_{\mathrm{QCD}}^2}{P_z^2}\right)$$

★ To obtain f^{TMD}, one computes $\tilde{f}^{\mathrm{TMD}}$ with lattice QCD

★ Also need non-perturbative calculation of

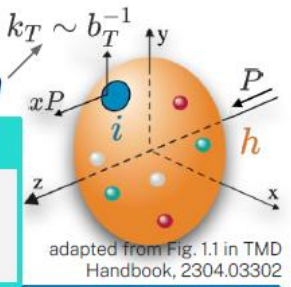
The Collins-Soper kernel, $K(b_T,\mu)$

The soft function, $g_S(b_T,\mu) \sim \sqrt{S_I(b_T,\mu)}$

Slide from D. Lin

Transverse Momentum Distributions

Collins-Soper (CS) kernel from lattice QCD + LaMET complements global analyses in nonperturbative region of *small transverse momentum*



adapted from Fig. 1.1 in TMD Handbook, 2304.03302

CS kernel governs RG evolution of **Transverse-Momentum-dependent Distributions (TMDs)**

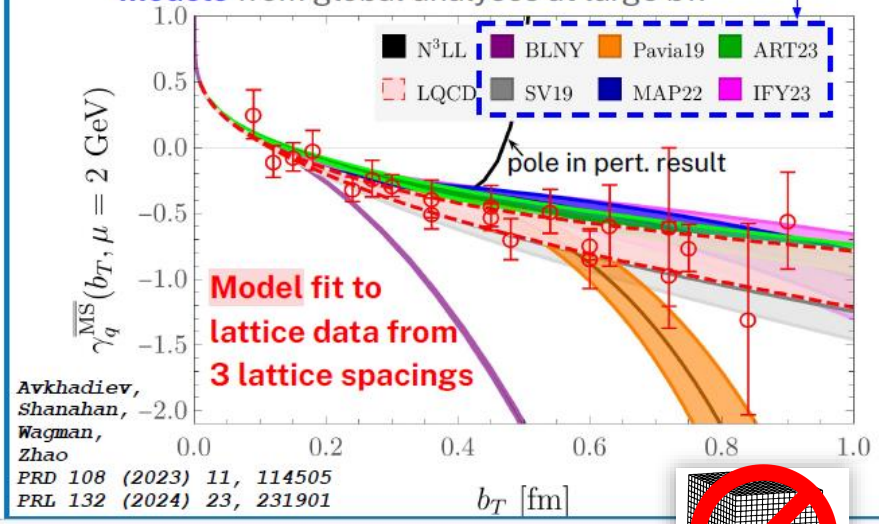
$$\gamma_i(b_T,\mu) = 1/\ln[\boldsymbol{P_1}/\boldsymbol{P_2}]\ln\left[\tilde{f}_{i/h}(x,b_T,\mu,\boldsymbol{P_1})\Big/\tilde{f}_{i/h}(x,b_T,\mu,\boldsymbol{P_2})\right] + \text{power corrections}$$

$i \in \{\text{quark, gluon}\}$

(space-like) **quasi-TMDs** computable in lattice QCD ⟷ LaMET ⟷ light-like TMDs

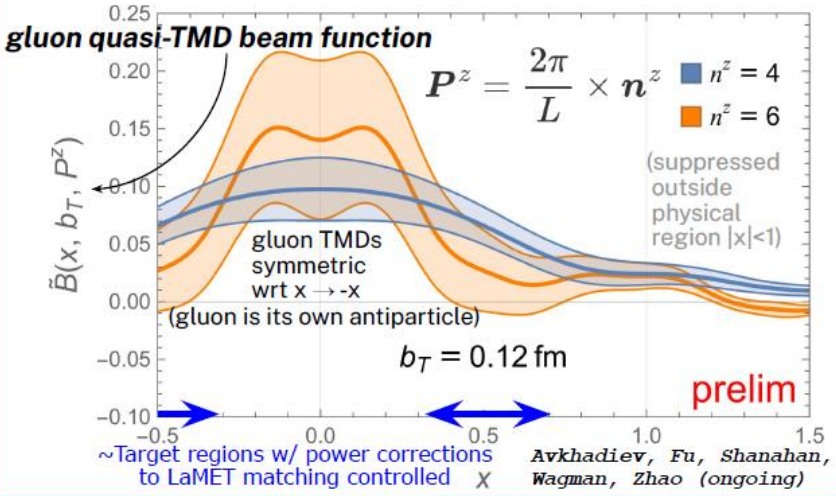
Quark CS kernel — completed

- Systematic control over quark mass, operator mixing, and discretization effects.
- **Sufficiently precise to discriminate between pheno models** from global analyses at large bT.



Gluon CS kernel — ongoing

- **Aiming for ~30x more stats** than in the quark project for comparable precision.
- **No global analysis results yet** (expected w/ EIC data) — lattice QCD + LaMET will provide a prediction.



Slide by G: Artur Avkhadiev (MIT)

Impact of Lattice-QCD PDFs on Global Fits

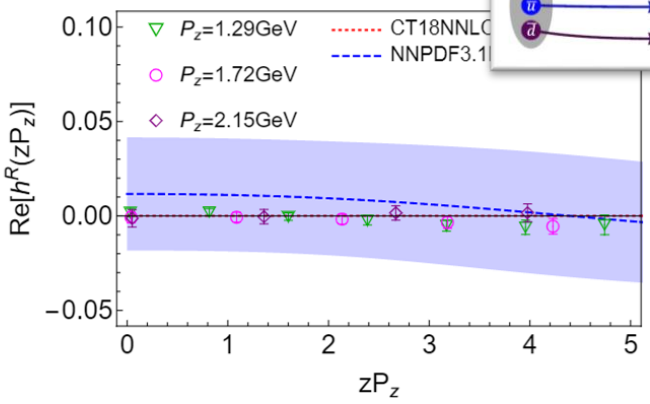
First Lattice Strange PDF

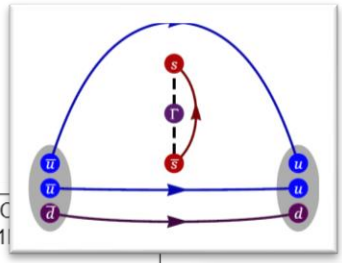
§ Results by MSULat/quasi-PDF method

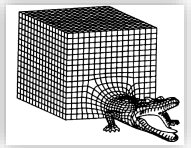
- Clover on 2+1+1 HISQ, 0.12-fm 310-MeV QCD vacuum
- Extrapolated to $M_\pi \approx 140$ MeV

R. Zhang et al (MSULat), 2005.01124

$$\mathrm{Re}[h(z)] \propto \int \mathrm{d}x \left(s(x) - \bar{s}(x)\right) \cos(xzP_z)$$





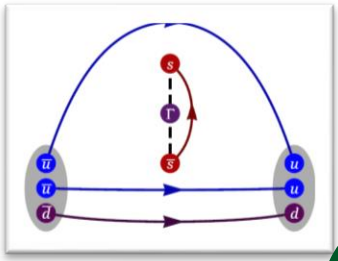


Quantities that can be calculated on the lattice

Lattice Strangeness Asymmetry Impact

§ Results by MSULat/quasi-PDF method

- Clover on 2+1+1 HISQ, 0.12-fm 310-MeV QCD vacuum
- Extrapolated to $M_\pi \approx 140$ MeV, $P_z \approx 1.7$ GeV



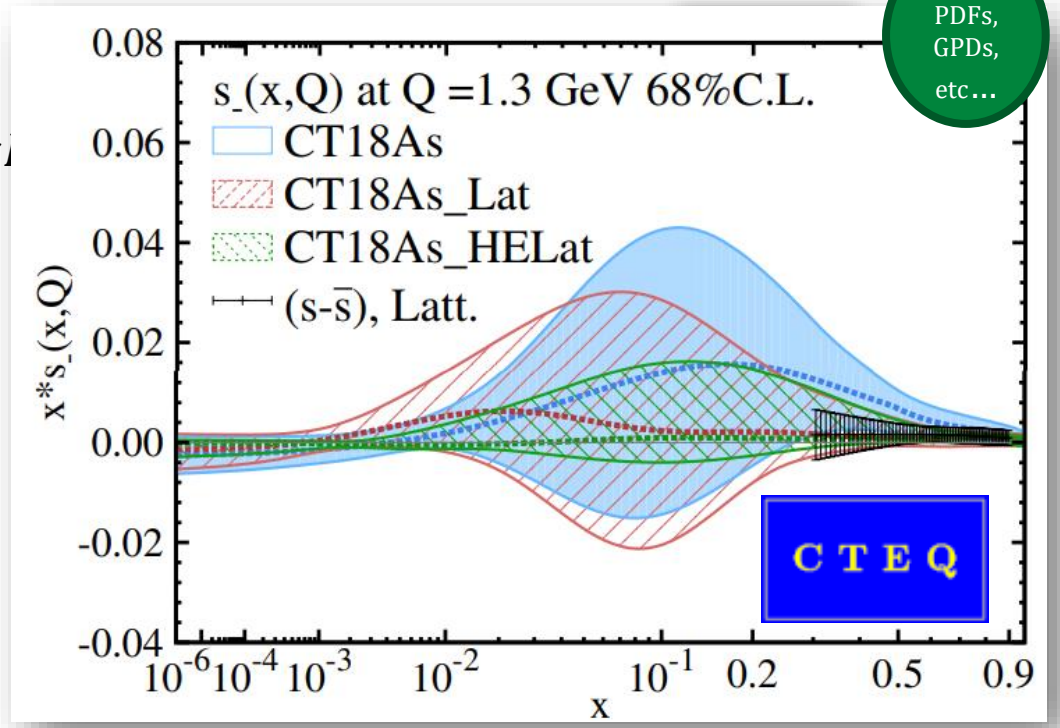
R. Zhang et al (MSULat), 2005.01124

$$\mathrm{Re}[h(z)] \propto \int \mathrm{d}x \left(s(x) - \bar{s}(x)\right) \cos(xz$$

§ From quasi-PDF to PDF

$$\tilde{f}_q(x, P_z) = \int_{-1}^{1} \frac{dy}{|y|} f_q(y) C_{q/q}(x, y, P_z, \mu) + O\left(\frac{\Lambda^2_{\mathrm{QCD}}}{x^2 P_z^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{(1-x)^2 P_z^2}\right)$$

T. Hou, HL, M. Yan, C. Yuan, 2211.11064



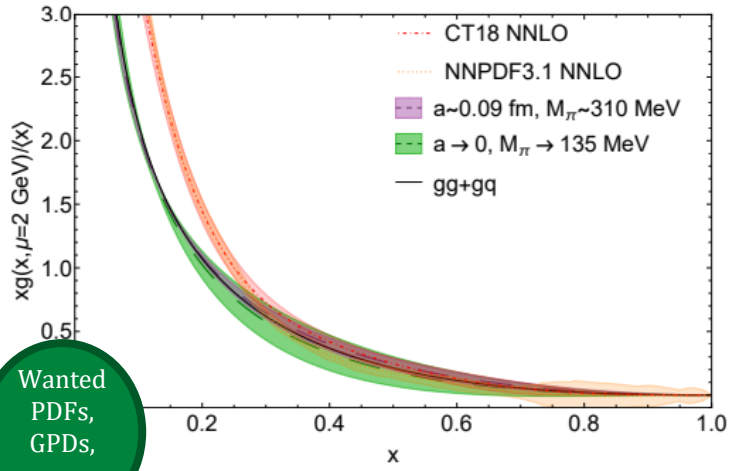
§ The strangeness asymmetry $s(x, Q) - \bar{s}(x, Q)$ at $x > 0.2$ is difficult to measure, but can be predicted in lattice QCD

Lattice Gluon PDF Impact

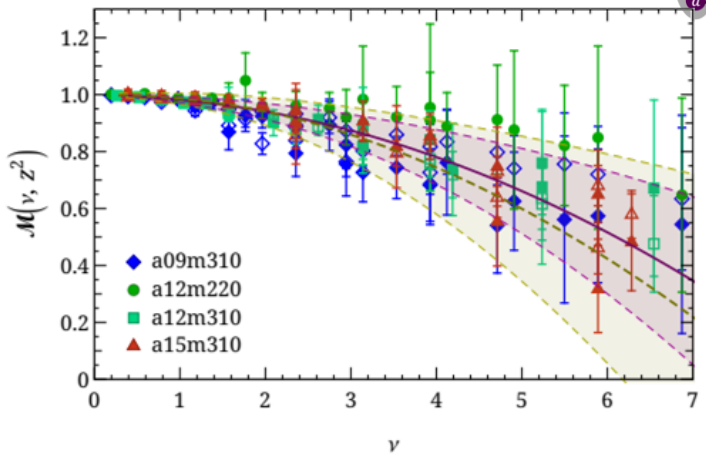
§ First continuum Gluon PDF w/ pseudo-PDF

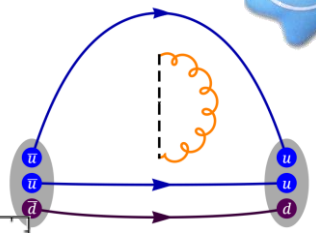
- 2+1+1 HISQ {0.09, 0.12, 0.15} fm, $P_z \approx 2.2$ GeV
  [220, 310, 700]-MeV pion, 10^5–10^6 statistics

2210.09985, W. Good et al (MSULat)

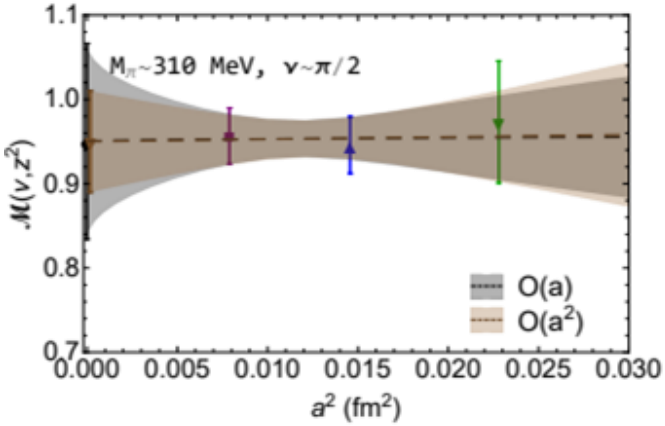


Wanted PDFs, GPDs, etc…





Quantities that can be calculated on the lattice



§ Can use $a \approx 0.09$ fm results as

- Best estimate of gluon PDF
- Aim for future precision

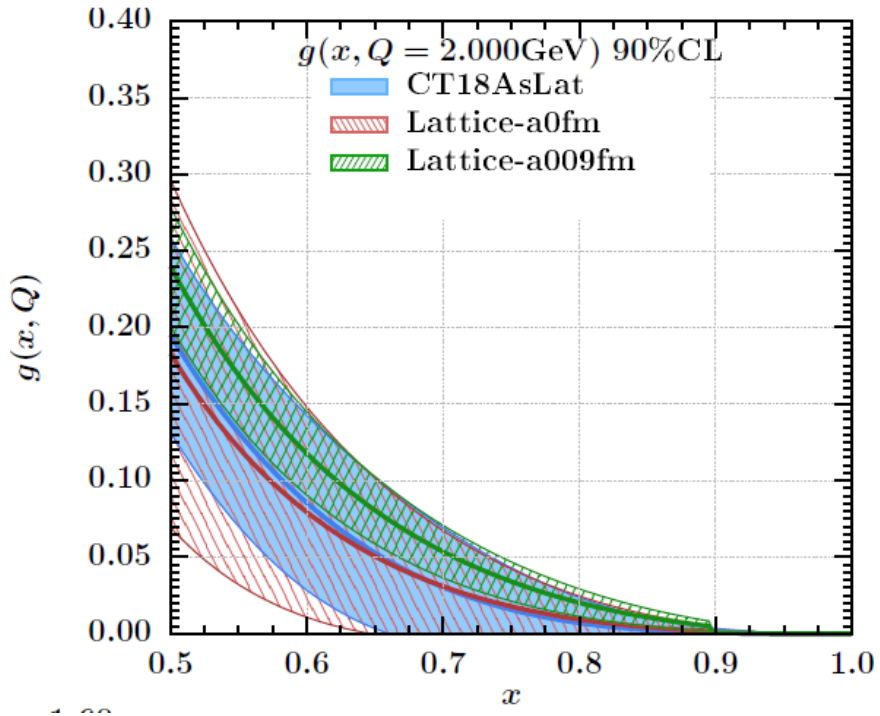
Lattice Gluon PDF Impact

§ Preliminary study with CTEQ-TEA analysis

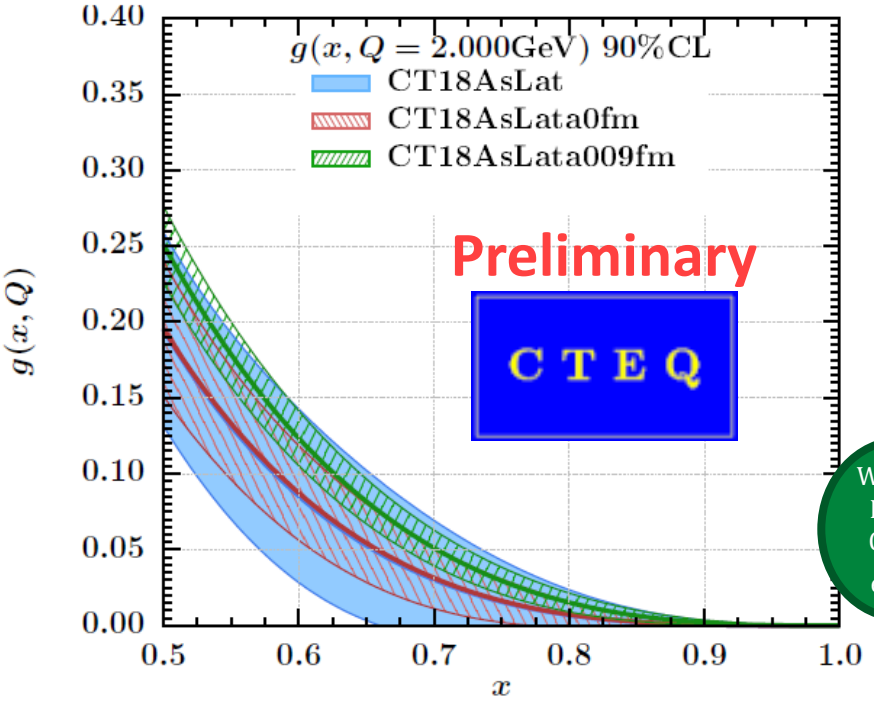
- Take lattice inputs in the region where no strong experimental data constraints, $x \in [0.4, 0.7]$
- Using e-pump for re-weighting

Plots by Alim Ablat (Xinjiang U.)

Before



After



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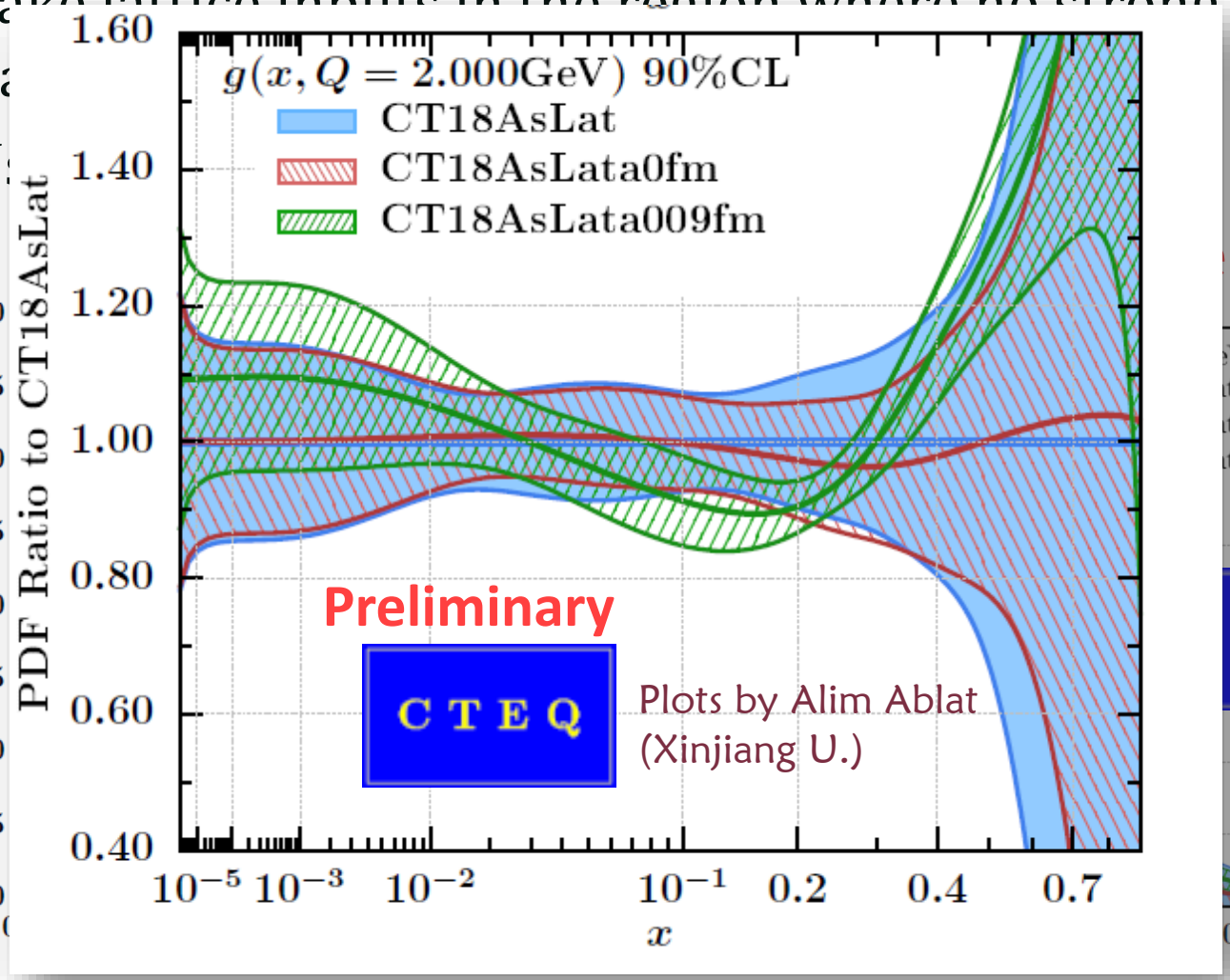
Lattice Gluon PDF Impact

§ Preliminary study with CTEQ-TEA analysis

- Take lattice inputs in the region where no strong experimental da
- Us



Wanted PDFs, GPDs, etc…

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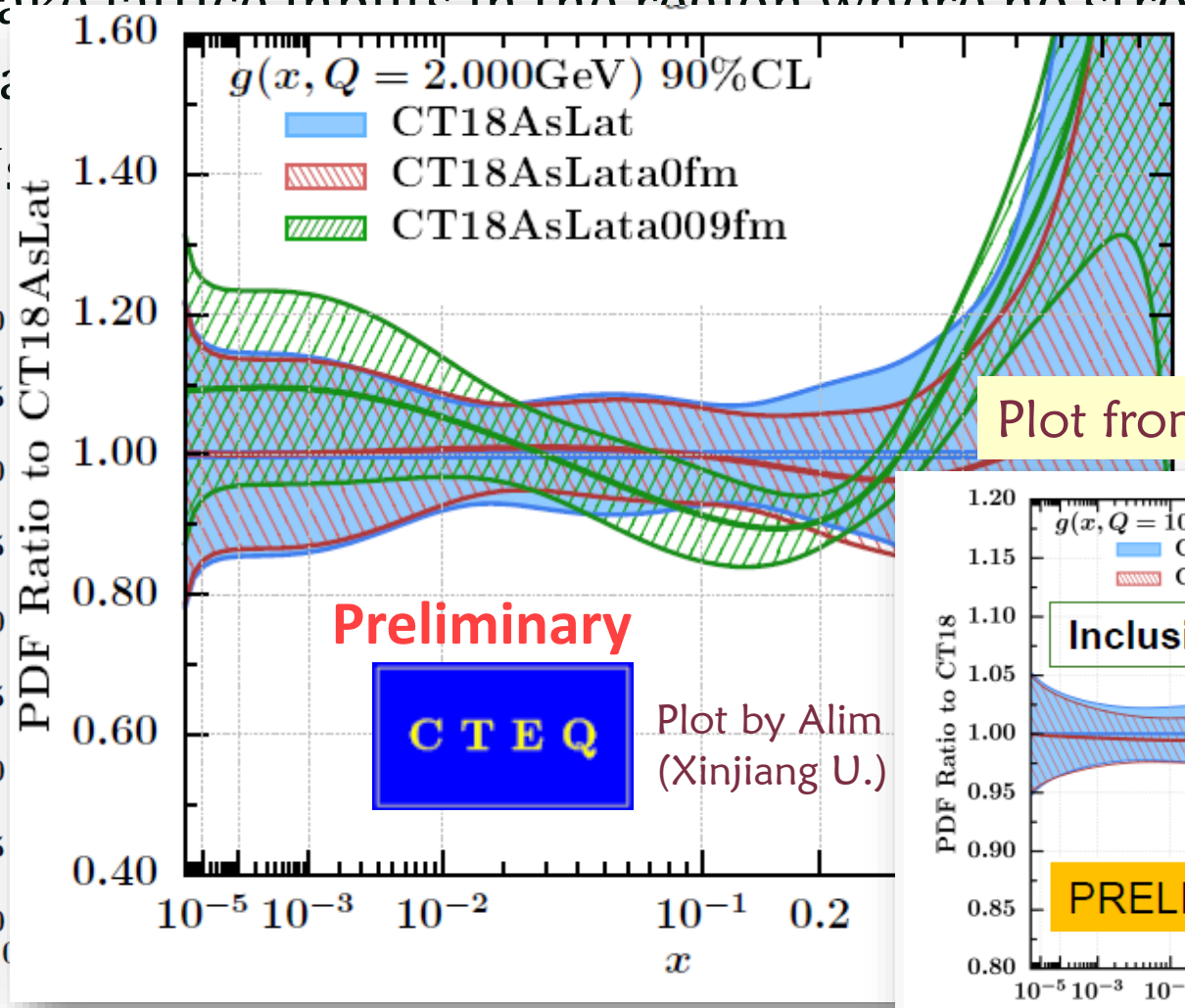
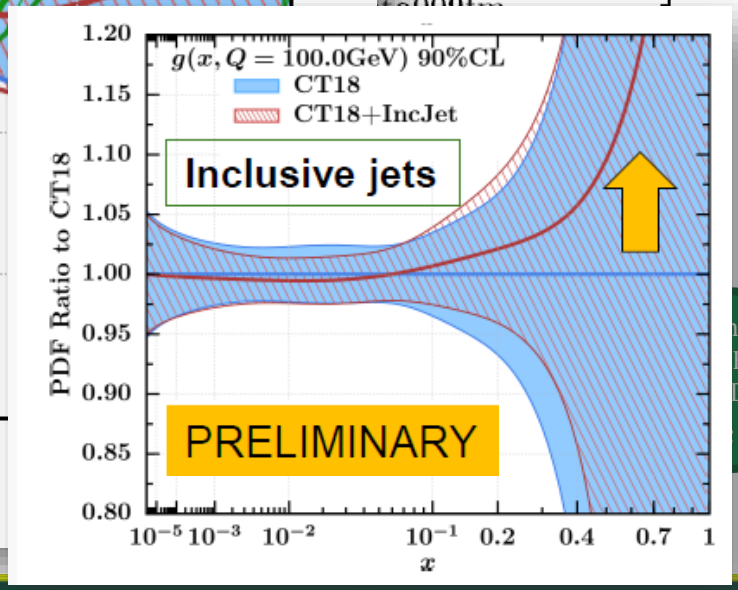


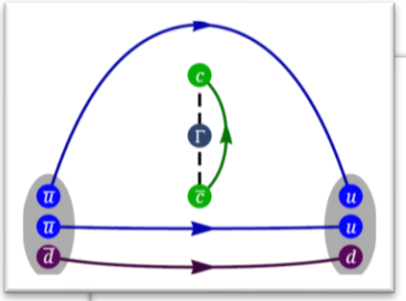
Plot from P. Nadolsky



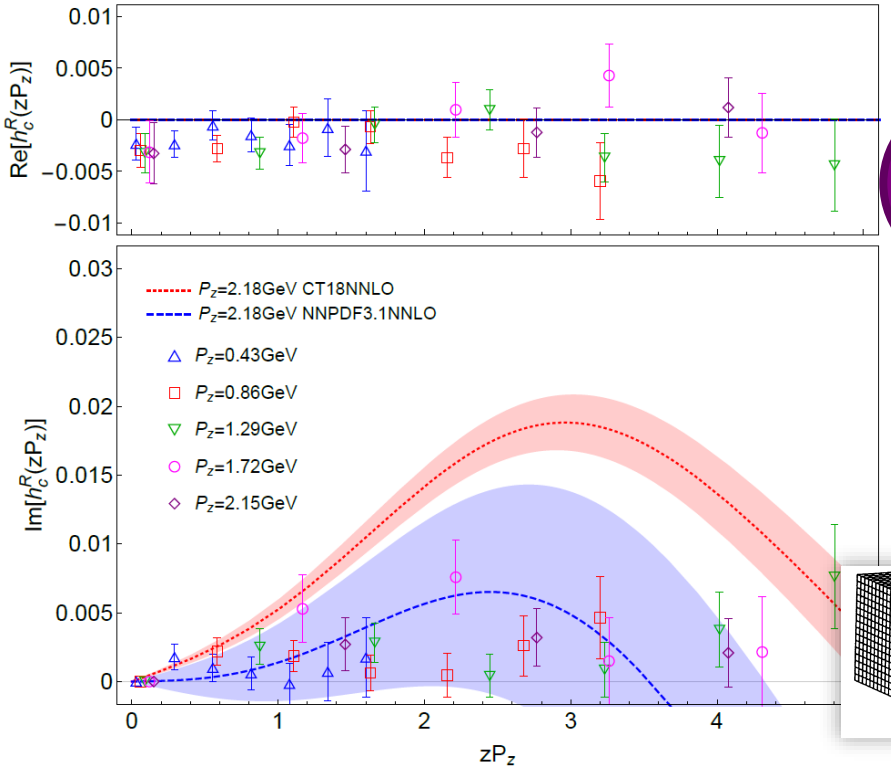
First Lattice Charm PDF

- § Large uncertainties in global PDFs
- § Results by MSULat/quasi-PDF method
  - Clover on 2+1+1 HISQ 0.12-fm 310-MeV QCD vacuum

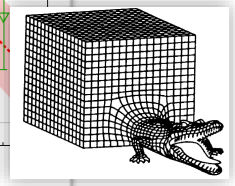
2005.01124, R. Zhang et al (MSULat)



- suggest a symmetric $c - \bar{c}$ distribution
- much smaller than strange PDF



Quantities that can be calculated on the lattice



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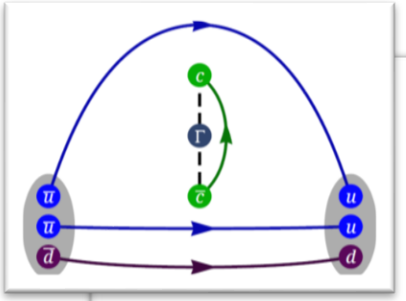

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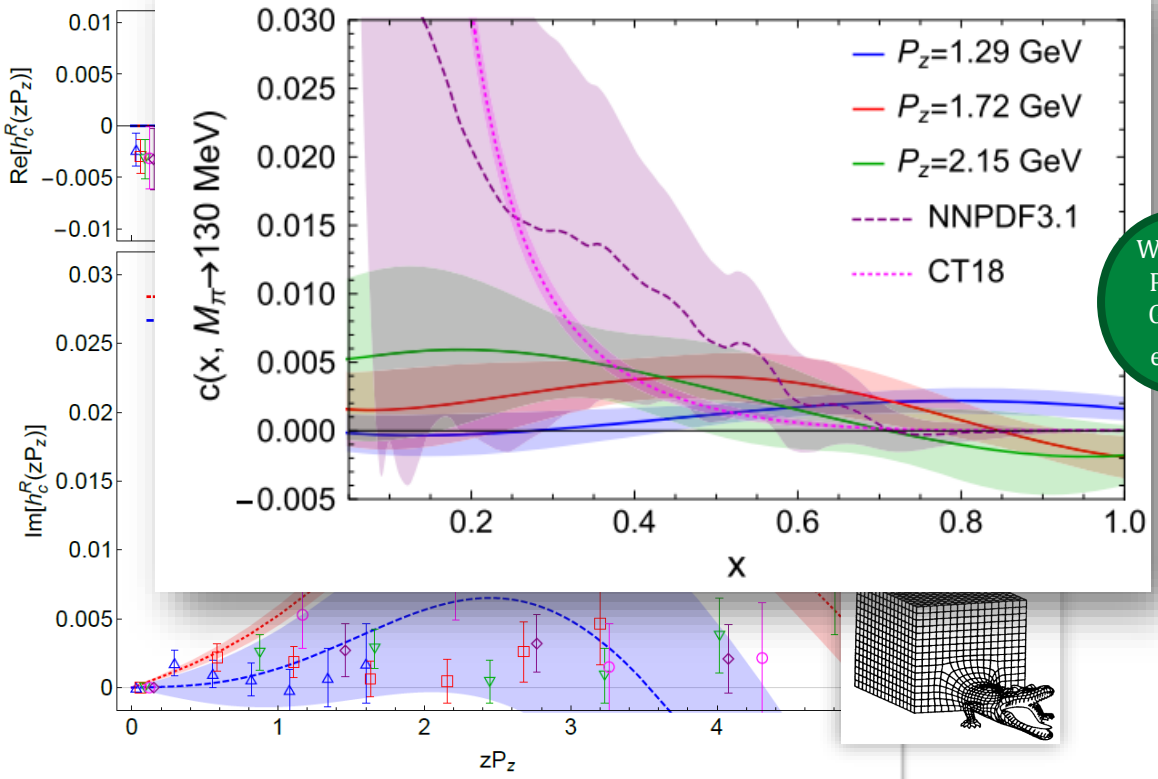


Wanted PDFs, GPDs, etc…

Challenges

Snowmass Whitepaper: Lattice QCD Calculations of Parton Physics, 2202.07193

§ Large momentum is essential

- With sufficient-statistics nucleons may reach 5 GeV

~~§ Renormalization of linear divergence~~

- ~~Wilson-line ops have linear divergences that must be subtracted~~

§ Methods for signal-to-noise improvement

- Gluonic observables, new ideas for even larger momentum

§ Inverse problems PDF extraction in SDF

- Remove the model/preconditioner-choice dependence

§ Reaching long-range correlations in LaMET

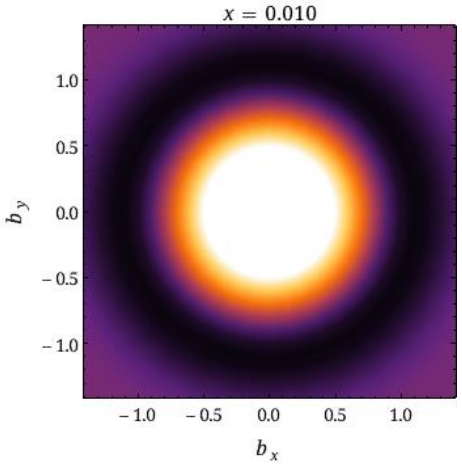
- For small-x physics, new methods for calculating longer-range correlations must be developed

Other LQCD Structure Talks

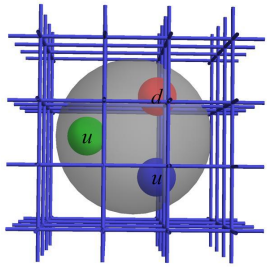
- James Zanotti @ Mon. 4:20PM
  - Constraining beyond the Standard Model nucleon isovector charges
- Ross Young @ Mon. 5:00PM
  - Revealing the transverse force distributions in the nucleon from lattice QCD
- Shoji Hashimoto @ Mon. 5:30PM
  - Inclusive processes from lattice QCD: problems and opportunities
- Andre Walker-Loud @ Wed. 10AM
  - Beta decay as probe of new physics
- Rajan Gupta @ Thur. 4:50/Fri. 3:30PM
  - Axial vector form factors for neutrino-nucleus scattering from lattice QCD
  - Nucleon charges from Lattice QCD and their Implications for BSM Physics

Summary and Outlook

- § Exciting era using LQCD to study *x*-dependent parton distributions
- § Overcoming longstanding limitations
  - Bjorken-*x* dependence of parton distributions now widely studied
  - More study of systematics planned for the near future
  - Did not cover advances in higher-twist GPDs, DAs, etc.



