Solving the strong CP problem

Alessandro Strumia, talk at QCS 2024/8/21

Introduction

The strong CP problem

$$\mathscr{L}_{\rm QCD} = \sum_q \bar{q}(i\not{D} - M_q)q - \frac{1}{4}{\rm Tr}\ G^2 + \theta_{\rm QCD}\frac{g_3^2}{32\pi^2}{\rm Tr}\ G\tilde{G}.$$

Data show:

- large CP violation in quark mixing, $\delta_{\rm CKM} = (\text{some phase in } M_q) \sim 1$;
- no CP violation in neutron dipole, $\bar{\theta} = \arg\det M_q + \theta_{\rm QCD} \lesssim 10^{-10}$.

Doubts, the total derivative in $\mathscr{L}$ has no classical effect, selects a state of H.

Solutions

Low energy:

1) Axion, $aG\tilde{G}$.

High energy:

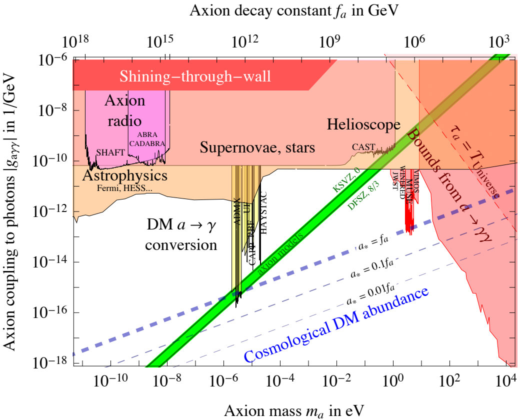
2) $G\tilde{G}$ is P-odd: P suitably broken by a real $\langle\text{scalar}\rangle$.
3) $G\tilde{G}$ is CP-odd: CP suitably broken by complex $\langle\text{scalars}\rangle$.

1) Cancel $\theta_{\rm QCD}$ via axion

Goldstone of **global** anomalous $U(1)_{\rm PQ}$ spontaneously broken at $f_a \gtrsim 10^9$ GeV:

$$\mathscr{L}^{\rm eff}_{\rm axion} = \frac{(\partial_\mu a)^2}{2} - \frac{a}{f_a}\left[\frac{\alpha_s}{8\pi} G^a_{\mu\nu}\tilde G^a_{\mu\nu} + c_\gamma \frac{\alpha}{8\pi} F_{\mu\nu}\tilde F_{\mu\nu}\right] + \cdots$$

- Adjusts $\bar\theta = 0$ acquiring $m_a \sim \Lambda^2_{\rm QCD}/f_a \approx 5.7\,\mu{\rm eV}\ (10^{12}\,{\rm GeV}/f_a)$.
- Can be DM: $\Omega_a^{\rm misalignement} \approx 0.15\left(f_a/10^{12}\,{\rm GeV}\right)^{7/6}(a_*/f_a)^2$ + defects.
- Main searches rely on DM detection via the coupling $g_{a\gamma\gamma} = c_\gamma \alpha/2\pi f_a$:



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- Main searches rely on DM detection via the coupling $g_{a\gamma\gamma} = c_\gamma \alpha/2\pi f_a$.
- Measuring m_a, c_γ tells charge/color of fermions in the anomaly loop:

$$\frac{c_{a\gamma\gamma}}{m_a} = \text{known} \times \left(\frac{E}{N} - 1.92\right), \qquad \frac{E}{N} = \frac{\sum_f q_f^{\rm PQ} q_f^2}{\sum_f q_f^{\rm PQ} T_f^2} = \frac{8}{3} \text{ in the SM.}$$

- **Quality problem**: gravity expected to break global symmetries, Planck-suppressed operators up to dimension $\gtrsim 9$ ruin the axion. Special models of accidental global symmetries e.g. SU(10) broken by S_{ij}.

2) Forbid $\theta_{\rm QCD}$ via P

Parity is badly broken in the SM. Big extensions needed.