- § Lattice strange and gluon PDFs can have impacts
  - Treat lattice matrix elements as expt inputs in the future
- § Precision and progress are limited by resources

Theory Input → **Global Analysis of PDFs/GPDs/TMDs** ← **Exp't Input** + 

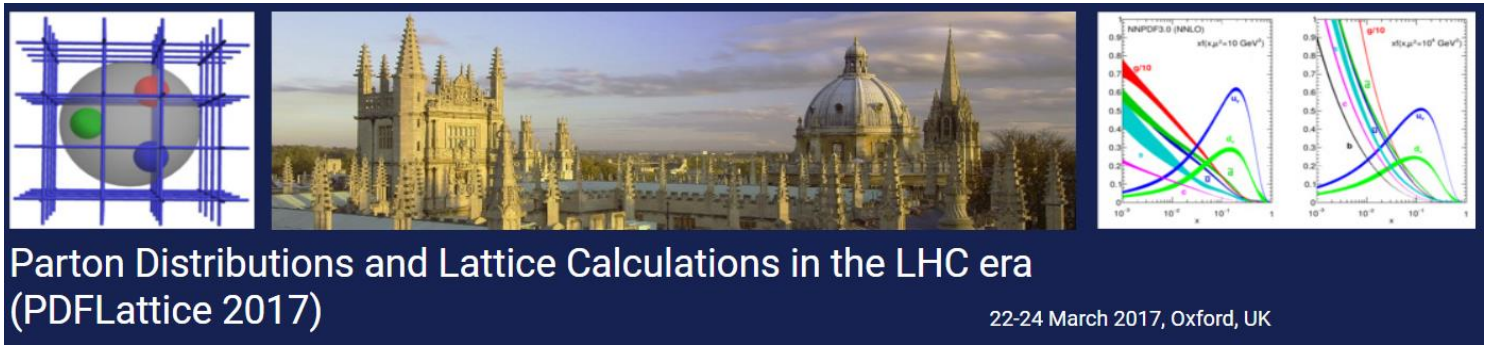
EXCLAIM

Thanks to MILC collaboration for sharing their 2+1+1 HISQ lattices & USQCD/NSF/DOE for computational resources
This work is partially sponsored by grants NSF PHY 1653405 & 1653405, DOE DE-SC0024053 & RCSA Cottrell Scholar Award

PDFLattice24 Workshop

§ November 18–20, 2024@Jefferson Lab, Newport News, VA, USA

- Joint community workshop between global-fit and lattice-QCD practitioners
- Theme: uncertainty quantification on nonperturbative correlator functions in phenomenology and lattice calculations



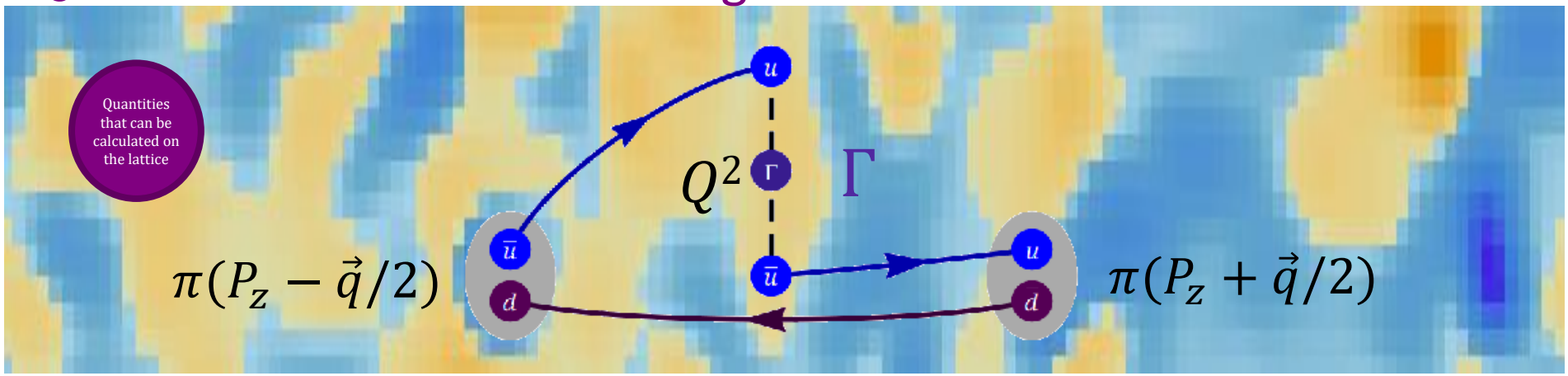
Students Wanted

LGT4HEP website: https://lgt4hep.github.io/

Backup Slides

Generalized Parton Distributions

- Nucleon/pion GPDs using quasi-PDFs at **physical pion mass** calculated at Breit Frame
- One calculates the following matrix elements on the lattice



pion unpolarized valence quark GPD: H^π

$$F^\pi\left(x,\tilde{\xi},t,\bar{P}_z\right)=\frac{\bar{P}_z}{\bar{P}_0}\int\frac{dz}{4\pi}e^{ixz\bar{P}_z}\left\langle P'\middle|\tilde{O}_{\gamma_0}(z)\middle|P\right\rangle=\frac{\bar{u}(P')}{2\bar{P}^0}\left(\boldsymbol{H}^\pi\left(x,\tilde{\xi},t,\bar{P}_z\right)\gamma^0\right)u(P'')$$

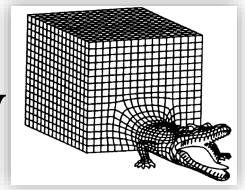
$$p^\mu=\frac{p''^\mu+p'^\mu}{2},\qquad \Delta^\mu=p''^\mu-p'^\mu,\qquad t=\Delta^2,\qquad \xi=\frac{p''^+-p'^+}{p''^++p'^+}$$

HL, Phys.Lett.B 846 (2023) 138181

Pion Tomography

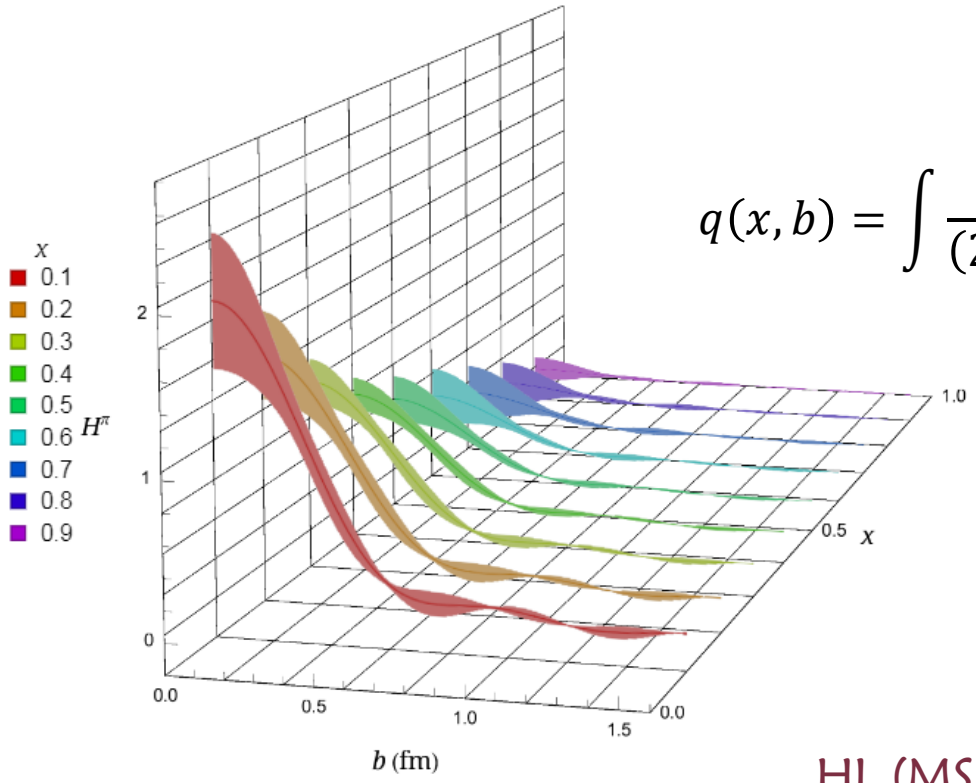
§ Pion GPD using quasi-PDFs at physical pion mass

- Lattice details: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 1.7$ GeV

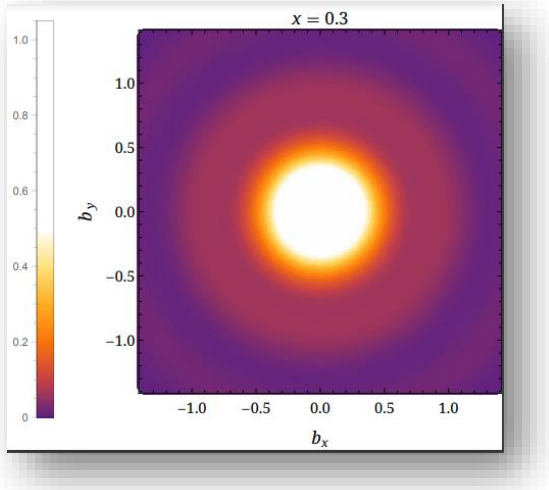


finite-volume, discretization,

- $\xi = 0$ valence-quark pion GPD results



$$q(x,b) = \int \frac{d\vec{q}}{(2\pi)^2} H(x, \xi = 0, t = -\vec{q}^{\,2}) e^{i\vec{q}\cdot\vec{b}}$$



HL (MSULat), Phys. Lett. B 846 (2023) 138181

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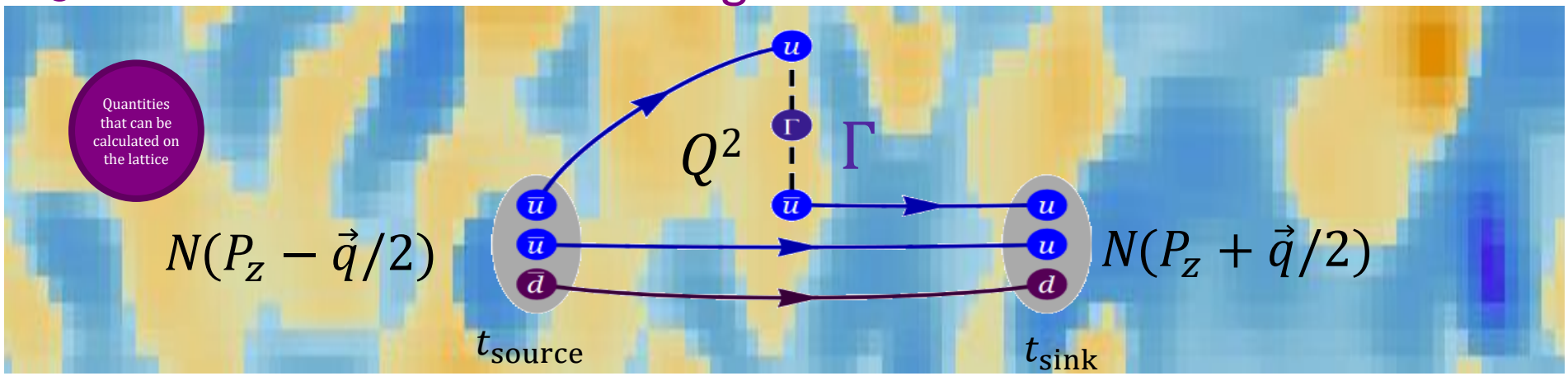


Nucleon longitudinally polarized GPDs: $\tilde{H}, \tilde{E}$

$$\tilde{F}\left(x,\tilde{\xi},t,\bar{P}_{\mathrm{Z}}\right) = \int \frac{dz}{4\pi} e^{ixz\bar{P}_Z}\left\langle P'\middle|\tilde{O}_{\gamma_z\gamma_5}(z)\middle|P\right\rangle = \frac{\bar{u}(P')}{2\bar{P}^{\mathrm{z}}}\left(\tilde{\boldsymbol{H}}\left(x,\tilde{\xi},t,\bar{P}_{\mathrm{Z}}\right)\gamma^{\mathrm{z}}\gamma^5 + \tilde{\boldsymbol{E}}\left(x,\tilde{\xi},t,\bar{P}_{\mathrm{Z}}\right)\frac{\gamma^5\Delta^z}{2M}\right)u(P'')$$

$$p^\mu = \frac{p''^\mu + p'^\mu}{2}, \qquad \Delta^\mu = p''^\mu - p'^\mu, \qquad t = \Delta^2, \qquad \xi = \frac{p''^+ - p'^+}{p''^+ + p'^+}$$

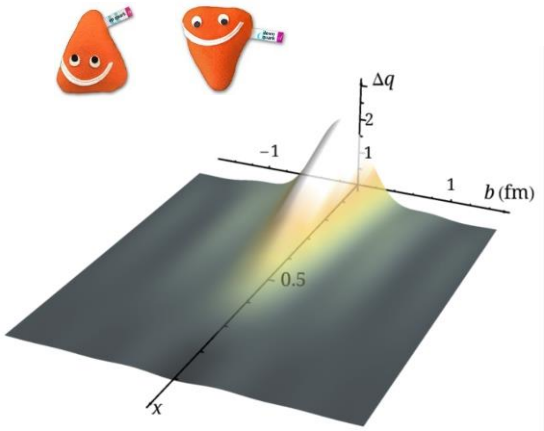
HL, Phys. Lett. B 824 (2022) 136821

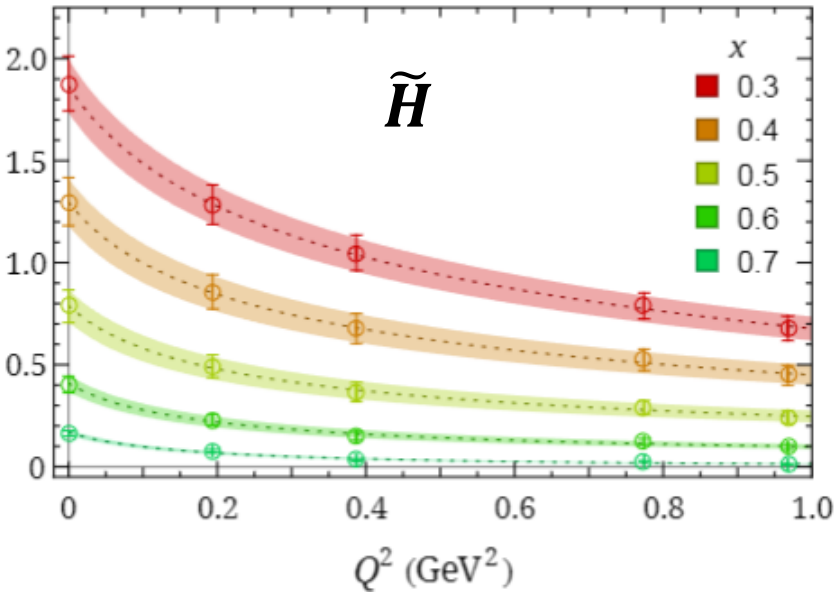
2021: Nucleon Polarized GPDs

§ Helicity GPD ($\widetilde{H}$) using quasi-PDFs at **physical pion mass**

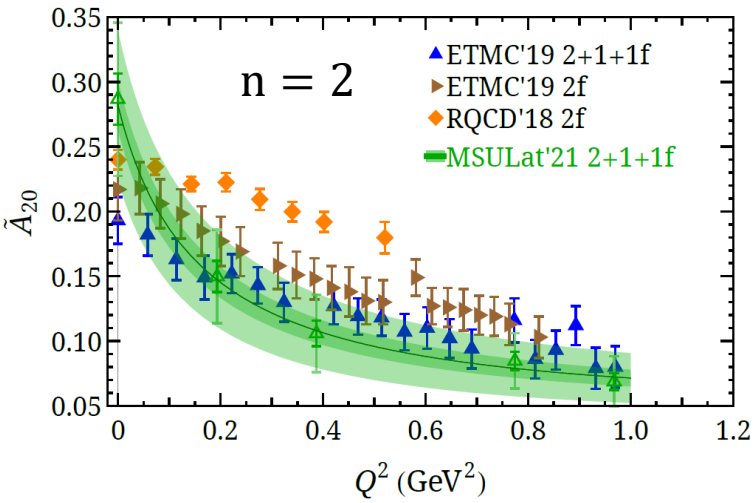
- MSULat: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon (quasi-)GPD results

HL (MSULat), Phys.Lett.B 824 (2022) 136821

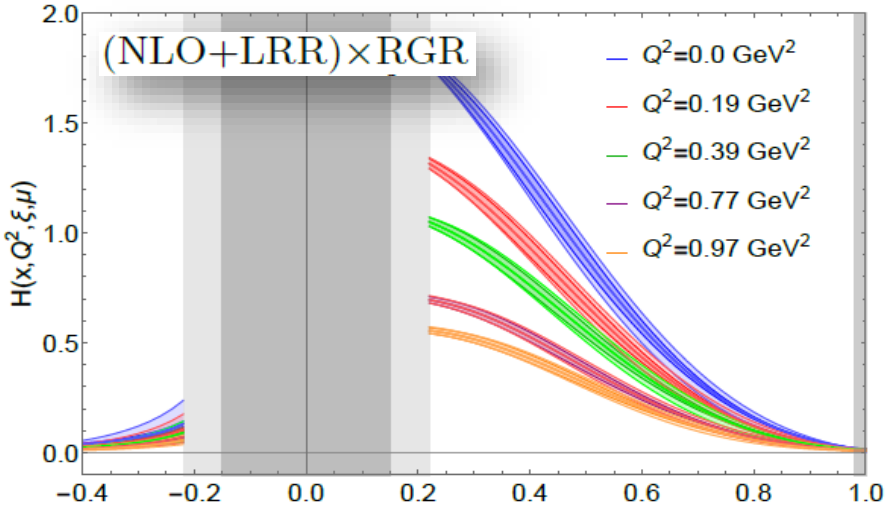


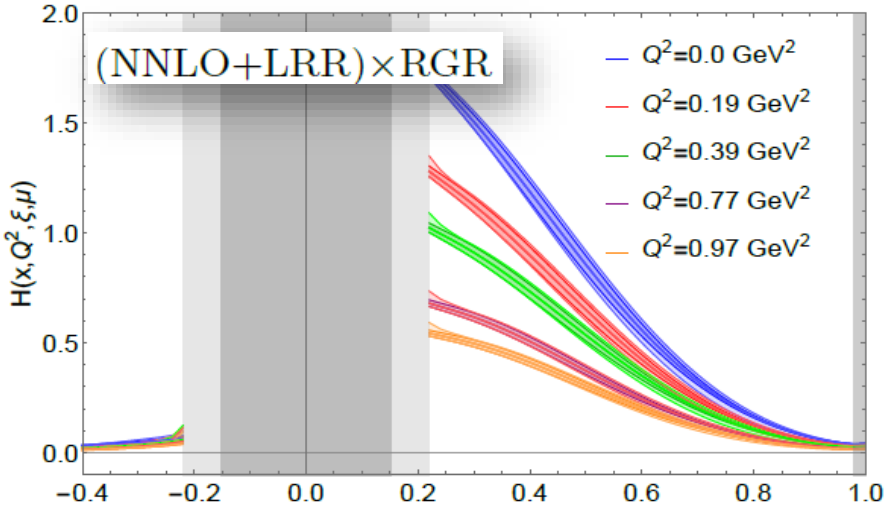


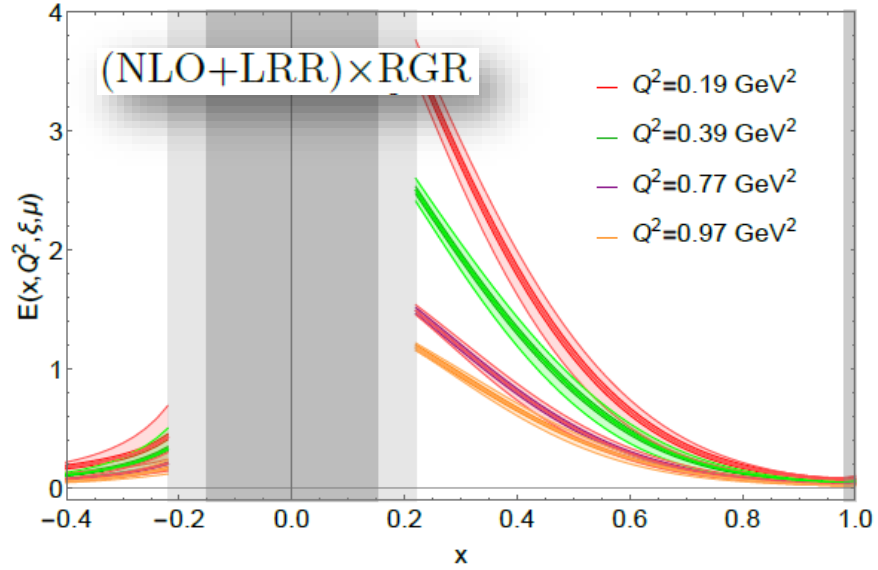
- Take the integral to form moments

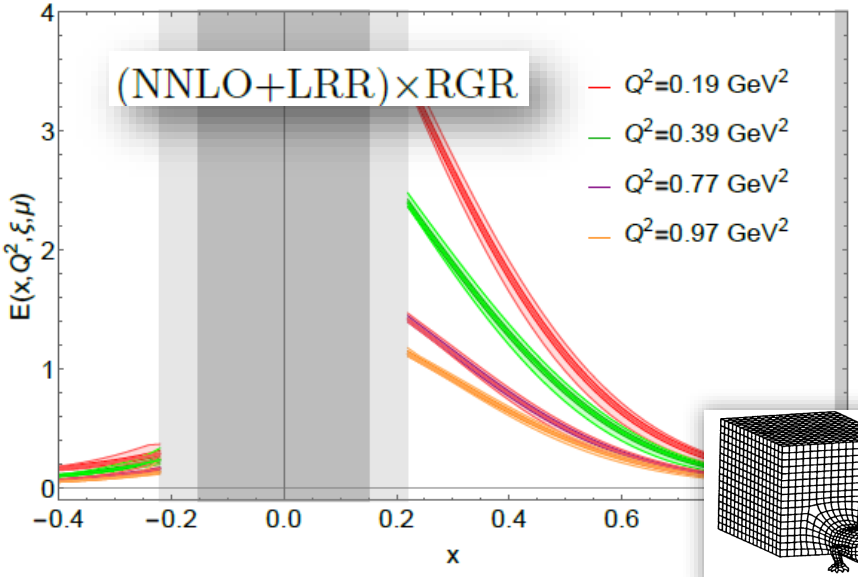


Systematic Improved $\xi=0$ GPDs





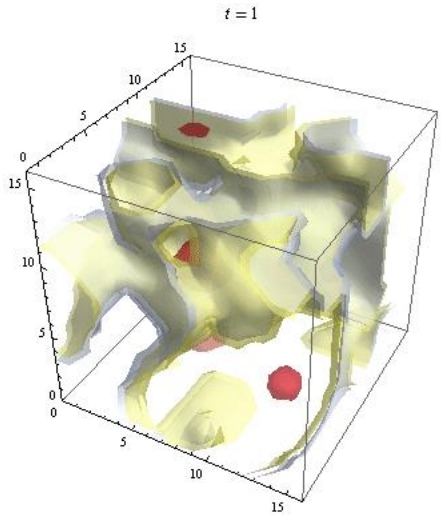


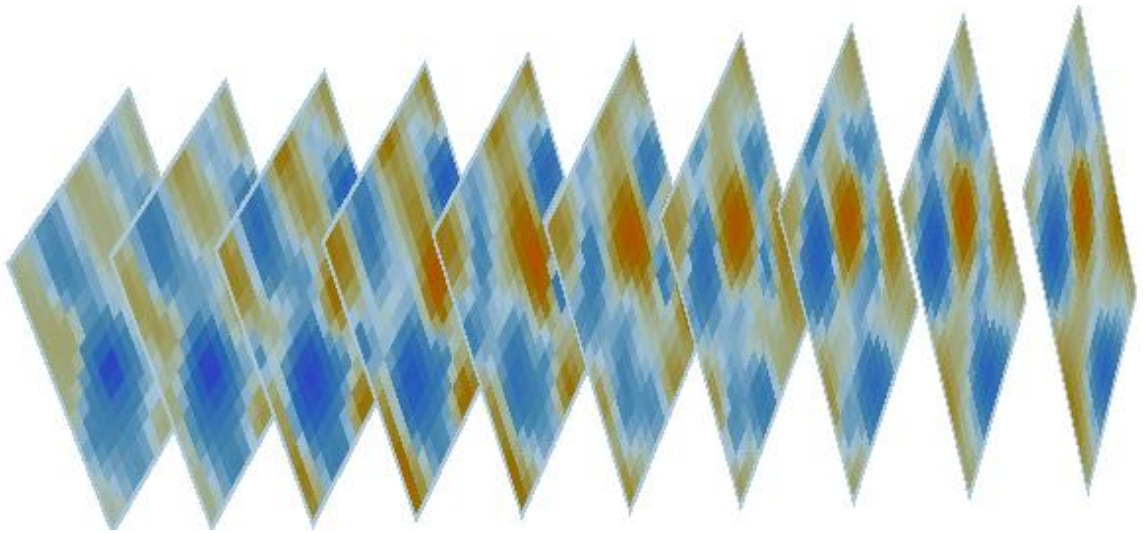


J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Anatomy of a Lattice Calculation

1. Start with QCD Vacuum (gauge configurations)





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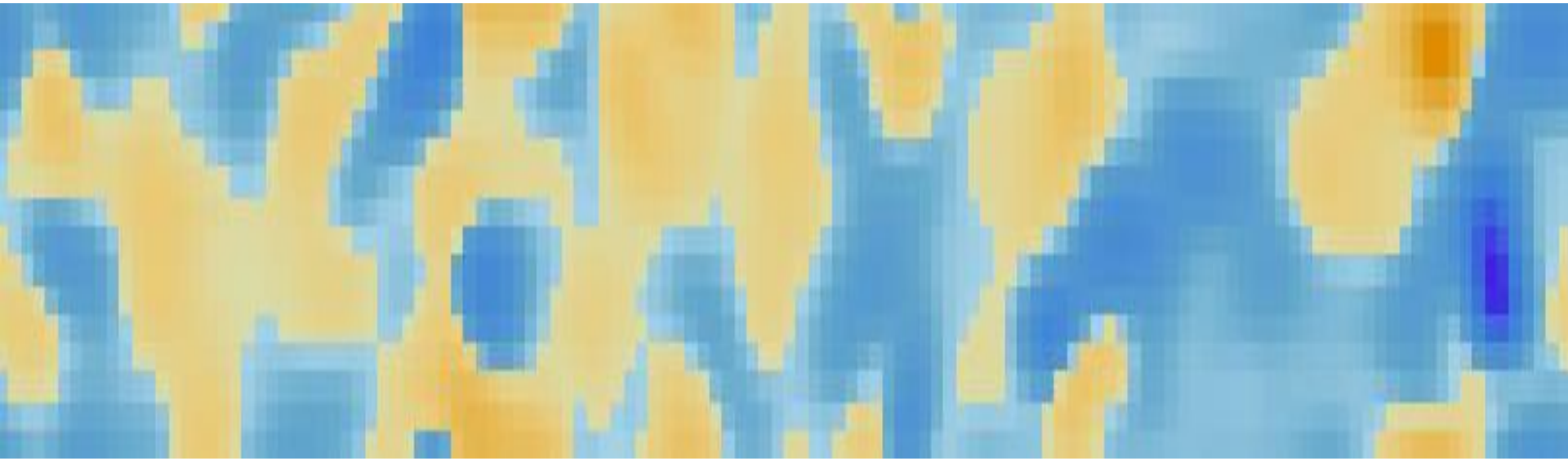
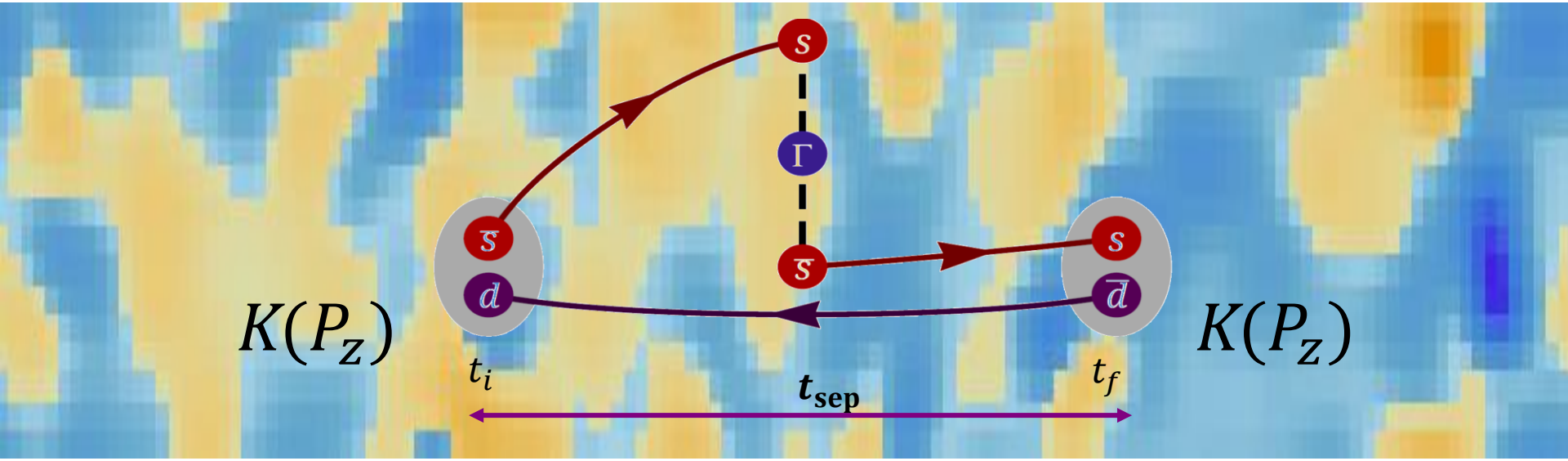


Thanks to MILC collaboration for sharing their 2+1+1 HISQ lattices

Anatomy of a Lattice Calculation

2. Correlators (hadronic observables)

- Invert Dirac operator matrix (rank 10^{12})
- Combine using color, spin and momentum into hadrons



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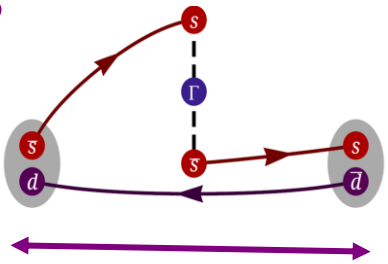
Anatomy of a Lattice Calculation

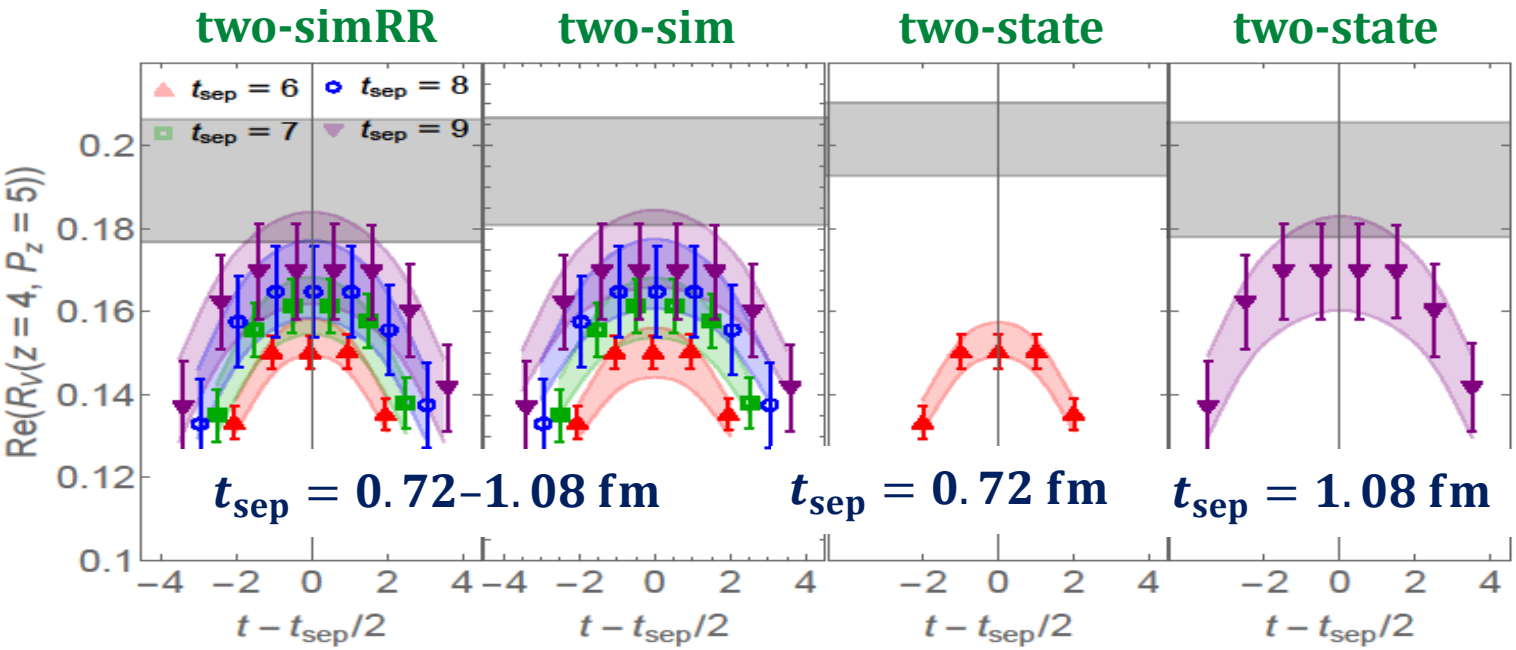
3. Extract reliable ground-state matrix elements

- Excited-state removal
  - For example, kaon matrix element at $M_\pi \approx 220$ MeV, $a \approx 0.12$ fm



t_{sep}

HL et al. (MSULat), 2003.14128



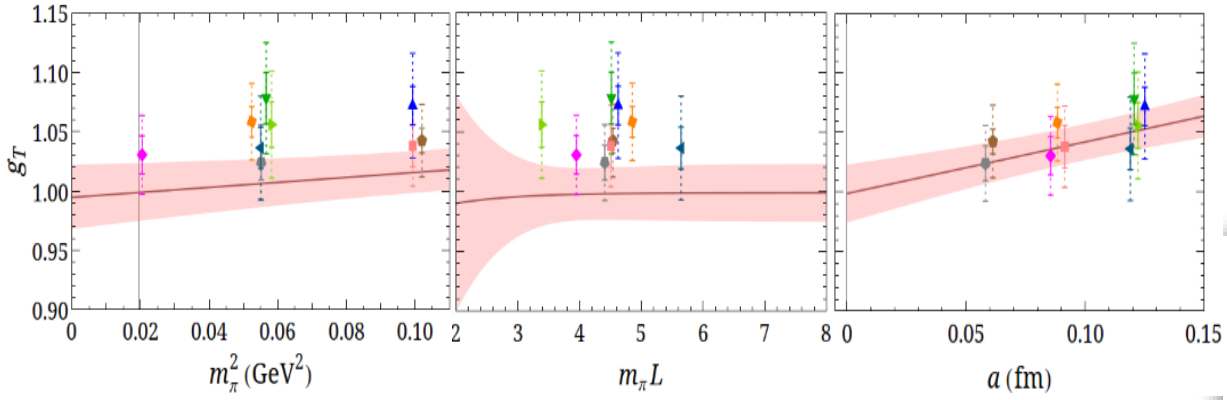
stability in extracting matrix elements

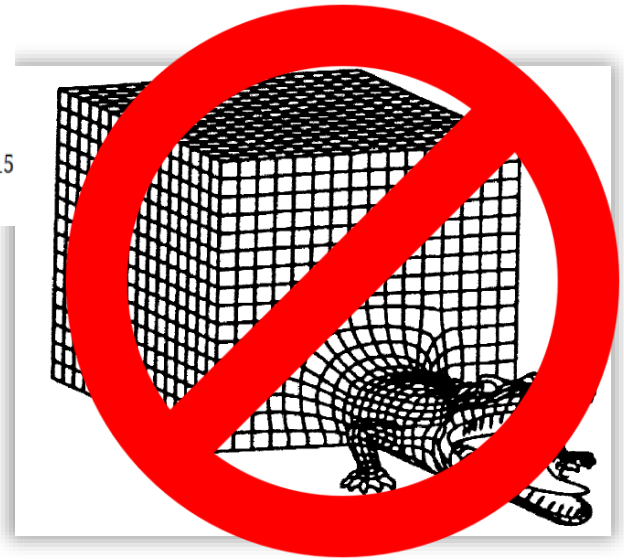
Anatomy of a Lattice Calculation

4. Systematic uncertainty (nonzero a, finite L, etc.)

- Nonperturbative renormalization, etc
- Extrapolation to the continuum limit

$$(m_\pi \to m_\pi^{\text{phys}},\ L \to \infty,\ a \to 0)$$

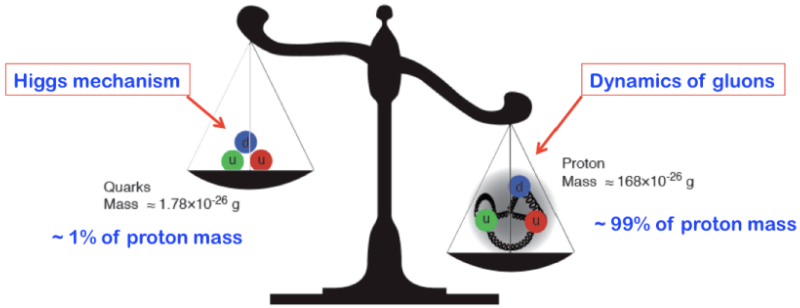




Backup Slides
Pion and Kaon PDFs

Motivation

- Meson structure is crucial to understand the mechanism of emergent hadron mass (EHM)
  - Help decode QCD origin of mass
- Experimentally, meson structure is harder to study
  - LQCD can provide predictions and better precision inputs
  - Quark and gluon parton distribution functions(PDFs), for example



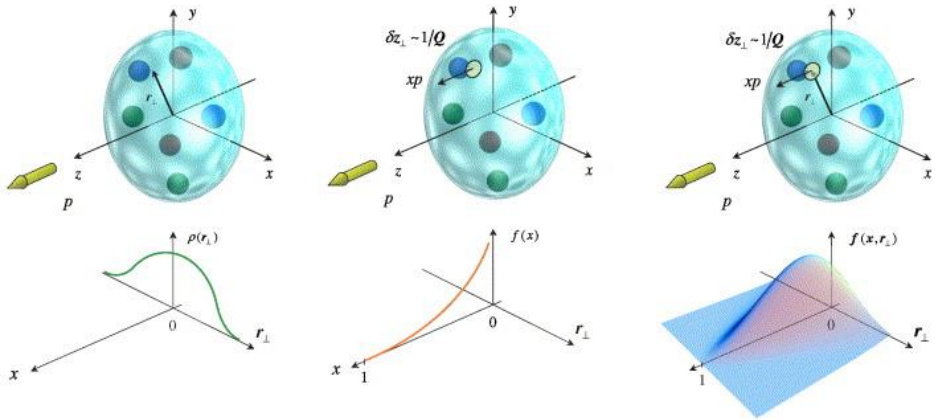


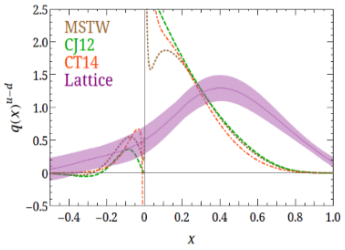
Image from A. Belitskya and A Radyushkin, Physics Report, 416 (2015)

- Generalized parton distributions (GPDs) encode information about the spatial structure & the partonic distribution of spin and orbital angular momenta
  - US EIC, EIcC, LHeC, ...

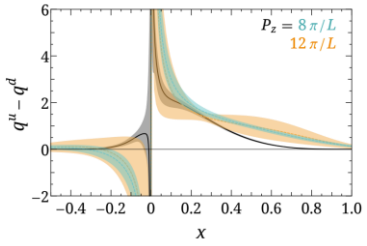
Lattice Parton Calculations

§ Physics quantity milestones

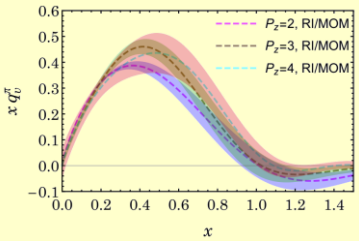
First unpol. lattice PDF



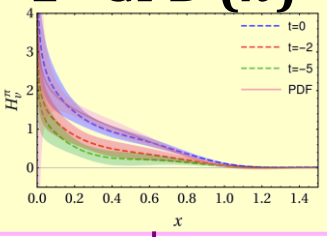
First PDFs at $M_\pi^{\rm phys}$



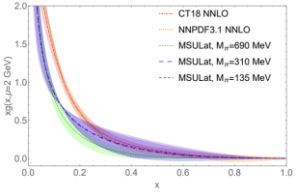
Pion v-PDF



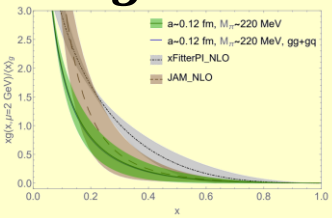
1st GPD (π)



N g-PDF

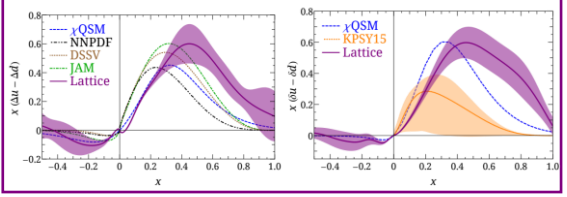


π g-PDF

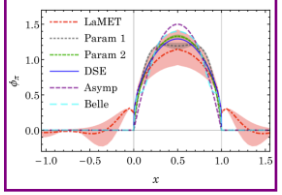


2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023

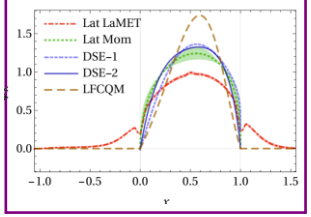
Pol. PDFs and mass corrections



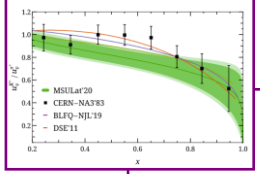
Pion DA



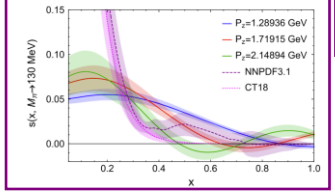
Kaon DA



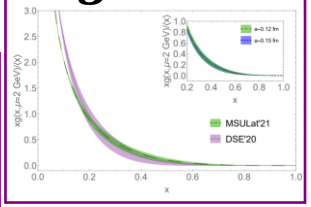
K PDF



s,c PDF



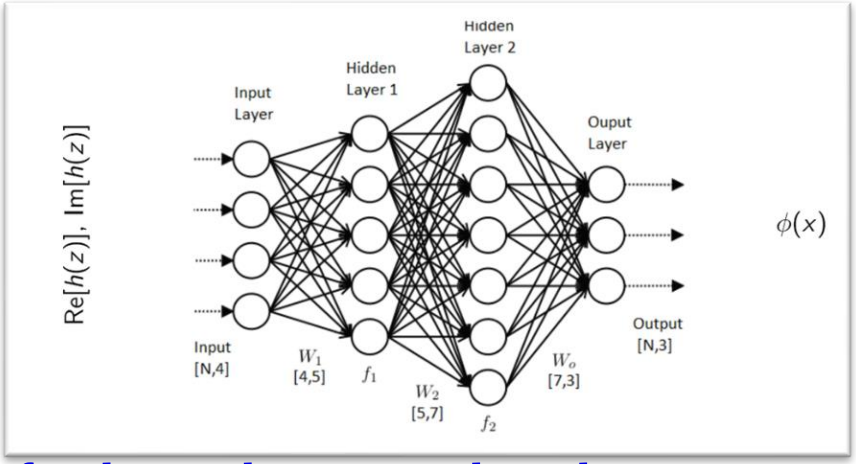
Kaon g-PDF



MSULat Pion/Kaon Structure

§ Meson distribution amplitude

- Pion Distribution Amplitude from Lattice,
  Phys. Rev. D 95 (2017) 9, 094514
- Kaon Distribution Amplitude from Lattice QCD and the Flavor SU(3) Symmetry, Nucl. Phys. B 939 (2019) 429-446
- Pion and kaon distribution amplitudes in the continuum limit,
  Phys. Rev. D 102 (2020) 9, 094519



- Precision control in lattice calculation of x-dependent pion distribution amplitude, Nucl. Phys. B 993 (2023) 116282

§ Miscellaneous

- Machine-learning prediction for quasiparton distribution function matrix elements, Phys. Rev. D 101 (2020) 3, 034516

MSULat Pion/Kaon Structure

§ Pion/kaon PDFs

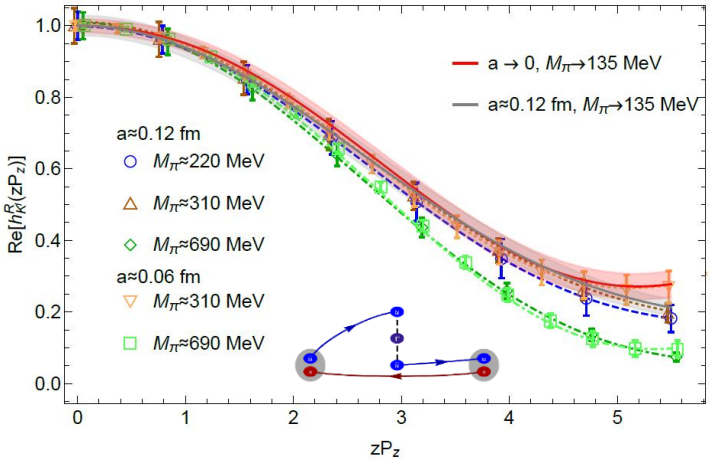
- First direct lattice-QCD calculation of the *x*-dependence of the pion parton distribution function, Phys. Rev. D 100 (2019) 3, 034505
- Valence-Quark Distribution of the Kaon and Pion from Lattice QCD, Phys. Rev. D 103 (2021) 1, 014516
- Gluon parton distribution of the pion from lattice QCD, Phys. Lett. B 823 (2021) 136778
- First glimpse into the kaon gluon parton distribution using lattice QCD, Phys. Rev. D 106 (2022) 9, 094510
- The Gluon Moment and Parton Distribution Function of the Pion from N_f=2+1+1 Lattice QCD, 2310.12034 [hep-lat]
- Pion valence quark distribution at physical pion mass of N_f=2+1+1 LQCD

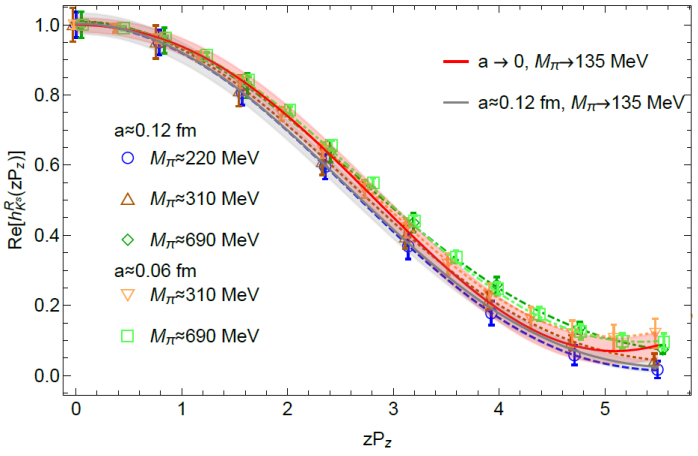
§ Pion GPD

- Pion generalized parton distribution from lattice QCD, Nucl. Phys. B 952 (2020) 114940
- Pion valence-quark generalized parton distribution at physical pion mass , Phys. Lett. B 846 (2023) 138181

Meson Valence-quark PDFs

§ Pion/kaon PDFs using quasi-PDF in the continuum limit

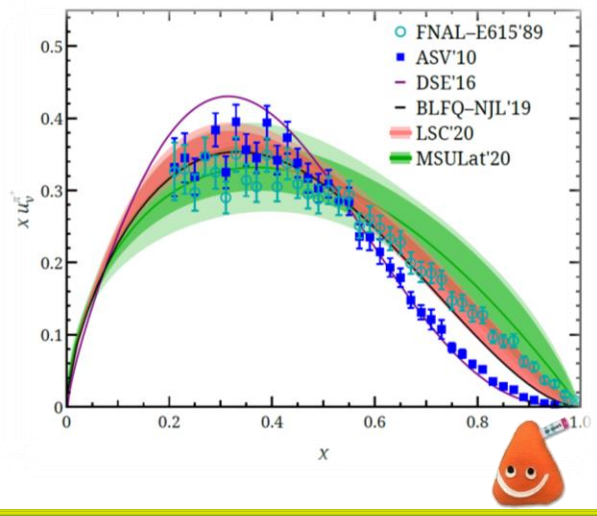


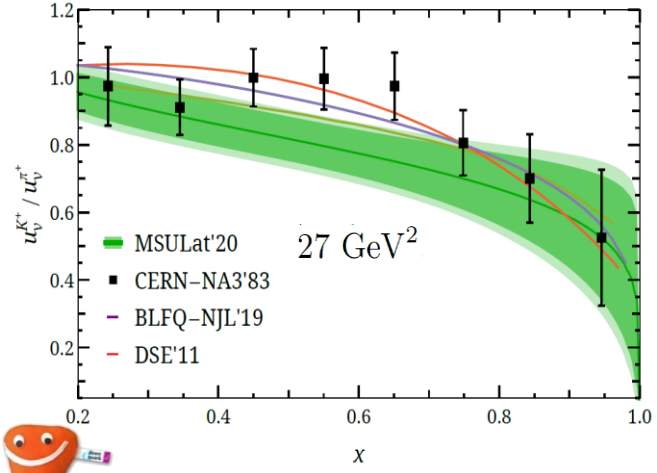


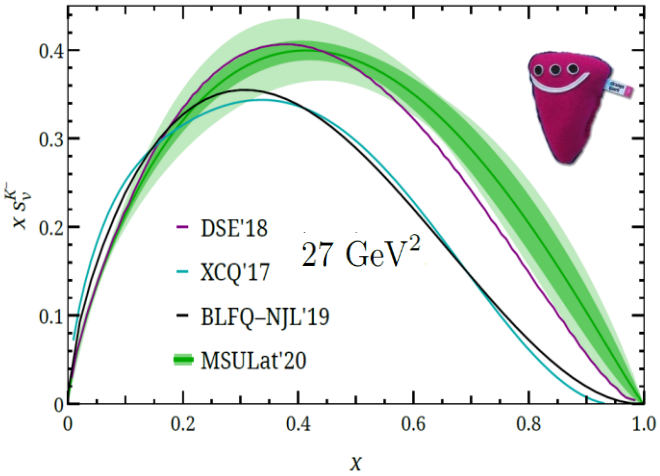
Quantities that can be calculated on the lattice

MSULat, 2003.14128

Wanted PDFs, GPDs, etc…



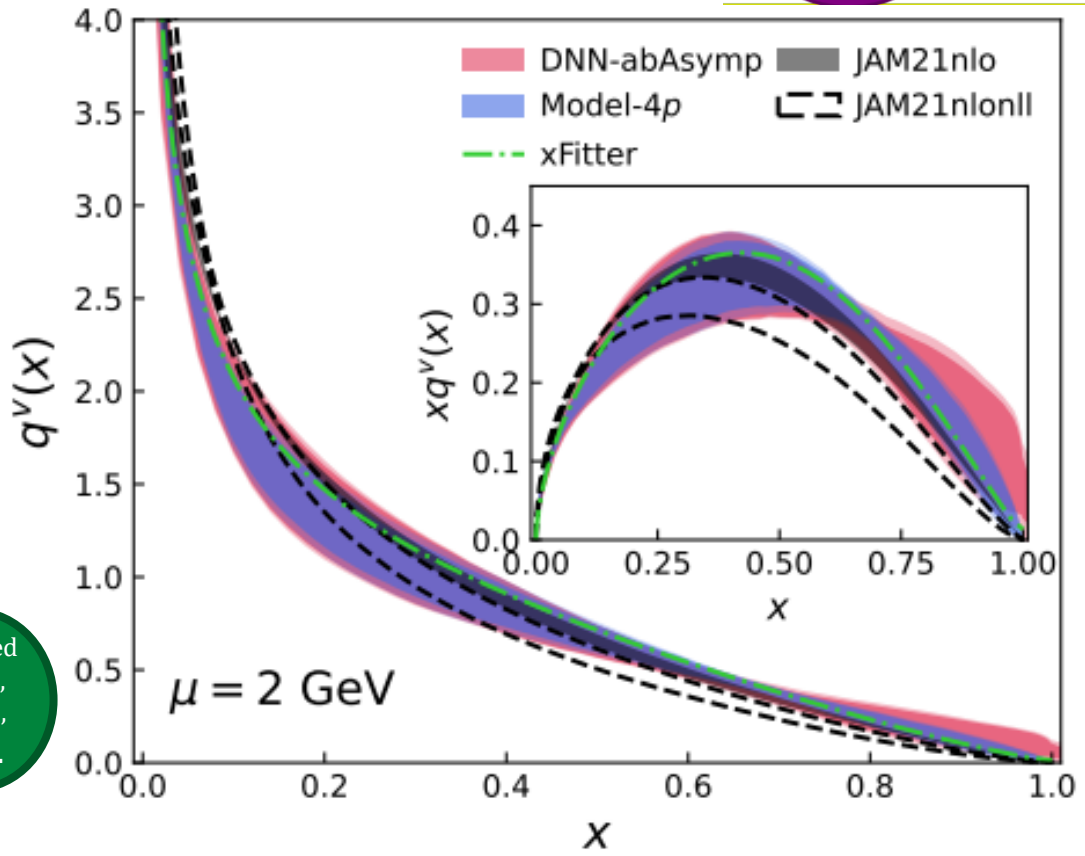




Valence-quark PDFs Update

§ Pion PDFs calculated directly at physical pion mass

- with NNLO matching



Wanted PDFs, GPDs, etc…

- N_f=2+1 clover/HISQ
  a~0.076 fm

ANL/BNL, Phys. Rev. D 106, 114510 (2022)

Valence-quark PDFs Update

§ Pion PDFs calculated directly at physical pion mass

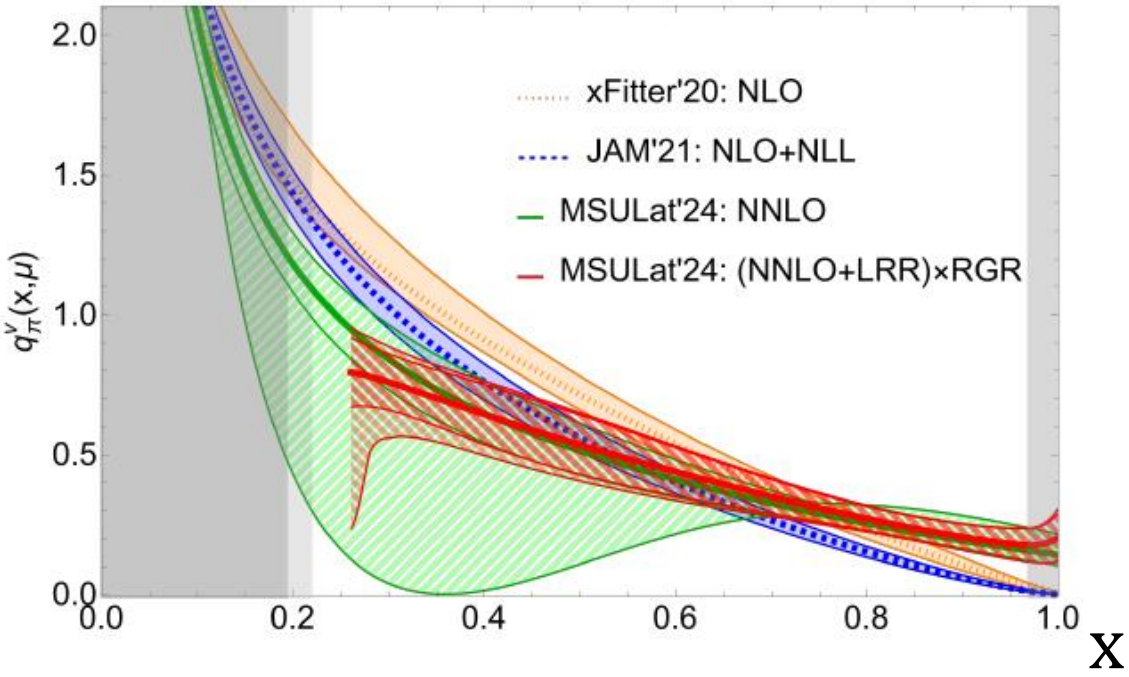
- NNLO matching & treat leading-renormalon effects
  - Leading-renormalon resummation (LRR)
  - Renormalization-group resummation (RGR)

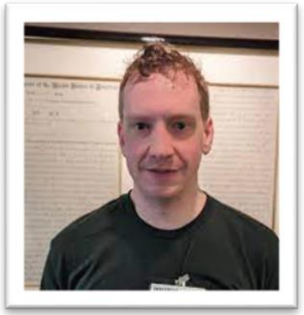
R. Zhang, et. al.
PLB 844, 138081 (2023)

- N_f=2+1+1 clover/HISQ, a~0.09 fm

J. Holligan, HL (MSULat), 10.1088/1361-6471/ad3162

Wanted PDFs, GPDs, etc…

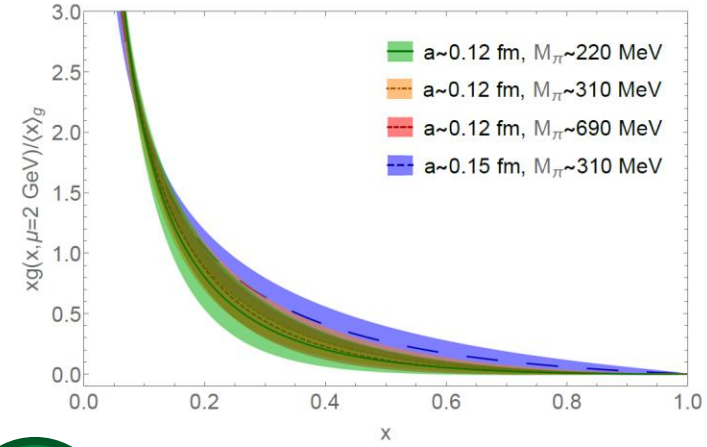


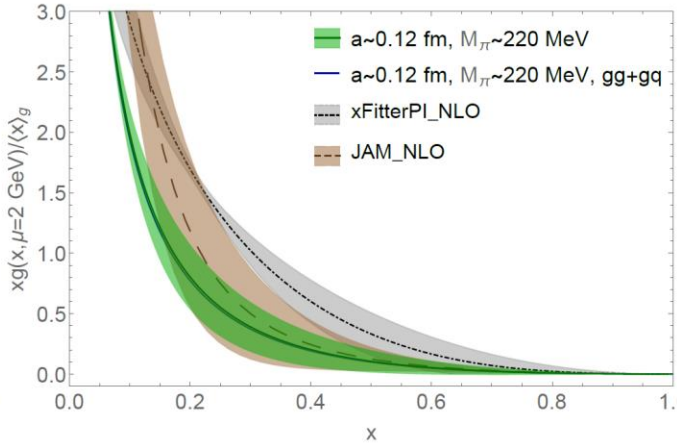


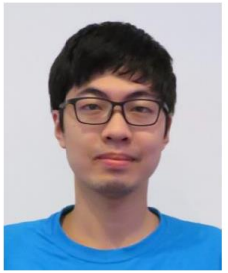
P: Jack Holligan

Meson Gluon PDFs

§ First pion and kaon gluon PDFs $g(x)/\langle x\rangle$ using pseudo-PDF



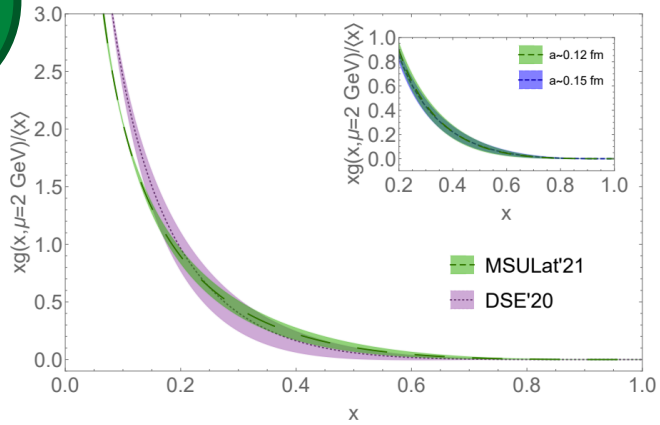




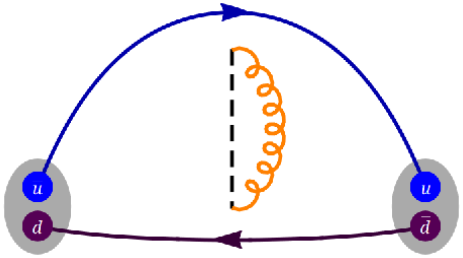
G: Zhouyou Fan

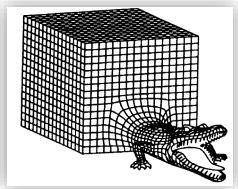
2104.06372, Fan et al. (MSULat); 2112.03124, Salas-Chavira et al. (MSULat)

Wanted PDFs, GPDs, etc…



G: Alejandro Salas-Chavira





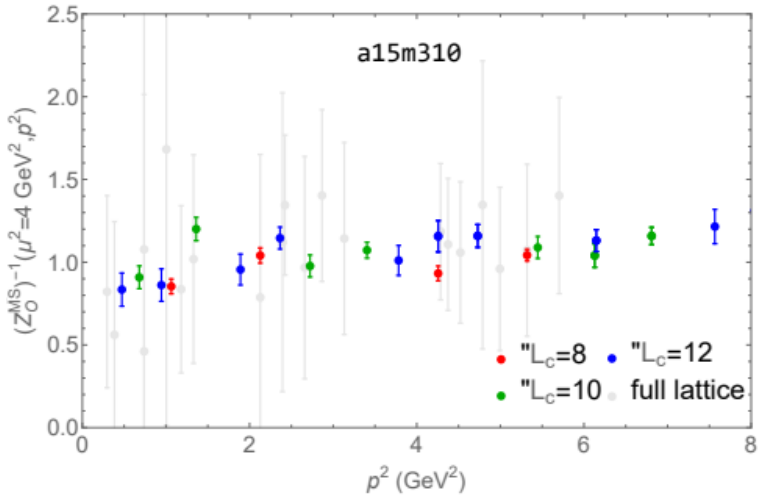
finite-volume, discretization, heavy quark mass, …

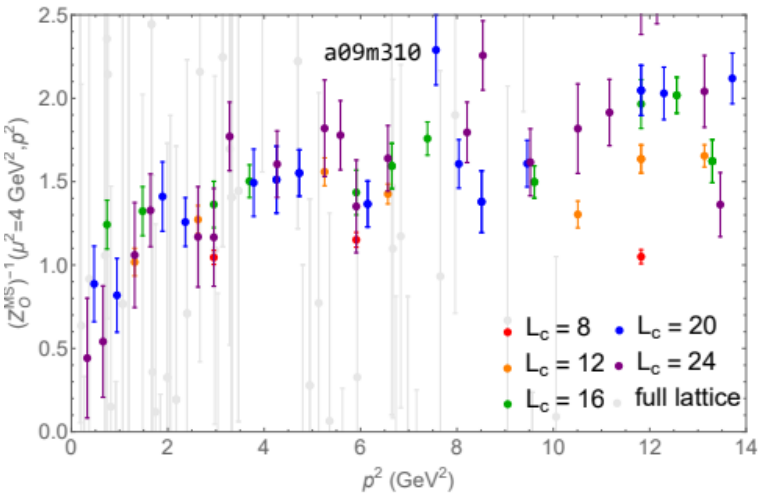
§ What does lattice QCD say about $g(x)$?