$\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ with $L \leftrightarrow R$ and extra heavy singlets Q such that [Babu 90]

$$M_q = \begin{array}{c} \\ q_L \\ Q_L \end{array} \begin{array}{c} \begin{array}{cc} q_R & Q_R \end{array} \\ \begin{pmatrix} 0 & y_q v_L \\ y_q^\dagger v_R & M \end{pmatrix} \end{array}$$

has real det if v_L, v_R are real.

But $V(H_L, H_R)$ can be complex: restrict V adding supersymmetry.

Loops small enough for $v_R \gtrsim 10\,\mathrm{TeV}$:

- one loop $d_n \sim e g^2 m_q/(4\pi v_R)^2$;
- two loop $\bar\theta \sim (y_t/4\pi)^4 v_L/v_R$.

Related models: mirror $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)'_L \ldots$

3) Forbid $\theta_{\rm QCD}$ via T $\sim$ CP

Imposing CP needs milder SM extensions and is interesting: why $\mathscr{L}$ is complex?
CP invariance spontaneously broken by 'complex' scalar singlets z.

Special mass matrices involving heavy quarks Q [Nelson-Barr]:

$$M_q \sim \begin{matrix} q_L \\ Q_R^c \end{matrix} \overset{\begin{matrix} q_R & \qquad Q_R \end{matrix}}{\begin{pmatrix} y_q v & 0 \\ y\langle z\rangle + y'\langle z\rangle^* & M \end{pmatrix}} \qquad \text{has real det.}$$

Extended into model [Bento-Branco-Parada]:

$$y_q H q_L q_R + (yz + y'z^*) Q_R^c q_R + M Q_R Q_R^c - V(H,z)$$

- $H q_L Q_R$ forbidden by imposing a $\mathbb{Z}_2$ that flips Q_R, Q_R^c, z.
- $\mathbb{Z}_2$ allows $S H q_L Q_R / M_{\rm Pl}$, needs $\langle z\rangle \lesssim \bar\theta M_{\rm Pl} \sim 10^8$ GeV.
- Loop: $\bar\theta \sim \lambda_{Hz} y^2/(4\pi)^2$, needs small couplings or SUSY.

New idea from 2305.08908 and 2406.01689 with Feruglio, Titov, Parriciatu.

3) Understand $\theta_{\rm QCD}$ with $\mathrm{U}(1)_{\rm CP}$

Assume that CP is a spontaneously broken flavour symmetry, U(1) or modular:

$$m = (\text{real constants } c) \times (\text{CP-breaking operators } z_a)^{(\text{powers } k_a)}$$

with positive powers $k_a \geq 0$ and no $z_a^\dagger$.

Toy example with one z and $N_g = 2$ generations:

$$M_q = \begin{array}{c} \\ q_{L1} \\ q_{L2} \end{array} \!\!\begin{array}{c} \begin{array}{cc} q_{R1} & \quad q_{R2} \end{array} \\ \begin{pmatrix} c_{11} z^{k_{11}} & c_{12} z^{k_{12}} \\ c_{21} z^{k_{21}} & c_{22} z^{k_{22}} \end{pmatrix} \end{array} \qquad \text{where} \qquad k_{ij} = \frac{k_{q_{Li}} + k_{q_{Rj}} + k_{H_q}}{k_z}$$

after matching the U(1) charges k or modular weights k. Then

$$\det M_q = c_{11} c_{22}\, z^{k_{11}+k_{22}} - c_{12} c_{21}\, z^{k_{12}+k_{21}} = (c_{11}c_{22} - c_{12}c_{21}) z^k$$

can be real for any c_{ij}, z if the total charge of fields involved in the det is

$$\boxed{k = \sum_{i=1}^{N_g=2} k_{q_{Li}} + k_{q_{Ri}} + k_{H_q} = 0}$$

$\delta_{\rm CKM} = 0$: one 'scalar' z only does not break CP, it can be U(1)-rotated to real.