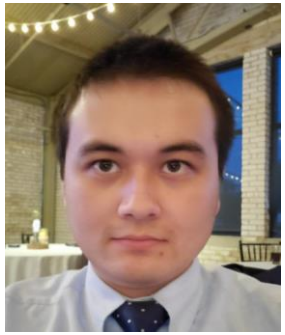
Pion Gluon PDF Update

§ Nonperturbatively renormalized $\langle x \rangle_{\{\pi,g\}}$ at the finer lattice spacing at lighter pion mass is nontrivial

- Using cluster-decomposition error reduction (CDER) to enhance the signal-to-noise ratio 1805.00531, Y. Yang et al. (χQCD)
- Lattice details: clover/HISQ, a ~{0,15, 0.12, 0.09} fm 2208.00980, Fan et al. (MSULat)

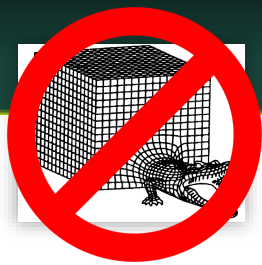




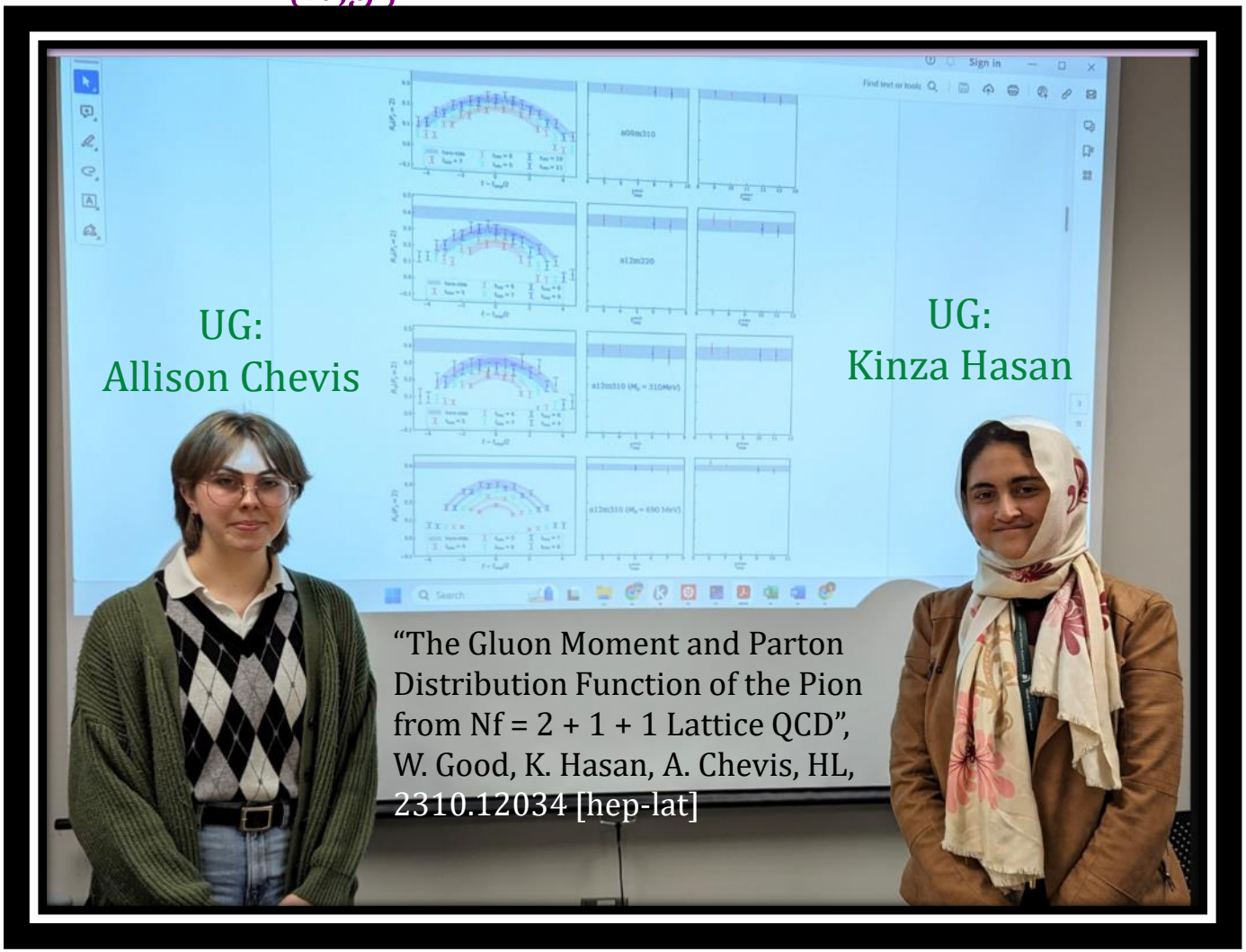


G: Matthew Zeilbeck

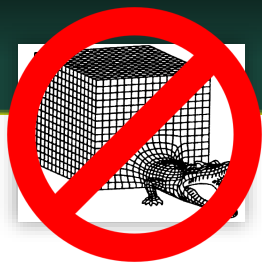
Pion Gluon PDF Update



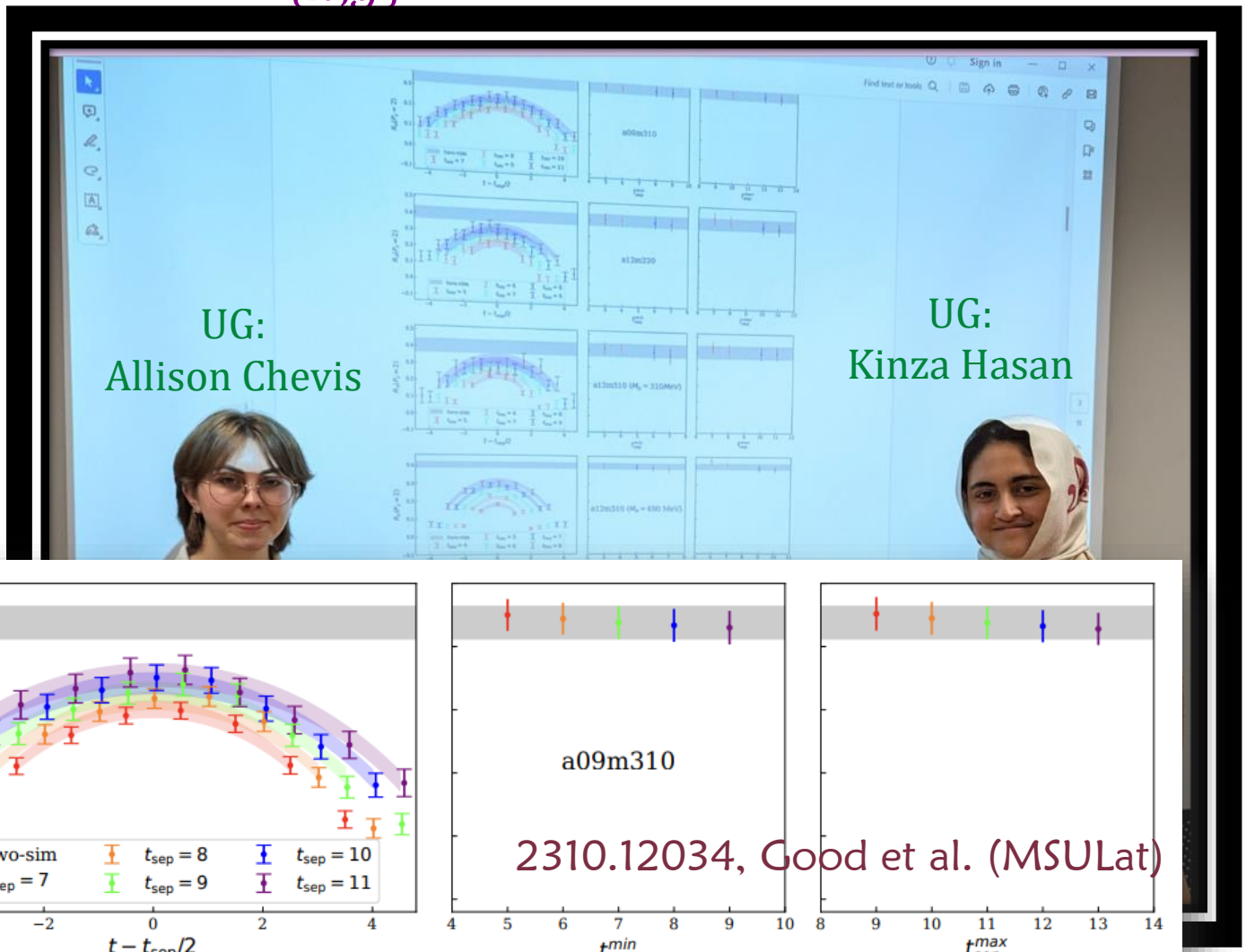
§ Study Bare $\langle x \rangle_{\{\pi,g\}}$



Pion Gluon PDF Update



§ Study Bare $\langle x \rangle_{\{\pi,g\}}$

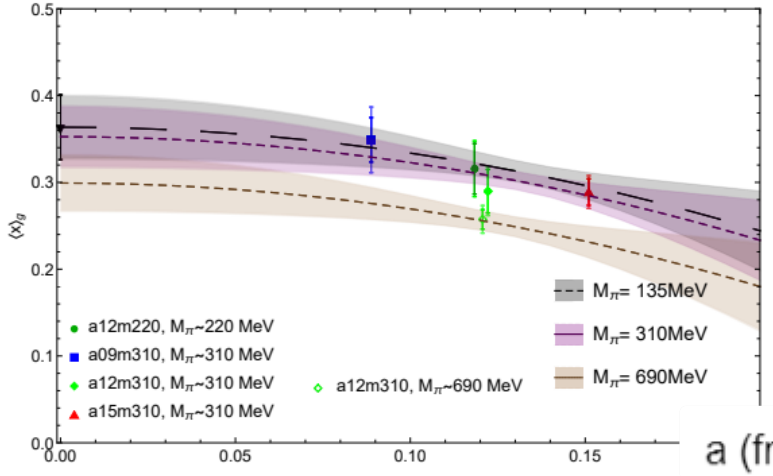


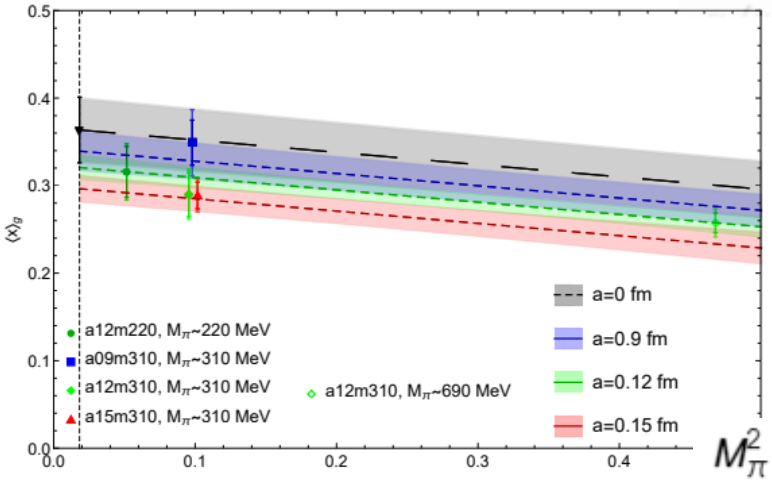
2310.12034, Good et al. (MSULat)

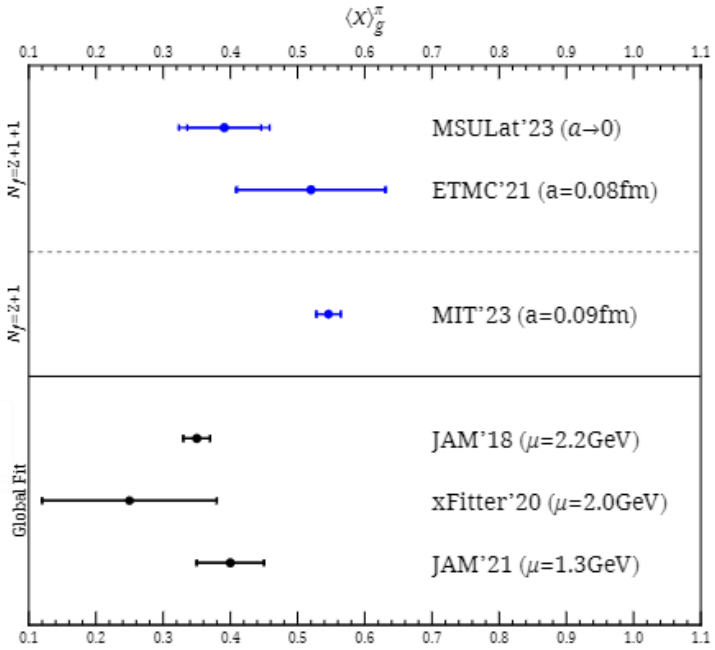
Pion Gluon PDF Update

§ Study discretization systematic in $\langle x \rangle_{\{\pi,g\}}$

- Lattice details: clover/HISO. HISO. a ~{0.15. 0.12. 0.09} fm







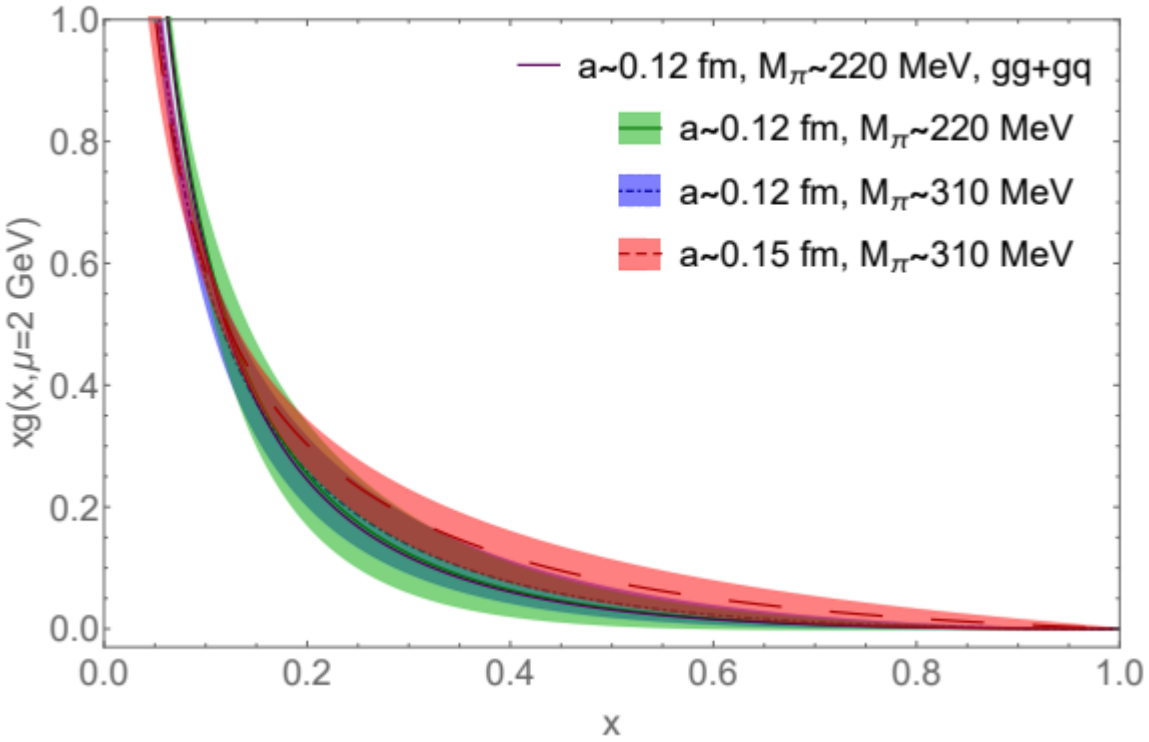
2310.12034, Good et al. (MSULat)

G: Bill Good

Pion Gluon PDF Update

§ Back to Pion gluon PDF $g(x)$

- Update previous calculated $g(x)/\langle x \rangle$ in 2021



2310.12034, Good et al. (MSULat)

G: Bill Good

Pion Gluon PDF Update

§ Back to Pion gluon PDF $g(x)$

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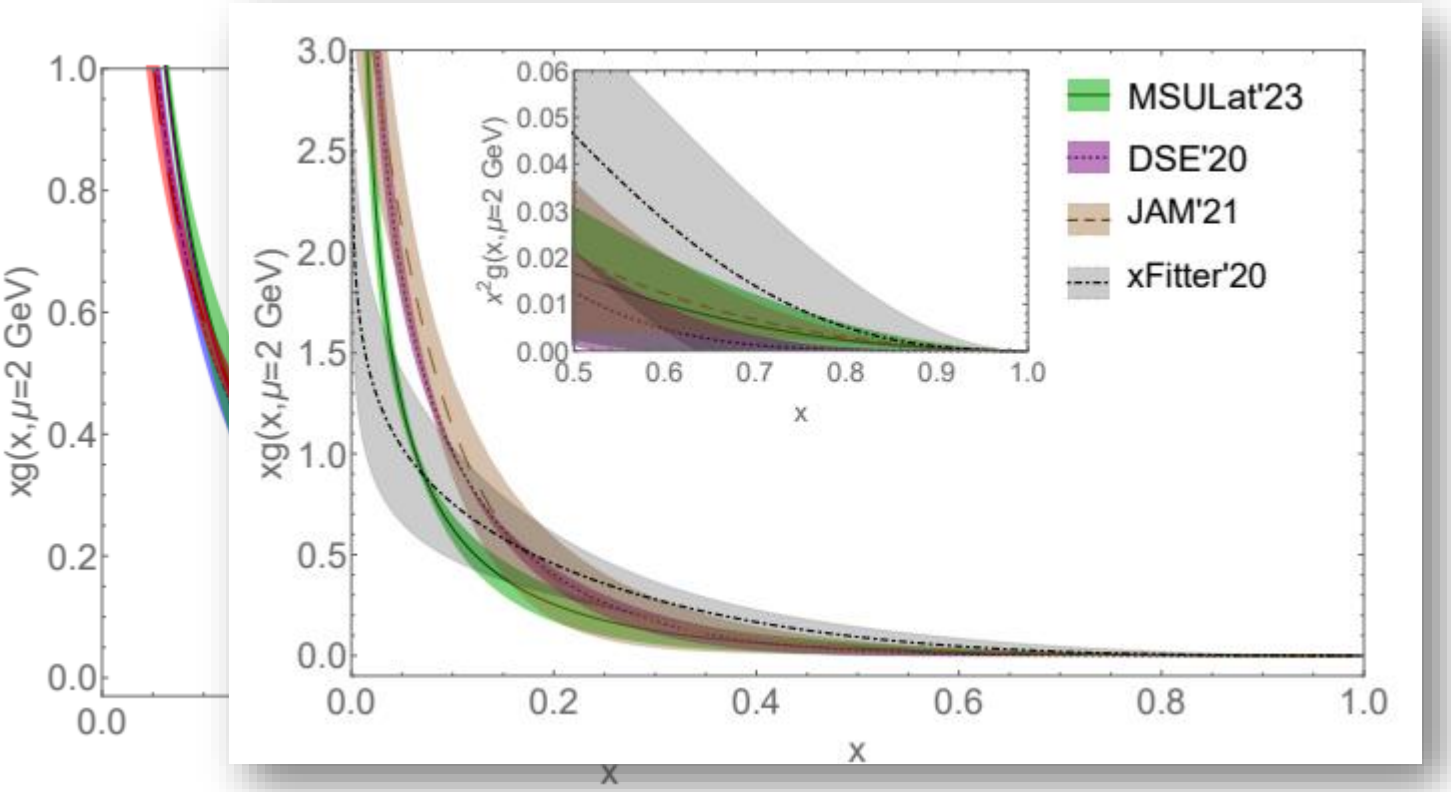
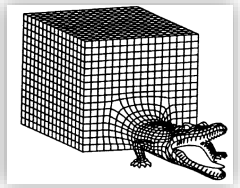


2310.12034, Good et al. (MSULat)

G: Bill Good

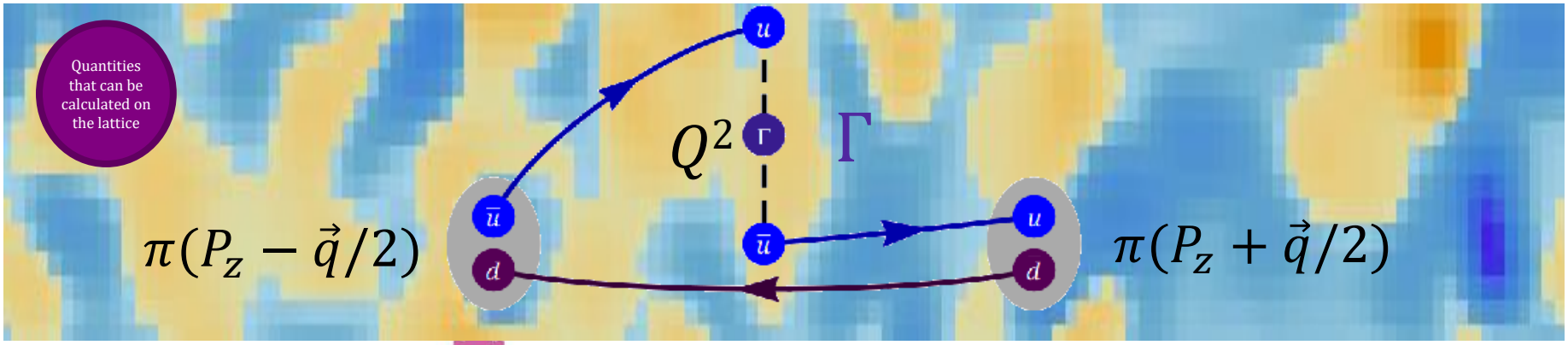
Generalized Parton Distributions

Single-ensemble result



finite-volume,
discretization,
heavy quark mass,
...

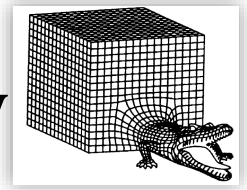
$$H_q^\pi(x,\xi,t,\mu) = \int \frac{d\eta^-}{4\pi} e^{-ix\eta^- P^+} \left\langle \pi(P+\Delta/2) \left| \bar{q}\left(\frac{\eta^-}{2}\right) \gamma^+ \Gamma \left(\frac{\eta^-}{2}, -\frac{\eta^-}{2}\right) q\left(-\frac{\eta^-}{2}\right) \right| \pi(P-\Delta/2) \right\rangle$$



Valence-Quark Pion GPD

§ Pion GPD (H^π) using quasi-PDFs at physical pion mass

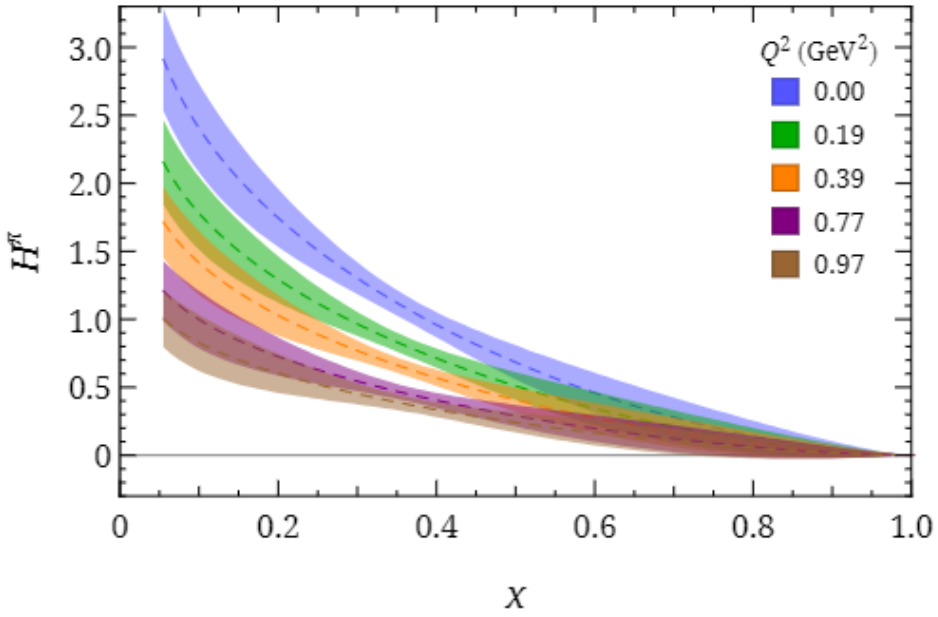
- Lattice details: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 1.7$ GeV

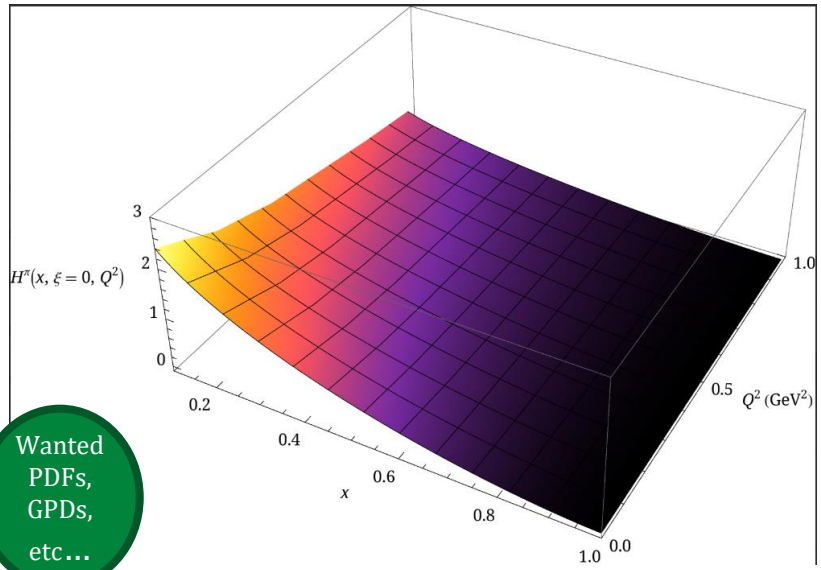


finite-volume, discretization,

- $\xi = 0$ valence-quark Pion GPD results

HL (MSULat), Phys. Lett. B 846 (2023) 138181



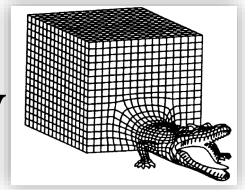


Wanted PDFs, GPDs, etc…

Valence-Quark Pion GPD

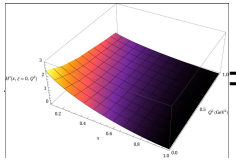
§ Pion GPD (H^π) using quasi-PDFs at physical pion mass