Get $\bar{\theta} = 0$ and $\delta_{\rm CKM} \neq 0$

CP-breaking arises if U(1) is broken by multiple scalars z_a with different phases.

The det identity $\det M_q(\lambda z_a) \propto \lambda^k$ still holds for multiple z_a and any N_g, even adding heavy quarks. Real det if the total charge is

$$k = \sum_i 2k_{Q_i} + k_{U_i} + k_{D_i} + k_{H_u} + k_{H_d} = 0$$

Now $\delta_{\rm CKM} \neq 0$ can be obtained. Even assuming the $N_g = 3$ generations and no heavy quarks. In such a case there is a unique Yukawa matrix:

$$Y = \begin{pmatrix} 0 & 0 & c_{13} \\ 0 & c_{22} & Y_{23} \\ c_{31} & Y_{32} & Y_{33} \end{pmatrix}, \qquad \det Y = c_{13}c_{22}c_{31}.$$

For example realised with $N = 2$ scalars z_+ and z_{++} with U(1) charge 1 and 2

$$Y = \begin{array}{c} \\ -1 \\ 0 \\ +1 \end{array} \begin{array}{c} \begin{array}{ccc} -1 & 0 & +1 \end{array} \\ \begin{pmatrix} 0 & 0 & c_{13} \\ 0 & c_{22} & c_{23}\, z_+ \\ c_{31} & c_{32} z_+ & c_{33}\, z_+^2 + c'_{33} z_{++} \end{pmatrix} \end{array} \qquad 1 + 1 = 2.$$

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For example realised with $N = 2$ scalars z_4 and z_6 with U(1) charge 4 and 6

$$Y = \begin{matrix} & \begin{matrix} -6 & 0 & +6 \end{matrix} \\ \begin{matrix} -6 \\ 0 \\ +6 \end{matrix} & \begin{pmatrix} 0 & 0 & c_{13} \\ 0 & c_{22} & c_{23}\, z_6 \\ c_{31} & c_{32} z_6 & c_{33}\, z_6^2 + c_{33} z_4^3 \end{pmatrix} \end{matrix}, \qquad 6 + 6 = 4 + 4 + 4.$$

More choices. More structures (including Nelson-Barr) adding heavy quarks.

QFT implementation

Must avoid $z_a^\dagger$: justified assuming **supersymmetry**, possibly broken at high scale.

A global supersymmetric theory is described by

- the holomorphic super-potential

$$W = Y^u_{ij}(z_a)\, U_i Q_j\, H_u + Y^d_{ij}(z_a)\, D_i Q_j\, H_d + \cdots$$

- the 'Kahler' kinetic term K can be general, as it does not contribute to $\bar\theta$.
- the gauge kinetic function f, real as we assume CP.
  For example the minimal form is $f = 1/g^2 - \theta/8\pi^2$ with $\theta = 0$.

Assume a **local** flavour symmetry to avoid Goldstones and troubles with gravity.
All **anomalies** must vanish, in particular the $\mathrm{U}(1)_{\mathrm{CP}} \cdot \mathrm{SU}(3)_c^2$ anomaly:

$$A = \sum_i 2k_{Q_i} + k_{U_i} + k_{D_i} = 0.$$

Its cancellation coincides with solving the QCD $\bar\theta = 0$ problem by imposing

$$k = \sum_i 2k_{Q_i} + k_{U_i} + k_{D_i} + k_{H_u} + k_{H_d} = 0$$

if $k_{H_u} + k_{H_d} = 0$ meaning that the Higgs $H_{u,d}$ do not break the flavour symmetry.

$\bar\theta = 0$ understood if CP is an anomaly-free flavour symmetry not broken by Higgs.