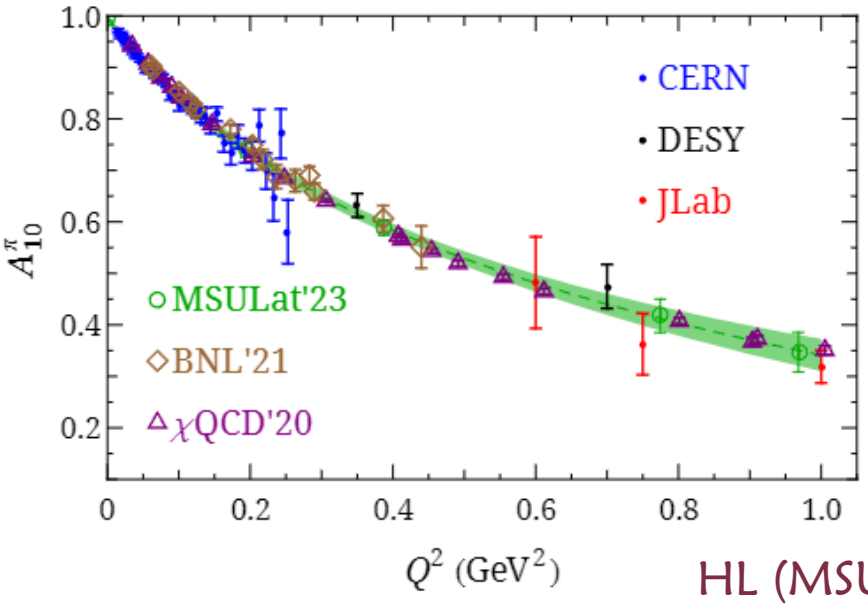
- Lattice details: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 1.7$ GeV



finite-volume, discretization,

- $\xi = 0$ valence-quark Pion GPD results

$$\int_{-1}^{+1} dx\, x^{n-1} \; \text{  } \; = A_{ni}^{\pi}(t)$$

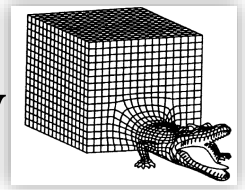


HL (MSULat), Phys. Lett. B 846 (2023) 138181

Pion Tomography

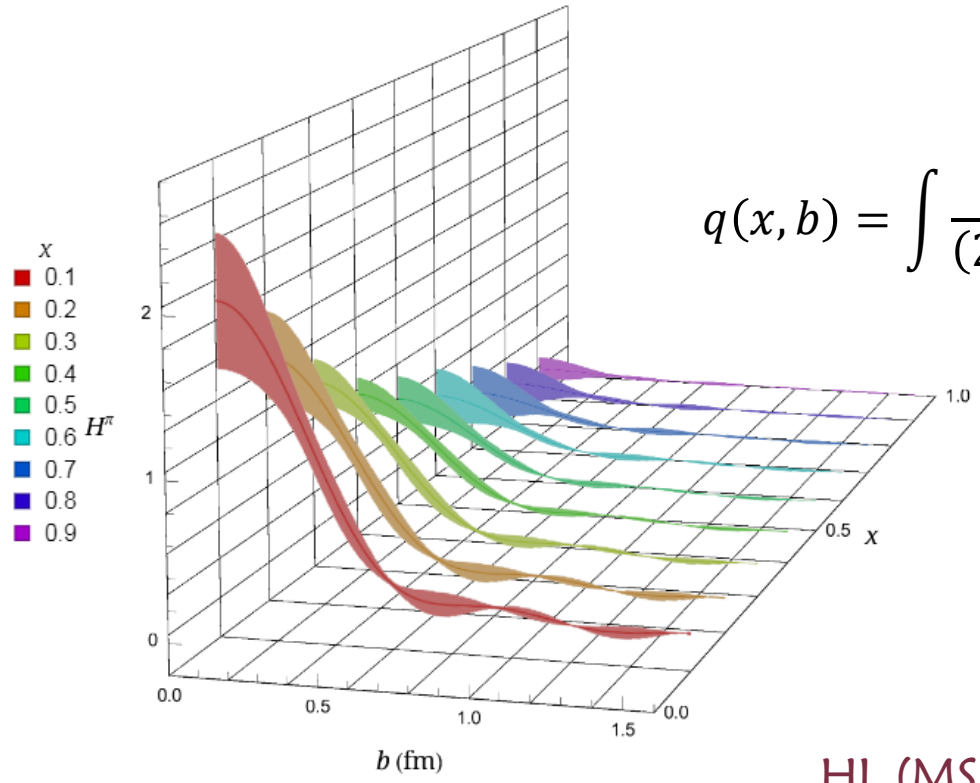
§ Nucleon GPD using quasi-PDFs at physical pion mass

- Lattice details: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 1.7$ GeV

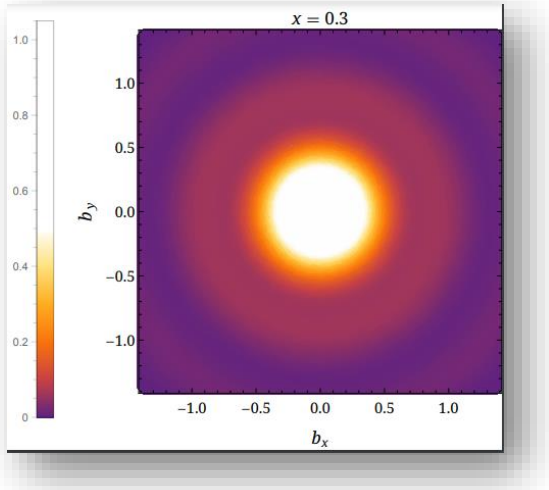


finite-volume, discretization,

- $\xi = 0$ valence-quark Pion GPD results



$$q(x,b) = \int \frac{d\vec{q}}{(2\pi)^2} H(x, \xi = 0, t = -\vec{q}^{\,2}) e^{i\vec{q}\cdot\vec{b}}$$



HL (MSULat), Phys. Lett. B 846 (2023) 138181

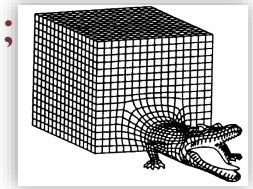
First Lattice GPDs

§ First glimpse into pion GPD using Quasi-PDF/LaMET

- Lattice details: clover/HISQ, **0.12fm**, **310-MeV** pion mass

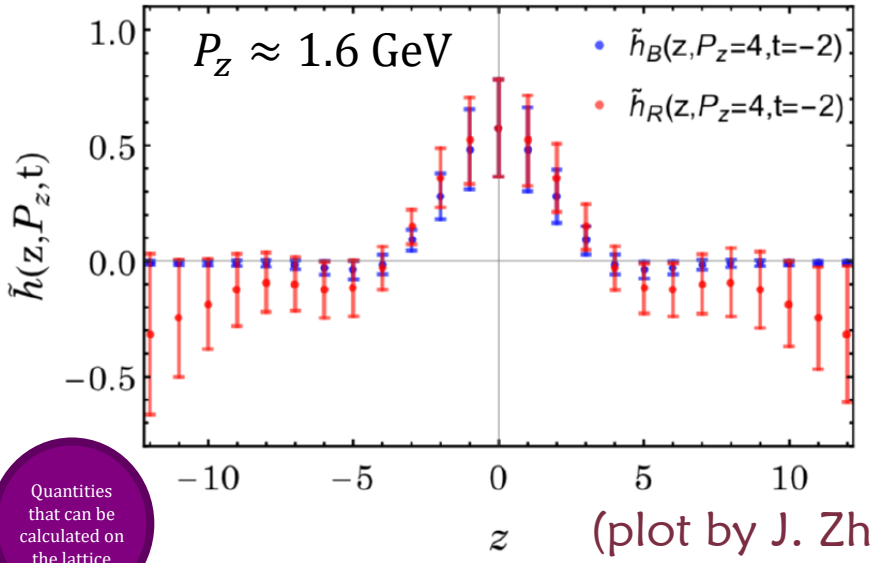
$P_z \approx 1.3, 1.6$ GeV

MILC, Phys. Rev. D, 82 (2010), 074501;
Phys. Rev. D, 87 (2013), 0545056

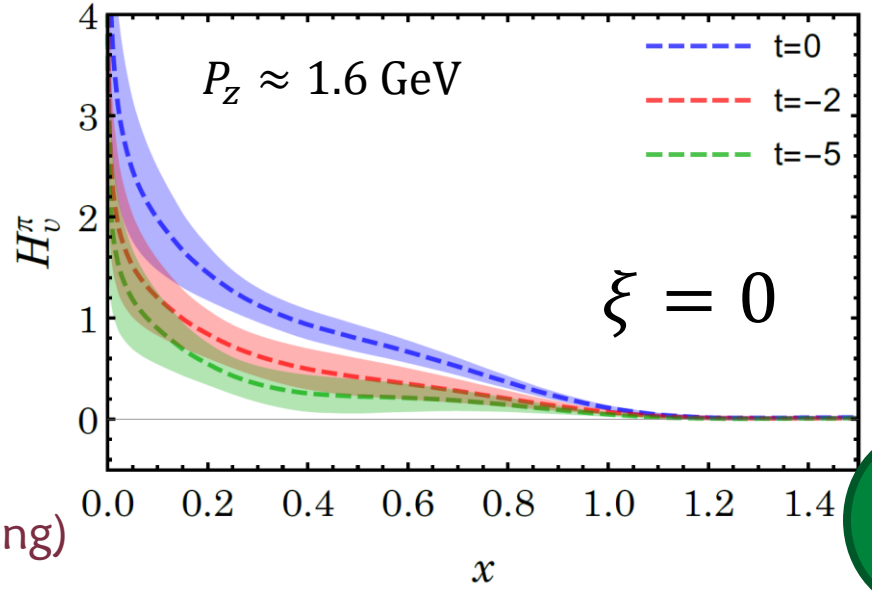


J. Chen, HL, J. Zhang, 1904.1237;

$$H_q^\pi(x,\xi,t,\mu) = \int \frac{d\eta^-}{4\pi} e^{-ix\eta^- P^+} \left\langle \pi(P+\Delta/2) \left| \bar{q}\left(\frac{\eta^-}{2}\right) \gamma^+ \Gamma \left(\frac{\eta^-}{2}, -\frac{\eta^-}{2}\right) q\left(-\frac{\eta^-}{2}\right) \right| \pi(P-\Delta/2) \right\rangle$$



(plot by J. Zhang)



Quantities that can be calculated on the lattice

Wanted PDFs, GPDs, etc…

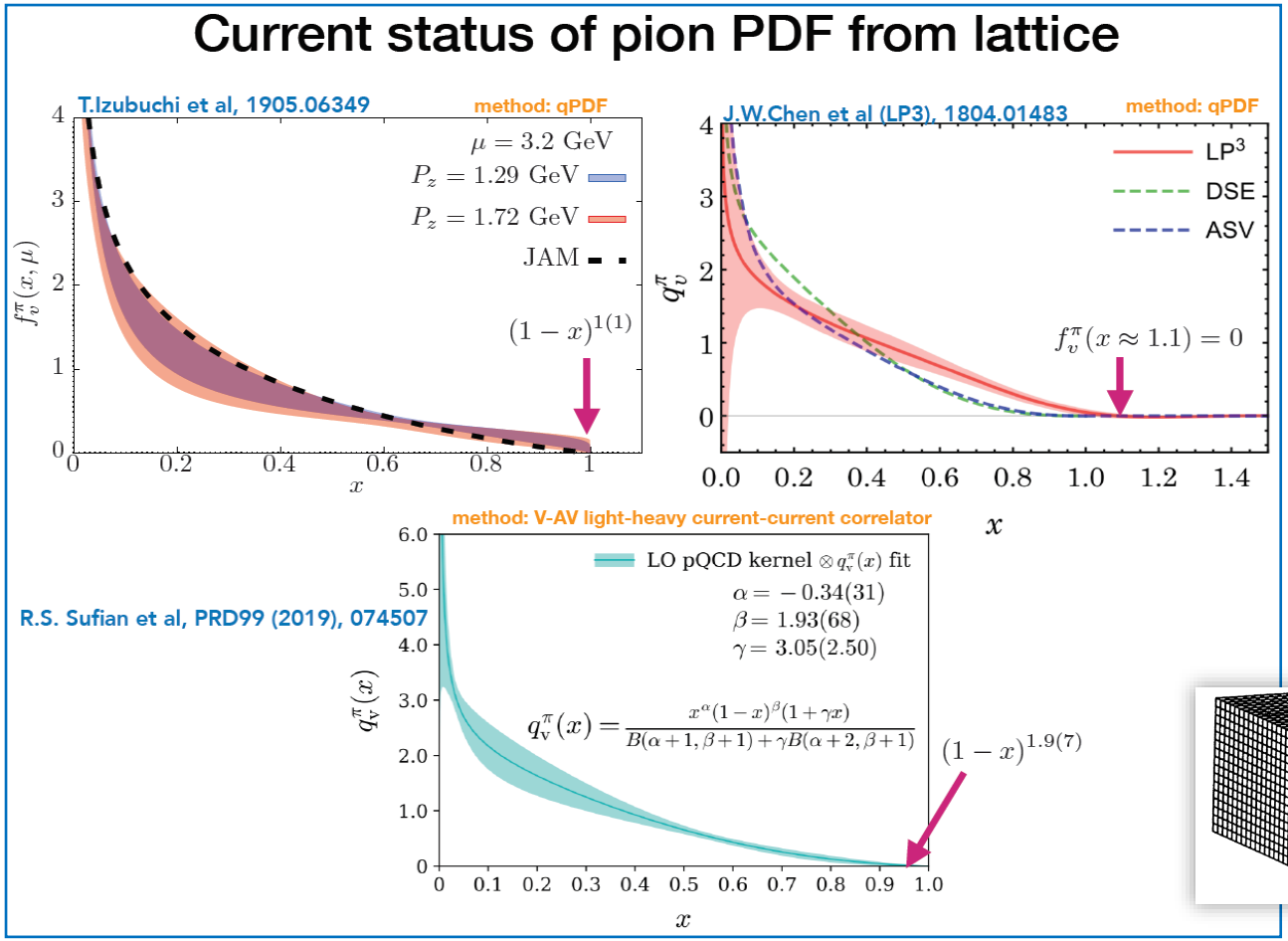
Pion Valence-Quark PDF

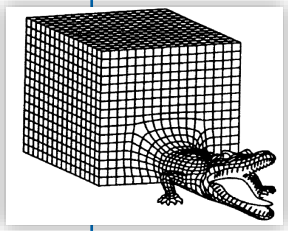
§ Status as of Summer 2019

Slide by Nikhil Karthik @ Lattice 2019

$M_\pi \approx 310$ MeV

$M_\pi \approx 426$ MeV





§ Single-ensemble calculation

- Non-physical pion mass, single lattice spacing, single volume

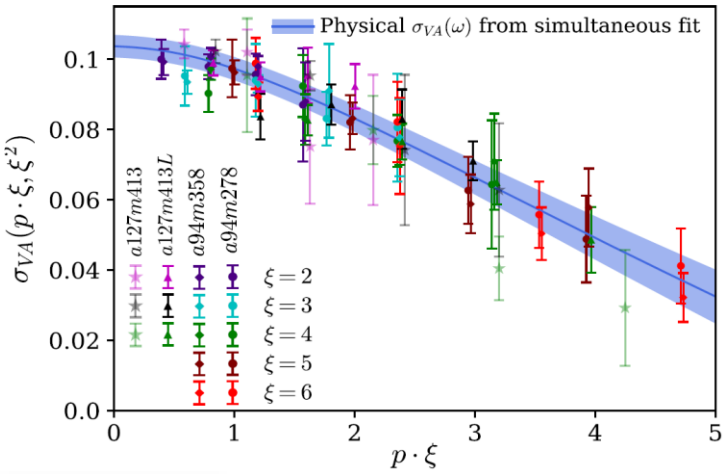
Pion Valence-Quark PDF

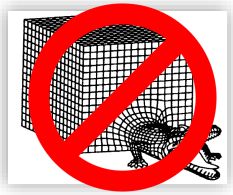
§ Results from JLab-W&M/ **LCS** method

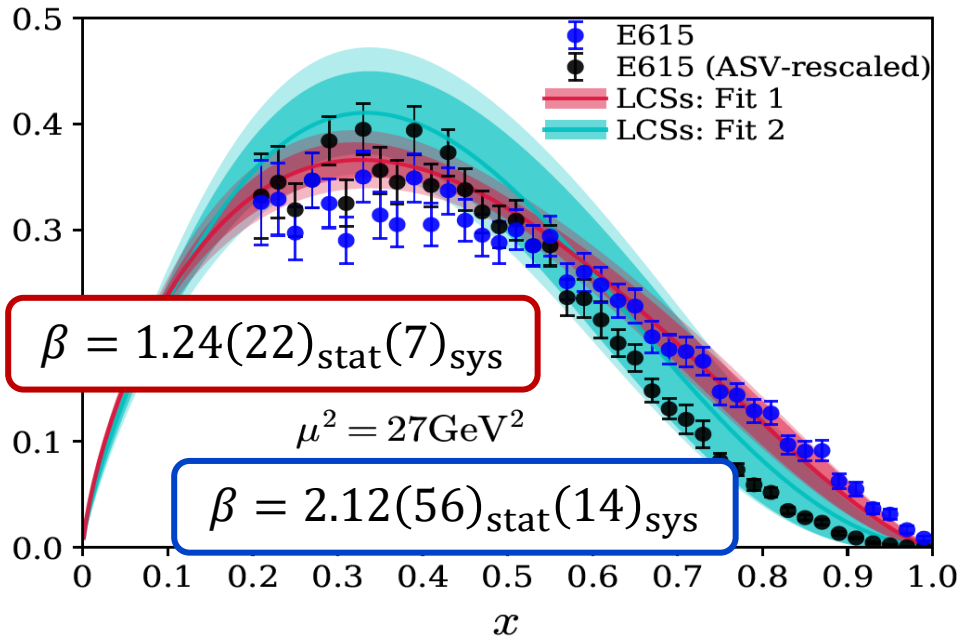
- $M_\pi = \mathbf{278}, 358, 413$ MeV with $a = 0.094, 0.127$ fm
- Extrapolated to physical limit (shown as blue band)
- Renormalized $Z_{V,A}$ in RI/MOM, matched to $\overline{\text{MS}}$, run to 27 GeV^2

R. S. Sufian, et al, 2001.04960

$$q_v^\pi(x) = \frac{x^\alpha (1-x)^\beta (1+\gamma x)}{B(\alpha+1,\beta+1)+\gamma B(\alpha+2,\beta+1)}$$





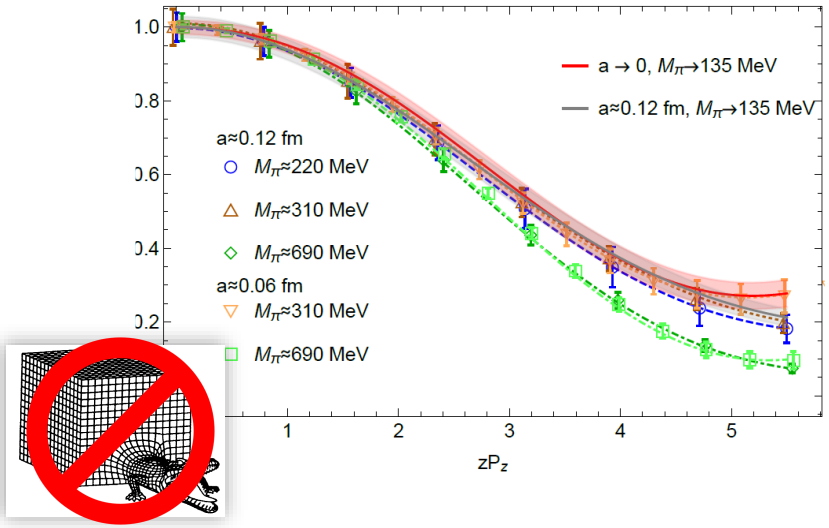


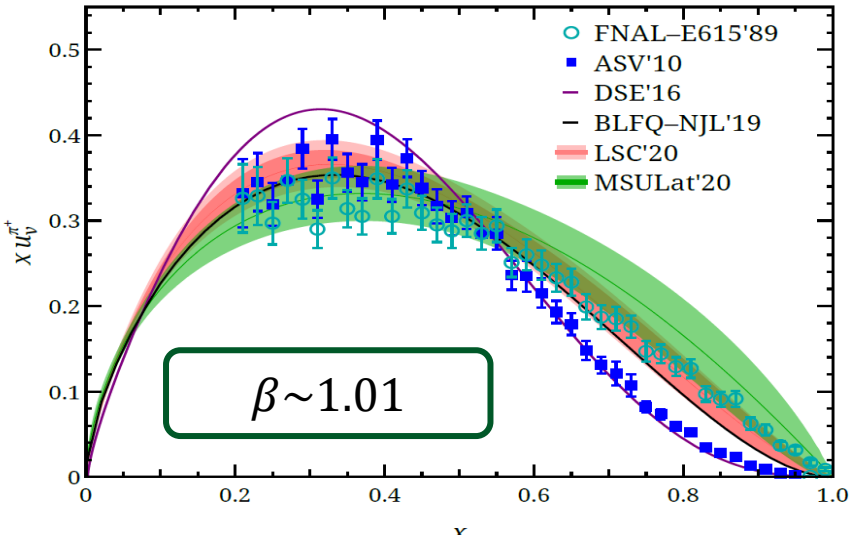
Pion Valence-Quark PDF

§ Results from MSULat/quasi-PDF method

- $M_\pi = \mathbf{220}, 310, 790$ MeV with $a = 0.06, 0.12$ fm
- Extrapolated to physical limit (shown as pink/green band)
- Renormalized in RI/MOM, matched to $\overline{\text{MS}}$, run to 27 GeV^2

H. Lin et al. (MSULat), 2003.14128





J. S. Conway et al., PRD39, 92 (1989).
M. Aicher et al, PRL105, 252003 (2010), 1009.2481.
C. Chen et al, PRD93, 074021 (2016), 1602.01502.

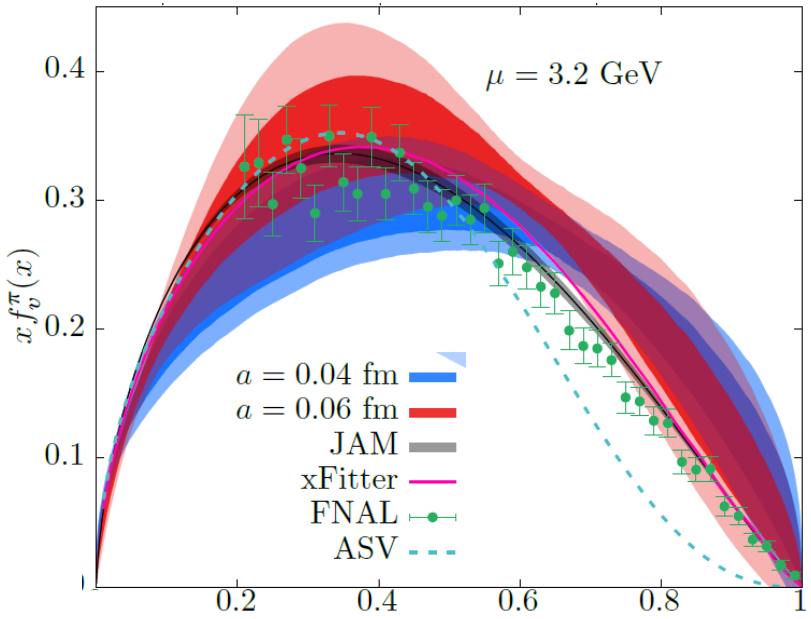
J. Lan, et al, PRL122, 172001 (2019), 1901.11430;
PRD101, 034024 (2020), 1907.01509.
R. S. Sufian, et al, 2001.04960

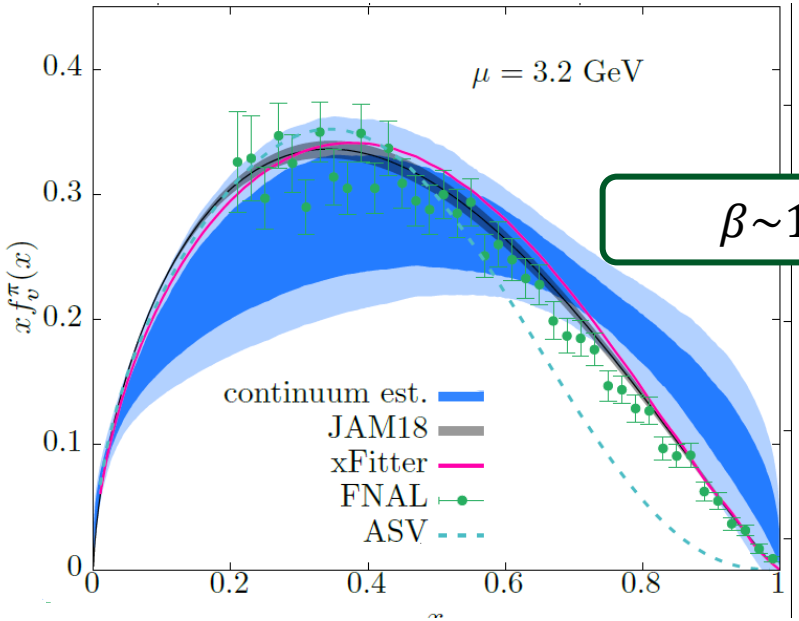
Pion Valence-Quark PDF

§ Results from BNL/quasi-PDF method

- $M_\pi = \mathbf{300}$ MeV with $a = 0.04, 0.06$ fm
- Extrapolated to continuum limit
- Renormalized in RI/MOM, matched to $\overline{\text{MS}}$ at 10 GeV^2

X. Gao et al. 2007.06590



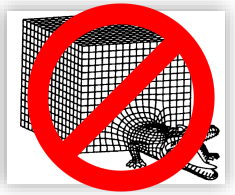


$\beta \sim 1$

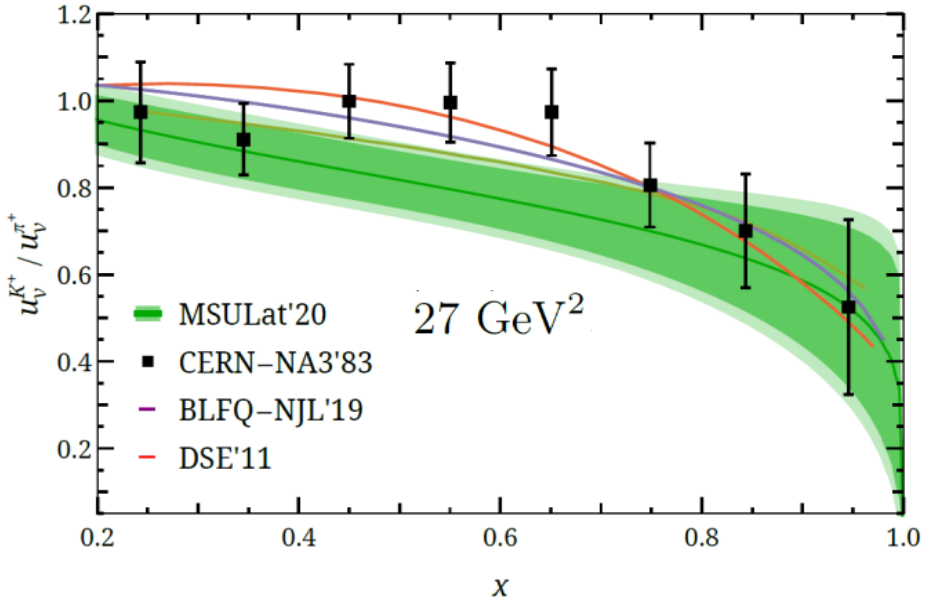
Kaon Valence-Quark PDFs

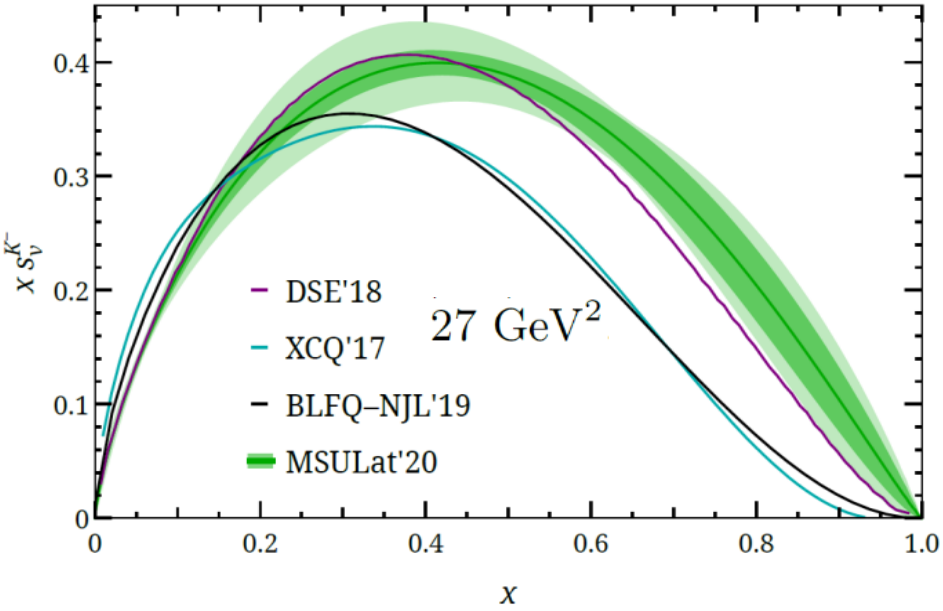
§ Pion/**kaon** PDFs using quasi-PDF in the continuum limit

- Lattice details: clover/2+1+1 HISQ (MSULat)
  $a \approx \{0.06, 0.12\}$ fm,
  $M_\pi \in \{220, 310, 690\}$-MeV pion
  $P_z \approx \{1.3, 1.7\}$ GeV



2003.14128 HL et al (MSULat)

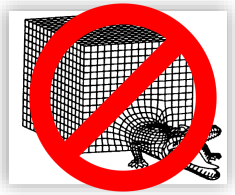




Kaon Valence-Quark PDFs

§ Pion/**kaon** PDFs using quasi-PDF in the continuum limit

- Lattice details: clover/2+1+1 HISQ (MSULat)
  $a \approx \{0.06, 0.12\}$ fm,
  $M_\pi \in \{220, 310, 690\}$-MeV pion
  $P_z \approx \{1.3, 1.7\}$ GeV



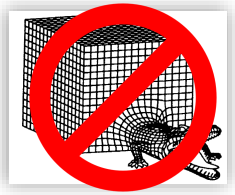
§ First LQCD calculation $\langle x^n \rangle$ of $u_v^{K^+}$ and $s_v^{K^-}$ 2003.14128 HL et al (MSULat)

n	$\langle x^n \rangle (u_v^{K^+})$	$\langle x^n \rangle (s_v^{K^-})$
1	$0.192(8)_{\text{stat}}(6)_{\text{syst}}$	$0.261(8)_{\text{stat}}(8)_{\text{syst}}$
2	$0.080(7)_{\text{stat}}(6)_{\text{syst}}$	$0.120(7)_{\text{stat}}(9)_{\text{syst}}$
3	$0.041(6)_{\text{stat}}(4)_{\text{syst}}$	$0.069(6)_{\text{stat}}(8)_{\text{syst}}$

Kaon Valence-Quark PDFs

§ Pion/**kaon** PDFs using quasi-PDF in the continuum limit

- Lattice details: clover/2+1+1 HISQ (MSULat)
  $a \approx \{0.06, 0.12\}$ fm,
  $M_\pi \in \{220, 310, 690\}$-MeV pion
  $P_z \approx \{1.3, 1.7\}$ GeV



§ First LQCD calculation $\langle x^n \rangle$ of $u_v^{K^+}$ and $s_v^{K^-}$

§ Later ETMC **260**-MeV results on $\langle x^n \rangle$ of $u_v^{K^+}$ and $s_v^{K^-}$

2003.14128 HL et al (MSULat)

n	$\langle x^n \rangle (u_v^{K^+})$	$\langle x^n \rangle (s_v^{K^-})$
1	$0.192(8)_{\text{stat}}(6)_{\text{syst}}$	$0.261(8)_{\text{stat}}(8)_{\text{syst}}$
2	$0.080(7)_{\text{stat}}(6)_{\text{syst}}$	$0.120(7)_{\text{stat}}(9)_{\text{syst}}$
3	$0.041(6)_{\text{stat}}(4)_{\text{syst}}$	$0.069(6)_{\text{stat}}(8)_{\text{syst}}$

2010.0349, 2104.02247

n	$\langle x^n \rangle (u_v^{K^+})$	$\langle x^n \rangle (s_v^{K^-})$
1	$0.246(2)_{\text{stat}}(2)_{\text{syst}}$	$0.317(2)_{\text{stat}}(1)_{\text{syst}}$
2	$0.093(5)_{\text{stat}}(3)_{\text{syst}}$	$0.134(5)_{\text{stat}}(2)_{\text{syst}}$
3	$0.035(6)_{\text{stat}}(3)_{\text{syst}}$	$0.075(5)_{\text{stat}}(1)_{\text{syst}}$

First Pion Gluon PDF

§ Pion GLUON PDFs using pseudo-PDF

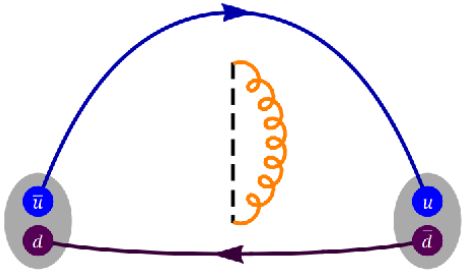
- Lattice details: clover/2+1+1 HISQ (MSULat)

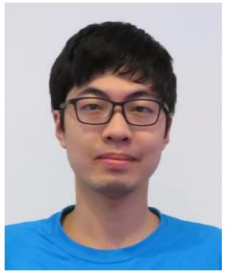
  $a \approx \{0.12, 0.15\}$ fm,

  $M_\pi \in \{220, 310, 690\}$-MeV pion

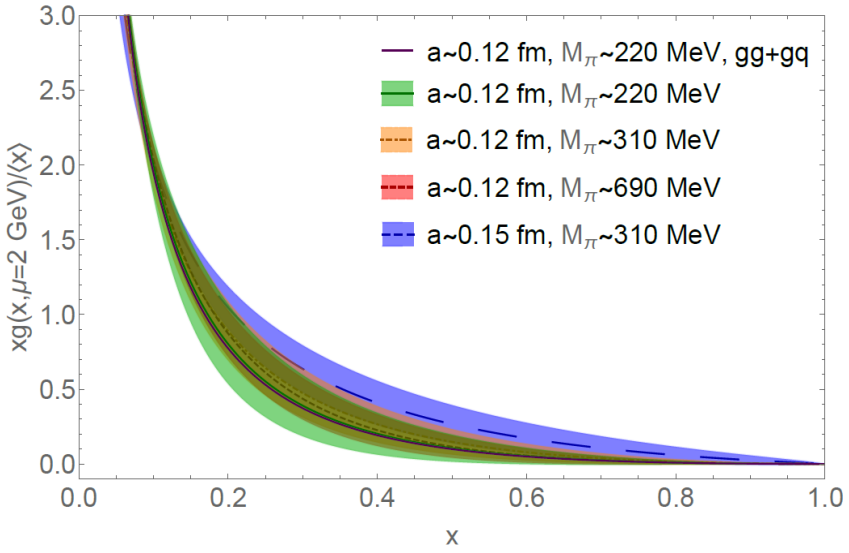
  $P_{z,\max} \approx 2.3$ GeV

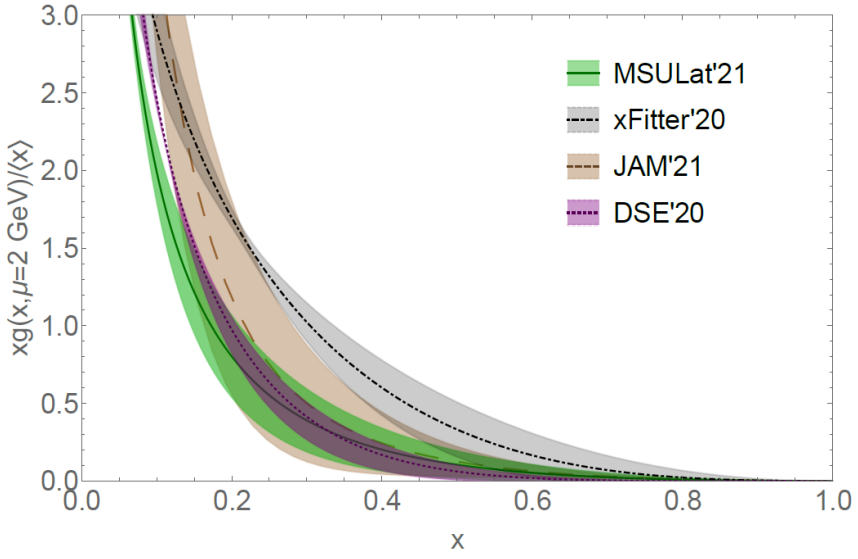
2104.06372, Fan, HL(MSULat)





Zhouyou Fan (MSU)



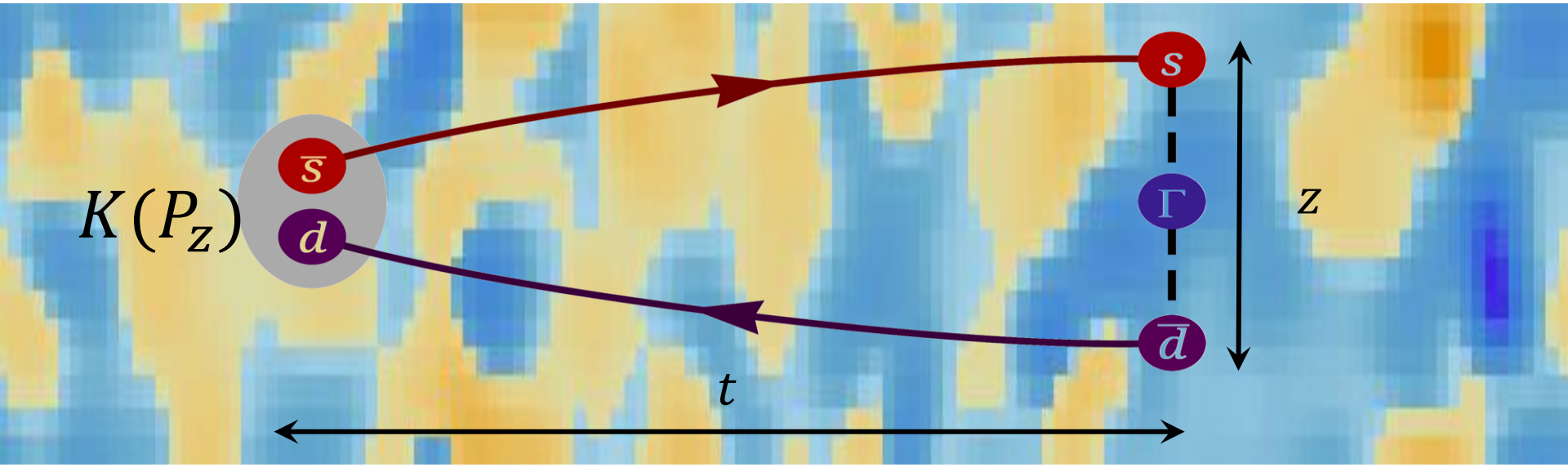


Pion and Kaon DA

§ The first continuum-limit study of *x*-dependent meson DA on the lattice

- $M_\pi \in \{\mathbf{310}, 690\ (\eta_s)\}$ MeV
- $a \in \{0.06, 0.09, 0.12\}$ fm
- $M_\pi^{\min} L = 4.5$

$$C_M^{\mathrm{DA}}(z,P,t) = \left\langle 0 \left| \int d^3y\, e^{i\,\vec{P}\cdot\vec{y}} \bar{\psi}_1(\vec{y},t)\gamma_z\gamma_5 U(\vec{y},\vec{y}+z\,\hat{z})\psi_2(\vec{y}+z\,\hat{z},t)\bar{\psi}_2(0,0)\gamma_5\psi_1(0,0) \right| 0 \right\rangle$$



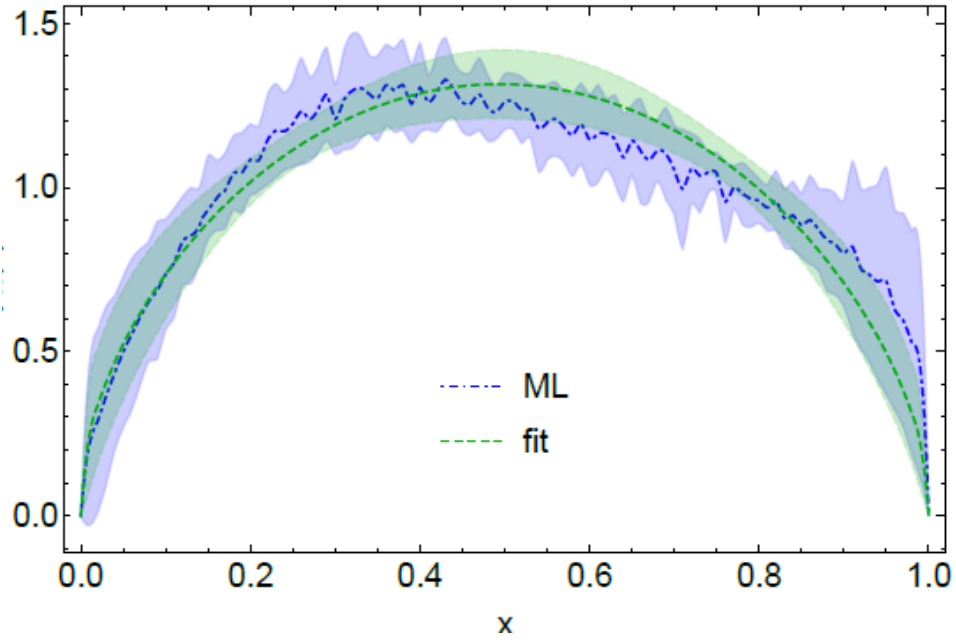
Pion and Kaon DA

§ Extract the DA distribution from the physical-continuum matrix elements

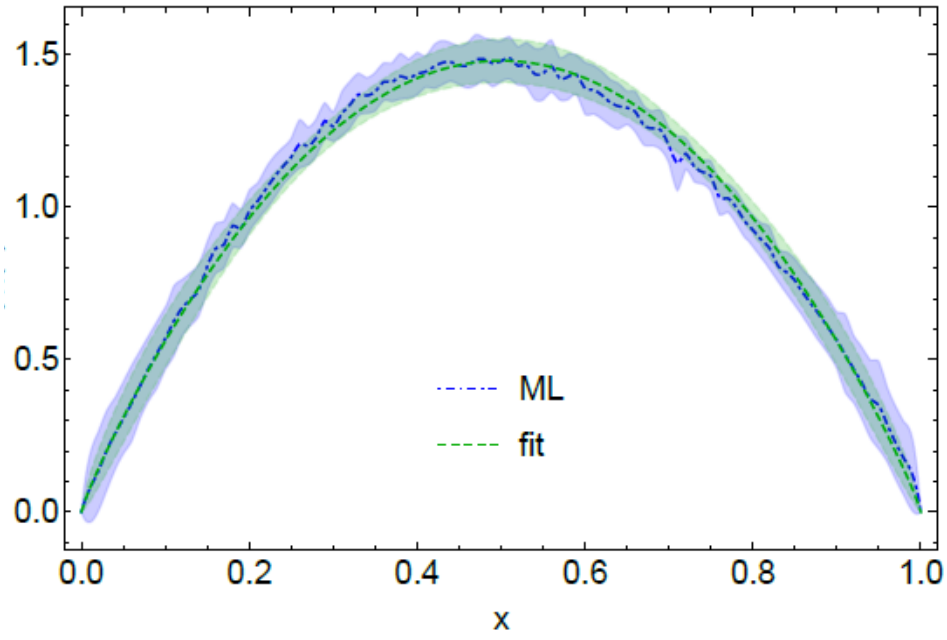
R. Zhang et al. (MSULat), 2005.13955

$$h(z,\mu^R,p_z^R,P_z)=\int_{-\infty}^{\infty}dx\int_0^1 dy\, C\left(x,y,\left(\frac{\mu^R}{p_z^R}\right)^2,\frac{P_z}{\mu^R},\frac{P_z}{p_z^R}\right)f_{m,n}(y)e^{i(1-x)zP_z}$$

Pion



Kaon



Pion and Kaon DA

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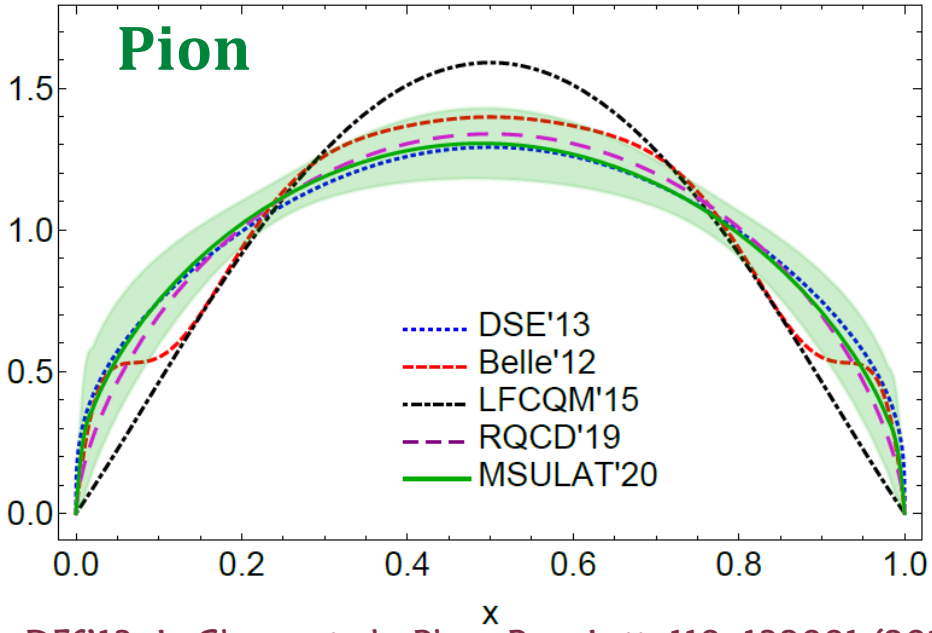
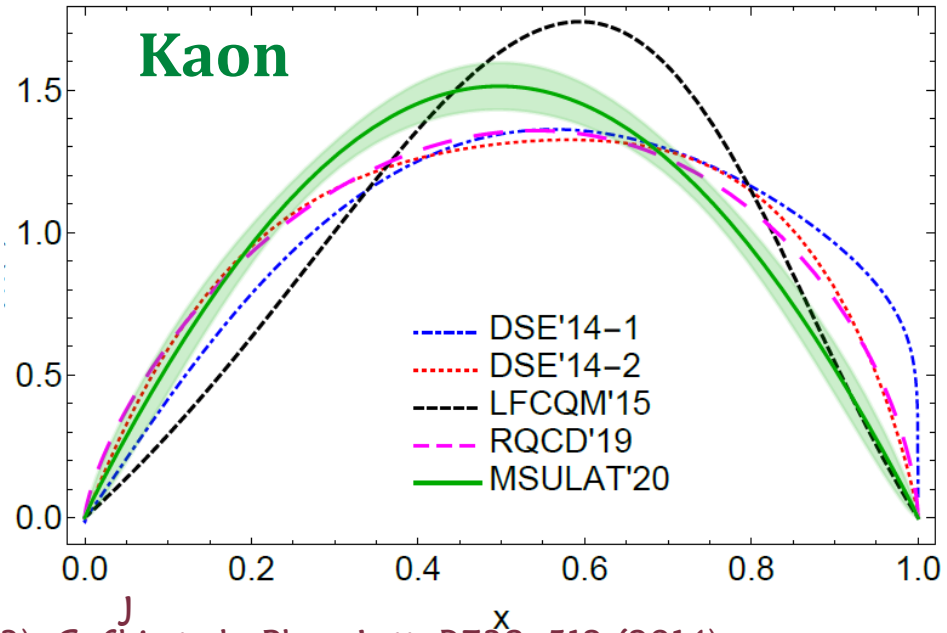
$$h(z,\mu^R,p_z^R,P_z)=\int_{-\infty}^{\infty}dx\int_0^1 dy\, C\left(x,y,\left(\frac{\mu^R}{p_z^R}\right)^2,\frac{P_z}{\mu^R},\frac{P_z}{p_z^R}\right)f_{m,n}(y)e^{i(1-x)zP_z}$$



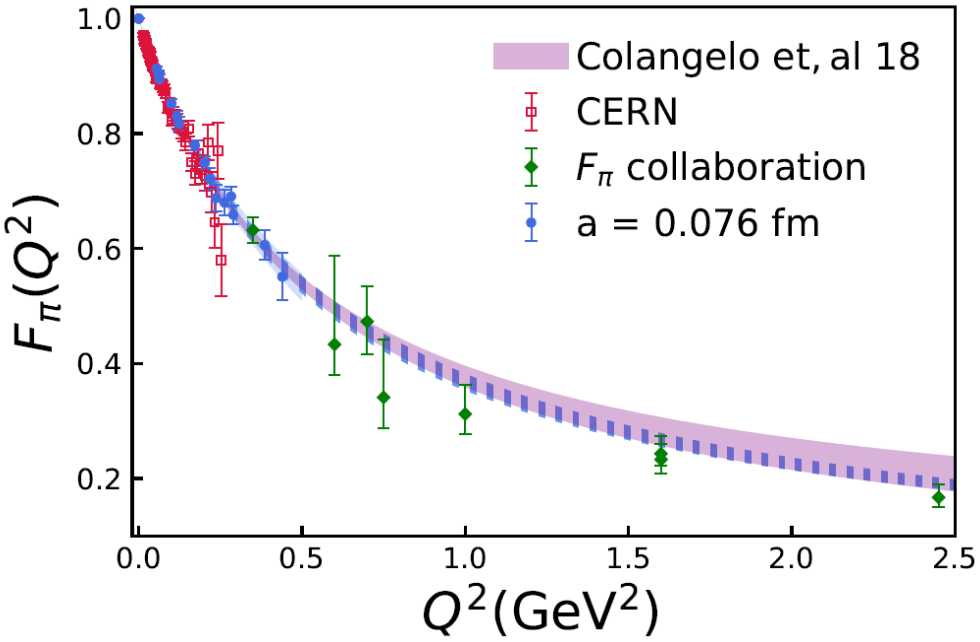


DES'13: L. Chang et al., Phys. Rev. Lett. 110, 132001 (2013); C. Shi et al., Phys. Lett. B738, 512 (2014)
Belle'12: S. Agaev et al., Phys. Rev. D86, 077504 (2012);
LFCQM'15:J. P. B. C. de Melo et al., AIP Conf. Proc. 1735, 080012 (2016);
RQCD'19: G. S. Bali et al.,J HEP 08, 065 (2019); DSE'14:

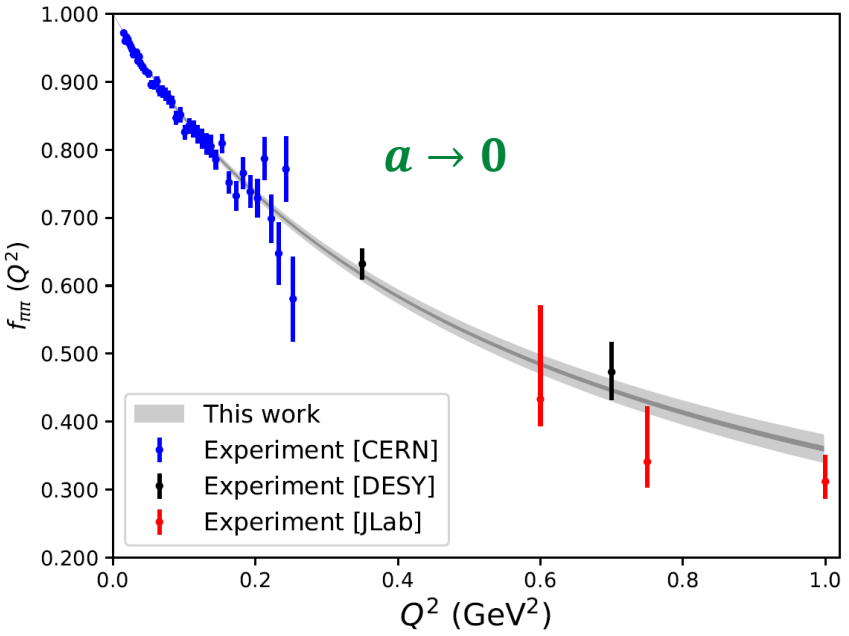
Pion Form Factors

§ Two new lattice pion form factors calcs at physical pion

- χQCD: 2+1f, overlap/DWF,
  $a \approx [0.08,0.2]$ fm, $M_\pi \in [139,340]$-MeV
- BNL: 2+1+1,clover/HISQ,
  $a \approx [0.04,0.08]$ fm, $M_\pi \in \{135, 300\}$-MeV



X. Gao et al (BNL), 2102.06047



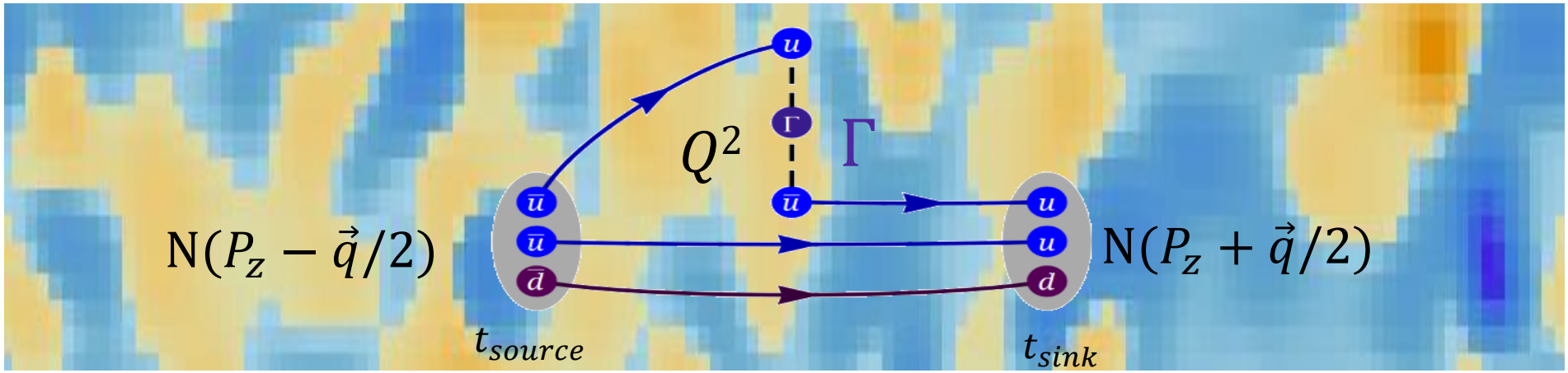
G. Wang et al (XQCD), 2006.05431

Backup Slides
Updates on Nucleon GPDs

2020: Isovector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at **physical pion mass**

- MSULat: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results



$$\tilde{F}(x,\tilde{\xi},t,\bar{P}_z) = \frac{\bar{P}_z}{\bar{P}_0}\int \frac{dz}{4\pi} e^{ixz\bar{P}_z}\langle P'|\tilde{O}_{\gamma_0}(z)|P\rangle = \frac{\bar{u}(P')}{2\bar{P}^0}\left(H(x,\tilde{\xi},t,\bar{P}_z)\gamma^0 + E(x,\tilde{\xi},t,\bar{P}_z)\frac{i\sigma^{0\mu}\Delta_\mu}{2M}\right)u(P'')$$

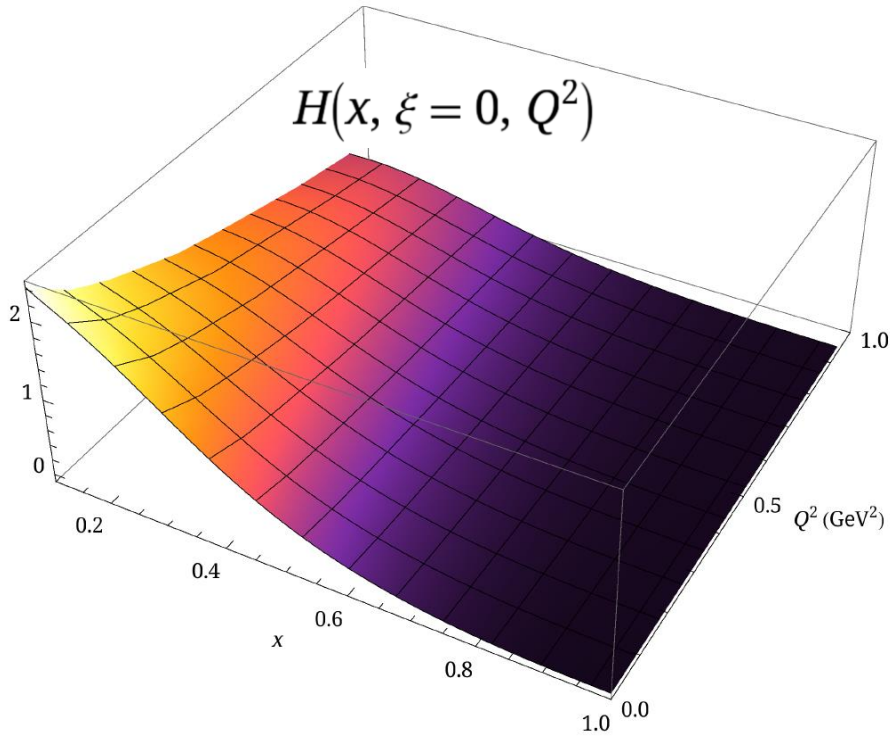
$$p^\mu = \frac{p''^\mu + p'^\mu}{2}, \qquad \Delta^\mu = p''^\mu - p'^\mu, \qquad t = \Delta^2, \qquad \xi = \frac{p''^+ - p'^+}{p''^+ + p'^+}$$

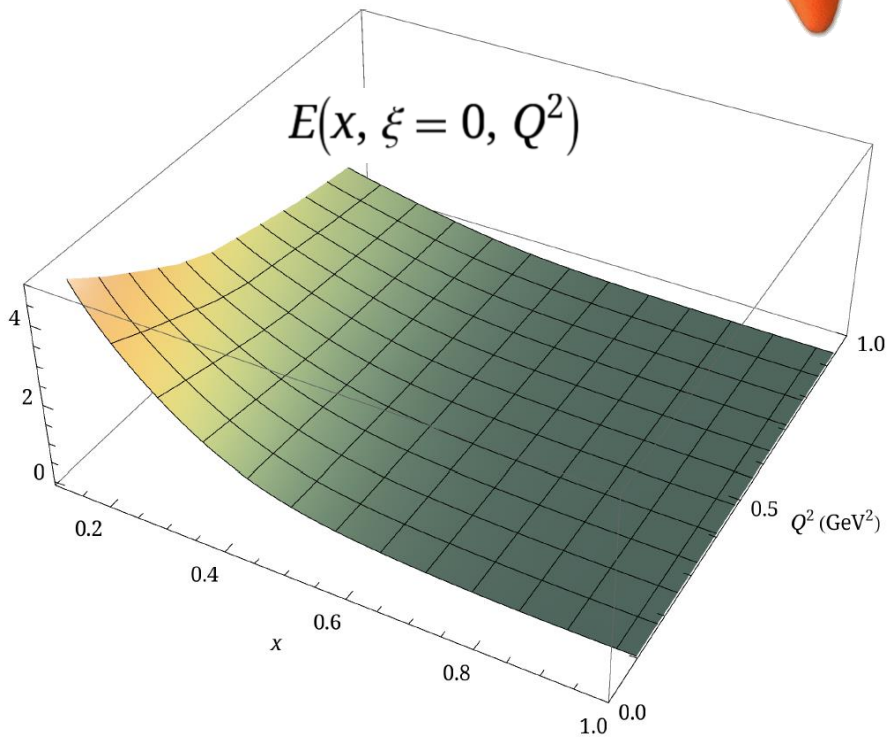
HL, Phys.Rev.Lett. 127 (2021) 18, 182001

2020: Isovector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at **physical pion mass**

- MSULat: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results





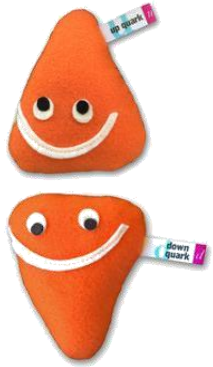
HL, Phys.Rev.Lett. 127 (2021) 18, 182001

2020: Isovector Nucleon GPDs

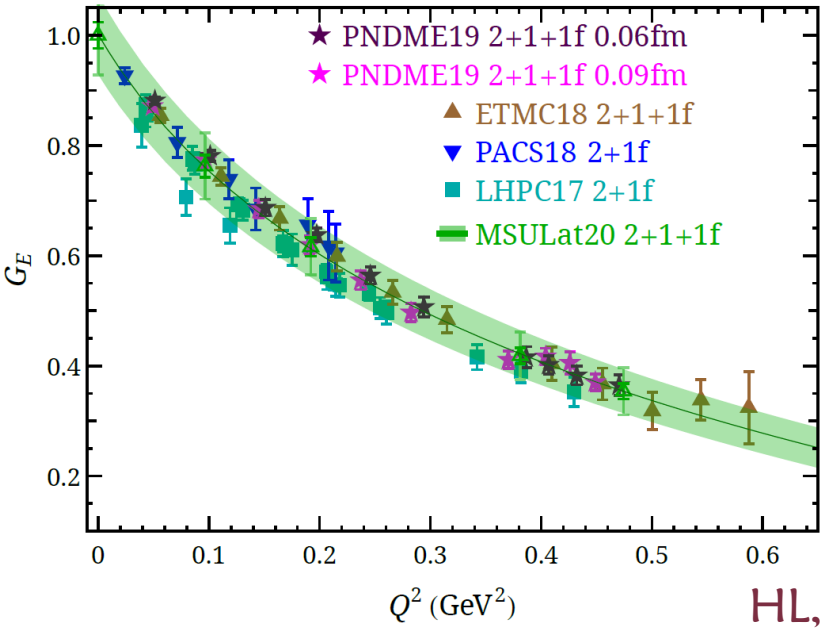
§ Nucleon GPD using quasi-PDFs at physical pion mass

- Lattice details: clover/2+1+1 HISQ (MSULat)
  0.09 fm, **135-MeV** pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results

$$\int_{-1}^{+1} dx\, x^{n-1} \; = \sum_{i=0,\text{even}}^{n-1} (-2\xi)^i A_{ni}^q(t) + (-2\xi)^n C_{n0}^q(t)\Big|_{n\ \text{even}}$$



$n = 1$



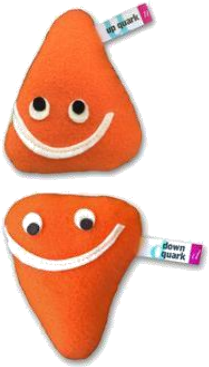
HL, Phys.Rev.Lett. 127 (2021) 18, 182001

2020: Nucleon GPDs

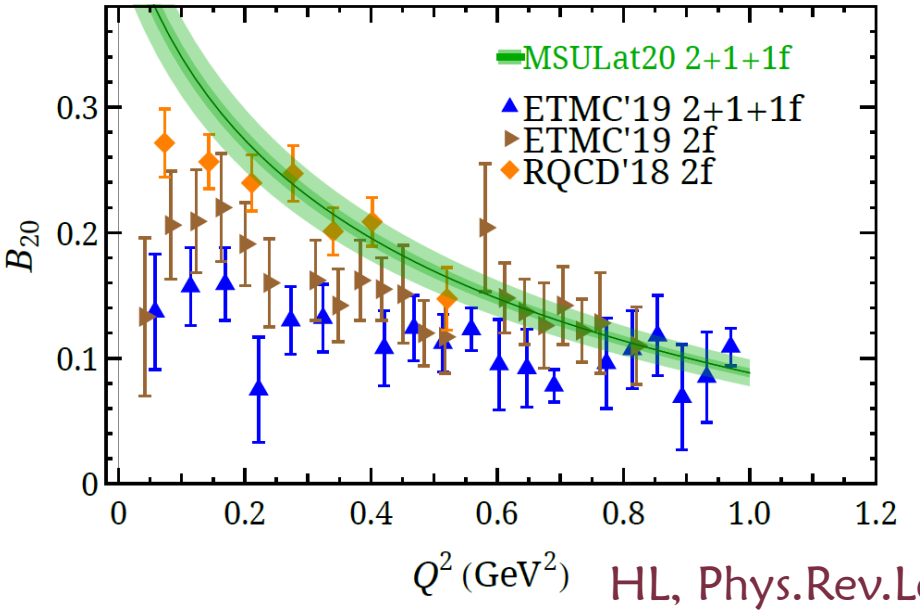
§ Nucleon GPD using quasi-PDFs at physical pion mass