Models with extra heavy quarks

In models with heavy quarks, for example $Q_R \oplus Q_R^c$, the mass matrix becomes

$$\mathcal{M} = \begin{matrix} & \begin{matrix} q_R & Q_R \end{matrix} \\ \begin{matrix} q_L \\ Q_R^c \end{matrix} & \begin{pmatrix} yv & y'v \\ \mu & M \end{pmatrix} \end{matrix}.$$

Nelson-Barr assume $y' = 0$, real y, M, complex μ. Realised assuming U(1) charges

$$\mathcal{M} = \begin{matrix} & \begin{matrix} 0 & -1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} yv & 0 \\ cz & M \end{pmatrix} \end{matrix}.$$

More general models have complex M, y', y and an anomalous light field content.
In the full theory $\bar{\theta} = 0$ as real $\det m_{\text{light}} M_{\text{heavy}}$. In the low-energy EFT $\bar{\theta} = 0$ as

- complex $\det m_{\text{light}}$ cancels with
- anomalous gauge kinetic function $f_{\text{EFT}} = f_{\text{UV}} - \ln\det M_{\text{heavy}}/8\pi^2$.

It's the anomaly cancellation mechanism in string models with anomalous EFT.

Completing to a full theory needs:

1) a potential $V(z)$ minimised by z_a with relative phases.
2) mediators that give $k_{ij\cdots} \geq 0$ only.

Not nice with U(1).

$\mathrm{U}(1)_{\mathrm{CP}} \to$ modular $\mathrm{SL}(2,\mathbb{Z})$

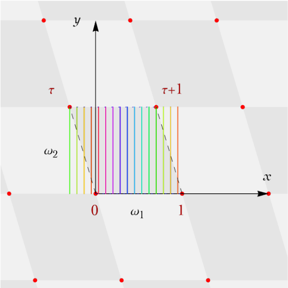
Better implementation with **modular invariance**. What's that? It's like a U(1) automatically broken in a predictive $k > 0$ way by 2 scalars with different phases.

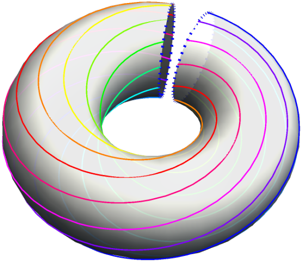
Modular invariance can be done as math independently from its string motivation.

Super-strings are real in $4 + 6$ dimensions. Chiral families of fermions can arise compactifying on spaces with a complex structure. So CP can be a geometric symmetry spontaneously broken by the compactification.

$N = 1$ supersymmetry needs a Ricci-flat compactification. Simplest: orbi-folded 6d flat tori. 2d torus obtained writing a 2d flat space as $z = x + iy$ and identifying

$$z = z + \omega_1 \qquad \text{and} \qquad z = z + \omega_2.$$





$\tau = \omega_1/\omega_2$ tells the geometry: $\mathrm{Im}\,\tau$ is the relative radius, $\mathrm{Re}\,\tau$ is the $\not{\mathrm{CP}}$ twisting.

Modular invariance

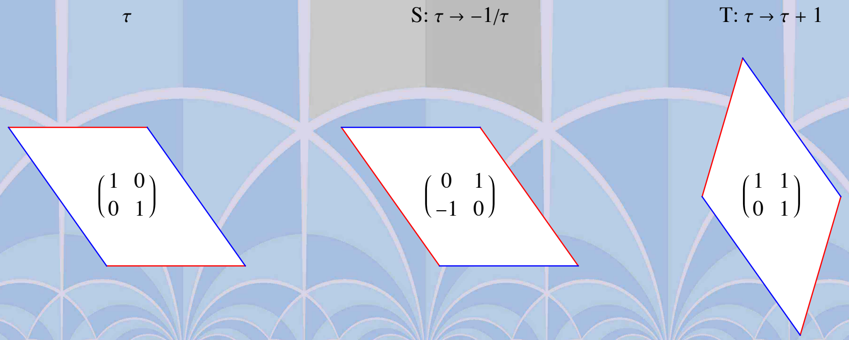
Modular invariance is a sub-group of discrete global reparametrizations, because

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

gives an equivalent lattice torus if a, b, c, d are **integers** with $ad - bc = 1$.

So the 4-dimensional EFT contains a modulus superfield τ rotating under $\mathrm{SL}(2, \mathbb{Z})$

$$\tau \to \frac{a\tau + b}{c\tau + d}.$$



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Unusual: appears integrating out infinite states, because of how strings experience the geometry (e.g. $R = 1/R$). Matter fields Φ transform as phase and scaling

$$\Phi \to (c\tau + d)^{-k_\Phi}\Phi \qquad \text{with 'weight' } k_\Phi.$$

The minimal global SUSY action with $h \ll \bar{M}_{\rm Pl}$

$$\begin{aligned} K &= -h^2 \ln(-i\tau + i\tau^\dagger) + \sum_\Phi \frac{\Phi^\dagger e^{2V}\Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}}, \\ W &= Y^u_{ij}(\tau) U_i Q_j\, H_u + Y^d_{ij}(\tau) D_i Q_j\, H_d \end{aligned}$$

is modular invariant if Yukawa couplings transform with definite weights

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}} Y^q_{ij}(\tau) \qquad k^q_{ij} = k_{q_{Ri}} + k_{q_{Lj}} + k_{H_q}.$$

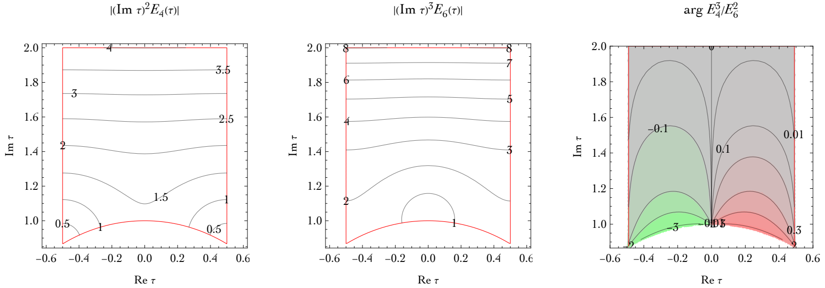
Modular invariance and CP

The only modular functions of τ with weight k and no singularity (‘*forms*’) are the Eisenstein series E_k, that transform nicely thanks to lattice summation

$$E_k(\tau) \equiv \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k} \qquad \text{finite and non-vanishing for even } k \geq 4.$$

Weight k	0	$1,2,3$	4	6	8	10	12	$\cdots$
Forms	1	$-$	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	$E_{12} \sim E_4^3 + E_6^2$	$\cdots$

- ✓ E_4 and E_6 are like two scalars with charge 4 and 6.
- ✓ They have different phases:



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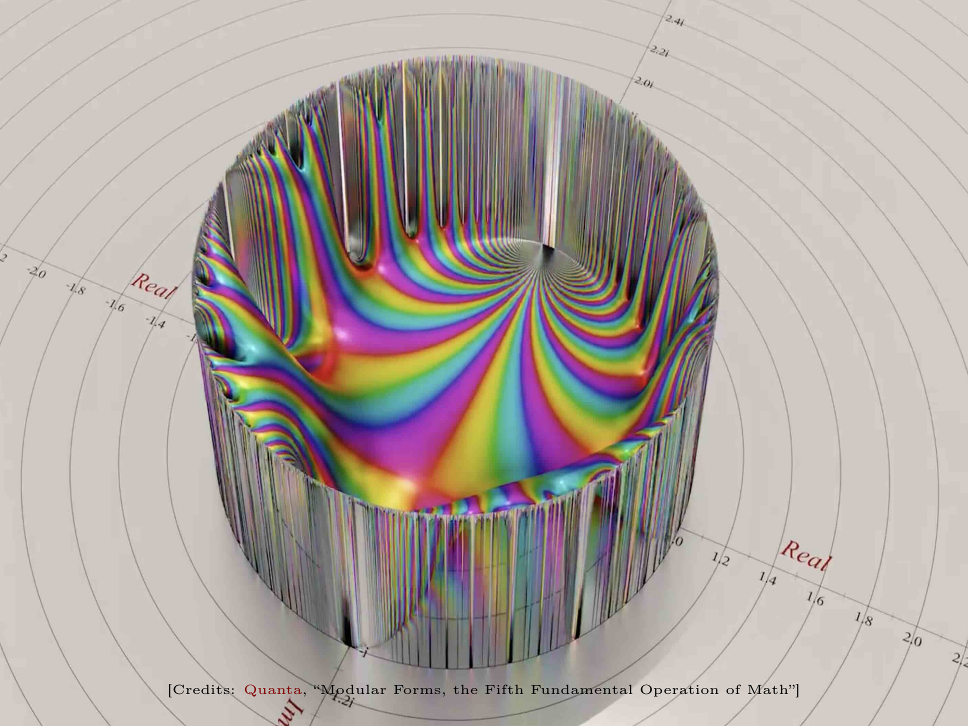
- ✓ E_4 and E_6 are like two scalars with charge 4 and 6.
- ✓ They have different phases.
- ✓ The modulus τ breaks CP, as $\tau \overset{\text{CP}}{\to} -\tau^\dagger$, $\Phi \overset{\text{CP}}{\to} \Phi^\dagger$.
- ✓ Forms forbid negative weight $k < 0$.
- ✓ Nicer than triangles.



[Credits: Quanta, “Modular Forms, the Fifth Fundamental Operation of Math”]

Recap: $\bar{\theta} = 0$ from modular invariance

Assume:

- CP broken by modulus $\mathrm{Re}\,\tau$ only.
- Supersymmetry, broken such that the gluino mass M_3 is real.
  E.g. gauge mediation. No weak-scale SUSY needed.
- Higgses don't break modular invariance, $k_{H_u} + k_{H_d} = 0$.

Then:

- $Y^q_{ij}(\tau) = c^q_{ij}\, F_{k^q_{ij}}(\tau)$ where c is real and F_k is a modular form with weight k.
- No anomalies, no QCD modular anomaly. E.g. with SM quarks only:

$$A = \sum_{i=1}^{3} (2k_{Q_i} + k_{U_i} + k_{D_i}) = 0.$$

- $\det M_q$ is a modular form with weight $A = 0$, so it's a real constant, so

$$\arg\det M_u M_d = 0 \qquad \theta_{\rm QCD} = 0.$$

- $\delta_{\rm CKM} \propto \mathrm{Im}\,\det[Y_u^\dagger Y_u, Y_d^\dagger Y_d] \sim 1$ has no special modular properties.
- Quark kinetic matrices Z_q can be made canonical via a quark linear transformation that affects q masses and mixing but not $\bar{\theta}$. Minimal Kähler:

$$Y^q_{ij}|_{\rm can} = c^q_{ij}(2\mathrm{Im}\,\tau)^{k^q_{ij}/2} F_{k^q_{ij}}(\tau).$$

The Minimal MSSM Model

Simplest model: modular weights $k_Q = k_U = k_D = \{-6, 0, +6\}$ so $\det Y_q$ is real:

$$Y_q|_{\rm can} = \begin{array}{c} \\ q_{R1} \\ q_{R2} \\ q_{R3} \end{array} \begin{array}{c} \begin{array}{ccc} q_{L1} & q_{L2} & q_{L3} \end{array} \\ \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23}(2{\rm Im}\,\tau)^3 E_6(\tau) \\ c^q_{31} & c^q_{32}(2{\rm Im}\,\tau)^3 E_6(\tau) & (2{\rm Im}\,\tau)^6 \left[c^q_{33} E_4^3(\tau) + c'^q_{33} E_6^2(\tau)\right] \end{pmatrix} \end{array}.$$