- Lattice details: clover/2+1+1 HISQ (MSULat)
  0.09 fm, **135-MeV** pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results

$$\int_{-1}^{+1} dx\, x^{n-1} \; = \sum_{i=0,\text{even}}^{n-1} (-2\xi)^i B^q_{ni}(t) - (-2\xi)^n C^q_{n0}(t)\Big|_{n \text{ even}}$$



$n = 2$

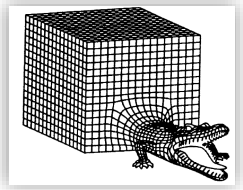


HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Nucleon Tomography

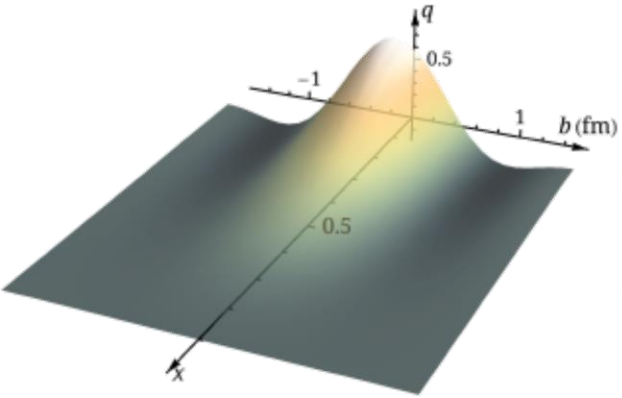
§ Nucleon GPD using quasi-PDFs at physical pion mass

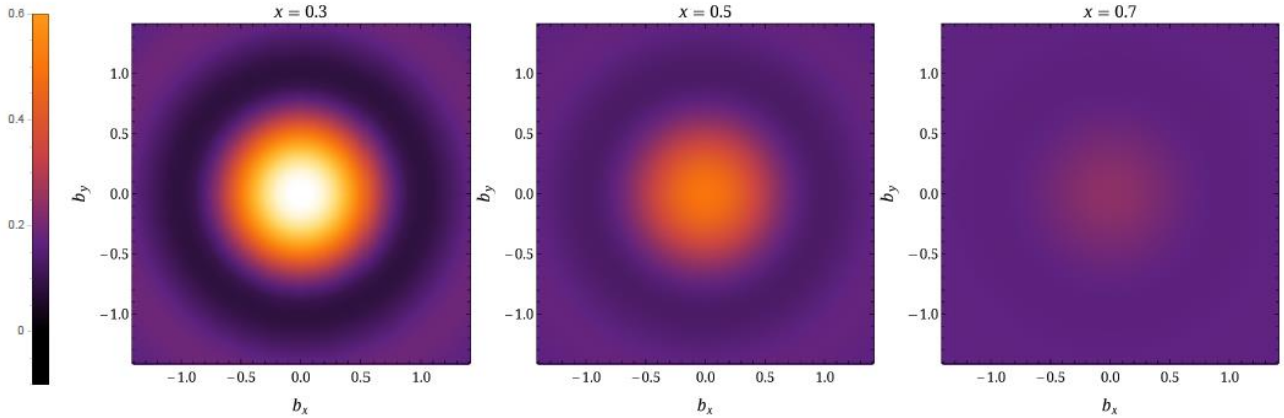
- Lattice details: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon GPD results



finite-volume, discretization,

$$q(x,b) = \int \frac{d\vec{q}}{(2\pi)^2} H(x, \xi = 0, t = -\vec{q}^2) e^{i\vec{q}\cdot\vec{b}}$$





HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Also see work done by ANL/BNL/ETMC, 2209.05373 ,2310.13114

Nucleon Polarized GPDs

§ Helicity GPD ($\widetilde{H}$)using quasi-PDFs at **physical pion mass**

- MSULat: clover/2+1+1 HISQ

  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV

$$\widetilde{F}^q(x,\xi,t) = \int \frac{\mathrm{d}z^-}{4\pi} e^{-ixP^+z^-} \langle p'|\bar{q}(z^-/2)\gamma^+\gamma_5 q(-z^-/2)|p\rangle$$

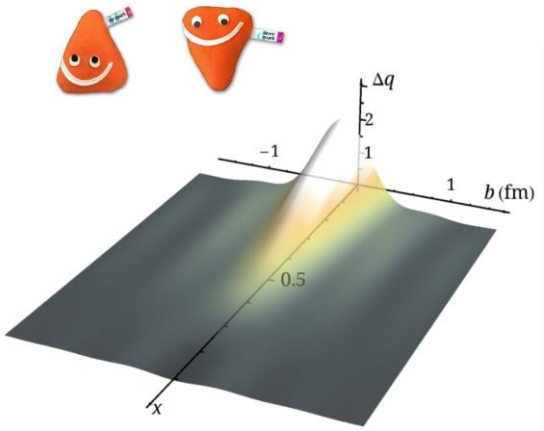
$$= \frac{1}{2P^+}\left[\widetilde{H}^q(x,\xi,t)\,\bar{u}(p')\,\gamma^+\gamma_5 u(p) - \widetilde{E}^q(x,\xi,t)\,\bar{u}(p')\,\frac{\gamma_5\Delta^+}{2m}u(p)\right]$$

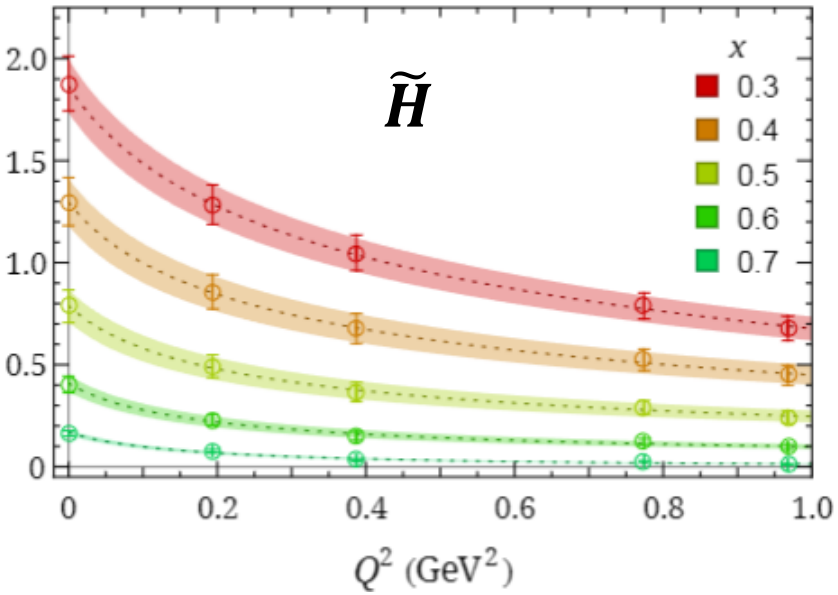
Nucleon Polarized GPDs

§ Helicity GPD ($\widetilde{H}$)using quasi-PDFs at **physical pion mass**

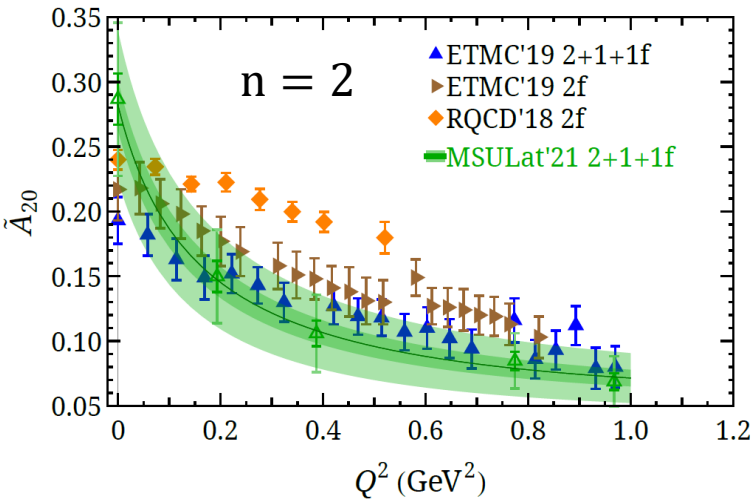
- MSULat: clover/2+1+1 HISQ
  0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- $\xi = 0$ isovector nucleon (quasi-)GPD results

HL (MSULat), Phys.Lett.B 824 (2022) 136821





- Take the integral to form moments



Caveats

§ Systematics in our earlier quasi-PDF calculation

- Renormalization: non-perturbative RI/MOM renormalization
  - State of the art: hybrid-ratio renormalization

    X. Ji et. al. NPB 964, 115311 (2021)
- Next-leading order (NLO) matching only
  - State of the art: NNLO matching kernel available

    X. Gao, PRL 128, 142003 (2022)
- Did not teat leading-renormalon effects
  - Leading-renormalon resummation (LRR)
  - Renormalization-group resummation (RGR)

    R. Zhang, et. al. PLB 844, 138081 (2023)
- For the rest of this presentation, we will focus on the uncertainties from the above (rather than typical lattice-calculation precision or systematics)

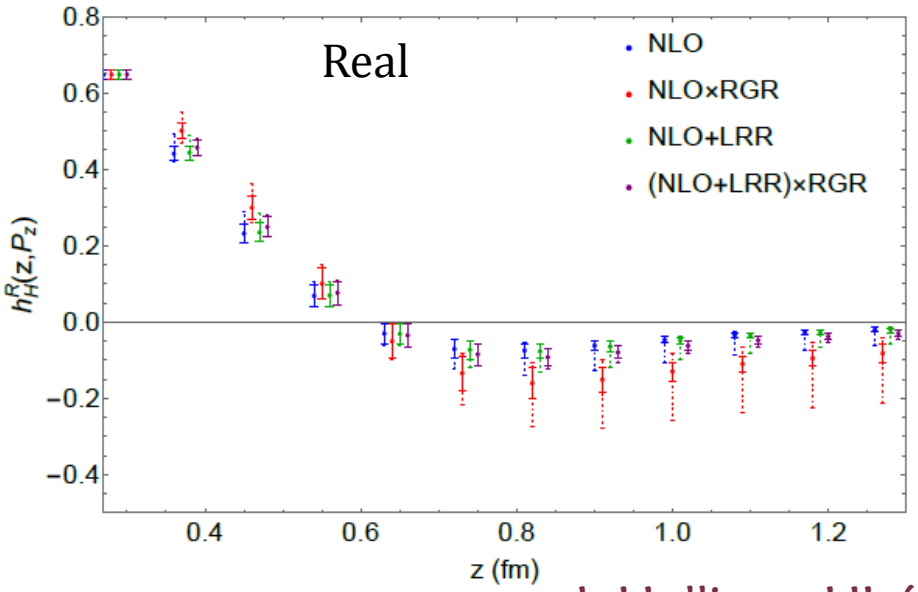
Forward-Limit Case: PDF

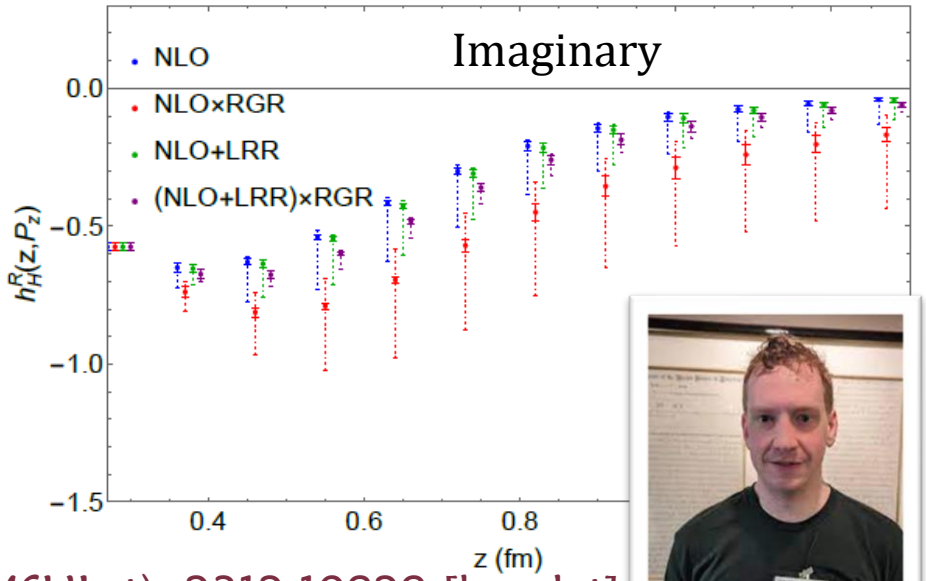
§ NLO hybrid-ratio renormalized matrix elements

$$h^R(z,P_z) = \begin{cases} N\dfrac{h^B(z,P_z)}{h^B(z,P_z=0)} & \text{for } z < z_s \\ Ne^{(\delta m+m_0)(z-z_s)}\dfrac{h^B(z,P_z)}{h^B(z,P_z=0)} & \text{for } z \geq z_s \end{cases}$$

Remove the linear divergence & renormalon ambiguity at large distances

- Vary the scale within $[0.75, 1.5]$: $\approx 15\%$ variation $\alpha_s(\mu = 2.0\ \text{GeV})$
- Systematic errors shown below:



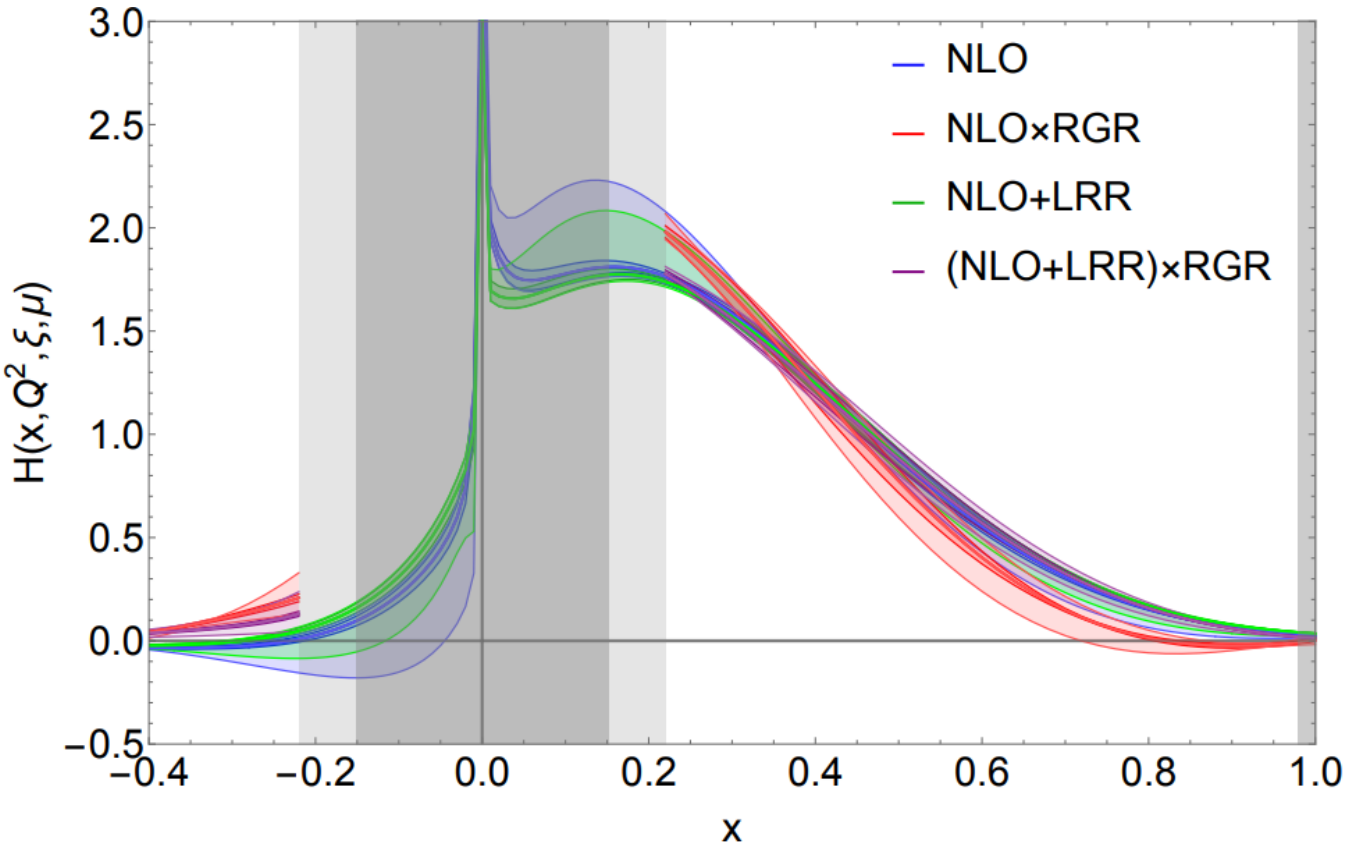


J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Forward Limit Case: PDF

§ NLO isovector nucleon $H(\xi = 0, Q^2 = 0, x)$

- RGR process: DGLAP equation breaks down for $|x| \lesssim 0.2$ with $\mu = c' \times 2xP_z$



J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

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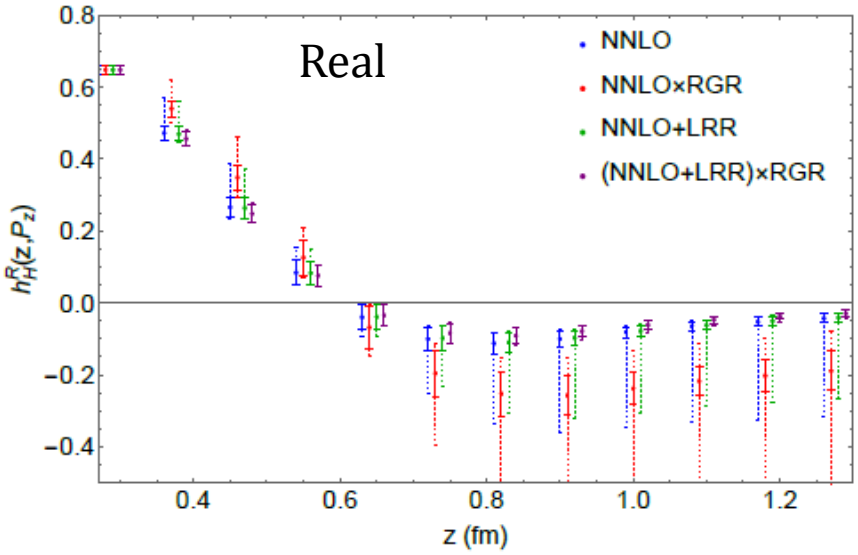

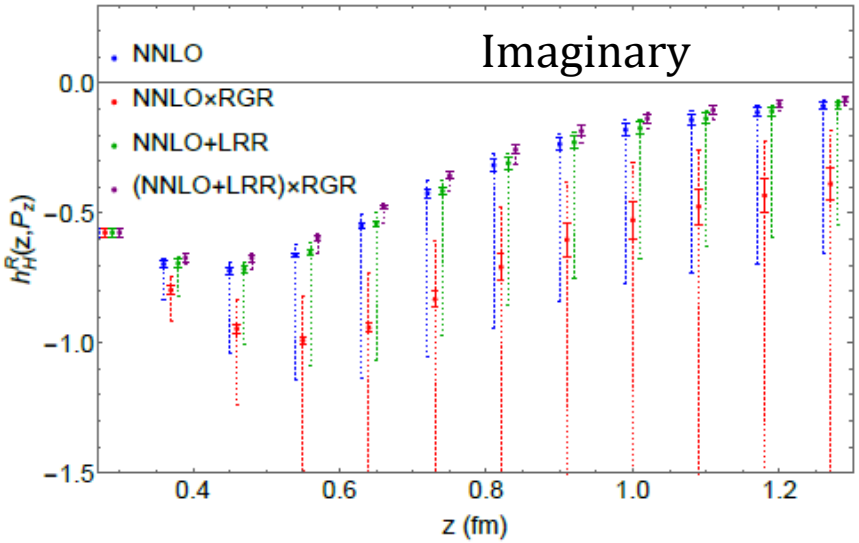

J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

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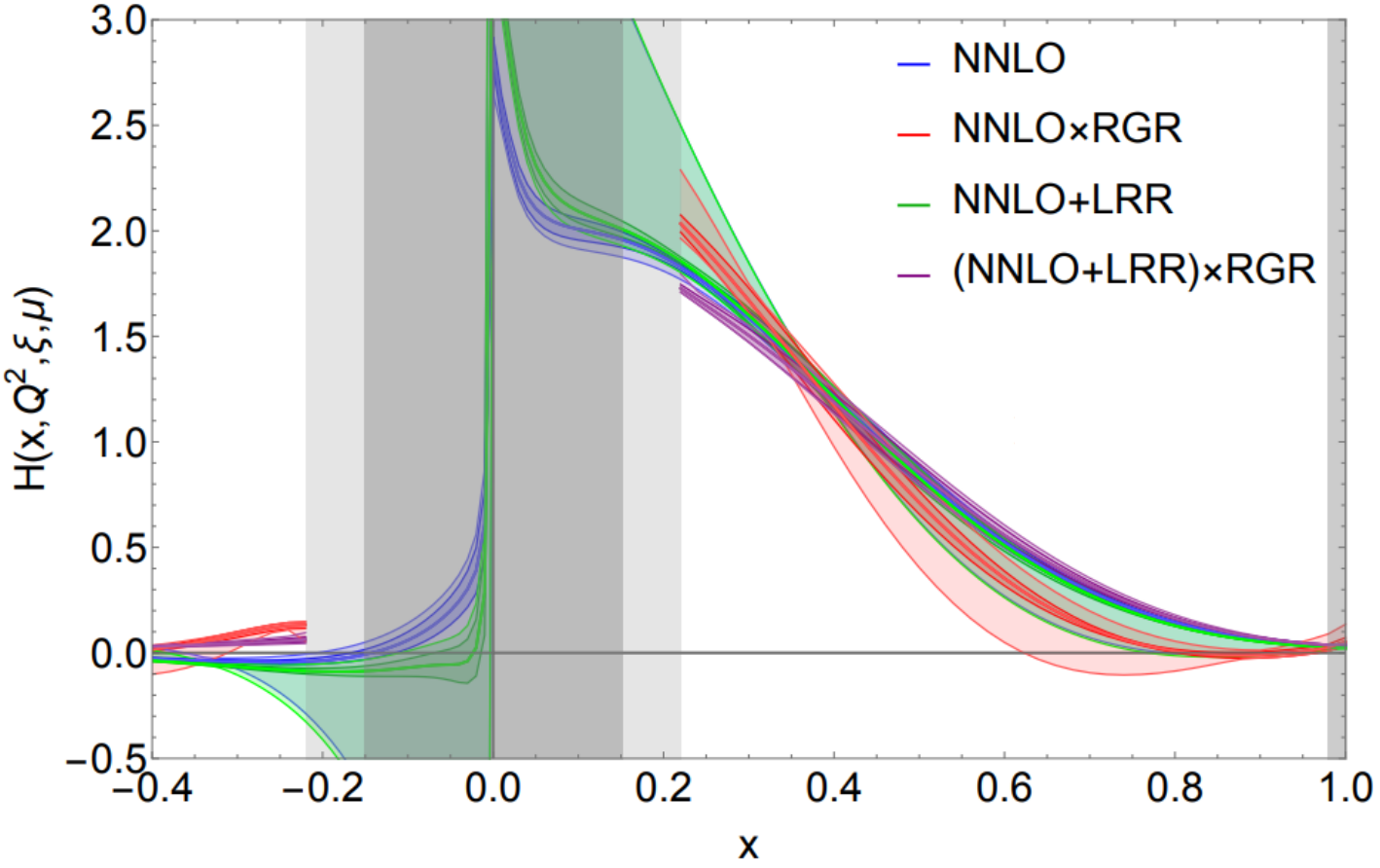
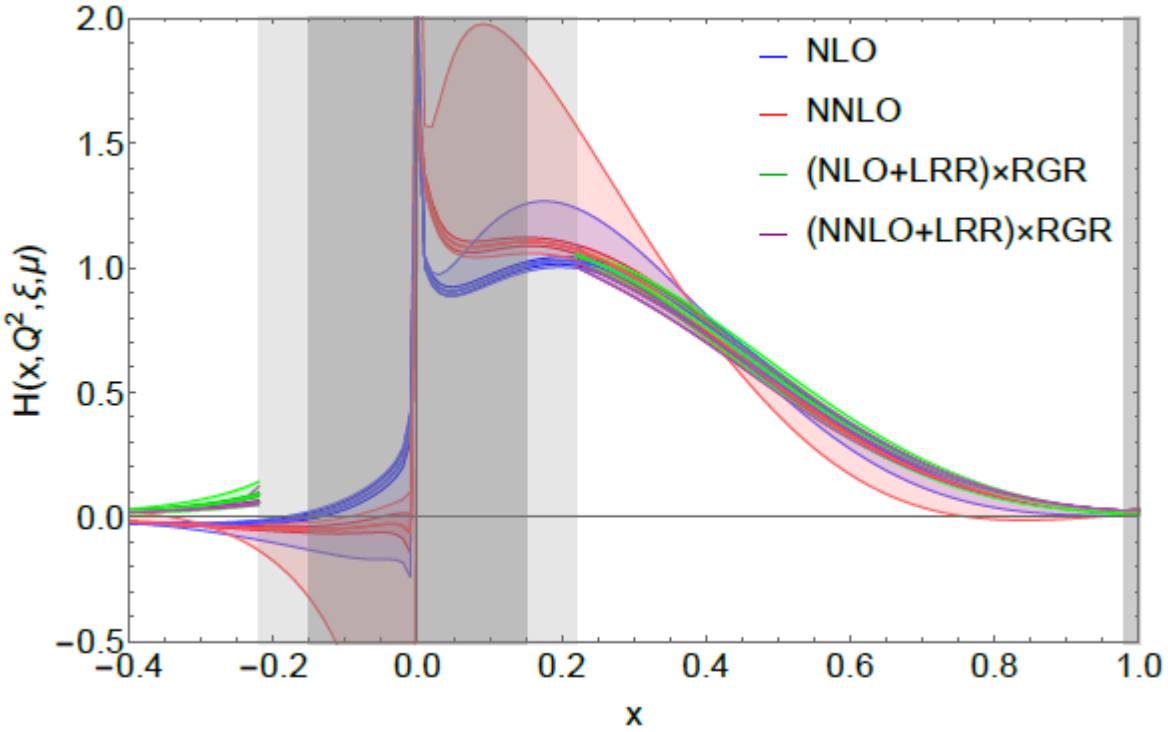


J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

$\xi=0$, $Q^2=0.39$ GeV^2 GPDs

§ Repeat the procedure for nonzero transfer momentum



J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

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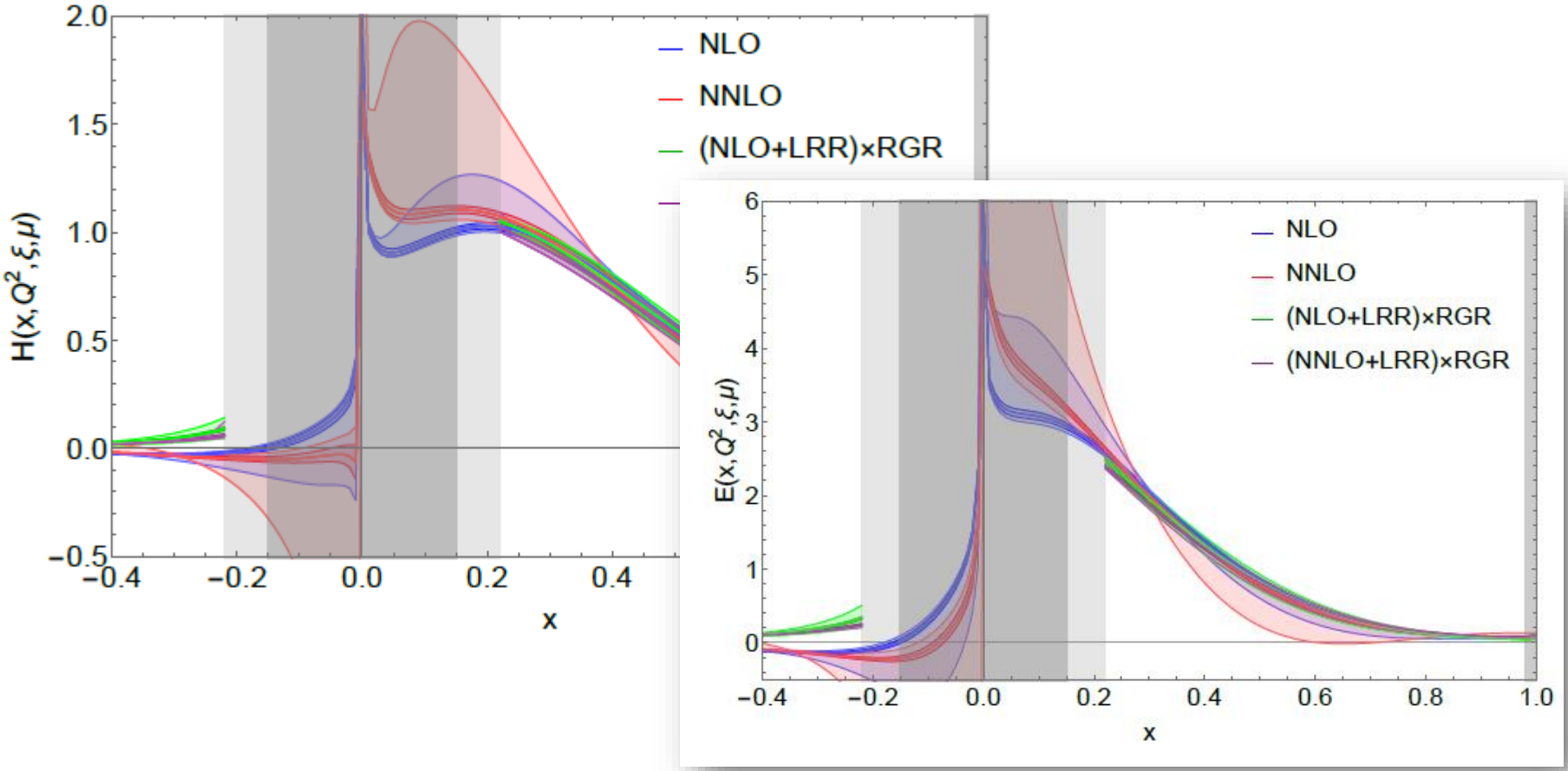
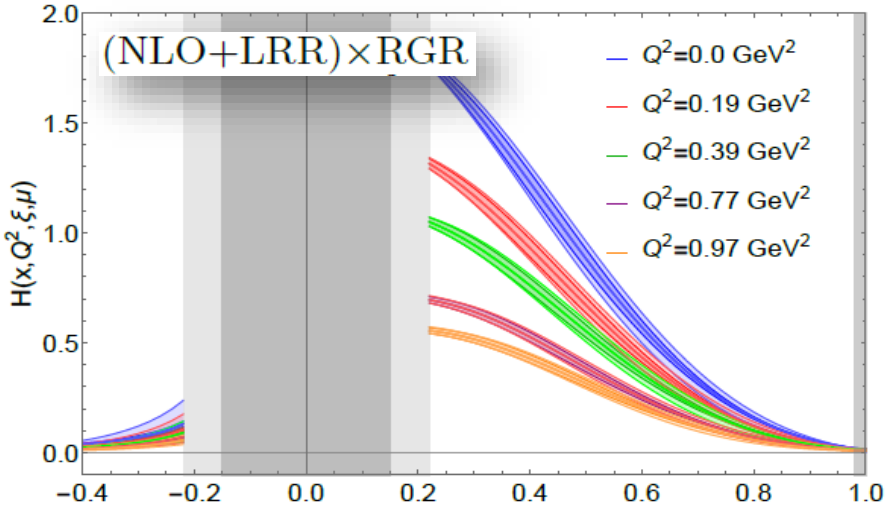
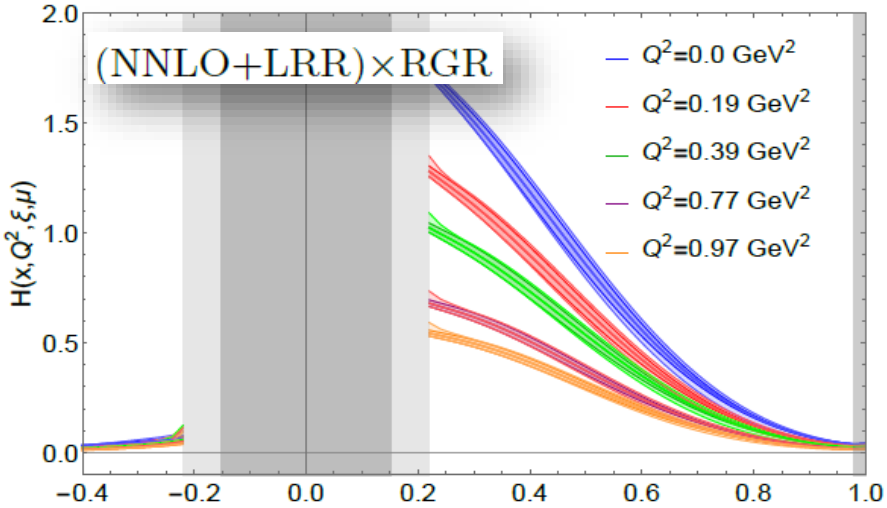
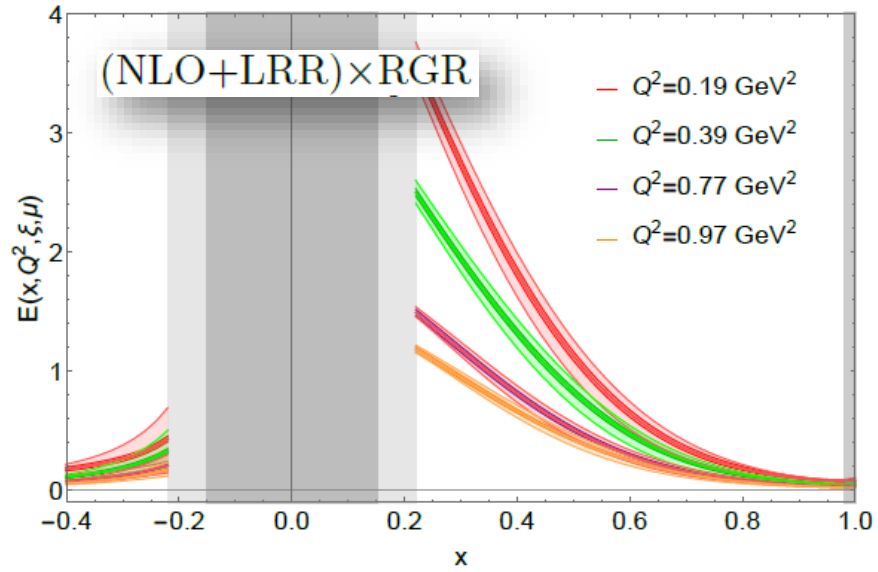