Onion-like form: a numerical or approximate diagonalisation

$$y_3 \simeq y_{33}, \quad y_2 \simeq y_{22}, \quad y_1 \simeq -\frac{y_{13}y_{31}}{y_{33}}, \qquad \theta_{23} \simeq \frac{y_{32}}{y_{33}}, \quad \theta_{13} \simeq \frac{y_{31}}{y_{33}}, \quad \theta_{12} \simeq \frac{y_{31}y_{23}}{y_{22}y_{33}}$$

shows that all quark masses and mixings can be reproduced with comparable c

$$c^u_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51, 1.40 \end{pmatrix} \qquad c^d_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734, 1.76 \end{pmatrix}$$

for $\tan\beta = 10$ and $\tau = 1/8 + i$. No predictions. The quark hierarchies are reproduced somehow like in $\mathrm{U}(1)_{\rm FN}$ Froggatt-Nielsen, thanks to the modular '6' e.g. $(2{\rm Im}\,\tau)^6 = 64$ for $\tau \sim i$.

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for $\tan\beta = 10$ and $\tau = 1/8 + i$. Leptons too assuming $k_L = k_E = k_Q$:

$$c^e_{ij} = 10^{-3} \begin{pmatrix} 0 & 0 & 1.29 \\ 0 & 5.95 & 0.35 \\ -2.56 & 1.47 & 1.01, 1.32 \end{pmatrix} \qquad c^\nu_{ij} = \frac{1}{10^{16}\,{\rm GeV}} \begin{pmatrix} 0 & 0 & 3.4 \\ 0 & 7.1 & 1.2 \\ 3.4 & 1.2 & 0.19, 0.95 \end{pmatrix}.$$

Modular sub-groups $\Gamma(N)$

Compactifications + orbifolds/branes give modular sub-groups at higher level N

$$\Gamma(N) \equiv \mathrm{SL}(2,\mathbb{Z}) \text{ subgroup with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N.$$

allowing models with SM quarks and lower weights as 'motivated' by strings.

$\Gamma(2)$ has two modular forms with weight $k = 2$.

$$Z_1^{(2)} = \frac{2i}{\pi}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - 8\frac{\eta'(2\tau)}{\eta(2\tau)}\right], \quad Z_2^{(2)} = \frac{2\sqrt{3}i}{\pi}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right]$$

Models with weights $k_{Q_i} = k_{U_i} = k_{D_i} = \{-2, 0, 2\}$ can fit data.

$$y_q = \begin{pmatrix} 0 & 0 & c_{13}^q \\ 0 & c_{22}^q & \mathrm{Im}\,\tau\left[c_{23}^q Z_1^{(2)} + c_{23}'^q Z_2^{(2)}\right] \\ c_{31}^q & \mathrm{Im}\,\tau\left[c_{32}^q Z_1^{(2)} + c_{32}'^q Z_2^{(2)}\right] & (\mathrm{Im}\,\tau)^2\left[c_{33}^q Z_1^{(4)} + c_{33}'^q Z_2^{(4)} + c_{33}''^q Z_3^{(4)}\right] \end{pmatrix}$$

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allowing models with SM quarks and lower weights as 'motivated' by strings.

$\Gamma(2)$ has two modular forms with weight $k = 2$.
Models with weights $k_{Q_i} = k_{U_i} = k_{D_i} = \{-2, 0, 2\}$ can fit data.

$\Gamma(3)$ has two modular forms with weight $k = 1$

$$Z_1^{(1)} = \sqrt{2}\ \frac{\eta^3(3\tau)}{\eta(\tau)}, \qquad Z_2^{(1)} = \frac{\eta^3(3\tau)}{\eta(\tau)} + \frac{\eta^3(\tau/3)}{3\ \eta(\tau)}.$$

Models with weights $k_{Q_i} = k_{U_i} = k_{D_i} = \{-1, 0, 1\}$ can fit data.

$$y_q = \begin{pmatrix} 0 & 0 & c_{13}^q \\ 0 & c_{22}^q & \sqrt{\mathrm{Im}\,\tau}\left[c_{23}^q Z_1^{(1)} + c_{23}'^q Z_2^{(1)}\right] \\ c_{31}^q & \sqrt{\mathrm{Im}\,\tau}\left[c_{32}^q Z_1^{(1)} + c_{32}'^q Z_2^{(1)}\right] & \mathrm{Im}\,\tau\left[c_{33}^q Z_1^{(2)} + c_{33}'^q Z_2^{(2)} + c_{33}''^q Z_3^{(2)}\right] \end{pmatrix}$$

Non-abelian modular representations

Generation number can be embedded in $\mathrm{SL}(2,\mathbb{Z})$ as multiplets of the finite group

$$\Gamma_N \equiv \frac{\mathrm{SL}(2,\mathbb{Z})}{\Gamma(N)} \qquad \text{with} \qquad \Gamma_2 = S_3, \qquad \Gamma_3 = T' \sim A_4.$$

Used to explain large neutrino mixings. Heavy quarks needed to get small mixings.

$N = 2$ allows models with doublets and low weights ± 2:

	SM quarks			Extra vector-like quarks			
	Q	D	U	D'	D'^c	U'	U'^c
Flavour Γ_2	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_1$	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_1$	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_0$
Weights k	-2	-2	-2	$+2$	$+2$	$+2$	$+2$

$N = 3$ could allow models with triplets and low weights ± 1

	SM quarks			Extra vector-like quarks			
	Q	D	U	D'	D'^c	U'	U'^c
Flavour Γ_3	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
Weights k	-1	± 1	± 1	$+1$	∓ 1	$+1$	∓ 1

but generic non-minimal Kahler are needed to fit data.

Supergravity and superstrings

Strings etc motivate a Planckian τ decay constant $h = n\bar{M}_{\rm Pl}$ with integer n.

If $h \sim \bar{M}_{\rm Pl}$ supergravity predicts new effects:

- W acquires modular weight $k_W = h^2/\bar{M}_{\rm Pl}^2 > 0$;
- The gluino phase rotates, so the modular anomaly becomes

$$A = A_{\rm quark} + A_{\rm gluino} = \sum_{i=1}^{3} (2k_{Qi} + k_{U_i} + k_{D_i} - 2k_W) + 3k_W .$$

- $A = 0$ again implies $\bar{\theta} \propto \arg M_3^3 \det M_q = 0$. But...
- Extra states needed to avoid massless quarks e.g. 8 of SU(3) with $k = -k_W$.

Could something similar happen in strings? Modular invariance is non-anomalous, but the QFT field content is. Strong CP problem solved if $A_{\rm quark} = 0$?

Conclusions

New solution to the QCD $\bar{\theta} \ll \delta_{\rm CKM} \sim 1$ problem

Assume: CP is part of a local flavour symmetry, spontaneously broken by multiple scalars z_a in a theory where Y_q are proportional to positive powers of z_a but no $z_a^\dagger$ (so, SUSY). Then $\det Y_q \propto z^k$ is real selecting charges such that $k = 0$, as demanded by anomaly cancellation in simpler models.

- Without heavy quarks: unique Y_q structure.
- With heavy quarks: justifies and extends Nelson-Barr models.
- Can be realized with U(1), up to complications.

Modular realization

Modular invariance $\mathrm{SL}(2, \mathbb{Z})$ as flavour symmetry avoids complications.

- $N = 1$ is like two scalars E_4, E_6, assume $k_{Q,U,D,L,E} = \{-6, 0, 6\}$, q and ℓ masses and mixings reproduced up to order one coefficients.
- $N = 2$ allows $k_{Q,U,D,L,E} = \{-2, 0, 2\}$. Or as $2 \oplus 1$, adding heavy Q.
- $N = 3$ allows $k_{Q,U,D,L,E} = \{-1, 0, 1\}$. Or as 3, adding heavy Q?

All can be heavy... how can this be ~~tested~~ confirmed?