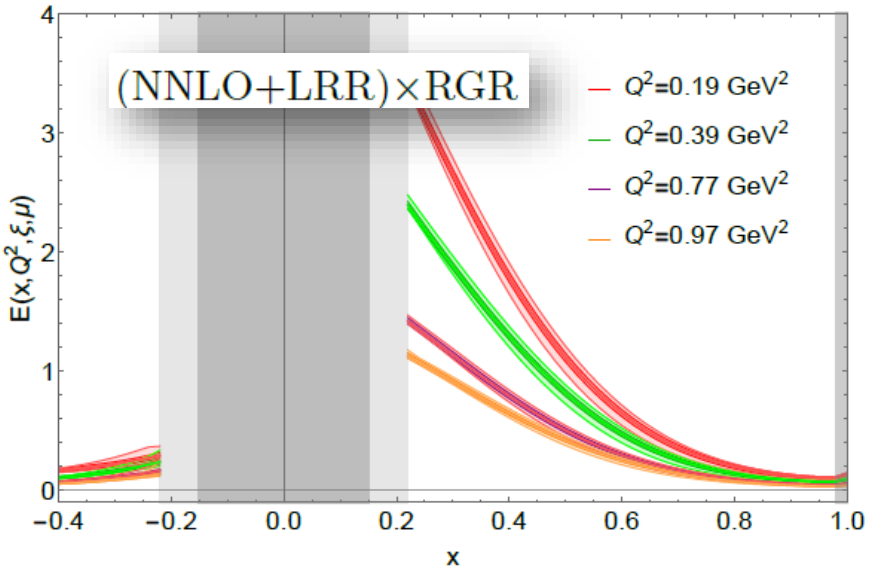
J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

$\xi=0$ GPDs









J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

$\xi \neq 0$ GPDs

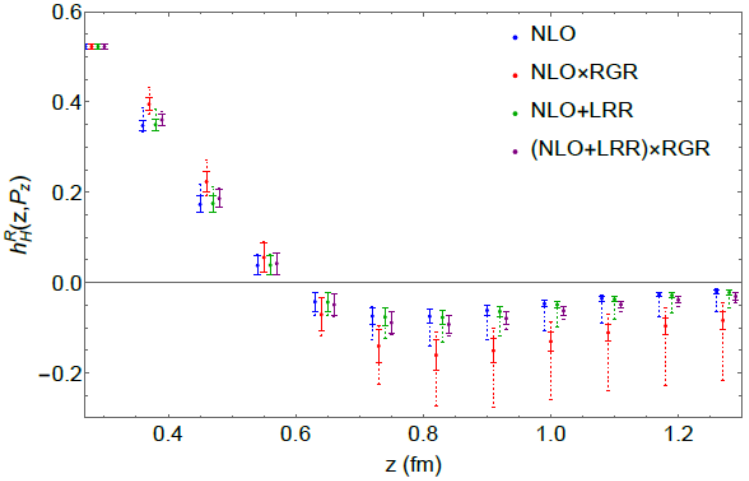
§ Only the NLO matching kernel is available

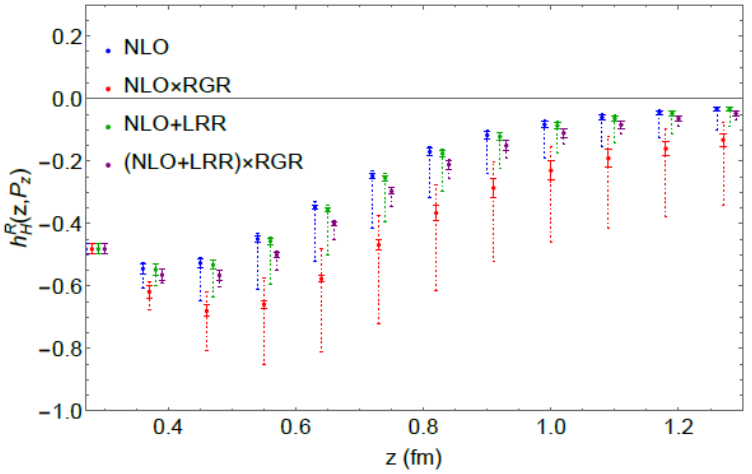
$$
\begin{aligned}
&\mathcal{K}(x, y, \mu, \xi, P_z) \\
&= \delta(x-y) \\
&+ \frac{\alpha_s C_F}{4\pi}\Bigg[\left(\frac{|\xi+x|}{2\xi(\xi+y)} + \frac{|\xi+x|}{(\xi+y)(y-x)}\right)\left(\ln\left(\frac{4y^2(\xi+x)^2P_z^2}{\mu^2}\right)-1\right) \\
&+ \left(\frac{|\xi-x|}{2\xi(\xi-y)} + \frac{|\xi-x|}{(\xi-y)(x-y)}\right)\left(\ln\left(\frac{4y^2(\xi-x)^2P_z^2}{\mu^2}\right)-1\right) \\
&+ \left(\left(\frac{\xi+x}{\xi+y} + \frac{\xi-x}{\xi-y}\right)\frac{1}{|x-y|} - \frac{|x-y|}{\xi^2-y^2}\right)\left(\ln\left(\frac{4y^2(x-y)^2P_z^2}{\mu^2}\right)-1\right)\Bigg]
\end{aligned}
$$

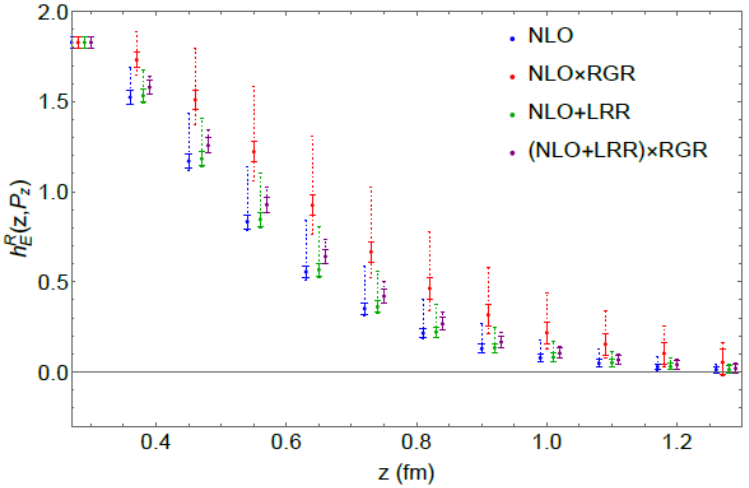
F. Yoa et al, JHEP 11(2023) 021

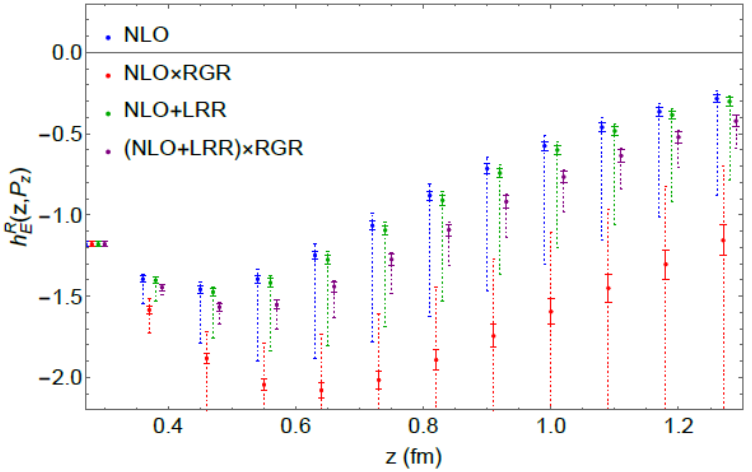
$\xi \neq 0$ GPDs

§ NLO $\xi = 0.1$, $Q^2 = 0.23\ \mathrm{GeV}^2$









J. Holligan, HL (MSULat), in preparation

$\xi \neq 0$ GPDs

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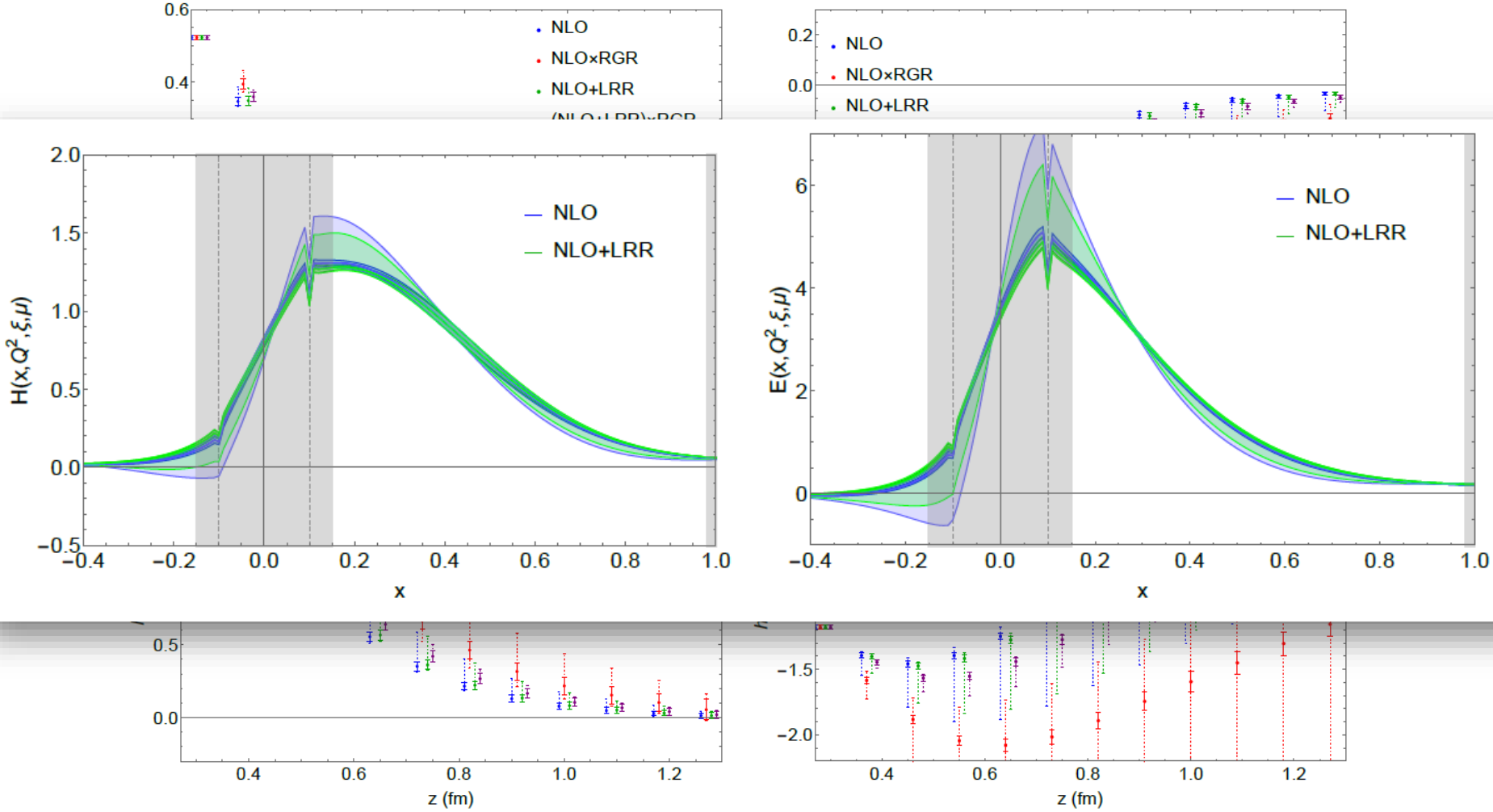
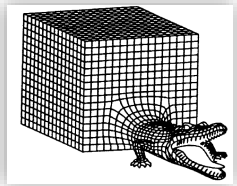


J. Holligan, HL (MSULat),2312.10829 [hep-lat]

Generalized Parton Distributions

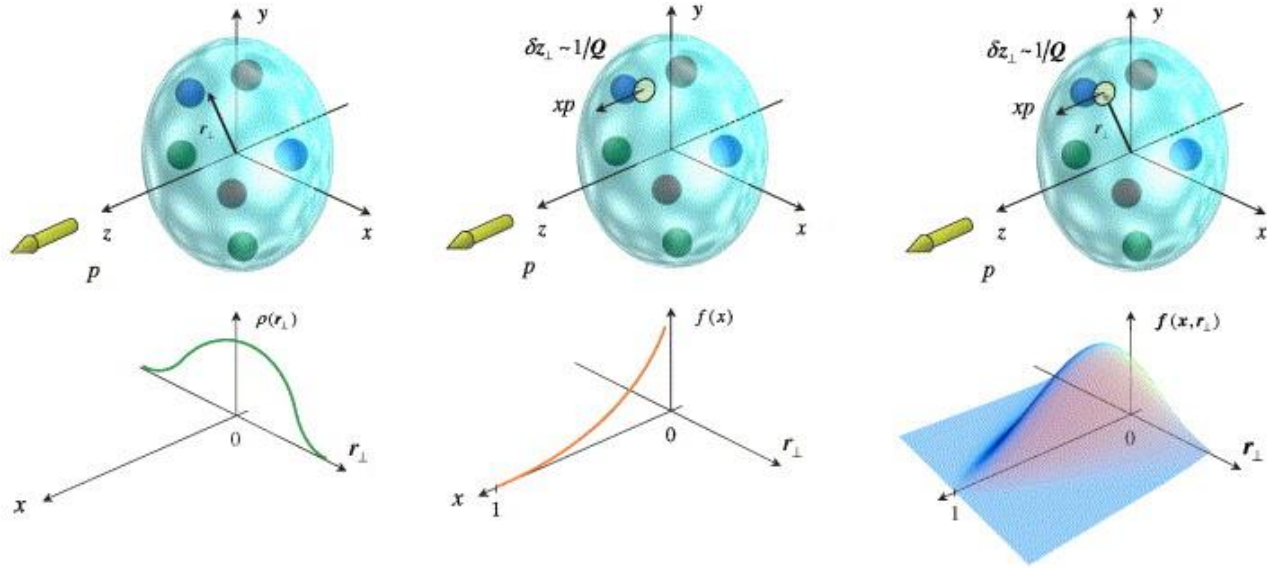
Single-ensemble result



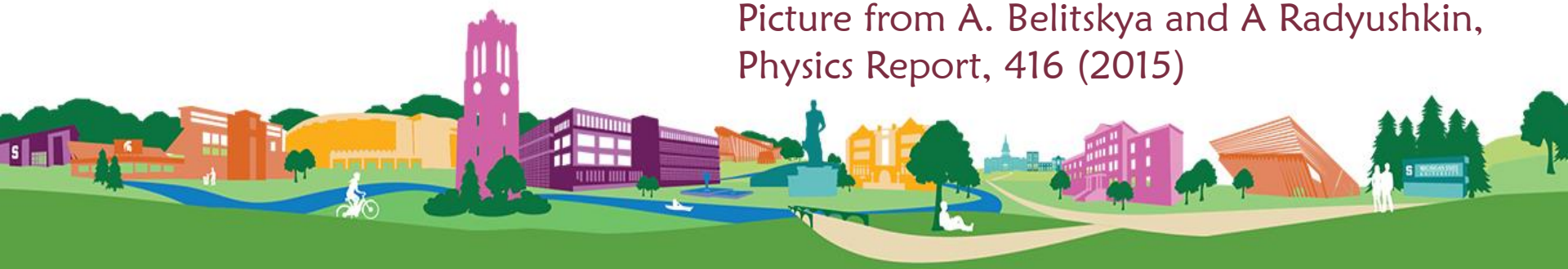
finite-volume,
discretization,
heavy quark mass,
...

Biased selected/highlighted results

Bjorken-x Dependent GPDs

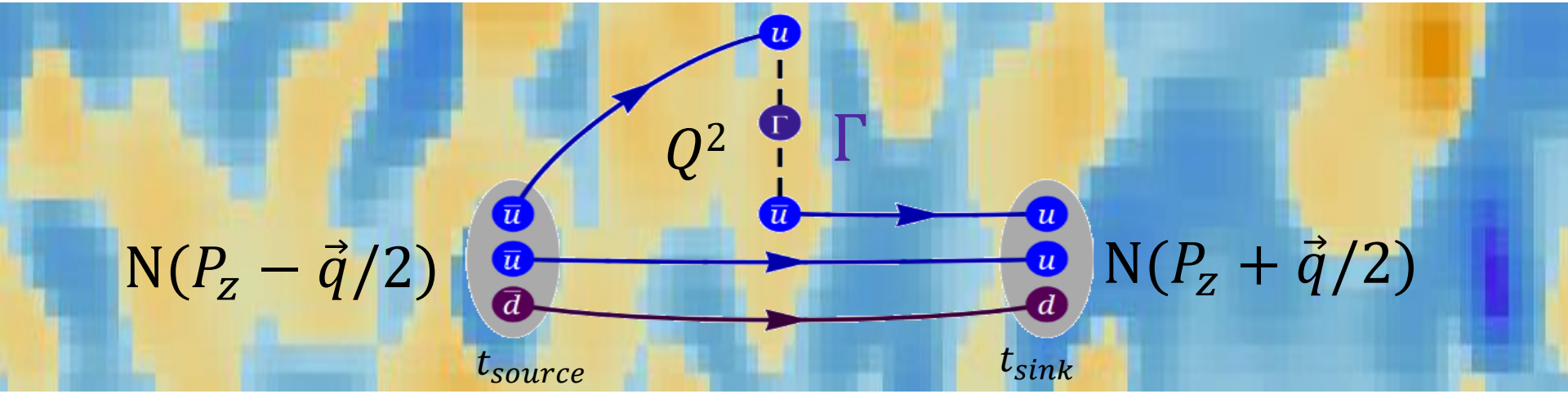


Picture from A. Belitskya and A Radyushkin, Physics Report, 416 (2015)



Generalized Parton Distributions

§ On the lattice, one needs to calculate the following (nucleon example)

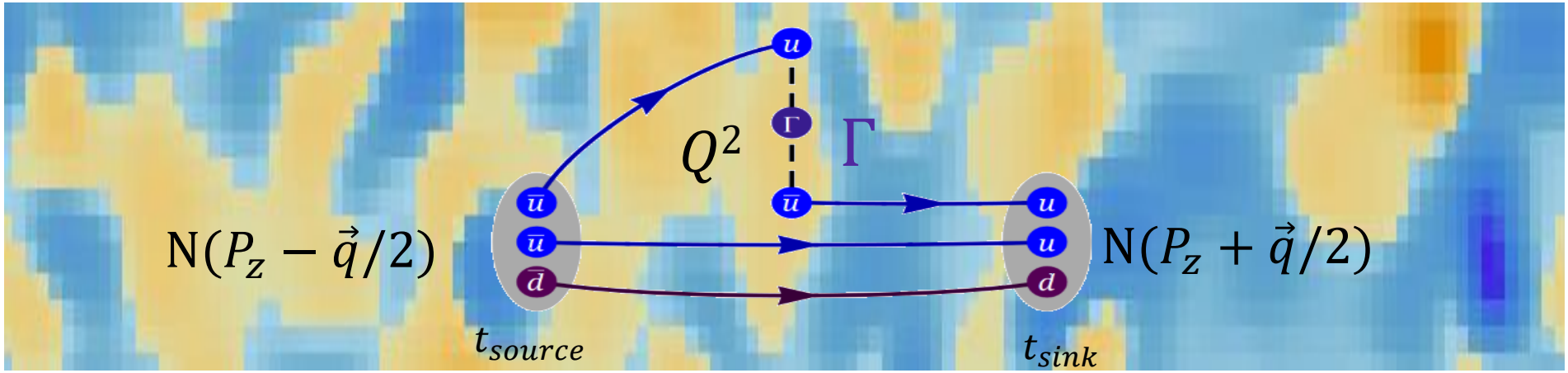


$$\tilde{F}\left(x,\tilde{\xi},t,\bar{P}_z\right) = \frac{\bar{P}_z}{\bar{P}_0}\int\frac{dz}{4\pi}e^{ixz\bar{P}z}\left\langle P'\middle|\tilde{O}_{\gamma_0}(z)\middle|P\right\rangle = \frac{\bar{u}(P')}{2\bar{P}^0}\left(H\left(x,\tilde{\xi},t,\bar{P}_z\right)\gamma^0 + E\left(x,\tilde{\xi},t,\bar{P}_z\right)\frac{i\sigma^{0\mu}\Delta_\mu}{2M}\right)u(P'')$$

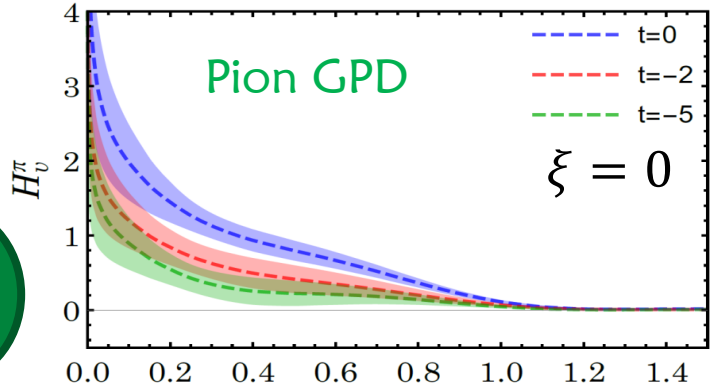
$$p^\mu = \frac{p''^\mu + p'^\mu}{2}, \qquad \Delta^\mu = p''^\mu - p'^\mu, \qquad t = \Delta^2, \qquad \xi = \frac{p''^+ - p'^+}{p''^+ + p'^+}$$

Generalized Parton Distributions

§ On the lattice, one needs to calculate the following

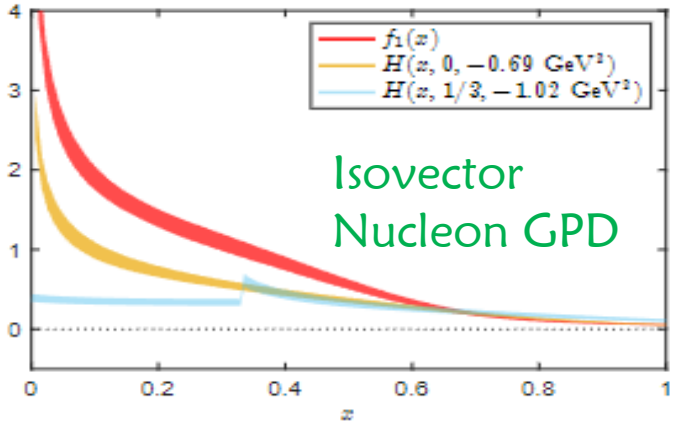


§ Heavy pion-mass results



Wanted PDFs, GPDs, etc…

J. Chen, HL, J. Zhang, 1904.12376



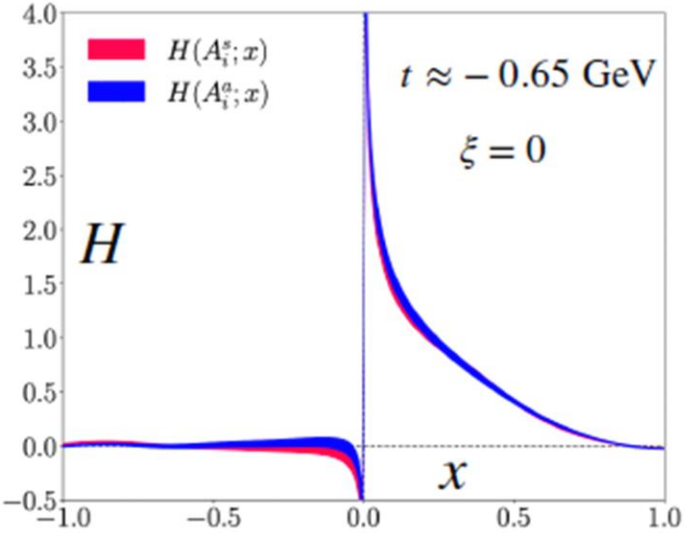
C. Alexandrou et al, 2008.10573

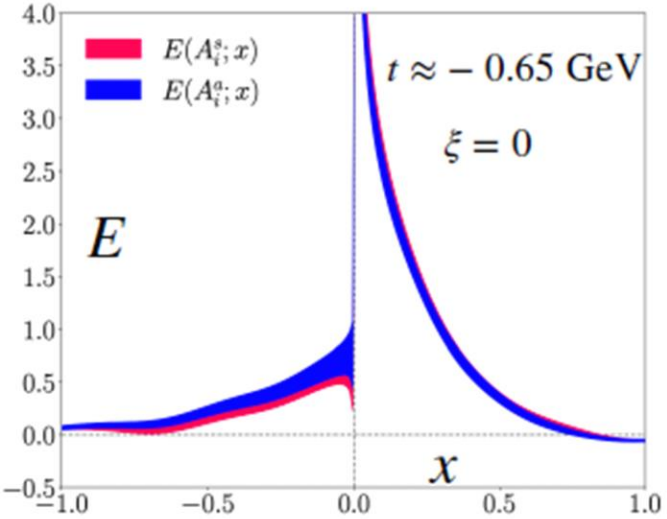
Ge

§ Or

See T

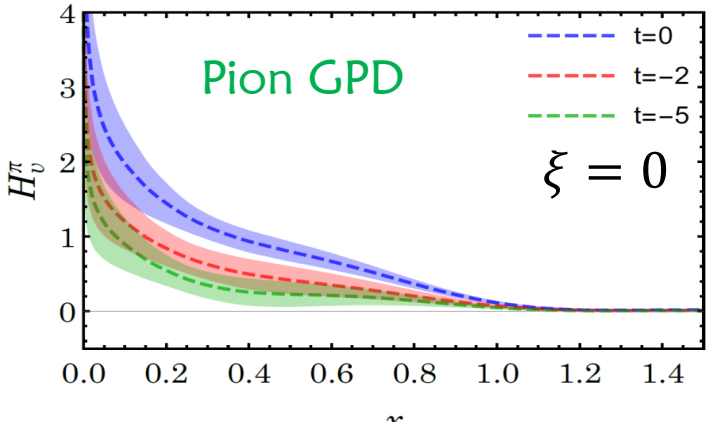
Shohini Bhattacharya *et al.*, arXiv:2209.05373 [hep-lat]



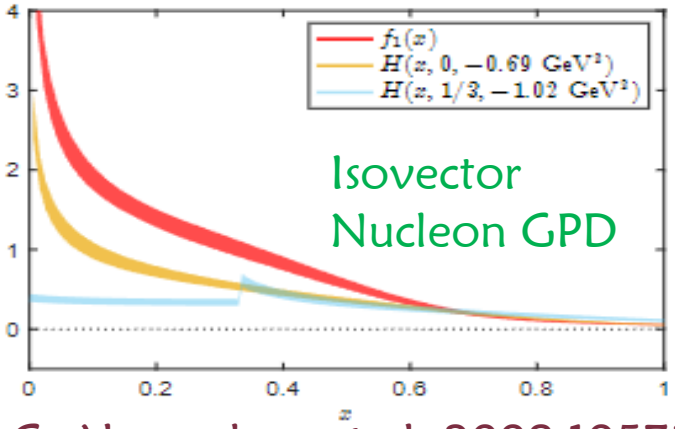


260-MeV results for asymmetric setup for GPD calculations to save computing time

§ Heavy pion-mass results



J. Chen, HL, J. Zhang, 1904.12376



C. Alexandrou et al, 2008.10573

2008.10573

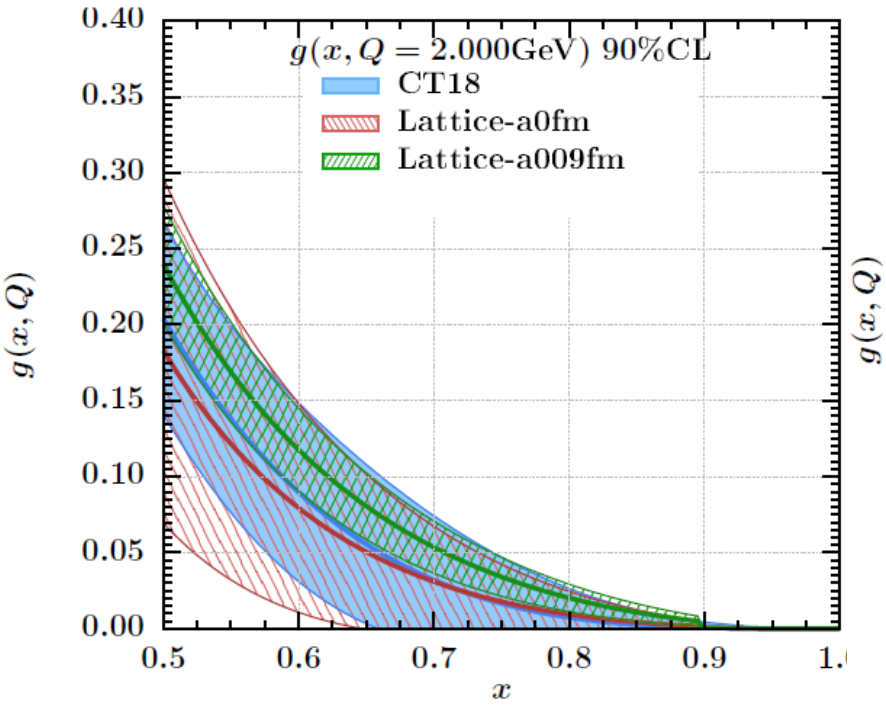
Lattice Gluon PDF Impact

§ Preliminary study with CTEQ-TEA analysis

- Take lattice inputs in the region where no strong experimental data constraints, x ∈ [0.4,0.7]
- Using e-pump for re-weighting

Plots by Alim Ablat (Xinjiang U.)

Before



After